A quasi steady state method for solving transient Darcy flow in complex 3D fractured networks accounting for matrix to fracture flow

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Modeling natural Discrete Fracture Networks (DFN) receives more and more attention in applied geosciences, from oil and gas industry, geothermal recovery. The fractures may be either natural, or artificial in case of well stimulation. Accounting for the flow inside the fracture network, and accounting for the transfers between the matrix and the fractures, with the same level of accuracy is an important issue for calibrating the wells architecture and for setting up optimal resources recovery strategies. Recently, we proposed an original method allowing to model transient pressure diffusion in the fracture network only. The matrix was assumed to be impervious. A systematic approximation scheme was built, allowing to model the initial DFN by a set of N unknowns located at the intersection between fractures. The higher N, the higher the accuracy of the model. The lowest order approximation $N = 1$ appears under the form of solving a transient problem in a resistor/capacitor network, a so-called pipe network. Its topology is the same as the network of geometrical intersections between fractures.

In this paper, we generalize this approach in order to account for fluxes from matrix to fractures. We show that in the case of well separated time scales between matrix and fractures, the preceding model need only to be slightly modified in order to incorporate these fluxes. The additional knowledge of the so called matrix to fracture transfer function allows to modify the mass matrix that becomes a time convolution operator. This is reminiscent of existing space averaged transient dual porosity models.

Keywords: Flow in Fractured media, Discrete fracture network, Low permeability matrix, Quasi steady state, Dual porosity, Transfer function, Laplace transform

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1. INTRODUCTION

Modeling of fluid flows including heat or chemical transfers into naturally fractured rocks using explicit descriptions of fractured media (DFN) is becoming increasingly popular among geoscientists. This growing interest is due to a wide range of applications in various industries, to a better characterization of fracture networks, and evidently to the increasing computing power. In the common practice, these detailed descriptions are used to build and to calibrate a so called "double porosity model" that is designed to manage field applications. This class of double porosity models corresponds to the large scale homogenized version of the Darcy equations in the fractured medium, coupled to a linear transfer model with the matrix that is acting as a reservoir. These models, which were proposed in the early 60's by Barenblatt et *al* [2] remain still the base of most industrial fluid flow simulators [3–10]. Homogenization techniques [11–14], or Volume averaging techniques [15–17] allow a formal derivation of the double porosity equations, starting from the detailed DFN, at least in the Darcy hypothesis, and in the case of a well connected fracture network. Numerical solution of the associated closure problems permits to evaluate the parameters of the dual porosity model as a function of the geometry of the DFN. Useful connections with random walk theory providing efficient computational tools were made by several authors [18–22]. In the case of badly connected networks, modelling approaches involving percolation theory background are more appropriate [27–29]. But a complete workflow remains to be developed, especially if strong couplings with the matrix are involved, and in situations in which non linear transfers, like multiphase flow, are to be accounted for [8–10]. Direct simulations of flows in 2D or 3D DFN were already performed by several groups ([5, 30–40]). The underlying numerical methods involve finite volume, finite elements techniques. Some groups intend to couple the high resolution DFN model with a flow in the matrix [41].

Here, we focus on the simplest problem: fractures (here $2D$ objects like closed polygons or ellipses of small thickness ε of high typical conductivity $C_f = k_f \times \varepsilon$ are embedded in a $3D$ matrix having a low permeability $k_m \ll k_f$ that will be supposed as being uniform for sake of simplicity. The fractures are supposed to be well connected (FIG. 1). Our goal is to solve linear diffusion equation within such a medium. Considering large cases involving thousands of intersecting fractures, the main difficulty of direct numerical solution techniques is to get an automated meshing fulfilling the quality requirements of the associated discretization scheme [5, 40]. Even if this practical question is solved, the overall number of degrees of freedom remain equal to the number of fractures, say N, times the typical number of cells N_{typ} used to mesh every fracture (typically $N_{typ} \backsimeq$ several hundred). The number of associated matrix elements should scale as $N \times N_{typ}^{3/2}$. Getting a numerical solution of a 10 millions fractures problem will imply thus solving close to several billion equations. This justifies developing approximation methods in which the number of degrees of

freedom remains close to the total number of connected intersections $N_\cap$. This was done in [1] assuming an impervious matrix with $k_m = 0$. In that paper, it was shown that a systematic approximation scheme can be built that involves N_p unknowns that describe the trace of pressure at the intersections between fractures with increasing accuracy. At first order, $N_p = 1$, which is equivalent to suppose that the pressure profiles at the intersection are uniform, the so called pipe network model is recovered. The resulting equations posses the structure of a resistor/capacitor network involving the total number of connected intersections $N_\cap$. The main physical assumption is that the considered time scales are much more greater that a typical diffusion time over one single fracture. The resulting set of equations reads:

$$\forall i = 1, N_\cap, m = 1, \infty, \quad \sum_{j \in J(i)} \sum_{n=1}^{\infty} K_{ij}^{mn} \times \frac{dP_j^n(t)}{dt} = \sum_{j \in J(i)} \sum_{n=1}^{\infty} T_{ij}^{mn} \times P_j^n. \tag{1}$$

Here, the integers i and j label the intersections, $J(i)$ is the ensemble of the labels of intersections directly connected to the i-th one. So as any intersection involves two different fractures, the set $J(i)$ involves the whole set of intersections belonging to both fractures intersecting at i. Labels m and n indicate the degrees of freedom corresponding to the order of approximation of the pressure profiles at intersections. $N_{\cap i}$ corresponds to all the other intersections than i belonging to the pair of fractures the intersection of which is the i th intersection. The mass matrix $\mathbf{K}$ and the transmissivity matrix $\mathbf{T}$ which are both symmetric positive can be related to L^2 scalar products of elementary mapping problems to be solved on each fracture domains independently of each other. In practice, truncating the above equations with $n = 1$ and $m = 1$ corresponds to building a so called pipe network model characterized by uniform pressure at the intersections. The model was successfully implemented for 3D DFN [42].

In the present paper, we generalize the method in order to account for the matrix to fracture flow. We show that this can be achieved by changing the form of the mass term $\sum_{j \in J(i)} \sum_{n=1}^{\infty} K_{ij}^{mn} \times \frac{dP_j^n(t)}{dt}$ under the form of a time convolution involving the average transfer function $f(t)$ between the matrix and the fracture:

$$\forall i = 1, N_\cap, m = 1, \infty, \quad \sum_{j \in J(i)} \sum_{n=1}^{\infty} K_{ij}^{mn} \times (V_f \delta(t) + V_m f(t)) * \frac{dP_j^n(t)}{dt} = \sum_{j \in J(i)} \sum_{n=1}^{\infty} V_f T_{ij}^{mn} \times P_j^n. \tag{2}$$

The $*$ symbol corresponds to a time convolution. V_f and V_m represent the volumic fractions of the fractures and of the matrix, $V_f + V_m = 1$. The transfer function $f(t)$ $[t^{-1}]$ appears as a time variable porosity. This function can be estimated by solving boundary value problem on the matrix blocks, or by alternative continuous time random walk

methods [20–22] that can avoid any explicit mesh of the matrix. Analytical forms $f(t)$ accounting from both short times and long times asymptotic behavior of $f(t)$ can be proposed. Finally, the resulting equations 2 may be solved in Laplace domain.

The paper is organized as follows: first, the Section 2 introduces the pressure diffusion model in the matrix and in the fractures considered having a small thickness ε. In next Section 3 3.1, we summarize with more details the techniques and results obtained in [1], that are also presented in more details in the appendices A,B,C. In section 4, we present our assumptions to account for the matrix to fracture flow. In order to proceed, we must come back about the mathematical treatment of the finite (not null) thickness of the fractures 4 4.1. We are thus in good position for introducing the so called exchange function $f(t)$ 4 4.2 at the level of a single fracture. The explicit coupling of the DFN to the matrix, as well as the adaptation of the projection formalism is presented in 4 4.3. Some general properties of the exchange function are presented in section 5. In section 6, we come back about our main assumption of quasi steady state flow inside the fractures, in order to check its consistency in the light of our findings. The application of the formalism to the resistor capacitor network that will be employed in practice is given in 7, before giving some comments and discussions.

2. MODEL PROBLEM, GEOMETRY AND NOTATIONS

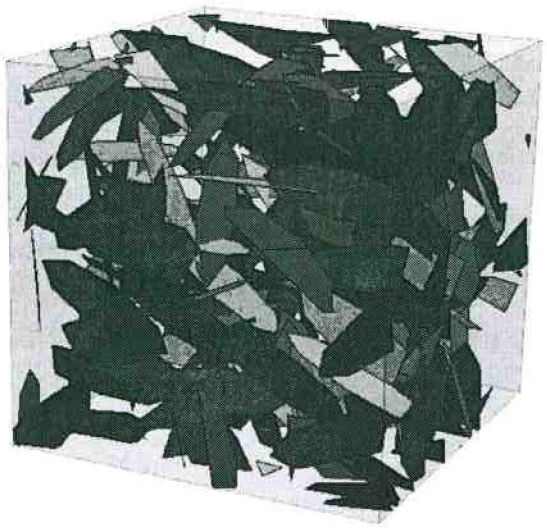

FIG. 1: $3D$ network of $2D$ polygonal fractures in a cubic box Ω.

We consider flow in a $3D$ cubic domain Ω containing N distinct permeable fractures (FIG. 1) embedded in a low permeability matrix. No flux boundary conditions will be considered first at the frontier of Ω. The individual I^{th}

fracture is considered as being a closed $2D$ object (e.g. polygonal or elliptic), the position of which can be given by the coordinates of its center, the orientation of its normal, and all the necessary parameters chosen by the geologist to characterize its detailed shape.

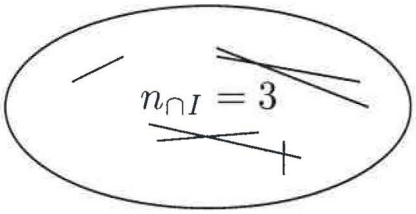

FIG. 2: An example of $2D$ elliptic fracture with a number of "cluster of intersections" $n_{\cap I} = 3$.

We consider a well connected network of N fractures, so each fracture is connected to all the others via at least one path. So each fracture intersects at least one other fracture. Let $n_{\cap I}$ denotes the number of disconnected cluster of intersections of the I^{th} fracture with the others. By the name "cluster of intersections", we mean that intersections between different fractures can intersect between each other (see FIG. 2), providing clusters that are not necessarily restricted as segments.

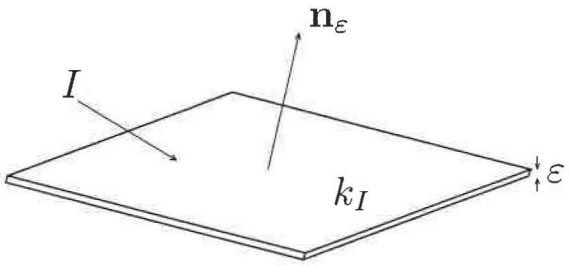

FIG. 3: $3D$ fracture of thickness ε and of permeability k_I.

On the hydrodynamic point of view, we consider that all the fractures share a small common thickness denoted by ε (FIG. 3). The permeability of the I^{th} fracture is denoted by k_I. This permeability may vary on the fracture's plane, but in order to simplify notations, this dependence will not be explained, although it will be accounted for in the ∇ operator manipulations.

Our main goal is to study the solution of the following diffusion problem when ε is small of:

$$\varphi \mu c_t \frac{\partial p_\varepsilon(\mathbf{r},t)}{\partial t} = \nabla \cdot (k(\mathbf{r}) \nabla p_\varepsilon(\mathbf{r},t)) + g(\mathbf{r}). \tag{3}$$

$$k(\mathbf{r}) = k_I if \mathbf{r} \in \Omega_I for some I = 1, ...N \tag{4}$$

$$k(\mathbf{r}) = k_m else \tag{5}$$

The source term $g(\mathbf{r})$ is arbitrary for the moment: it can be a bulk source term. Here, we do not have to add boundary conditions at the fractures boundaries in contact with the matrix, but we may recall a normal flux continuity condition that will be ensured :

$$(k_I \nabla p_\varepsilon(\mathbf{r},t)) \cdot \mathbf{n}_\varepsilon = (k_m \nabla p_\varepsilon(\mathbf{r},t)) \cdot \mathbf{n}_\varepsilon$$

We denote by D_f and D_m the associated diffusion coefficients.

$$D_f = \frac{k_I}{\varphi \mu c_t} \tag{6}$$

$$D_m = \frac{k_m}{\varphi \mu c_t} \tag{7}$$

$$D_m \ll D_f \tag{8}$$

Here φ is the porosity and c_t is the compressibility of the fluid, both are supposed to share the same value between matrix and fractures, an hypothesis that can be easily relaxed. μ is the fluid viscosity. Finally, in order to obtain a well-posed evolution problem, we assume that initial value data at $t = 0$ are provided.

3. THE PROJECTION FORMALISM

3.1. The projection formalism in the case of an impervious matrix

In that section, we recall the results obtained in [1]. We consider first the steady state problem corresponding to the long time limit of 3 with an impervious matrix $k_m = 0$. More details are given in the appendix A.

$$\nabla \cdot (k_I \nabla p_\varepsilon(\mathbf{r})) = g_I(\mathbf{r}),$$

The source term $g_I(\mathbf{r})$ corresponds to the restriction of $g(\mathbf{r})$ in the I th fracture domain. Notice that the Neumann' boundary conditions at the frontier of Ω, and the well connectivity of the network, give a perfectly well posed problem in the fracture domain as far as the thickness ε is not equal to zero. In [1], is was shown that the solution of Laplace equation in the fractured domain can be reconstructed once the trace of the pressure at the intersections between fractures is known. These intersections are generally segments. This trace may be in turn decomposed by projection on a complete set of basic function A. The n-th components of pressure on the j th intersection is denoted by P_j^n. In

order to determine P_j^n, one needs boundary conditions at every intersection. It was shown that a correct boundary condition is that the sum of the (generally four) fluxes converging at a given point of the considered intersection is equal to zero (see appendix A and [1]). Projecting thus this condition of the same set of basis function gives thus the following linear system:

$$\forall i = 1, N_\cap; m = 1, \infty, \sum_{j \in J(i)} \sum_{n=1}^{\infty} T_{ij}^{mn} \times P_j^n \; = \; B_i^m, \tag{9}$$

Explicit expressions for T_{ij}^{mn} are given in the appendix A. The right hand side B_i^m are linear forms involving the source term g_I are also given in the same appendix.

3.2. Generalization to the transient case: the quasi steady state approximation

The projection formalism can be adapted in order to solve the transient diffusion equation 3 with a source term in the fracture domain only. The proposed expression is a faithful approximation if the characteristic diffusion time over one fracture (of typical value $\tau \simeq \varphi \mu c_t L^2 / k_f \ll \tau_{g_I}$) is smaller than the characteristic time of variation of the source term. this hypothesis is not restrictive at all and may be fulfilled in most practical cases. In that situation, the latter can appear as being stationary. The net result is a generalization of 9 that reads:

$$\forall i = 1, N_\cap; m = 1, \infty, \quad \sum_{j \in J(i)} \sum_{n=1}^{\infty} K_{ij}^{mn} \times \frac{dP_j^n(t)}{dt} = \sum_{j \in J(i)} \sum_{n=1}^{\infty} T_{ij}^{mn} \times P_j^n + B_i^m. \tag{10}$$

The notations are essentially the same. This set of first order differential equations may be solved once an initial condition is fulfilled.

Explicit evaluation methods of T_{ij}^{mn} and K_{ij}^{mn} are given in the appendix A and B. Both matrices $\mathbf{K}$ and $\mathbf{T}$ are symmetric positive.

$$T_{ij}^{mn} \; = \; T_{ji}^{nm}, \tag{11}$$

$$K_{ij}^{mn} \; = \; K_{ji}^{nm}, \tag{12}$$

for all labels i, j, m and n.

4. INTRODUCING THE MATRIX TO FRACTURE FLOW

4.1. Basic assumptions, accounting for the finite thickness of the fractures

We are now in position to couple the fracture network with the matrix. In order to fix the ideas, we solve the initial value problem 3. The initial value data at $t = 0$ is $p(\mathbf{r}, \mathbf{t} = \mathbf{0}) = \mathbf{0}$ if $\mathbf{r} \in$ **matrix**. Our first assumption is to consider than the ratio of typical diffusion time over an elementary matrix block having a characteristic size of L (that can be considered as of the same order of magnitude of a fracture length) over a characteristic diffusion time over a single fracture that can be estimated as $\frac{k_f}{k_m}$ is very large. So, the pressure inside the blocks can be considered as slowly varying in the time domain. This observation permits us to use the preceding projection formalism at quasi steady state. At a given time and at a given location $\mathbf{r}$ inside a fracture, say the I-th fracture, one can compute a matrix to fracture flux $f_{Imf}(\mathbf{r}, \mathbf{t})$ given by

$$f_{Imf}(\mathbf{r}, \mathbf{t}) = \frac{k_m}{\mu}(\nabla p_m^+(\mathbf{r}, \mathbf{t}) - \nabla \mathbf{p_m^-}(\mathbf{r}, \mathbf{t})) \cdot \mathbf{n_I} \tag{13}$$

Here, $\mathbf{n_I}$ is a vector normal to the I th fracture, and the + or -- signs correspond to both sides of the fracture. At

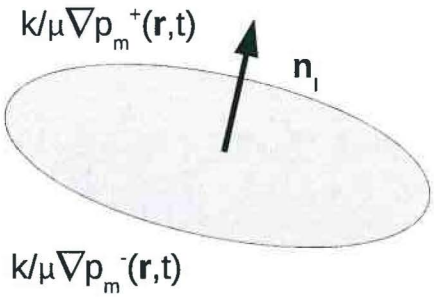

FIG. 4: Notations for the exchange flux

this stage, one must be careful with the small thickness ε of the fracture, by considering as an intermediate step the full 3D diffusion problem 3 involving both fracture and matrix. In order to get meaningful results, we impose that

137 $k_f \times \frac{|\Omega_f|}{|\Omega|} >> k_m$, that states that the overall permeability of the fracture network dominates the matrix permeability.
138 Since the fracture volume $|\Omega_f|$ is proportional to the fracture thickness ε, the limit $\varepsilon \to 0$ can causes some technical
139 difficulties. Keeping k_f constant and letting $\varepsilon \to 0$ should lead to a vanishing influence of the fractures, that is not of
140 interest for our purposes. So, in the present situation in which we want to account for the matrix, it is more adapted
141 to consider that the overall fracture conductivity proportional to $k_f \times \varepsilon$ is maintained fixed as ε tends to zero.
142 In order to decouple the fracture and matrix problems, one can integrate the local equation over the 3D volume V_ε of
143 thickness ε bounded by a surface $\partial V_\varepsilon = S_\varepsilon \cup S_+ \cup S_-$, as shown in FIG. 5. The surface S_ε is thus essentially a narrow band of thickness ε. One obtains:

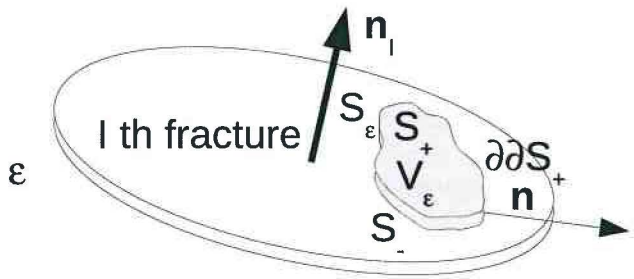

FIG. 5: Arbitrary integration domain V_ε over the I th fracture

144

$$\int_{V_\varepsilon} d^3(\mathbf{r})\varphi\mu c_t \frac{\partial p_\varepsilon(\mathbf{r},t)}{\partial t} = \int_{V_\varepsilon} d^3(\mathbf{r})\nabla\cdot\left(k(\mathbf{r})\nabla p_\varepsilon\left(\mathbf{r},t\right)\right) \tag{14}$$

145 Using Green's theorem on the right hand side of this equation, one obtains:

$$\int_{V_\varepsilon} d^3\mathbf{r}\varphi\mu c_t \frac{\partial p_\varepsilon(\mathbf{r},t)}{\partial t} = \int_{\partial V_\varepsilon} d^2\mathbf{r}\mathbf{n}\cdot\left(k(\mathbf{r})\nabla p_\varepsilon\left(\mathbf{r},t\right)\right) \tag{15}$$

146 So, using the decomposition $\partial V_\varepsilon = S_\varepsilon \cup S_+ \cup S_-$, one gets

$$\int_{V_\varepsilon} d^3\mathbf{r}\varphi\mu c_t \frac{\partial p_\varepsilon(\mathbf{r},t)}{\partial t} = \int_{\partial S_\varepsilon} d^2\mathbf{r}\mathbf{n}\cdot\left(k(\mathbf{r})\nabla p_\varepsilon(\mathbf{r},t)\right) + \int_{\partial S_+} d^2\mathbf{r} f_{Imf}(\mathbf{r},\mathbf{t}) \tag{16}$$

147 One can exploit the smallness of ε to estimate the various integrals and in order to decrease the order of integration.

$$\varepsilon\int_{S_+} d^2\mathbf{r}\varphi\mu c_t \frac{\partial p_\varepsilon(\mathbf{r},t)}{\partial t} \simeq \varepsilon\int_{\partial\partial S_+} d\mathbf{r}\mathbf{n}\cdot\left(k(\mathbf{r})\nabla p_\varepsilon(\mathbf{r},t)\right) + \int_{\partial S_+} d^2\mathbf{r} f_{Imf}(\mathbf{r},\mathbf{t}) \tag{17}$$

148 The $2D$ integration was transformed into a $1D$ one along the curve denoted by $\partial\partial S_+$. The normal vector $\mathbf{n}$ is the
149 normal vector to the 1D curve $\partial\partial S_+$ belonging to the I th fracture plane. As the preceding equality is exact for any
150 $\partial\partial S_+$, and using Green's theorem, one gets the local $2D$ equation valid only on the fracture plane:

$$\varepsilon\varphi\mu c_t \frac{\partial p_\varepsilon(\mathbf{r},t)}{\partial t} = \varepsilon\nabla\cdot\left(k(\mathbf{r})\nabla p_\varepsilon(\mathbf{r},t)\right) + f_{Imf}(\mathbf{r},\mathbf{t}) \tag{18}$$

151 In present form, the limit $\varepsilon \to 0$ can be evaluated safely. The differential operators are defined as in the preceding
152 sections on the considered fracture only. In practice, we have essentially to solve the same equations, up to a factor
153 ε. The projection formalism can be used on the fracture domain using the following correspondence:

$$g_I(\mathbf{r}) = \frac{1}{\varepsilon} f_{Imf}(\mathbf{r},\mathbf{t}) \tag{19}$$

154 The next issue will be to relate the interporosity flux $f_{Imf}(\mathbf{r},\mathbf{t})$ to the fracture pressure $p_\varepsilon(\mathbf{r},t) = p(\mathbf{r},t)$. The subscript
155 ε can be suppressed now to simplify notations.

156 4.2. About the matrix to fracture flow, the exchange function

157 In order to specify a workable form of $f_{Imf}(\mathbf{r},\mathbf{t})$, a possible option is to relate $f_{Imf}(\mathbf{r},\mathbf{t})$ to the pressure map $p_I(\mathbf{r},\mathbf{t})$
158 of the I th fracture. The basic assumption is to consider a time convolution form that keeps the causality and linearity
159 of the underlying equations:

$$\frac{k_m}{\mu}\nabla p_m^+(\mathbf{r},\mathbf{t})\cdot\mathbf{n_I} = -\int_0^t \mathbf{dt'} f_{\mathbf{Im}}^+(\mathbf{t}-\mathbf{t'})\frac{\partial \mathbf{p_I}(\mathbf{r},\mathbf{t'})}{\partial \mathbf{t'}} \tag{20}$$

and so for the total source term accounting for both sides of the fracture:

$$g_I(\mathbf{r}) \;=\; -\frac{1}{\varepsilon}\int_0^t dt' f_{Im}(t-t')\frac{\partial p_I(\mathbf{r},\mathbf{t}')}{\partial t'} \tag{21}$$

This form may be justified by the following arguments: let Ω_m be a matrix block surrounded by several fractures. Let $P_f(t)$ be the pressure of these fractures, assumed to be spatially uniform. We consider a solution of the diffusion equation inside the matrix block without any source term:

$$\varphi\mu c_t\frac{\partial p_m(\mathbf{r},t)}{\partial t} = \nabla\cdot(k_m\nabla\delta p_m(\mathbf{r},t)) = 0, \tag{22}$$

$$p_m(\mathbf{r},t) = P_f(t) \quad if \quad \mathbf{r}\in\partial\Omega_m \tag{23}$$

$$\forall\mathbf{r}\in\Omega_m, p_m(\mathbf{r},t=0)=0$$

The present goal is to relate the flux $\frac{k_m}{\mu}\nabla p_m^+(\mathbf{r},\mathbf{t})\cdot\mathbf{n}$, or at least its average $\frac{1}{|\partial\Omega_m|?}\int_{\partial\Omega_m} d^2\mathbf{r}\frac{k_m}{\mu}\nabla p_m^+(\mathbf{r},\mathbf{t})\cdot\mathbf{n}$ to the variations of the pressure at the boundary $P_f(t)$. Applying the divergence theorem, we obtain:

$$\int_{\partial\Omega_m} d^2\mathbf{r}\frac{k_m}{\mu}\nabla p_m^+(\mathbf{r},\mathbf{t})\cdot\mathbf{n} = -\varphi\mathbf{c_t}|\Omega_\mathbf{m}|\frac{\mathbf{d}\langle\mathbf{P_m}\rangle(\mathbf{t})}{\mathbf{dt}} = -\varphi\mathbf{c_t}\frac{\mathbf{d}\int_{\Omega_\mathbf{m}}\mathbf{d^3rP_m}(\mathbf{r},\mathbf{t})}{\mathbf{dt}}. \tag{24}$$

Here $\langle\cdots\rangle$ denotes a volume average of the pressure P_m over the matrix block. The minus sign comes from the normal $\mathbf{n}$ orientation.

It is now possible to search for a relation between $\langle P_m\rangle(t)$ and $P_f(t)$ under the form of a convolution product:

$$\langle P_m\rangle(t) = \int_0^t dt' f(t-t')P_f(t'), \tag{25}$$

The mapping function $f(t)$, homogeneous to an inverse of time is the solution of a well posed boundary value problem that will be discussed in more details in 5. Coming back to the average flux, using 24 and 25, and elementary properties of convolution products, we get the following form:

$$\int_{\partial\Omega_m} d^2\mathbf{r}\frac{k_m}{\mu}\nabla p_m^+(\mathbf{r},\mathbf{t})\cdot\mathbf{n} = -\varphi\mathbf{c_t}|\Omega_\mathbf{m}|\int_\mathbf{0}^\mathbf{t}\mathbf{dt'f(t-t')}\frac{\mathbf{dP_f(t')}}{\mathbf{dt'}},$$

172 Coming back to the local interporosity flux $\frac{k_m}{\mu}\nabla p_m^+(\mathbf{r},\mathbf{t})\cdot\mathbf{n_I}$ in the general case of a non uniform pressure in the
173 fractures, the preceding developments suggest the proposed form that leads to 21.

$$\frac{k_m}{\mu}\nabla p_m^+(\mathbf{r},\mathbf{t})\cdot\mathbf{n_I} = -\int_0^{\mathbf{t}} \mathbf{dt'}\varphi\mathbf{c_t}\frac{|\Omega_\mathbf{m}|}{|\partial\Omega_\mathbf{m}|}\mathbf{f}^+(\mathbf{t}-\mathbf{t'})\frac{\partial\mathbf{p_I}(\mathbf{r},\mathbf{t'})}{\partial\mathbf{t'}} \tag{26}$$

174 The subscript + recalls that any location in a fracture is in contact with two matrix blocks. Adding both contri-
175 butions, we get:

$$g_I(\mathbf{r}) = -\varphi c_t\frac{|\Omega_m|}{\varepsilon|\partial\Omega_m|}\int_0^t dt' f(t-t')\frac{\partial p_I(\mathbf{r},\mathbf{t'})}{\partial t'} \tag{27}$$

$$= -\varphi c_t\frac{V_m}{V_f}\int_0^t dt' f(t-t')\frac{\partial p_I(\mathbf{r},\mathbf{t'})}{\partial t'} \tag{28}$$

176 A more detailed discussion about the $f(t)$ functions will be provided in section 5. We now turn our attention about
177 the final closure of the problem, and the resulting consequences on the projection formalism.

178 ### 4.3. Final closure and projection formalism

179 Combining 3 and 28, we get an equation driving $p(\mathbf{r},t)$ inside any fracture:

$$\varphi\mu c_t\frac{\partial p(\mathbf{r},t)}{\partial t} = \nabla\cdot(k_I\nabla p(\mathbf{r},t)) - \frac{V_m}{V_f}\varphi\mu c_t\int_0^t dt' f(t-t')\frac{\partial p(\mathbf{r},\mathbf{t'})}{\partial t'}$$

180 When it is possible, we suppress the subscript I because we are considering local pressure without any ambiguity on
181 the I th fracture. As the local pressure $p(\mathbf{r},t)$ appears in both members, the original diffusion equation becomes an
182 integral equation which can be rewritten under the alternative form:

$$\int_0^t dt'\varphi\mu c_t(V_f\delta(t-t') + V_m f(t-t'))\frac{\partial p(\mathbf{r},t')}{\partial t'} = V_f\nabla\cdot(k_I\nabla p(\mathbf{r},t))$$

As in [1], we introduce the following pressure decomposition, see appendices A and B for the details.

$$p(\mathbf{r},t) = \delta p(\mathbf{r},t) + \sum_{j\in J_I}\sum_{n=1}^{\infty} P_j^n(t)\hat{P}_j^n(\mathbf{r})$$

The pressure fluctuation $\delta p(\mathbf{r},t)$ follows the equation:

$$\varphi\mu c_t \int_0^t dt'(V_f\delta(t-t') + V_m f(t-t')\frac{\partial\delta p(\mathbf{r},t')}{\partial t} = V_f\nabla\cdot(k_I\nabla\delta p(\mathbf{r},t)) \quad (29)$$

$$- \sum_{j\in J_I}\sum_{n=1}^{\infty}\int_0^t \varphi\mu c_t(V_f\delta(t-t') + V_m f_{Im}(t-t'))\dot{P}_j^n(t')\hat{P}_j^n(\mathbf{r}), \quad (30)$$

with an additional condition:

$$\delta p(\mathbf{r},t) = 0 \quad if \quad \mathbf{r}\in \cup\cap_i .$$

Here, the $P_j^n(t)$ are assumed to be "slowly varying" if compared to typical diffusion time over one fracture. The steady state assumption assumes that the residual term $\delta p(\mathbf{r},t)$ obeys a steady state equation in which the left hand side is considered as negligible. The validity of this major assumption will be discussed in more details in section 6. So, we assume that

$$V_f\nabla\cdot(k_I\nabla\delta p(\mathbf{r},t)) = \sum_{j\in J_I}\sum_{n=1}^{\infty}\int_0^t \varphi\mu c_t(V_f\delta(t-t') + V_m f(t-t'))\dot{P}_j^n(t')\hat{P}_j^n(\mathbf{r}),$$

We can combine B2 and 19 to get:

$$\forall\, i,m, \quad \sum_{j\in J(i)}\sum_{n=1}^{\infty} K_{ij}^{mn}\times\int_0^t (V_f\delta(t-t') + V_m f_{Im}(t-t'))\frac{dP_j^n(t')}{dt} = \sum_{j\in J(i)}\sum_{n=1}^{\infty} V_f T_{ij}^{mn}\times P_j^n \quad (31)$$

This is the proposed equation 2. Further insights can be given introducing time domain Laplace transform defined by

$$g(s) = \int_O^{\infty} exp - stg(t)dt \quad (32)$$

With the property for the Laplace transform of the time derivative of a function $g(t)$:

$$[\frac{dg(t)}{dt}](s) = sg(s) - g(t=0) \tag{33}$$

By convention, throughout the rest of the paper, employing a s argument corresponds to using the Laplace transform of any function $g(t)$.

The Laplace transform of 31 gives :

$$\forall\, i, m, \quad \sum_{j\in J(i)} \sum_{n=1}^{\infty} K_{ij}^{mn} \times (V_f + V_m f(s)(sP_j^n(s) - P_j^n(t=0)) = \sum_{j\in J(i)} \sum_{n=1}^{\infty} V_f T_{ij}^{mn} \times P_j^n(s) \tag{34}$$

In the Laplace domain, the net effect of the matrix is a modification of the porosity by a s dependent porosity. Setting $V_m = 0$, one recovers the impervious matrix case equations 1. It appears that having a solution of the corresponding impervious matrix problem given by 1 using Laplace transforms, and replacing the Laplace argument s by $s(V_f + V_m f(s))$ will provide the solution of 34. Numerical Laplace inversion can be performed with accuracy by Stehfest algorithm [45]. The net result is that using Laplace transform techniques, the additional computational cost relies mainly in the determination of the exchange function $f(.)$. This observation was already highlighted by several authors in the context of the averaged continuous double porosity descriptions with transient interporosity flow [24–26].

Finally, for small s, as $f(s) \sim 1$ (section 5) corresponding to long time relaxation or low frequencies forcing, one obtains, coming back to the time domain:

$$\forall\, i, m, \quad \sum_{j\in J(i)} \sum_{n=1}^{\infty} K_{ij}^{mn} \frac{dP_j^n}{dt} \simeq \sum_{j\in J(i)} \sum_{n=1}^{\infty} V_f T_{ij}^{mn} \times P_j^n(t) \tag{35}$$

This corresponds to the original set of equations, up to a V_f factor that appears as a retention factor that will hinder diffusion in the fracture domain.

209 ## 5. ABOUT THE EXCHANGE FUNCTION

210 ### 5.1. Numerical evaluation of the exchange function

211 We can now study the exchange function as well as its practical evaluation. We recall 25 relating the volume average
212 $\langle P_m \rangle(t) =$ of the pressure in the matrix to the forcing imposed by the boundary condition in the fractures $P_f(t)$:

$$\langle P_m \rangle(t) = \int_0^t dt' f(t-t') P_f(t'),$$

213 Or, equivalently using Laplace transforms:

$$\langle P_m \rangle(s) = f(s) P_f(s) \tag{36}$$

214 Choosing as $P_f(t)$ a Heaviside function gives: $\langle P_{Hm} \rangle(t) = \int_0^t dt' f(t')$, from which $f(t)$ can be obtained by direct time
215 derivative evaluation. So, solving the following boundary value problem:

$$\varphi \mu c_t \frac{\partial p_{Hm}(\mathbf{r},t)}{\partial t} = \nabla \cdot (k_m \nabla \delta p_{Hm}(\mathbf{r},t)), \tag{37}$$

$$p_m(\mathbf{r},t) = 1 \quad if \quad \mathbf{r} \in \partial\Omega_m, t > 0 \tag{38}$$

$$p_m(\mathbf{r},t=0) = 0 \quad \forall \mathbf{r} \in \Omega_m \tag{39}$$

216 and computing the average $\langle P_{Hm} \rangle(t)$ yields the exchange function. This evolution equation can be solved by several
217 numerical methods. The main task is to mesh the matrix. Note that any explicit meshing of the fractures is avoided
218 because the fractures enter only via a Dirichlet boundary condition. The resulting linear systems to be solved will not
219 contain highly contrasted coefficients, because k_f does not enter in the problem, so correct preconditioning properties
220 can be expected. A useful alternative interpretation of the exchange function in terms of random walks can be
221 proposed [20, 22]. Alternative numerical techniques such as MINC approaches can also be employed [37].

222 ### 5.2. Properties of the exchange function

223 Equations 25 and 39 permits to obtain some analytical solutions in simple cases for $f(t)$ or $f(s)$. We can consider
224 1D blocks (the associated coordinate $x \in [0, 2\ell]$ perpendicular to the plane of the fracture). The potential $p_{Hm}(x,t)$

depends on x and t, So one can use the $1D$ solution:

$$\frac{\partial p_{Hm}(x,t)}{\partial t} = D_m \frac{\partial^2 p_{Hm}(x,t)}{\partial x^2}, \tag{40}$$

$$p_m(x=0,t) = 1 for \quad t>0$$

$$\frac{\partial p_{Hm}}{\partial x}(x=\ell,t) = 0, for \quad t>0 \quad by \quad symmetry$$

$$\forall x \neq 0, p_m(x,t=0) = 0$$

this equation can be solved using time domain Laplace transform:

$$sp_{Hm}(x,s) = D_m \frac{\partial^2 p_{Hm}(x,s)}{\partial x^2}, \tag{41}$$

$$p_m(x=0,s) = 1/s$$

$$\frac{\partial p_{Hm}}{\partial x}(x=\ell,s) = 0, for \quad t>0 \quad by \quad symmetry$$

(42)

This single variable differential equation can be solved easily. One obtains finally:

$$f(s) = \frac{\sqrt{D_m}}{\ell\sqrt{s}} \times th(\sqrt{\frac{s}{D_m}}\ell) \tag{43}$$

At short times t, when the potential in the fractures corresponding to the boundaries of the matrix blocks, is set to 1, the diffusion in the matrix takes place only in a small boundary layer close to the fractures. One can adopt two point of views, in the first one, one can write $f(s) = \frac{\sqrt{D_m}}{\ell\sqrt{s}} \times th(\sqrt{\frac{s}{D_m}}\ell) \simeq \frac{\sqrt{D_m}}{\ell\sqrt{s}}$. In the second point of view, one can consider that the matrix blocks are infinite, $\ell = \infty$, which does not permit using 43 directly because the average pressure on the matrix is not well defined. But, one can use directly the evaluation 20 of the matrix to fracture flux.

$$sp_{Hm}(x,s) = D_m \frac{\partial^2 p_{Hm}(x,s)}{\partial x^2}, \tag{44}$$

$$p_m(x=0,s) = 1/s$$

and to compute $\varphi c_t D_m \frac{\partial p_{Hm)}}{\partial x}(x=0,s)$ for $x>0$ by symmetry to obtain:

$$f(s) = \frac{\sqrt{D_m}}{\sqrt{s}} \times \frac{|\partial\Omega_m|}{|\Omega_m|} \tag{45}$$

In the real time domain, this corresponds to

$$f(t) = \frac{\sqrt{D_m}}{\sqrt{\pi}\sqrt{t}} \times \frac{|\partial\Omega_m|}{|\Omega_m|} \tag{46}$$

Comparing both results shows that for consistency, $\ell = \frac{|\Omega_m|}{|\partial\Omega_m|}$. This formula can be interpreted as follows: at short times t, the characteristic diffusion length is of the order of $\sqrt{D_m t}$. So the corresponding flux is given by 46 up to numerical constants. This corresponds to the large s asymptotics of $f(s)$.

Several other general properties of $f(t)$ can be attained by studying the limit $s \to 0$. One has, using a Taylor expansion of the *th* function:

$$f(s) = 1 - \frac{1}{3}\frac{s}{D_m}\ell^2 + ... \tag{47}$$

We observe that $f(s=0) = 1$. This is a general equality that occurs because using 36 and remarking that both fracture and matrix potential equalize at the long times (or low frequency) limit so $f(s=0) = 1$. In next paragraph, we show that the linear term in s is closely related to the so called "exchange coefficient" $\alpha_\infty = 3\frac{D_m}{V_m \ell^2}$ in present case. This coefficient arises from large scale averaging theories that yields homogenized form of double porosity equations [21] valid at long times, long distances. A useful interpretation of $f(t)$ in terms of escape time pdf from the matrix was derived in [20, 22]. It corresponds to the exit time distribution from the matrix blocks of a particle undergoing brownian motion of diffusion coefficient $\frac{k_m}{\phi\mu c_t}$. In particular, the average exit time may be directly related to the so called exchange coefficient or "shape factor" that enter in classical dual porosity models [17, 21]. Continuous time random walk techniques can thus be set-up to determine this exit time distribution. This can provide techniques avoiding any explicit meshing of the matrix. Detailed expressions of exchange functions using Laplace transforms are given for several block geometries by de Swann [23–25], that can be useful for testing numerical solutions or analytical parameterizations.

5.3. The steady state double porosity case

It can be shown that choosing the following form for $f(s)$

$$f(s) \approx \frac{\alpha_\infty}{V_m s + \alpha_\infty} \tag{48}$$

corresponding to an exponential relaxation in the time domain is equivalent to consider a steady state double porosity model [22]. Using 48 and 24, it is possible to show that at a given time, the flux between matrix and the fractures is given by:

$$\frac{k_m}{\mu}\nabla p_m(\mathbf{r},\mathbf{t}).\mathbf{n} = \frac{|\Omega_\mathbf{m}|}{|\partial\Omega_\mathbf{m}|}\varphi\mathbf{c_t}\alpha_\infty(\langle\mathbf{P_m}\rangle(\mathbf{t}) - \mathbf{P_I}(\mathbf{t})) \tag{49}$$

This corresponds to a steady state double porosity model with a particular choice of the so called shape factor α_∞ [17]. The flux is proportional to the difference between the pressure of the matrix and the fracture. The reader should note that in [22], the exchange function $f(s)$ corresponds to $V_f + V_m f(s)$.

6. TESTING THE SELF CONSISTENCY OF THE ASSUMPTIONS

In that section, we verify on a simplified test problem whether the quasi steady state assumption of section 4 4.3 is consistent with the subsequent findings. In other words, we check if the source term arising from the matrix does not modify drastically the pressure diffusion inside a fracture, that could lead to fracture relaxation time comparable with the matrix relaxation time. We come back about a simplified form of 30 on a single fracture, keeping our notations:

$$\begin{aligned}\varphi\mu c_t \int_0^t dt'(V_f\delta(t-t') + V_m f(t-t')\frac{\partial\delta p(\mathbf{r},t')}{\partial t} &= V_f\nabla\cdot(k_I\nabla\delta p(\mathbf{r},t)) \\ \delta p(\mathbf{r},t=0) \quad fixed &\end{aligned} \tag{50}$$

It corresponds to an initial value problem on the fracture domain, without source term. We want to check if the relaxation time associated with the operator $\varphi\mu c_t \int_0^t dt'(V_f\delta(t-t') + V_m f(t-t')\frac{\partial\delta p(\mathbf{r},t')}{\partial t}$ is small compared with the diffusion time in the matrix $\simeq D_m/\ell^2$. In order to proceed, we consider that $f(s)$ is given by the steady state double porosity model 48. We replace also the Laplace operator $V_f\nabla\cdot(k_I\nabla\delta p(\mathbf{r},t))$ by its smallest eigenvalue corresponding to the larger relaxation time of the fracture: $V_f\nabla\cdot(k_I\nabla\delta p(\mathbf{r},t)) \sim -V_f\varphi\mu c_t\lambda\delta p(\mathbf{r},t)$. Here, $\lambda \sim D_f/\ell^2 \gg \alpha_\infty \sim D_m/\ell^2$.

The negative sign was chosen to recall that the Laplace operator has negative eigenvalues. Using Laplace transform, and the property 33, we get:

$$(V_f + V_m \frac{\alpha_\infty}{V_m s + \alpha_\infty}) s\delta p(\mathbf{r}, s) = -V_f \lambda \delta p(\mathbf{r}, s) + V_f \delta p(\mathbf{r}, t = 0)$$

or, equivalently:

$$\delta p(\mathbf{r}, s) = \frac{(V_m s + \alpha_\infty)}{(V_f V_m s^2 + (\alpha_\infty + \lambda V_f V_m) s + V_f \lambda \alpha} V_f \delta p(\mathbf{r}, t = 0)$$

Recalling the boundary conditions for $\delta p(\mathbf{r}, s) = 0$ at the fractures intersections, it is clear that for large time $\delta p(\mathbf{r}, t) \to 0$ for $t \to \infty$. The convergence is exponential and in order to estimate the relaxation time, the roots of the denominator has to be evaluated. The resulting expressions can be simplified using the fact that $\alpha_\infty \ll \lambda$, and we obtain two roots, up to terms of order $\frac{\alpha_\infty}{\lambda} \ll 1$, which are $-\lambda$ and $-\frac{\alpha_\infty}{\varphi_m}$. Finally, after simplification with the numerator, we get:

$$\delta p(\mathbf{r}, s) = \frac{1}{s + \lambda} \delta p(\mathbf{r}, t = 0)$$

equivalent to:

$$\delta p(\mathbf{r}, t) = exp - (\lambda t) \delta p(\mathbf{r}, t = 0)$$

This confirms the fast relaxation of the transient and the justification of the quasi steady state approximation.

7. APPLICATION TO THE RESISTOR CAPACITOR, OR PIPE NETWORK

In that subsection, we restrict the problem to the case $m = 1$ and $n = 1$ used in practice [42]. In practice it means physically that we estimate only the average pressure along each intersection, and that the mass conservation equation at the intersection is only fulfilled globally. Assuming that all $P_i^m = 0$ if $m \geq 2$, the differential system to be solved is:

$$\forall\, i, \quad \sum_{j\in J(i)} K_{ij}^{11} \times (V_f\delta(t) + V_m f(t)) * \frac{dP_j^1(t)}{dt} = \sum_{j\in J(i)} V_f T_{ij}^{11} \times P_j^1. \tag{51}$$

We recall the following relation [1]:

$$\sum_{ij\in\Omega I} K_{ij}^{11} \;=\; \varphi\mu c_t \int_{\Omega_I} d^2\mathbf{r} = \varphi\mu c_t S_I,$$

where S_I is the total area of the fracture.

The resulting time discretization scheme of the differential equations (51) can be rather time consuming in the case of large fracture networks. It is thus appealing to use a mass condensation (or mass lumping) scheme by acknowledging that pressure variations of neighboring nodes will be very close together. We replace $\sum_{j\in J(i)} K_{ij}^{11} \times \frac{dP_j^1(t)}{dt}$ by $\sum_{j\in J(i)} K_{ij}^{11} \times \frac{dP_i^1(t)}{dt}$. So we get:

$$\forall\, i, \quad M_i \times (V_f\delta(t) + V_m f(t)) * \frac{dP_i^1(t)}{dt} \;=\; \sum_{j\in J(i)} V_f T_{ij}^{11} \times (P_j^1 - P_i^1), \tag{52}$$

$$with \quad M_i \;=\; \sum_{j\in J(i)} K_{ij}^{11}. \tag{53}$$

Indeed, these equations (52) possess the structure of the equations driving the variations of the node potentials of a resistor/capacitor network. The main difference is that the capacity term appears under the form of convolution products, that are simple products in the Laplace domain.

The numerical determination of the T_{ij}^{11} can be done by solving $n_\cap$ elementary Laplace problems with the boundary conditions at the intersections, and computing next the scalar products (A15). Thus, the masses M_i given by (53) are obtained using a suitable surface integration scheme. Fast evaluations of these quantities avoiding solving local Laplace problems on each fracture are proposed in [42].

8. FINAL COMMENTS AND DISCUSSIONS

In this paper, we generalize a method that permits to solve diffusion problems in complex $3D$ fracture networks using a relatively small number of degrees of freedom. The generalization enables us to consider flows coupled with a low permeability matrix acting as a reservoir. The flow exchanges with the matrix can be modeled using the so called

exchange function $f(.)$. The main assumption is that diffusion is so fast in the fractures that the matrix blocks are bounded by essentially spatially uniform boundary conditions that are quasi steady state. The second assumption is to replace the local matrix to fracture flux by its average. Both assumptions permit to define $f(.)$ as a volume average of a solution of a well posed boundary value problem. This function can be determined by existing numerical techniques that avoid a complex meshing of the DFN and solving a badly conditioned problem. It posses also a probabilistic interpretation as it represents the pdf of escape time of diffusing particle in the matrix. Alternatively, it can be parameterised using generic analytical forms fulfilling asymptotic requirements at short and long times. These forms permit to capture the essential features of the matrix: The surface to volume ratio, the typical size L of the blocks, and a shape factor. Very ramified DFN with many dead ends having fractal like structures could be described using a power law $f(.)$ function accounting for scale dependent surface to volume ratio. Using Laplace transform techniques, we show that the effect of the matrix can be modeled at a small extra cost once a previous modeling of potential diffusion in the DFN with an impervious matrix is available. No major extra computing cost can be expected. Numerical tests have to be carried out in order to test the accuracy of the approach, and especially of the limitation to the pipe network approximation (n, m) restricted to 1. Another issue is the generalization of the present formalism to other transport equations, such as convection diffusion equations in the fracture network, coupled with purely diffusive transport in the matrix. This could be done following works of [27] and [28].

APPENDIX A: PROJECTION FORMALISM STEADY STATE CASE

1. Small fracture thickness limit

The projection method follows several steps. The first one is to account for the small thickness ε of the fractures in order to be able to treat the intersection between fractures as $1D$ objects, and the fractures as $2D$ objects embedded in a $3D$.

Some geometrical quantities and several notations are presented in (FIG. 6). Let $\cap_{IJ}$ be an intersection between the two fractures I and J; a $3D$ volume having the shape of a match. As ε tends to zero, this volume becomes a $1D$ segment corresponding to the intersection of the two planes containing fractures I and J. In order to simplify the discussion, and this changes nothing to the global solution, we consider that this segment does not intersect a third fracture. Let x denote a coordinate along this segment. The point $\mathbf{r}_{\cap_{IJ}}(x)$ denotes in a rather natural way the generic point of this segment labeled by x (in practice, the three coordinates of $\mathbf{r}_{\cap_{IJ}}(x)$ may depend linearly on x).

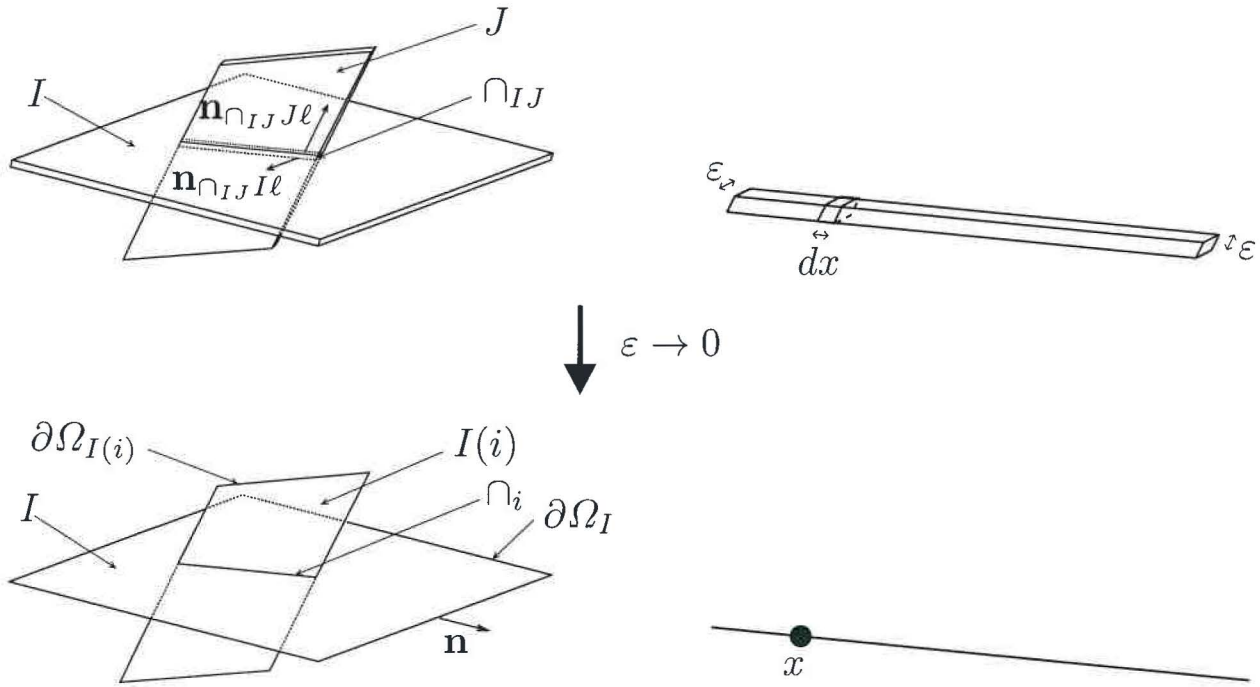

FIG. 6: Geometry and notation two rectangular fractures and the associated intersection, and limit $\varepsilon \to 0$.

As ε tends to zero, we can consider that close to the point $\mathbf{r}_{\cap_{IJ}}(x)$, the intersection separates locally the I^{th} fracture (resp J^{th}) in two halves denoted arbitrary by the suffix ℓ and r (for left and right). We introduce also the normal $\mathbf{n}_{\cap_{IJ}I\ell}$ as being the normal to intersection $\cap_{IJ}$ pertaining to the plane of the I^{th} fracture, pointing in the ℓ direction. In addition, it is possible to introduce the ∇_I gradient operator as being the $2D$ gradient operator operating only in the I^{th} fracture plane. When there is no ambiguity, we will remove the index I to this operator.

We denote the considered limit as $lim_{\varepsilon\to 0}(p_\varepsilon(\mathbf{r})) = p(\mathbf{r})$. We argue that $p(\mathbf{r})$ is the solution of the following problem:

$$\forall\, I = 1, \cdots, N, \quad \nabla_I \cdot (k_I \nabla_I p(\mathbf{r})) = g_I(\mathbf{r}). \tag{A1}$$

The notation $\nabla_I ...$ corresponds to the 2D gradient operator defined in the fracture. In order to get a meaningful limit, we must specify boundary conditions at the frontiers of the computational domain, at the boundary of each fracture, and finally at the intersections between fractures.

The boundary condition at the frontier Ω of the computational domain remains essentially unchanged (notice that the same reasoning should hold also when using mixed Dirichlet Neumann conditions). Considering now each fracture, the boundary value problem to be solved is $2D$. The position vector $\mathbf{r}$ is essentially $2D$. In particular, the boundary condition at the border of each fracture, say $\partial\Omega_I$ in the I^{th} fracture plane, (a $1D$ curve, corresponding for example

342 to an ellipse in the case of elliptic fractures) can be written as:

$$k_I \nabla p(\mathbf{r}) \cdot \mathbf{n} = 0,$$

343 where $\mathbf{n}$ is the outward normal to the boundary (FIG. 6). Notice that in the present formulation, the two initial faces
344 of the fracture in direct contact with the matrix do not play any role.

345 In order to get a well-defined problem, a boundary condition must be specified at every intersection between
346 fractures. In [1], the following condition was proposed:

$$\mathbf{n}_{\cap_{IJ}I\ell} \cdot k_I [\nabla_I p(\mathbf{r}_{\cap_{IJ}}(x))_\ell - \nabla_I p(\mathbf{r}_{\cap_{IJ}}(x))_r] + \mathbf{n}_{\cap_{IJ}J\ell} \cdot k_J [\nabla_J p(\mathbf{r}_{\cap_{IJ}}(x))_\ell - \nabla_J p(\mathbf{r}_{\cap_{IJ}}(x))_r] = 0. \tag{A2}$$

347 It means physically that at each location of the intersection, the four fluxes converging at the considered position
348 must balance. The subscript $_\ell$ and $_r$ (left and right) account for the two sides of the intersection. The normal $\mathbf{n}_{\cap_{IJ}I}$
349 is one normal vector to the intersection under consideration lying in the plane of the I th fracture involved in the
350 intersection. This condition reflects that due to the small transverse area of the intersection, longitudinal flow in the
351 intersection will become negligible as $\varepsilon \to 0$, independently on the value of permeability value at the intersection.
352 Note that the same argument can be followed in the case of a transient problem, because the volume integral of the
353 accumulation term inside the intersection will also become negligible.

354 **2. The projection method.**

355 We are in position to build an approximation scheme allowing us to eliminate internal degrees of freedom inside
356 each fracture. In a finite element solution framework, using an explicit mesh of each fracture, these degrees of freedom
357 will correspond to the unknowns associated with the generic element inside each fracture. The basic idea is to express
358 these lumped degrees of freedom as a function of the pressure trace at intersections. Thus, using the boundary
359 condition (A2), we get equations coupling only degrees of freedom attached to intersections. In order to proceed, we
360 focus our attention over the I^{th} fracture called Ω_I of the set. In order to simplify the analysis, we suppose that this
361 fracture intersects $n_{\cap I}$ other fractures by simple intersections restricted to be segments. So, the $n_{\cap I}$ intersections are
362 non-intersecting segments denoted by $\cap_j$ of arbitrary lengths, the label of which belong to a subset of the $N_\cap$ labels
363 denoted J_I, such that $Card J_I = n_{\cap I}$. For the intersection labeled by i, we call $I(i)$ the label of the other fracture

that is intercepted by I (FIG. 6). So we have $J(i) = J_I \cup J_I(i)$. By hypothesis, this fracture is well defined.

For each intersection segment, we introduce a complete set of basis functions, denoted by $\Phi_j^n(x)$. Here, the integer $n = 1, \cdots \infty$ labels the function, while $j = 1, \cdots, n_{\cap I}$ labels the intersection number. We add two conditions:

$$\Phi_j^1(x) = 1, \tag{A3}$$

$$\int_{\cap_j} \Phi_j^n(x)dx = 0 \quad if \quad n \geqslant 2. \tag{A4}$$

Possible choices could involve sine and cosine functions, or polynomial families like Legendre or Tchebychev. Notice that up to a dilatation due to the varying length of the intersection, the same set of functions can be retained for every intersection between any fractures. We introduce elementary solutions defined by $\hat{P}_j^n(\mathbf{r})$, solution of the following boundary value problem:

$$\begin{cases} \nabla_I \cdot (k_I \nabla \hat{P}_j^n(\mathbf{r})) = 0, \\ k_I \nabla \hat{P}_j^n(\mathbf{r}) \cdot \mathbf{n} = \mathbf{0} \quad on \quad \partial\Omega_I, \\ \hat{P}_j^n(\mathbf{r}) = \delta_{ij} \times \Phi_j^n(\mathbf{r}) \quad if \quad \mathbf{r} \in \cap_i, \quad \forall\, i = 1, n_\cap. \end{cases} \tag{A5}$$

Here, δ_{ij} is the Kronecker symbol. These functions are perfectly defined, as being the unique solution of a Dirichlet Neumann problem. We can decompose the pressure along the j^{th} intersection:

$$p_{\cap_j}(x) = \sum_{n=1}^{\infty} P_j^n \Phi_j^n(x), \tag{A6}$$

and we define $p_\cap(\mathbf{r})$ as the solution of the Laplace equation, without source term but with the imposed profiles at the intersections. We have:

$$p_\cap(\mathbf{r}) = \sum_{j \in J_I} \sum_{n=1}^{\infty} P_j^n \hat{P}_j^n(\mathbf{r}). \tag{A7}$$

We consider now the complete Laplace limit problem (A1). We suppose that the pressures profiles at all the existing intersections are known: $P_j(\mathbf{r})$. Using the linearity of the Laplace equation, and the boundary conditions, we showed

377 that:

$$p(\mathbf{r}) = \sum_{j\in J_I}\sum_{n=1}^{\infty} P_j^n \hat{P}_j^n(\mathbf{r}) + \int_{\Omega_I} d^2\mathbf{r}' B(\mathbf{r},\mathbf{r}') \times g_I(\mathbf{r}'). \tag{A8}$$

378 Here the Green's function $B(\mathbf{r},\mathbf{r}')$ is an elementary solution of the Laplace problem, with a source term $\delta(\mathbf{r}-\mathbf{r}')$:

$$\forall\, I = 1,\cdots,N, \quad \nabla_{\mathbf{r}}\cdot(k_I \nabla_{\mathbf{r}} B(\mathbf{r},\mathbf{r}')) = \delta(\mathbf{r}-\mathbf{r}'). \tag{A9}$$

379 to be solved with the following boundary conditions:

$$\begin{aligned} k_I \nabla_{\mathbf{r}} B(\mathbf{r},\mathbf{r}')\cdot\mathbf{n} &= 0 \quad for \quad \mathbf{r}\in\partial\Omega_I, \\ B(\mathbf{r},\mathbf{r}') &= 0 \quad if \quad \mathbf{r}\in\cap_i, \quad \forall\, i = 1, n_\cap. \end{aligned}$$

380 One must remember that it is a $2D$ Green's function, because the working space is the space of the fracture. This is
381 the general form of the solution with source term, but we still need relations to determine the set of P_j^n values. This
382 will be done by using the boundary condition (A2) in next subsection.

383 In order to get equations allowing to determine the unknowns P_i^m, we use the boundary condition (A2), conveniently
384 projected on the basis function $\Phi_i^m(x)$. The projection gives the following relation: $\forall\, i = 1,\cdots,n_\cap, \quad \forall\, m = 1,\cdots,\infty$,

$$\begin{aligned} \int_{\cap_i} dx \mathbf{n}_{\cap_{IJ}I\ell}\cdot\ & k_I\ (\mathbf{r}(x))[\nabla_I p(\mathbf{r}_{\cap_{IJ}}(x))_\ell - \nabla_I p(\mathbf{r}_{\cap_{IJ}}(x))_r)] \times \Phi_i^m(x) \\ &+ \int_{\cap_i} dx \mathbf{n}_{\cap_{IJ}J\ell}\cdot k_J(\mathbf{r}(x))[\nabla_J p(\mathbf{r}_{\cap_{IJ}}(x))_\ell - \nabla_J p(\mathbf{r}_{\cap_{IJ}}(x))_r] \times \Phi_i^m(x) = 0. \end{aligned} \tag{A10}$$

385 This allows us to get an infinite set of relations, by inserting (A8) in the projection of the boundary conditions (A10)
386 : $\forall\, i,\cdots,m$,

$$\sum_{j\in J(i)}\sum_{n=1}^{\infty} T_{ij}^{mn} \times P_j^n + \int_{\Omega_I} d^2\mathbf{r}' B_i^m(\mathbf{r}') \times g_I(\mathbf{r}') + \int_{\Omega_{I(i)}} d^2\mathbf{r}' B_i^m(\mathbf{r}') \times g_{I(i)}(\mathbf{r}') = 0. \tag{A11}$$

387 or, equivalently, introducing $B_i^m = -\int_{\Omega_I} d^2\mathbf{r}' B_i^m(\mathbf{r}') \times g_I(\mathbf{r}') - \int_{\Omega_{I(i)}} d^2\mathbf{r}' B_i^m(\mathbf{r}') \times g_{I(i)}(\mathbf{r}')$:

$$\sum_{j\in J(i)}\sum_{n=1}^{\infty} T_{ij}^{mn}\times P_j^n \;=\; B_i^m, \tag{A12}$$

In order to emphasize the overall linearity of the problem, we have introduced the following quantities T_{ij}^{nm} and B_i^m as:

$$T_{ij}^{mn} \;=\; \int_{\cap_i} dx\mathbf{n}_{\cap_{IJ}I\ell}\cdot k_I(\mathbf{r}(x))[\nabla_I\hat{P}_j^n(\mathbf{r}(x))_\ell - \nabla_I\hat{P}_j^n(\mathbf{r}(x))_r]\times\Phi_i^m(x), \tag{A13}$$

$$B_i^m(\mathbf{r}') \;=\; \int_{\cap_i} dx\mathbf{n}_{\cap_{IJ}I\ell}\cdot k_I(\mathbf{r}(x))[\nabla_I B(\mathbf{r}(x),\mathbf{r}')_\ell - \nabla_I B(\mathbf{r}(x),\mathbf{r}')_r]\times\Phi_i^m(x). \tag{A14}$$

It was shown in [1], and the proof is presented in the appendix C that T_{ij}^{nm} and $B_i^m(\mathbf{r}')$ may be written under a much more simple and explicit form:

$$T_{ij}^{mn} \;=\; \int_{\Omega_I} d^2\mathbf{r}k_I(\mathbf{r})\nabla_I\hat{P}_i^m(\mathbf{r})\cdot\nabla_I\hat{P}_j^n(\mathbf{r}), \tag{A15}$$

$$B_i^m(\mathbf{r}') \;=\; \hat{P}_i^m(\mathbf{r}'). \tag{A16}$$

The notation $\sum'$ means that the summation must be performed over both fractures involved by the i^{th} intersection of the I^{th} fracture. Here, the notation $I(i)$ denotes that we are considering the moment of the solution over the i^{th} intersection, between the I^{th} fracture, and the $I(i)^{th}$ intersection. It is this summation over all the involved fractures that permits to ensure mass conservation at the intersections. In order to simplify the presentation, we did not reintroduce the labels of I and J th fractures. In practice, one will truncate the order n of the approximation by restricting $m,n\leqq n_0$. At the end of the process, we will have to solve a linear system of $N_\cap\times n_0$ equations. To the lowest order approximation $n_0=1$, will correspond $N_\cap$ equations to be solved. This corresponds well to our initial program, this approximation will be studied in more details in Section 7.

The algebraic form (A15) permits to check by direct inspection that we have the general symmetry relation:

$$T_{ij}^{mn} = T_{ji}^{nm},$$

for all labels i, j, m and n.

It can be checked by inspection of this formula that the matrix T_{ij}^{11} is symmetric, positive. It is not definite because

403 we have the general relation:

$$\sum_j T_{ij}^{11} = 0. \tag{A17}$$

404 This equality can be derived by noticing that we have the general sum rule:

$$P(\mathbf{r}) = \sum_{j=1,n_{\cap I}} \hat{P}_j^1(\mathbf{r}) = 1. \tag{A18}$$

405 ## APPENDIX B: PROJECTION FORMALISM QUASI STEADY STATE CASE

406 We come back to the full transient diffusion problem 3 on the fracture network. Our present goal is to build an
407 approximation scheme valid for time scales greater to a typical diffusion time over one fracture. For a typical fracture
408 I of permeability k_I and of size L_I, this time scale is of order ${L_I}^2/D_I$, here the diffusion coefficient is given by
409 $D_I = k_I/\varphi\mu c_t$. Let $p(\mathbf{r},t)$ be the solution of the full diffusion problem. We write $p(\mathbf{r},t)$ under the following form:

$$p(\mathbf{r},t) = \delta p(\mathbf{r},t) + \sum_{j\in J_I}\sum_{n=1}^{\infty} P_j^n(t)\hat{P}_j^n(\mathbf{r}) + \int_{\Omega_I} d^2\mathbf{r}' B(\mathbf{r},\mathbf{r}') \times g_I(\mathbf{r}').$$

410 Here, the $P_j^n(t)$ are assumed to be "slowly varying" if compared to typical diffusion time over one fracture. The
411 steady state assumption assumes that the residual term $\delta p(\mathbf{r},t)$ obeys a steady state equation. It means physically
412 that internal degrees of freedom inside a fracture are driven by the value of the potential at the intersections.
413 $\delta p(\mathbf{r},t)$ obeys the following equation:

$$\varphi\mu c_t \frac{\partial \delta p(\mathbf{r},t)}{\partial t} = \nabla\cdot(k_I \nabla \delta p(\mathbf{r},t)) - \varphi\mu c_t \sum_{j\in J_I}\sum_{n=1}^{\infty} \dot{P}_j^n(t)\hat{P}_j^n(\mathbf{r}), \quad with \quad \dot{P}_j^n(t) = \frac{dP_j^n(t)}{dt},$$

414 with an additional condition:

$$\delta p(\mathbf{r},t) = 0 \quad if \quad \mathbf{r} \in \cup\, \cap_i .$$

415 The definition of $\hat{P}_j^n(\mathbf{r})$ explains the overall simplification. Now, we can use the pseudo steady state assumption
416 to drop the partial time derivative in the LHS. It means physically that the pressure inside a given fracture follows a

steady state problem with a source term given by the term $\left(\varphi\mu c_t \sum_j^{n\cap I} \sum_{n=1}^{\infty} \dot{P}_j^n(t)\hat{P}_j^n(\mathbf{r})\right)$. We get:

$$\nabla\cdot(k_I\nabla\delta p(\mathbf{r},t)) = \varphi\mu c_t \sum_{j\in J_I}\sum_{n=1}^{\infty} \dot{P}_j^n(t)\hat{P}_j^n(\mathbf{r}).$$

In this expression, the term $\left(\varphi\mu c_t \sum_{j\in J_I} \sum_{n=1}^{\infty} \dot{P}_j^n(t)\hat{P}_j^n(\mathbf{r})\right)$ is a surface source term that appears due to the changing forcing term at the intersections. It specifies the form of the $g_I(\mathbf{r})$. Using the general solution with source term (A8), we get thus the following solution:

$$p(\mathbf{r},t) = \sum_{j\in J_I}\sum_{n=1}^{\infty} P_j^n(t)\hat{P}_j^n(\mathbf{r}) + \int_{\Omega_I} d^2\mathbf{r}' B(\mathbf{r},\mathbf{r}')\times g_I(\mathbf{r}') + \int_{\Omega_I} d^2\mathbf{r}' B(\mathbf{r},\mathbf{r}')\times \sum_{j\in J_I}\sum_{n=1}^{\infty}\varphi\mu c_t \dot{P}_j^n(t)\hat{P}_j^n(\mathbf{r}'). \quad \text{(B1)}$$

Using thus the boundary condition (A2) that remains fulfilled also in transient cases, we can follow the same analysis than in the steady state case with source term to get equations relating the pressure and its time derivatives. We obtain:

$$\begin{aligned}\forall\, i,m,\quad \sum_{j\in J(i)}\sum_{n=1}^{\infty} T_{ij}^{mn}\times P_j^n + \int_{\Omega_I} d^2\mathbf{r} B_i^m(\mathbf{r})\times g_I(\mathbf{r}') + \int_{\Omega_I} d^2\mathbf{r} B_i^m(\mathbf{r}) \;\times \sum_{j\in J(i)}\sum_{n=1}^{\infty}\varphi\mu c_t \dot{P}_j^n(t)\hat{P}_j^n(\mathbf{r}) \\ + \int_{\Omega_{I(i)}} d^2\mathbf{r} B_i^m(\mathbf{r})\times g_I(\mathbf{r}') + \int_{\Omega_{I(i)}} d^2\mathbf{r} B_i^m(\mathbf{r}) \;\times \sum_{j\in J(i)}\sum_{n=1}^{\infty}\varphi\mu c_t \dot{P}_j^n(t)\hat{P}_j^n(\mathbf{r}) = 0. \quad \text{(B2)}\end{aligned}$$

Let us introduce the "mass matrix" K_{ij}^{mn} by means of the definition:

$$K_{ij}^{mn} = \int_{\Omega_I} d^2\mathbf{r} B_i^m(\mathbf{r})\times\varphi\mu c_t\hat{P}_j^n(\mathbf{r}) = \varphi\mu c_t\int_{\Omega_I} d^2\mathbf{r}\hat{P}_i^m(\mathbf{r})\hat{P}_j^n(\mathbf{r}). \quad \text{(B3)}$$

Here, we have used directly the equality (A16). The set of equations can be rewritten under a more synthetic form:

$$\forall\, i,m,\quad \sum_{j\in J(i)}\sum_{n=1}^{\infty} K_{ij}^{mn}\times\frac{dP_j^n(t)}{dt} = \sum_{j\in J(i)}\sum_{n=1}^{\infty} T_{ij}^{mn}\times P_j^n + B_i^m. \quad \text{(B4)}$$

Summarizing the mass and transmissibility matrices, we get remarkable expressions involving surfaces integrals of

the base solutions:

$$T_{ij}^{mn} = \int_{\Omega_I} d^2\mathbf{r} k_I(\mathbf{r}) \nabla_I \hat{P}_i^m(\mathbf{r}) \cdot \nabla_I \hat{P}_j^n(\mathbf{r}), \tag{B5}$$

$$K_{ij}^{mn} = \varphi \mu c_t \int_{\Omega_I} d^2\mathbf{r} \hat{P}_i^m(\mathbf{r}) \hat{P}_j^n(\mathbf{r}). \tag{B6}$$

T_{ij}^{mn} and K_{ij}^{mn} appear as "scalar products" of the basic solutions or of their gradients. These expressions appear also in a finite element context [43]. The symmetry and positiveness of the matrices $\mathbf{K}$ and $\mathbf{T}$ can be checked by direct inspection:

$$T_{ij}^{mn} = T_{ji}^{nm}, \tag{B7}$$

$$K_{ij}^{mn} = K_{ji}^{nm}, \tag{B8}$$

for all labels i, j, m and n. Direct numerical methods can be set up to determine the transmissitivity T_{ij}^{mn} and the mass matrix K_{ij}^{mn} . Each Laplace boundary value problem (A5) can be solved on each fracture independently of the others once the intersection segments have been determined. This means that internal degrees of freedom between different fractures are not directly coupled. This is quite natural, as all the information must be carried by the intersections. The boundary value problem (A5) can be solved using for example a finite element code by meshing only the I^{th} fracture, once for all i, j, m and n, plus a Laplace equation solver. High values of m and n will probably need highly refined meshes, corresponding to having a high level of details. The same procedure must evidently be repeated for every fracture, leading to a numerical cost proportional to twice the total number of intersections. Finding a method to control the accuracy of the method as a function of m and n would be of theoretical interest. Fast evaluations methods of $\mathbf{K}$ and $\mathbf{T}$ remain an open area of work followed by Khvoenkova and Delorme [42]. The similar approach followed in $2D$, with fracture intersections that degenerate as single points is exact, and corresponds to a so called resistor/network model developed by [31].

APPENDIX C: DERIVATION OF EQ. (A15) AND EQ. (A16)

We want to show that:

$$T_{ij}^{mn} = \int_{\Omega_I} d^2\mathbf{r} k_I(\mathbf{r}) \nabla_I \hat{P}_i^m(\mathbf{r}) \cdot \nabla_I \hat{P}_j^n(\mathbf{r}), \tag{C1}$$

$$B_i^m(\mathbf{r}') = \hat{P}_i^m(\mathbf{r}'). \tag{C2}$$

We start from the definitions:

$$\begin{aligned} T_{ij}^{mn} &= \int_{\cap_i} dx \mathbf{n}_{\cap_I J I \ell} \cdot k_I(\mathbf{r}(x)) [\nabla_I \hat{P}_j^n(\mathbf{r}(x))_\ell - \nabla_I \hat{P}_j^n(\mathbf{r}(x))_r] \times \Phi_i^m(x), \\ B_i^m(\mathbf{r}') &= \int_{\cap_i} dx \mathbf{n}_{\cap_I J I \ell} \cdot k_I(\mathbf{r}(\mathbf{x})) [\nabla_I B(\mathbf{r}(x), \mathbf{r}')_\ell - \nabla_I B(\mathbf{r}(x), \mathbf{r}')_r] \times \Phi_i^m(x), \end{aligned} \tag{C3}$$

with $P_i^m(\mathbf{r})$ which is solution of the following boundary value problem:

$$\begin{cases} \nabla_I \cdot (k_I \nabla \hat{P}_i^m(\mathbf{r})) = 0, \\ k_I \nabla \hat{P}_i^m(\mathbf{r}) \cdot \mathbf{n} = 0 \quad on \quad \partial\Omega_I, \\ \hat{P}_i^m(\mathbf{r}) = \delta_{ij} \times \Phi_i^m(\mathbf{r}) \quad if \quad \mathbf{r} \in \cap_i, \quad \forall j = 1, n_\cap. \end{cases}$$

These equalities may be derived by remarking that T_{ij}^{mn} and $B_i^m(\mathbf{r}')$ can be rewritten under a slightly different form:

$$\begin{aligned} T_{ij}^{mn} &= \int_{\cap_i} dx \mathbf{n} \cdot k_I(\mathbf{r}(x)) [\nabla_I \hat{P}_j^n(\mathbf{r}(x)) - \nabla_I \hat{P}_j^n(\mathbf{r}(x))_r] \times \Phi_i^m(x) \\ &= \int_{\cap_i} dx \mathbf{n} \cdot k_I(\mathbf{r}(x)) \nabla_I \hat{P}_j^n(\mathbf{r}(x)) \times \hat{P}_i^m(\mathbf{r}(x)) \\ &= \int_{\bigcup \cap_k \bigcup \partial\Omega_I} dr \mathbf{n} \cdot k_I(\mathbf{r}(x)) \nabla_I \hat{P}_j^n(\mathbf{r}(x)) \times \hat{P}_i^m(\mathbf{r}(x)) \\ B_i^m(\mathbf{r}') &= \int_{\bigcup \cap_k \bigcup \partial\Omega_I} dx \mathbf{n} \cdot k_I(\mathbf{r}(x)) \nabla_I B(\mathbf{r}(x), \mathbf{r}') \times \hat{P}_i^m(\mathbf{r}(x)). \end{aligned}$$

In order to understand the second and fourth equalities, the reader must imagine that the intersection contour is a closed 2D contour of small thickness allowing to replace the contribution of left and right fluxes on the two sides of the intersection by a contour integral that allows to use Green's theorem. We did not change the notations in order to simplify the presentation. In present form, the boundary integrals are extended on all the frontiers associated with the I^{th} fracture, so we can use Green's theorem in order to transform the contour integral on a surface integral over

453 the fracture domain. We obtain:

$$T_{ij}^{mn} = \int_{\Omega_I} d^2\mathbf{r}\nabla\cdot[k_I(\mathbf{r})\nabla_I\hat{P}_j^n(\mathbf{r}) \times \hat{P}_i^m(\mathbf{r})],$$

454 from which the desired identity follows, using once again the local equation obeyed by $\hat{P}_j^n(\mathbf{r})$:

$$T_{ij}^{mn} = \int_{\Omega_I} d^2\mathbf{r}k_I(\mathbf{r})\nabla_I\hat{P}_j^n(\mathbf{r})\cdot\nabla\hat{P}_i^m(\mathbf{r}).$$

455 For $B_i^m(\mathbf{r}')$ we need some additional manipulations in the same style:

$$\begin{aligned} B_i^m(\mathbf{r}') &= \int_{\bigcup\cap_k\bigcup\partial\Omega_I} dx\mathbf{n}\cdot k_I(\mathbf{r})\nabla_I B(\mathbf{r}(x),\mathbf{r}') \times \hat{P}_i^m(\mathbf{r}(x)) \\ &= \int_{\Omega_I} d^2\mathbf{r}\nabla\cdot[k_I(\mathbf{r})\nabla_I B(\mathbf{r},\mathbf{r}') \times \hat{P}_i^m(\mathbf{r})]. \end{aligned} \tag{C4}$$

456 Using the equation defining $B(\mathbf{r},\mathbf{r}')$, we obtain:

$$B_i^m(\mathbf{r}') = \hat{P}_i^m(\mathbf{r}') + \int_{\Omega_I} d^2\mathbf{r}k_I(\mathbf{r})\nabla_I B(\mathbf{r},\mathbf{r}')\cdot\nabla\hat{P}_i^m(\mathbf{r}).$$

457 The second term of the RHS may be written under a more explicit form:

$$\begin{aligned} \int_{\Omega_I} d^2\mathbf{r}k_I(\mathbf{r})\nabla_I B(\mathbf{r},\mathbf{r}')\cdot\nabla\hat{P}_i^m(\mathbf{r}) &= \int_{\Omega_I} d^2\mathbf{r}k_I(\mathbf{r})\nabla_I B(\mathbf{r},\mathbf{r}')\cdot\nabla\hat{P}_i^m(\mathbf{r}) \\ &\quad + \int_{\Omega_I} d^2\mathbf{r}B(\mathbf{r},\mathbf{r}')\nabla_I\cdot[k_I(\mathbf{r})\nabla\hat{P}_i^m(\mathbf{r})] \\ &= \int_{\Omega_I} d^2\mathbf{r}\nabla_I\cdot[k_I(\mathbf{r})B(\mathbf{r},\mathbf{r}')\nabla\hat{P}_i^m(\mathbf{r})] \\ &= \int_{\bigcup\cap_k\bigcup\partial\Omega_I} drk_I(\mathbf{r})B(\mathbf{r},\mathbf{r}') \times \nabla_I\hat{P}_i^m(\mathbf{r})\cdot\mathbf{n} \end{aligned}$$

458 But the last expression is equal to zero, thanks to the boundary conditions on $B(\mathbf{r},\mathbf{r}')$ and $k_I(\mathbf{r})\nabla_I\hat{P}P_i^m(\mathbf{r})\cdot\mathbf{n}$ on

459 $\bigcup\cap_k\bigcup\partial\Omega_I$.

$$\int_{\bigcup\cap_k\bigcup\partial\Omega_I} drk_I(\mathbf{r})B(\mathbf{r},\mathbf{r}') \times \nabla_I\hat{P}_i^m(\mathbf{r})\cdot\mathbf{n} = 0.$$

460 This provides the announced result.

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