High-gain observer for a class of time-delay nonlinear systems

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Abstract

This paper proposes a high gain observer design for a class of nonlinear systems with multiple known time-varying delays intervening in the states and the inputs. In the free delay case, the class of systems under consideration coincides with a canonical form characterizing a class of multi outputs nonlinear systems which are observable for any input. The underlying high gain design has been mainly motivated by its inherent simplicity from both design and implementation points of view. Indeed, the observer gain is determined from an explicit resolution of a time-invariant Lyapunov algebraic equation up to the specification of a single design parameter. An academic observation problem is addressed to illustrate the effectiveness of the proposed observer.

Keywords : Delay nonlinear systems, high gain observers, Lyapunov algebraic equation.

1 Introduction

Time delay is an inherent property of various engineering systems such as biochemical processes, population dynamics and communication and information systems ([22], [23]). Time delays are troublesome in that they could cause oscillations or instability of the systems (see [17] for example). As a result, an intensive research activity has been devoted in the last few years to study the stability, control and state estimation for systems with time delays.

A particular attention has been paid to the case of linear systems (See for instance [7, 20, 6, 14, 16, 3, 12, 15, 21] and references therein) whereas only few results have been established in the nonlinear case (see for instance [28, 1, 27, 29]). Moreover, most of the underlying works are based on LMI techniques where the gain of the observer is designed through the resolution of a LMI problem and as a consequence an observer exits only if the considered LMI problem is feasible. As for systems with no delay and as noticed in [2], the feasibility of the LMI problems considered in observer design is generally not known *a priori* and is to be determined numerically.

Of fundamental interest, one should notice that the problem of observer design still be open in the multi-outputs case even in the case of free delay systems. This is mainly due to the lack of observable canonical forms. Even in the case of uniformly observable systems (systems which are observable for any inputs), there is not a canonical form which characterizes all uniformly observable systems in the multi-outputs case. Some canonical forms characterizing some classes of MIMO uniformly observable systems have been proposed in [26] and [13]. More recently, the authors in [19] have proposed a new canonical form for classes of systems that are larger than those considered in [26] and [13]. Moreover, the authors in [19] have used the so proposed canonical form in order to synthesize a high gain observer.

The look for general canonical forms that characterize some classes of time-delay systems is not the aim of this work. Rather, one shall consider a specific canonical form with time-varying delays, which is uniformly observable. This canonical form is the natural extension of that one proposed in [13]. Indeed, the form proposed in [13] is modified in such a way the modified version is a delayed system where the delayed states intervene in a triangular manner. This paper is concerned with state observer design for the so obtained form. More specifically, one will show that the general high gain observer design framework established in [4], [5], [11] and [13] for free delay systems can be properly extended to this class of time-delay systems. This makes it possible to derive an observer that shares all the appealing features of the high gain concept, namely an exponential convergence with an easy implementation. The easiness of implementation is mainly due to the fact that the observer gain is issued from the resolution of a time-invariant Lyapunov algebraic equation and it is explicitly given. Moreover, its tuning is performed through the choice a single design parameter whatever is the dimension of the considered system.

This paper is organized as follows. Section 2 is devoted to the problem formulation through the introduction of the class of nonlinear systems which will subject to the observer synthesis. In section 3, the observer design is given in the case of a single time-varying delay and a full convergence analysis is provided. An example with simulation results is given in section 4 for illustration purposes. Concluding remarks are given in section 5 together with some perspectives.

2 Problem formulation

The aim of this paper is to design a high-gain observer to estimate the state of the following class of systems

$$\left\{\begin{array}{lcl} \dot{x}(t) & = & Ax(t) + g\left(u(t), u_{\tau_1}(t), \ldots, u_{\tau_m}(t), x(t), x_{\tau_1}(t), \ldots, x_{\tau_m}(t)\right) \\ y(t) & = & Cx(t) = x^1(t) \\ x(s) & = & \varphi(s)\ \forall s \in [-\tau^\star, 0] \end{array}\right. \tag{1}$$

with

$$A = \begin{pmatrix} 0 & I_p & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & I_p \\ 0 & \ldots & \ldots & \ldots & 0 \end{pmatrix} \tag{2}$$

$$C = \begin{pmatrix} I_p & 0 & \ldots & 0 \end{pmatrix} \tag{3}$$

the state $x = \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^q \end{pmatrix}$; $x^k = \begin{pmatrix} x_1^k \\ x_2^k \\ \vdots \\ x_p^k \end{pmatrix}$; $x_{\tau_i} = \begin{pmatrix} x_{\tau_i}^1 \\ x_{\tau_i}^2 \\ \vdots \\ x_{\tau_i}^q \end{pmatrix}$; $x_{\tau_i}^k = \begin{pmatrix} x_{1\tau_i}^k \\ x_{2\tau_i}^k \\ \vdots \\ x_{p\tau_i}^k \end{pmatrix}$; the $\tau_i(t)$'s, $i = 1, \ldots, m$ are positive real-valued known functions that denote the time delays affecting both the state variables and the inputs and they are assumed to be bounded by $\tau^\star > 0$; $u_{\tau_i}(t)$ and $x_{\tau_i}(t)$ respectively denote the delayed inputs and states, i.e $u_{\tau_i}(t) = u(t-\tau_i)$ *and* $x_\tau(t) = x(t-\tau_i)$. Notice that for simplicity of presentation and without loss of generality, one assumes that the delays affecting the states and the inputs are the same; the output $y \in \mathbb{R}^p$; the state $x \in \mathbb{R}^n$ with $x^k \in \mathbb{R}^p$ and $x_i^k \in \mathbb{R}$, $k = 1, \ldots, q$ and $i = 1, \ldots, p$; the input $u \in \mathbb{R}^s$. The nonlinearities $g^k\left(u(t), u_{\tau_1}(t), \ldots, u_{\tau_q}(t), x(t), x_{\tau_1}(t), \ldots, x_{\tau_q}(t)\right)$, $k = 1, \ldots, q$, assume a triangular structure with respect to $x, x_{\tau_1}, \ldots, x_{\tau_q}$, i.e.

$$\begin{aligned}
&g^k\left(u(t), u_{\tau_1}(t), \ldots, u_{\tau_m}(t), x(t), x_{\tau_1}(t), \ldots, x_{\tau_m}(t)\right) = \\
&g^k\left(u(t), u_{\tau_1}(t), \ldots, u_{\tau_m}(t), x^1(t), \ldots, x^k(t), x^1_{\tau_1}(t), \ldots, x^k_{\tau_1}(t), \ldots, x^1_{\tau_m}(t), \ldots, x^k_{\tau_q}(t)\right)
\end{aligned}$$

System (1) may seem as being very particular since it assumes a non prime dimension and all sub-blocks x^k have the same dimension. In fact, it has been shown in [13] that in the free delay case, system (1) (with no delay) is a canonical form that characterizes the following class of uniformly observable nonlinear systems (systems which are observable for any input):

$$\left\{\begin{array}{l} \dot{x} = f(u,x) \\ y = \bar{C}x = x^1 \end{array}\right. \tag{4}$$

with $x = \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^q \end{pmatrix}$, $f(u,x) = \begin{pmatrix} f^1(u,x^1,x^2) \\ f^2(u,x^1,x^2,x^3) \\ \vdots \\ f^{q-1}(u,x) \\ f^q(u,x) \end{pmatrix}$ and $\bar{C} = [I_{n_1}, 0_{n_1\times n_2}, 0_{n_1\times n_3}, \ldots, 0_{n_1\times n_q}]$ where the state $x \in \mathbb{R}^n$ with $x^k \in \mathbb{R}^{n_k}$, $k = 1, \ldots, q$ and $n_1 \geq n_2 \geq \ldots \geq n_q$, $\sum_{k=1}^{q} n_k = n$; the input $u(t) \in \mathcal{U}$ the set of bounded absolutely continuous functions with bounded derivatives from $\mathbb{R}^+$ into U a compact subset of $\mathbb{R}^s$; the output $y \in \mathbb{R}^{n_1}$ and $f(u,x) \in \mathbb{R}^n$ with $f^k(u,x) \in \mathbb{R}^{n_k}$. The functions f^k are assumed to satisfy the following condition:

(C) For $k = 1, \ldots, q-1$, the map $x^{k+1} \mapsto f^k(u, x^1, \ldots, x^k, x^{k+1})$ is one tone from $\mathbb{R}^{n_k}$ into $\mathbb{R}^{n_k}$. Moreover, $\exists \alpha_f, \beta_f > 0$ such that for all $k \in \{1, \ldots, q-1\}$, $\forall x \in \mathbb{R}^n$, $\forall u \in U$,

$$\alpha_f^2 I_{n_{k+1}} \leq \left(\frac{\partial f^k}{\partial x^{k+1}}(u,x)\right)^T \frac{\partial f^k}{\partial x^{k+1}}(u,x) \leq \beta_f^2 I_{n_{k+1}}$$

Now, consider the following injective map: $\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}^{n_1 q}$, $x = \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^q \end{pmatrix} \mapsto z = \begin{pmatrix} z^1 \\ z^2 \\ \vdots \\ z^q \end{pmatrix}$ with

$$z = \Phi(u,x) = \begin{pmatrix} x^1 \\ f^1(u,x^1,x^2) \\ \dfrac{\partial f^1}{\partial x^2}(u,x^1,x^2) f^2(u,x^1,x^2,x^3) \\ \vdots \\ \left(\displaystyle\prod_{k=1}^{q-2} \frac{\partial f^k}{\partial x^{k+1}}(u,x)\right) f^{q-1}(u,x) \end{pmatrix} \tag{5}$$

where $z^k \in \mathbb{R}^{n_1}$, $k = 1, \ldots, q$. This transformation puts the original system under the following form (see [13] for more details):

$$\left\{ \begin{array}{l} \dot{z} = \tilde{A}z + \tilde{\varphi}(v, z) \\ y = \tilde{C}z = z^1 \end{array} \right. \tag{6}$$

where the state $z \in \mathbb{R}^{n_1 q}$, $v = [u^T, \dot{u}^T]^T$, $\tilde{\varphi}(v, z)$ has a triangular structure with respect to z and the matrices $\tilde{A}$ and $\tilde{C}$ are: $\tilde{A} = \begin{bmatrix} 0 & I_{(q-1)n_1} \\ 0 & 0 \end{bmatrix}$ and $\tilde{C} = [I_{n_1}, 0_{n_1}, \ldots, 0_{n_1}]$. It is clear that in the free delay case, system (6) is under form (1) with $n_1 = p$.

Now, it is easy to see that one can derive a canonical form for time delay systems by introducing delayed inputs and states in system (4) in such a way that the delayed states intervene in a triangular manner. Indeed, let us modify system (4) as follows:

$$\left\{ \begin{array}{l} \dot{x} = f(u, u_{\tau_1}, \ldots, u_{\tau_{m'}}, x, x_{\tau_1}, \ldots, x_{\tau_{m'}}) \\ y = \bar{C}x = x^1 \end{array} \right. \tag{7}$$

where the k^{th} component, f^k, of the function f has now the following structure:

$$\begin{array}{l} f^k\left(u(t), u_{\tau_1}(t), \ldots, u_{\tau_{m'}}(t), x(t), x_{\tau_1}(t), \ldots, x_{\tau_{m'}}(t)\right) = \\ f^k\left(u(t), u_{\tau_1}(t), \ldots, u_{\tau_q}(t), x^1(t), \ldots, x^{k+1}(t), x^1_{\tau_1}(t), \ldots, x^k_{\tau_1}(t), \ldots, x^1_{\tau_q}(t), \ldots, x^k_{\tau_{m'}}(t)\right) \end{array}$$

where τ_i, $i = 1, \ldots, m'$ are the time delays and the other variables still have the same meaning as in system (4). The functions f^k are always supposed to satisfy the condition (C). It is easy to see that the injective map introduced for system (4) puts system (7) under the form considered in this paper with view to observer design, ie. system (1). As a result, system (1) can be interpreted as a canonical form which characterize a class of uniformly observable delayed state system.

Let us now come back to system (1). The design of the observer requires the adoption of the following assumptions :

A1. The state $x(t)$ and the input $u(t)$ are bounded, *i.e.* there exist compact sets $X \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^m$ such that for all $t \in \mathbb{R}$, $x(t) \in X$ and $u(t) \in U$.

A2. The function g is Lipschitz with respect to $x, x_{\tau_1}, \ldots, x_{\tau_q}$, uniformly in $u, u_{\tau_1}, \ldots, u_{\tau_q}$.

A3. The delay functions fulfill the following properties:

$$\exists\ \tau^\star > 0\ /\ \sup_{t \geq 0} \tau_i(t) \leq \tau^\star\ \forall i \in \{1, \ldots, q\} \tag{8}$$

$$\exists\ \varepsilon > 0\ /\ \sup_{t \geq 0} \dot{\tau}_i(t) \leq 1 - \varepsilon\ \forall i \in \{1, \ldots, q\} \tag{9}$$

Please notice that Assumption **A2** is classical in the high gain design framework. It is worth mentioning that as the state is supposed to lie in a bounded set X, it is possible to define an extension $\tilde{g}$ of g which coincide with g on X and which is global Lipschitz on $\mathbb{R}^n$ (see e.g. [25]).
For the sake of clarity, the observer design will be detailed in the single-delay case. As one shall point it out later, the generalization to the multiple delays case is straightforward.

3 Observer design

In this section one assumes $m = 1$ and for writing convenience one shall denote τ_1 by τ. System (1) specializes as follows :

$$\left\{ \begin{array}{l} \dot{x}(t) = Ax(t) + g(u(t), u_\tau(t), x(t), x_\tau(t)) \\ y(t) = Cx(t) \end{array} \right. \tag{10}$$

A high gain candidate observer for the above class of systems is given by

$$\dot{\hat{x}}(t) = A\hat{x}(t) + g(u(t), u_\tau(t), \hat{x}(t), \hat{x}_\tau(t)) - \theta\Delta_\theta^{-1}S^{-1}C^T C(\hat{x}(t) - x(t)) \tag{11}$$

where Δ_θ is the diagonal matrix defined by

$$\Delta_\theta = diag \left[\begin{array}{cccc} I_p, & \frac{1}{\theta}I_p, & \ldots, & \frac{1}{\theta^{q-1}}I_p \end{array} \right] \tag{12}$$

where $\theta > 0$ is a real parameter and S is the unique solution of the following algebraic Lyapunov equation:

$$S + A^T S + SA - C^T C = 0 \tag{13}$$

Remark 1. *It has been shown that S is symmetric positive definite (see e.g. [11, 10]) and that one particularly has :*

$$S^{-1}C^T = \begin{bmatrix} C_q^1 I_p \\ C_q^2 I_p \\ \vdots \\ C_q^q I_p \end{bmatrix} \quad \textit{with } C_q^k = \frac{q!}{k!(q-k)!} \quad \textit{for} \quad k = 1, \ldots, q$$

The following fundamental result provides the performance of the above candidate observer.

Theorem 2. *Under assumptions **A1** to **A3**, system* (11) *is an exponential observer for system* (10). *More precisely, one has :*

$$\exists \alpha_1, \alpha_2 > 0; \; \exists \theta_0 > 0; \; \forall \theta > \theta_0; \; \exists c(\theta) \textit{ such that}$$
$$\|\hat{x}(t) - x(t)\| \leq \theta^{q-1}\sqrt{\frac{\alpha_1}{\alpha_2}}e^{-c(\theta)t} \max_{s \in [-\tau^\star, 0]} \|\hat{x}(s) - x(s)\| \tag{14}$$

with $\lim\limits_{\theta \to +\infty} c(\theta) = +\infty$

Proof. Let $\tilde{x}(t) = \hat{x}(t) - x(t)$ be the estimation error. From (10) and (11), one has:

$$\dot{\tilde{x}} = A\tilde{x} + g(u, u_\tau, \hat{x}, \hat{x}_\tau) - g(u, u_\tau, x, x_\tau) - \theta\Delta_\theta^{-1}S^{-1}C^T C\tilde{x} \tag{15}$$

Let us now introduce the following change of variable :

$$\bar{x} = \Delta_\theta \tilde{x} \tag{16}$$

where Δ_θ is defined by (12). One can easily check the following identities: $\Delta_\theta A \Delta_\theta^{-1} = \theta A$ and $C\Delta_\theta^{-1} = C$. Combining (15) and (16) yields

$$\dot{\bar{x}} = \theta A\bar{x} + \Delta_\theta \left(g(u, u_\tau, \hat{x}, \hat{x}_\tau) - g(u, u_\tau, x, x_\tau)\right) - \theta S^{-1}C^T C\bar{x} \tag{17}$$

Moreover, using the Mean Value Theorem (see Appendix), one gets:

$$\begin{aligned} g(u, u_\tau, \hat{x}, \hat{x}_\tau) - g(u, u_\tau, x, x_\tau) &= g(u, u_\tau, \hat{x}, \hat{x}_\tau) - g(u, u_\tau, x, \hat{x}_\tau) \\ &\quad + g(u, u_\tau, x, \hat{x}_\tau) - g(u, u_\tau, x, x_\tau) \\ &= \frac{\partial g}{\partial x}(u, u_\tau, \xi, \hat{x}_\tau)\tilde{x} + \frac{\partial g}{\partial x_\tau}(u, u_\tau, x, \zeta_\tau)\tilde{x}_\tau \end{aligned} \tag{18}$$

where $\xi, \zeta \in X \subset \mathbb{R}^n$.

Using (18), equation (17) can be rewritten as follows :

$$\dot{\bar{x}} = \theta A\bar{x} + \Delta_\theta \frac{\partial g}{\partial x}(u, u_\tau, \xi, x_\tau)\Delta_\theta^{-1}\bar{x} + \Delta_\theta \frac{\partial g}{\partial x_\tau}(u, u_\tau, \hat{x}, \zeta_\tau)\Delta_\theta^{-1}\bar{x}_\tau - \theta S^{-1}C^T C\bar{x} \tag{19}$$

Consider the Lyapunov-Krasovskii candidate functional:

$$V(\bar{x}) = \bar{x}^T S\bar{x} + \theta^{-\frac{t}{2\tau^\star}} \int_{t-\tau}^{t} \theta^{\frac{s}{2\tau^\star}} \bar{x}^T(s)\bar{x}(s)ds \tag{20}$$

where $\tau^\star$ is the upper bound of the delay as given in Assumption (A3).

The derivative of V along the trajectories of (17) can be expressed as:

$$\dot{V} = 2\bar{x}^T S\dot{\bar{x}} - \frac{\ln\theta}{2\tau^\star}\left(V - \bar{x}^T S\bar{x}\right) + \theta^{-\frac{t}{2\tau^\star}}\left(\theta^{\frac{t}{2\tau^\star}}\bar{x}^T\bar{x} - (1-\dot{\tau})\theta^{\frac{t-\tau}{2\tau^\star}}\bar{x}_\tau^T\bar{x}_\tau\right) \tag{21}$$

Using (19) and (21), one obtains:

$$\begin{aligned} \dot{V} + \frac{\ln\theta}{2\tau^\star}V &= 2\theta\bar{x}^T SA\bar{x} + 2\bar{x}^T S\Delta_\theta \frac{\partial g}{\partial x}(u, u_\tau, \xi, x_\tau)\Delta_\theta^{-1}\bar{x} \\ &\quad + 2\bar{x}^T S\Delta_\theta \frac{\partial g}{\partial x_\tau}(u, u_\tau, \hat{x}, \zeta_\tau)\Delta_\theta^{-1}\bar{x}_\tau - 2\theta\bar{x}^T C^T C\bar{x} \\ &\quad + \frac{\ln\theta}{2\tau^\star}\bar{x}^T S\bar{x} + \bar{x}^T\bar{x} - \frac{1-\dot{\tau}}{\sqrt{\theta^{\tau/\tau^\star}}}\bar{x}_\tau^T\bar{x}_\tau \end{aligned} \tag{22}$$

On the one hand, since $\dfrac{\tau}{\tau^\star} \le 1$, one has

$$\frac{1}{\sqrt{\theta^{\tau/\tau^\star}}} \ge \frac{1}{\sqrt{\theta}} \; for\ \theta \ge 1 \tag{23}$$

Making use of (13), (9) and (23), one obtains for $\theta \geq 1$:

$$\begin{aligned}\dot{V} + \frac{\ln\theta}{2\tau^\star}V \leq \quad & -\theta\bar{x}^T S\bar{x} - \theta\bar{x}^T C^T C\bar{x} \\ & +2\bar{x}^T S\Delta_\theta \frac{\partial g}{\partial x}(u, u_\tau, \xi, x_\tau)\Delta_\theta^{-1}\bar{x} \\ & +2\bar{x}^T S\Delta_\theta \frac{\partial g}{\partial x_\tau}(u, u_\tau, \hat{x}, \zeta_\tau)\Delta_\theta^{-1}\bar{x}_\tau \\ & +\frac{\ln\theta}{2\tau^\star}\bar{x}^T S\bar{x} + \bar{x}^T\bar{x} - \frac{\varepsilon}{\sqrt{\theta}}\bar{x}_\tau^T\bar{x}_\tau\end{aligned} \tag{24}$$

On the other hand, since g has a triangular structure with respect to x and x_τ, the matrices $\frac{\partial g}{\partial x}(\cdot)$ and $\frac{\partial g}{\partial x_\tau}(\cdot)$ are lower triangular. Moreover and according to Assumption (A2), these matrices are uniformly bounded. As a result, each entry of the matrices $\Delta_\theta \frac{\partial g}{\partial x}(\cdot)\Delta_\theta^{-1}$ and $\Delta_\theta \frac{\partial g}{\partial x_\tau}(\cdot)\Delta_\theta^{-1}$ is polynomial in $\frac{1}{\theta}$ and is uniformly bounded by a constant which does not depend on θ for $\theta \geq 1$. This means that:

$$\begin{aligned}\|\Delta_\theta \frac{\partial g}{\partial x}(u, u_\tau, \xi, x_\tau)\Delta_\theta^{-1}\| \leq k_1 \\ \|\Delta_\theta \frac{\partial g}{\partial x_\tau}(u, u_\tau, \hat{x}, \zeta_\tau)\Delta_\theta^{-1}\| \leq k_1\end{aligned} \tag{25}$$

where k_1 is a positive constant which does not depend on θ for $\theta \geq 1$. Making use of (24) and (25), one gets :

$$\dot{V} + \frac{\ln\theta}{2\tau^\star}V \leq \quad -(\theta - c_1 - \frac{\ln\theta}{2\tau^\star})\bar{x}^T S\bar{x} + c_2\|\bar{x}\|\|\bar{x}_\tau\| - \frac{\varepsilon}{\sqrt{\theta}}\bar{x}_\tau^T\bar{x}_\tau \tag{26}$$

with

$$c_1 = \frac{1}{\lambda_m(S)}\left(2k_1\|S\| + 1\right) \; and \; c_2 = 2k_1\|S\|$$

where $\lambda_m(S)$ stands for the minimum eigenvalue of S.
Now, let us choose θ high enough such that

$$k(\theta) \triangleq \theta - c_1 - \frac{\ln\theta}{2\tau^\star} > 0 \tag{27}$$

and set

$$V_1 = \bar{x}^T S\bar{x}, \;\; V_2 = \bar{x}_\tau^T\bar{x}_\tau, V_1^\star = k(\theta)V_1, \;\; V_2^\star = \frac{\varepsilon}{\sqrt{\theta}}V_2, V^\star = V_1^\star + V_2^\star, \;\;\; c_3 = \frac{c_2}{\sqrt{\lambda_m(S)}} \tag{28}$$

Using the above notations, inequality (26) can be written as follows:

$$\begin{aligned}\dot{V} + \frac{\ln\theta}{2\tau^\star}V \;\; & \leq \;\; -k(\theta)V_1 - \frac{\varepsilon}{\sqrt{\theta}}V_2 + c_3\sqrt{V_1}\sqrt{V_2} \\ & = \;\; -V_1^\star - V_2^\star + c_3\sqrt{\frac{\theta^{1/2}}{\varepsilon k(\theta)}}\sqrt{V_1^\star}\sqrt{V_2^\star} \\ & \leq \;\; -\left(1 - \frac{c_3}{2}\sqrt{\frac{\theta^{1/2}}{\varepsilon k(\theta)}}\right)V^\star\end{aligned} \tag{29}$$

Now, it suffices to choose θ such that:

$$1 - \frac{c_3}{2}\sqrt{\frac{\theta^{1/2}}{\varepsilon k(\theta)}} > 0 \tag{30}$$

which is equivalent, when replacing $k(\theta)$ by its expression (27), to

$$\theta - c_1 - \frac{\ln\theta}{2\tau^\star} - \frac{c_3^2}{4\varepsilon}\sqrt{\theta} > 0 \tag{31}$$

Combining (29) and (30) gives :

$$\dot{V} + \frac{\ln\theta}{2\tau^\star}V \leq 0 \tag{32}$$

Set $c(\theta) = \dfrac{\ln\theta}{4\tau^\star}$. Then, inequality (32) is equivalent to:

$$\dot{V} \leq e^{-2c(\theta)t} \max_{s\in[-\tau^\star,0]} V(\bar{x}(s)) \tag{33}$$

Hence,

$$\|\bar{x}(t)\| \leq \sqrt{\frac{\alpha_1}{\alpha_2}}e^{-c(\theta)t} \max_{s\in[-\tau^\star,0]} \bar{x}(s) \tag{34}$$

with

$$\alpha_1 = \lambda_M(S) + 1 \ \textit{and}\ \alpha_2 = \lambda_m(S)$$

where $\lambda_M(S)$ stand for the maximum eigenvalue of S.

Since $\theta \geq 1$, one has

$$\|\bar{x}(t)\| \leq \|x(t)\| \leq \theta^{q-1}\|\bar{x}(t)\| \tag{35}$$

and inequality (34) can then be expressed as follows:

$$\|\tilde{x}(t)\| \leq \theta^{q-1}\sqrt{\frac{\alpha_1}{\alpha_2}}e^{-c(\theta)t} \max_{s\in[-\tau^\star,0]} \tilde{x}(s) \tag{36}$$

This ends the proof.

Remark 3.

1 *In the case when an original system $\dot{x} = f(u, u_\tau, x, x_\tau)$, $y = Cx$ can be put under form (1) through a diffeomorphiesm $z = \Phi(u, u_\tau, x, , x_\tau)$, then the observer (11) can be written in the original coordinates, x as follows:*

$$\dot{\hat{x}}(t) = f(u, u_\tau, x, x_\tau) - \theta\left(\frac{\partial\Phi}{\partial x}(u, u_\tau, x, , x_\tau)\right)^{-1}\Delta_\theta^{-1}S^{-1}C^TC(\hat{x}(t) - x(t)) \tag{37}$$

2 *The generalization to the case of multiple is straightforward. Indeed, the equation of the observer is identical to that of system (11). For the proof, an appropriate Lyapunov function is: $V(\bar{x}) = \bar{x}^TS\bar{x} + \theta^{-\frac{t}{2\tau^\star}}\sum_{i=1}^{m}\int_{t-\tau_i}^{t}\theta^{\frac{s}{2\tau^\star}}\bar{x}^T(s)\bar{x}(s)ds$ where m is the number of considered time delays.*

□

4 Example

Consider the following time-delay dynamical system:

$$\left\{\begin{array}{rcl}\dot{x}_1 & = & 0.1\left(x_2 + x_2^3 - 0.1x_1(t-\tau) - u(t-\tau)x_1(t-\tau)\right) \\ \dot{x}_2 & = & -0.1x_2(t-\tau) - \dfrac{x_2}{1+x_2^2(t-\tau)} + sin(x_2(t-\tau)) \\ y & = & x_1\end{array}\right. \tag{38}$$

One can check that the following transformation puts system (38) under form (10) :

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} z_1 & = & x_1 \\ z_2 & = & 0.1(x_2 + x_2^3) \end{pmatrix}$$

The delay value has been set to the constant value $\tau = 1.5\ s$. With $x(t) = [0; cos(t)]$ for $t \in [-\tau, 0]$ and $u(t) = cos(0.1t)$, the solution trajectory of system (38) is bounded. In these settings, Assumptions (A1) to (A3) hold for system (38). An observer of the form (11) can then be synthesized and its equations in the original coordinates can be written as follows (see point 1 of remark 3):

$$\left\{\begin{array}{rcl}\dot{\hat{x}}_1 & = & 0.1\left(\hat{x}_2 + \hat{x}_2^3 - 0.1\hat{x}_1(t-\tau) - u(t-\tau)\hat{x}_1(t-\tau)\right) - 2\theta(\hat{x}_1 - x_1) \\ \dot{\hat{x}}_2 & = & -0.1\hat{x}_2(t-\tau) - \dfrac{\hat{x}_2}{1+x_2^2(t-\tau)} + sin(\hat{x}_2(t-\tau)) - \theta^2\dfrac{10}{1+3\hat{x}_2^2}(\hat{x}_1 - x_1)\end{array}\right. \tag{39}$$

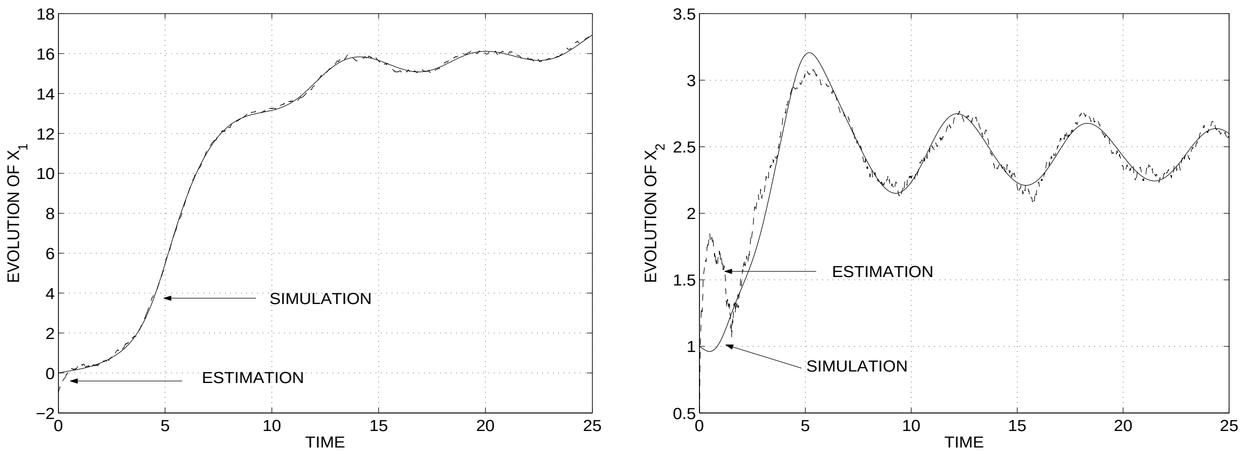

Figure 1: Time evolution of the state variables x_1 and x_2

Before being used by the observer, the system output x_1 has been corrupted by an additive uniformly distributed random signal produced by MATLAB with zero mean value and a relatively high standard deviation equal to 0.5. The observer simulation was performed under similar operating conditions as the model.

The following initial conditions have been used for the observer:

$$\left\{\begin{array}{c}\hat{x}_1(t) = -1 \ \text{ for } t \in [-\tau, 0] \\ \hat{x}_2(t) = 0 \ \text{ for } t \in [-\tau, 0]\end{array}\right.$$

The value of the observer design parameter θ has been set to 2.

Notice that the state estimates quickly converge to the true values (provided by the model). Moreover, the obtained results show the good behaviour of the proposed observer in dealing with the the relatively high noise.

5 Conclusion

A simple nonlinear observer has been proposed for a class of time-delay nonlinear systems. In the absence of delay, the considered class of systems coincides with an observable canonical form that characterizes a class of uniformly observable systems which are observable for any input. The considered time-varying delays are assumed to be known and bounded. Similarly to free delay systems, the gain of the proposed observer does not require the resolution of any dynamical system. It is rather issued from the resolution of a Lyapunov time-invariant algebraic equation and is explicitly given. Its tuning is achieved through the choice of a single design parameter, namely θ. It is established that the estimation error converges exponentially to zero as in the free delay case but with a decay rate proportional to $\ln\theta$ and not to θ.

The observer design described in the paper deals with a particular canonical form that characterizes a class of uniformly observable systems. Its generalization to a nonlinear multi-output systems with non triangular coupled structure characterized by a canonical form similar to that proposed in [19] has just been established and will be presented elsewhere. Moreover, we have extended an adaptive observer synthesis for a class of free delay nonlinear systems with nonlinear parameterization [9] to a similar class of time delay systems where the delayed states intervene in a triangular manner [24]. Current works are concerned with the extension to time delay systems the synthesis of unknown input observers [8, 18].

6 Appendix

The Mean Value Theorem can be stated as follows in the case of a real-valued function of several variables.

Theorem: *Let $f_i : I\!\!R^n \to I\!\!R$ be a $\mathcal{C}^1$ function and let $\hat{x}, x \in I\!\!R^n$. Then, there exists $t_i \in [0,\ 1]$ such that:*

$$\begin{aligned} f_i(\hat{x}) &= f_i(x) + f_i'(x + t_i(\hat{x} - x))(\hat{x} - x) \\ &= f_i(x) + \left(\nabla f_i(x + t_i(\hat{x} - x))\right)^T (\hat{x} - x) \\ &\triangleq f_i(x) + \left(\nabla f_i(\xi_i)\right)^T (\hat{x} - x) \end{aligned} \tag{40}$$

where $\nabla f_i(x + t_i(\hat{x} - x))$ is the gradient of f_i evaluated at $\xi_i = x + t_i(\hat{x} - x)$.

This theorem can be extended to the case of functions that take values in $\mathbb{R}^n$ as follows. Let

$f = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} : \mathbb{R}^n \to \mathbb{R}^n$ be a function of class $\mathcal{C}^1$. Then, there exist $\xi_1, \xi_2, \ldots, \xi_n \in \mathbb{R}^n$ such that:

$$\begin{aligned} f(\hat{x}) &= f(x) + f'(\xi_1, \ldots, \xi_n)(\hat{x} - x) \\ &= f(x) + \frac{\partial f}{\partial x}(\xi_1, \ldots, \xi_n)(\hat{x} - x) \end{aligned} \tag{41}$$

where $\frac{\partial f}{\partial x}$ is the jacobian of f.

More precisely, one has

$$\frac{\partial f}{\partial x}(\xi_1, \ldots, \xi_n) = \begin{pmatrix} (\nabla f_1(\xi_1))^T \\ \vdots \\ (\nabla f_n(\xi_n))^T \end{pmatrix} = \begin{pmatrix} (\nabla f_1(x + t_1(\hat{x} - x)))^T \\ \vdots \\ (\nabla f_n(x + t_n(\hat{x} - x)))^T \end{pmatrix}$$

where $\xi_i = x + t_i(\hat{x} - x)$ with $t_i \in [0, 1]$; $i = 1, \ldots, n$.
For writing convenience, one shall use the following notation:

$$\begin{aligned} \frac{\partial f}{\partial x}(\xi_1, \ldots, \xi_n) = \begin{pmatrix} (\nabla f_1(\xi_1))^T \\ \vdots \\ \left(\nabla f_n^T(\xi_n)\right)^T \end{pmatrix} &= \begin{pmatrix} (\nabla f_1(x + t_1(\hat{x} - x)))^T \\ \vdots \\ (\nabla f_n(x + t_n(\hat{x} - x)))^T \end{pmatrix} \\ &\triangleq \begin{pmatrix} \nabla f_1^T \\ \vdots \\ \nabla f_n^T \end{pmatrix} (x + \Theta(\hat{x} - x)) \\ &= \begin{pmatrix} \nabla f_1^T \\ \vdots \\ \nabla f_n^T \end{pmatrix} (\xi) = \frac{\partial f}{\partial x}(\xi) \end{aligned} \tag{42}$$

where $\xi = x + \Theta(\hat{x} - x)$ and $\Theta = diag(t_1, \ldots, t_n)$.

Since the function f is Lipschitz and using the adopted notation, one has:

$$\begin{aligned} \|\frac{\partial f}{\partial x}(\xi_1, \ldots, \xi_n)\| &\triangleq \|\frac{\partial f}{\partial x}(\xi)\| \\ &\leq L \end{aligned}$$

where L is the Lipschitz constant. From (41) , one obtains:

$$\|f(\hat{x}) - f(x)\| \leq L\|\hat{x} - x\|$$

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