LOWER BOUNDS FOR THE EIGENVALUES OF THE Spin c DIRAC OPERATOR ON MANIFOLDS WITH BOUNDARY

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ROGER NAKAD AND JULIEN ROTH

ABSTRACT. We extend the Friedrich inequality for the eigenvalues of the Dirac operator on Spin^c manifolds with boundary under different boundary conditions. The limiting case is then studied and examples are given.

Version francaise abrégée: En 1980, Th. Friedrich [3] a minoré la première valeur propre λ_1 de l'opérateur de Dirac D défini sur une variété riemannienne compact Spin à courbure scalaire positive R. En effet, il a montré que

$$\lambda_1^2 \geq \frac{n}{4(n-1)} \inf_M \mathrm{R}. \tag{1}$$

Le cas limite est caractérisé par l'existence d'un spineur de Killing. Plus tard, cette minoration a été établie [8, 9] pour la première valeur propre de l 'opérateur de Dirac défini sur une variété compacte Spin à bord et sous différents types de conditions à bord. Dans cette note, on établit l'inégalité de Friedrich dans le cas des variétés compactes Spin^c à bord. En effet, on montre le théorème suivant:

Théorème. On considère une variété riemannenne Spin^c à bord M de dimension n et on note par $i\Omega$ la courbure du fibré en droites associé à la structure Spin^c. On suppose que sous les conditions à bord gAPS donnée par $P_{\geqslant b}$ pour $b \leqslant 0$, qu 'il existe 2 fonctions a et u tel que $b + adu(\nu) \leqslant \frac{1}{2}H$, sur ∂M. Sous les conditions mgAPS $P^m_{\geqslant b}$ pour $b \leqslant 0$ ou CHI, on suppose qu'il existe 2 fonctions a et u tel que $adu(\nu) \leqslant \frac{1}{2}H$, sur ∂M. Alors toute valeur propre de l'opérateur de Dirac D de M satisfait

$$\lambda^2 \geqslant \frac{n}{4(n-1)} \sup_{a,u} \inf_M \left(R_{a,u} - c_n|\Omega|\right).$$

Ici $c_n = 2[\frac{n}{2}]^{\frac{1}{2}}$, $R_{a,u} = R - 4a\Delta u + 4 < \nabla a, \nabla u > -4(1-\frac{1}{n})a^2|\nabla u|^2$, R est la courbure scalaire. Sous les conditions mgAPS et CHI, le cas limite est caractérisé par l'existence d un spineur de Killing sur M et le bord ∂M est minimal. Pour la condition gAPS, le cas limite ne peut être atteint.

Enfin pour la condition MIT, nous démontrons la minoration optimale suivante:

$$|\lambda|^2 \geqslant \frac{n}{4(n-1)} \inf_M \left(R - c_n|\Omega|\right) + nH_0\Im m(\lambda),$$

où H_0 est l'infimum de la courbure moyenne sur M et $\Im(\lambda)$ la partie imaginaire de λ. De plus l'égalité a lieu si et seulement si M admet un spineur de Killing imaginaire et ∂M est totalement ombilique .

1. INTRODUCTION

The spectrum of the Dirac operator on compact Spin manifolds with or without boundary has been extensively studied over the past three decades. First, the intrinsic aspect has been systematically studied then, the extrinsic aspect has been intensively exploited by many

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authors in order to study the geometry and the topology of submanifolds in general, and hypersurfaces in particular (including boundaries of domains). In [3], Friedrich proved that the first eigenvalue of the Dirac operator on a closed manifold (M^n, g) of positive scalar curvature R satisfies

$$\lambda_1^2 \geq \frac{n}{4(n-1)} \inf_M \mathrm{R}. \tag{2}$$

The equality case is characterized by the existence of a real Killing spinor. The existence of such a spinor leads to geometric restrictions on the manifold. For example, the manifold is Einstein and in dimension 4, it has constant sectional curvature. The classification of simply connected Riemannian Spin manifolds carrying real Killing spinors gives, in some dimensions, other examples than the sphere. These examples are relevant to physicists in general relativity where the Dirac operator plays a central role.
In this note, we establish the lower bound (2) for the first eigenvalue of the Dirac operator defined on manifolds with boundary under different boundary conditions. In fact, we prove

Theorem 1.1. *Let (M^n, g) be a Riemannian* $\mathrm{Spin^c}$ *manifold with non empty boundary ∂M and line bundle curvature Ω. Let λ be an eigenvalue of the Dirac operator. Under the gAPS boundry condition $P_{\geqslant b}$ for some $b \leqslant 0$, we assume that there exists some real functions a and u such that*

$$b + a\mathrm{d}u(\nu) \leqslant \frac{1}{2}H,$$

on ∂M. Under the mgAPS boundary condition $P^m_{\geqslant b}$ for some $b \leqslant 0$ or under the CHI boundry condition P_{CHI}, we assume that there exists some real functions a and u such that

$$a\mathrm{d}u(\nu) \leqslant \frac{1}{2}H,$$

on ∂M. Then

$$\lambda^2 \geq \frac{n}{4(n-1)} \sup_{a,u} \inf_M \left(R_{a,u} - c_n|\Omega|\right). \tag{3}$$

Under the gAPS boundary condition, the equality case cannot occur. Under the mgAPS or the CHI boundary conditions, equality occurs if and only if M carries a non trivial real Killing spinor with Killing constant $-\frac{\lambda}{n}$ and the boundary ∂M is minimal.

Under the MIT bag condition P_{MIT}, we prove the following

Theorem 1.2. *Let (M^n, g) be a Riemannian* $\mathrm{Spin^c}$ *manifold with non empty boundary. Let $i\Omega$ be the curvature of the auxiliary Line bundle associated with the* $\mathrm{Spin^c}$ *structure. Let λ be an eigenvalue of the Dirac operator under the MIT bag condition P_{MIT}. Assume that the mean curvature H of ∂M is strictly positive, then*

$$|\lambda|^2 \geqslant \frac{n}{4(n-1)} \inf_M \left(R - c_n|\Omega|\right) + nH_0\Im m(\lambda), \tag{4}$$

where H_0 is the infimum of H on ∂M. When equality holds, the eigenspinor ψ is an imaginary $\mathrm{Spin^c}$ *Ω-Killing spinor and the boundary ∂M is totally umbilical with constant mean curvature equal to $H_0 = \dfrac{2\Im m(\lambda)}{n}$.*

At the end, we focus on examples satisfying the limiting case in (4) and (3), which are Spin^c but not Spin .

2. Manifolds with boundary.

Let (M^n, g) be a Riemannian $\mathrm{Spin^c}$ manifold with nonempty compact boundary ∂M. We denote by ∇ the Levi Civita $\mathrm{Spin^c}$ connection of M, $< .,. >$ denotes the Hermitian scalar product on the $\mathrm{Spin^c}$ bundle ΣM and "γ" the Clifford multiplication on M. We denote by L the auxiliary line bundle associated to the $\mathrm{Spin^c}$ structure and $i\Omega$ its curvature imaginary

2-form of some hermitian connection (see [4]). We will consider boundary conditions in order to generalize the Friedrich eigenvalue estimate for the spectrum of the Spin^c Dirac operator on M.

APS and gAPS boundary conditions: The well-known APS boundary condition [1] was introduced by Athiyah, Patodi and Singer. Since the boundary is a closed manifold, its Dirac operator has a real discrete spectrum and we defined the projection $\pi_+ : \Gamma(\Sigma M) \longrightarrow \Gamma(\Sigma M)$ onto the subspace of $\Gamma(\Sigma M)$ spanned by the eigenspinors associated with nonnegative eigenvalues. It is a classical fact ([9]) that this gives a self-adjoint elliptic boundary condition for the Dirac operator. and so it has a real discrete spectrum. The generalized Athiyah-Patodi-Singer condition, denoted gAPS is a generalization of the APS condition: For any real number b, we consider the projection $P_{\geqslant b}$ onto the subspace of $\Gamma(\Sigma M)$ spanned by the eigenspinors φ_k associated with eigenvalues $\lambda_k \geqslant b$. As mentionned in [2], this is also a self-adjoint elliptic boundary condition for any nonpositive b. We remark that the gAPS boundary condition for $b = 0$ is just the standard APS condition. Moreover, for more convenience, we will use the following useful notations, $P_{>b}$ is defined in the same way but the projection is on the subspace spanned by the eigenspinors φ_k associated with eigenvalues $\lambda_k > b$ We also define $P_{<b} = Id - P_{\geqslant b}$.

mAPS and mgAPS boundary conditions: The mAPS and mgAPS boundaryionditons are modifications of the APS and gAPS, respectively, in the following way. For $\varphi \in \Gamma(\Sigma M)$, we have $P^m_{\geqslant b}\varphi = P_{\geqslant b}(Id + \gamma(\nu))\varphi$. For $b = 0$, this condition is just the modified APS condition (mAPS) introduced by Hijazi, Montiel and Roldan [9].

Boundary condition associated with a chirality operator: In contrast with the above boundary conditions, we consider the following local boundary condition associated with a chirality operator which is subject to the existence of such an operator. So we consider an linear map $G : \Gamma(\Sigma M) \longrightarrow \Gamma(\Sigma M)$ such that

(5) $$G^2 = Id, \quad \langle G\varphi, G\psi\rangle = \langle \varphi, \psi\rangle\,, \quad \nabla_X(G\varphi) = G\nabla_X\varphi, \quad \gamma(X)G\varphi = -G\gamma(X)\varphi$$

for any vector X tangent to M. Such an operator is called a *chirality* operator because in the even dimensional case, an example is $G = \gamma(\omega_n)$ the Clifford multiplication by the complex volume element which gives the chirality decomposition of the spinor bundle. The boundary condition associated with this operator is defined by: $P_{CHI} = \frac{1}{2}(Id - \gamma(\nu)G)$. As proved in [9], this condition is self-adjoint and elliptic and so, under this boundary condition, the Dirac operator has a real discrete spectrum.

MIT boundary conditions: The MIT boundary condition is also a local boundary condition. It is defined as follows: For any spinor field φ on ∂M, $P_{MIT}\varphi = \frac{1}{2}(\varphi - i\gamma(\nu)\varphi)$. It is an elliptic condition and the spectrum of the Dirac operator is discrete. However, the Dirac operator is not self-adjoint anymore and its spectrum consists of complex eigenvalues whose imaginary part is strictly positive.

Lemma 2.1. *Let $b \leqslant 0$. We denote by D the Dirac operator on the boundary ∂M of M. Then,*

$$\int_{\partial M} \langle D\varphi, \varphi\rangle \quad \begin{cases} \leqslant b\int_M |\varphi|^2 & \text{under the gAPS condition,} \quad P_{\leq b}\varphi = 0 \\ = 0 & \text{under the mgAPS condition,} \quad P^m_{\leq b}\varphi = 0. \end{cases}$$

Under the CHI and MIT conditions, then, point wise on ∂M we have $\langle D\varphi, \varphi\rangle = 0$.

Proof: Let $(\varphi_k, \lambda_k)_{k\in\mathbb{Z}}$ be a spectral resolution of D on the boundary ∂M. Any spinor φ of $\Gamma(\Sigma\partial M)$ expresses as follows $\varphi = \sum_k a_k\varphi_k$ with $a_k = \int_{\partial M} < \varphi, \varphi_k > ds$. Under the gAPS condition, we have $P_{\geqslant b}\varphi = 0$, that is, $\varphi = \sum_{\lambda_k<b} a_k\varphi_k$. Then, ve have

$$\int_{\partial M} < D\varphi, \varphi > ds = \sum_{\lambda_k<b} \lambda_k|a_k|^2 \leqslant b\sum_{\lambda_k<b} |a_k|^2 = b\int_{\partial M} |\varphi|^2 ds.$$

Let φ such that $P^m_{\geqslant b}\varphi = 0$, for $b \leqslant 0$, that is, $P_{\geqslant b}(\varphi + \gamma(\nu)\varphi) = 0$. From this, we can see easily that $P_{>-b}(\varphi + \gamma(\nu)\varphi) = 0$. Moreover, from the relation $D(\gamma(\nu)) = -\gamma(\nu)D$, we see that for any b and any spinor ψ, $P_{<b}\gamma(\nu)\psi = \gamma(\nu)P_{>-b}\psi$. Hence, we have

$$\begin{aligned} P_{>-b}(\gamma(\nu)\varphi - \varphi) &= \gamma(\nu)P_{<b}\varphi - P_{>-b}(\varphi) = \gamma(\nu)\Big(P_{<b}\varphi + \gamma(\nu)P_{>-b}(\varphi)\Big) \\ &= \gamma(\nu)P_{<b}(\varphi + \gamma(\nu)\varphi) = \gamma(\nu)\Big[\varphi + \gamma(\nu)\varphi - \underbrace{P_{\geqslant b}(\varphi + \gamma(\nu)\varphi}_{=0}\Big] \\ &= \gamma(\nu)\varphi - \varphi. \end{aligned}$$

Now, using that $< D\varphi, \varphi >= \frac{1}{2} < D(\varphi+\gamma(\nu)\varphi), \varphi-\gamma(\nu)\varphi >$, the fact that $\gamma(\nu)\varphi-\varphi = P_{>-b}(\gamma(\nu)\varphi - \varphi)$ and $P_{>-b}(\varphi + \gamma(\nu) = 0$, we deduce that

$$\int_{\partial M} < D\varphi, \varphi >= \int_{\partial M} \frac{1}{2} < D(\varphi + \gamma(\nu)\varphi), \varphi - \gamma(\nu)\varphi >= 0.$$

Now, we observe that from (5), we have the following pointwise equality $< D\varphi, \varphi >=< \gamma(\nu)GD\varphi, \gamma(\nu)G\varphi >$. Moreover, we have $DG = GD$ and since $P_{CHI}\varphi = 0$, then $\gamma(\nu)G\varphi = \varphi$. So, we get

$$< D\varphi, \varphi >=< \gamma(\nu)GD\varphi, \varphi >=< \gamma(\nu)DG\varphi, \varphi >= - < D\gamma(\nu)G\varphi, \varphi >= - < D\varphi, \varphi > .$$

Finally, we have $< D\varphi, \varphi >= 0$.
The proof for the MIT condition is similar.

3. Eigenvalue estimates for manifolds with boundary.

First, for any real functions a and u, we consider the following modified connection $\nabla^{a,u}$ on M

$$\nabla^{a,u}_X\varphi = \nabla_X\varphi + a\nabla_X u \cdot \varphi + \frac{a}{n}X \cdot \nabla u \cdot \varphi + \frac{\lambda}{n}X \cdot \varphi,$$

where $X \in \mathfrak{X}(M)$ and $\varphi \in \Gamma(\Sigma M)$. A simple calculation, using the Spin^c Reilly inequality (see [8, 12]), we have

$$\begin{aligned} \int_M |\nabla^{a,u}\varphi|^2 &= \int_M \left[\left(1 - \frac{1}{n}\right)\lambda^2 - \frac{R_{a,u}}{4}\right]|\varphi|^2 dv_g + \int_M < \frac{i}{2}\Omega^{\mathcal{Z}} \cdot \psi, \psi > dv_g \\ &+ \int_{\partial M}\left(< D\varphi, \varphi > + \left[a\mathrm{d}u(\nu) - \frac{H}{2}\right]\right)|\varphi|^2 ds, \end{aligned} \tag{6}$$

where $R_{a,u}$ is defined by

$$R_{a,u} = R - 4a\Delta u + 4 < \nabla a, \nabla u > -4\left(1 - \frac{1}{n}\right)a^2|\nabla u|^2.$$

We have [6] that $< i\Omega \cdot \varphi, \varphi >\geqslant -\frac{c_n}{2}|\Omega|_g|\varphi|^2$, where $|\Omega|_g$ is the norm of Ω with respect to the metric g given by $|\Omega|^2_g = \sum_{i<j}\Omega^2_{ij}$ in any orthornomal frame and $c_n = 2[\frac{n}{2}]^{1/2}$. Moreover, equality occurs if and only if $\Omega \cdot \varphi = i\frac{c_n}{2}|\Omega|_g\varphi$. Using this and the fact that $|\nabla^{a,u}\varphi|^2 \geqslant 0$, Inequality (6) becomes

$$\int_M \left[\left(1 - \frac{1}{n}\right)\lambda^2 - \frac{R_{a,u}}{4} - \frac{c_n}{4}|\Omega|_g\right]|\varphi|^2 dv_g \geqslant \int_{\partial M}\left(< D\varphi, \varphi > + \left[a\mathrm{d}u(\nu) - \frac{H}{2}\right]|\varphi|^2\right) ds \tag{7}$$

Now, we can prove theorem 1.1.
Proof of theorem 1.1: From Inequality (7), Lemma 2.1 and the assumption $a + \mathrm{d}u(\nu) \leqslant \frac{1}{2}H$, we get immediately that

$$\lambda^2 \geqslant \frac{n}{4(n-1)}\sup_{a,u}\inf_M\left(R_{a,u} - c_n|\Omega|\right).$$

We just have to prove that inequality can not occur under the gAPS boundry condition. For this, we need the following lemma, generalizing Lemma 3 in [7] to the case of Spin^c manifolds.

Lemma 3.1. *Suppose that there exists a spinor field φ satisfying*

$$\nabla^{a,u}\varphi = 0 \quad \textit{and} \quad \Omega\cdot\varphi = i\frac{c_n}{2}|\Omega|\varphi, \tag{8}$$

for some real number λ and real functions a and u. Then $a = 0$ or $u = 0$, that is, φ is a Killing spinor.

Assuming this lemma, then φ is a non-trivial real Killing spinor, and so $|\varphi|$ is a positive constant. Let $(\varphi_k)_{k\in\mathbb{Z}}$ be an hilbertian basis of eigenspinors for the Dirac operator of the boundary, associated to the eigenvalues $(\lambda_k)_{k\in\mathbb{Z}}$. Under the gAPS condition, we have $P_{\geqslant b}\varphi = 0$, that is, $\varphi = \sum_{\lambda_k > b} a_k\varphi_k$, where $a_k = \int_{\partial M} <\varphi, \varphi_k> ds$. Then, ve have

$$\begin{aligned} 0 &= \int_{\partial M} < D\varphi, \varphi > ds - \int_{\partial M}\frac{H}{2}|\varphi|^2 ds = \sum_{\lambda_k<b}\lambda_k a_k^2 - \frac{1}{2}\sum_{\lambda_j,\lambda_k<b} a_j a_k\int_{\partial M} <\varphi_j,\varphi_k> ds \\ &\leqslant \sum_{\lambda_k<b}(\lambda_k - b)|a_k|^2 < 0, \end{aligned}$$

since $H \geqslant 2b$. This is a contradiction and so equality cannot occur.

Proof of Lemma 3.1: First, we observe that (8) implies that $D\varphi = \lambda\varphi$. Now, we use the Ricci identity and we get

$$\frac{1}{2}e_k\cdot Ric(e_k)\cdot\varphi = \frac{i}{2}e_k\cdot(e_k\lrcorner\Omega)\cdot\varphi + e_k\cdot\sum_{j=1}^{n} e_j\cdot\mathcal{R}(e_j, X)\varphi$$

Hence, by summing on k from 1 to n, we have

$$\begin{aligned} \sum_{j,k}\frac{1}{2}R_{kj}e_k\cdot e_j\varphi &= \frac{i}{2}\Omega\cdot\varphi - D^2\varphi + \sum_k e_k\cdot\nabla_{e_k}(D\varphi) \\ &= \Big(\frac{c_n}{4}|\Omega| - \frac{2(1-n)}{n}\lambda^2 + \frac{2a}{n}\Delta u - \frac{2(2-n)}{n} < \nabla u, \nabla a > \Big)\varphi \\ &\quad -\frac{2}{n}\nabla u\cdot\nabla a\cdot\varphi + \frac{4a\lambda}{n^2}\nabla u\cdot\varphi \end{aligned}$$

From this, we deduce that the term $\frac{4a\lambda}{n^2}\nabla u\cdot\varphi$ necessarily vanishes and so $a = 0$ or $\nabla u = 0$, which implies that φ is a Killing spinor with Killing constant λ. □

In a similar way, we can prove Theorem 1.1

Examples: Let $M = (\mathbb{S}^3, g_{\kappa,\tau})$ be the sphere endowed with the Berger metric. For $\kappa > 0$ and $\tau \neq 0$ this metric is defined by $g_{(\kappa,\tau)}(X,Y) = \frac{\kappa}{4}\Big(g(X,Y) + (\frac{4\tau^2}{\kappa} - 1)g(X,\xi)g(Y,\xi)\Big)$, where g is the round metric and ξ the Killing vector tangent to the fibers of the Hopf fibration of $\mathbb{S}^3$. For $\kappa = 4\tau^2$, we found the round sphere of curvature κ. Berger spheres are also embedded spheres of constant mean curvature in the complex space forms of constant holomorphic sectional curvature $1 - \tau^2$. In [11], we proved that M has a canonical Spin^c structure carrying a Killing spinor of Killing constant $-\frac{\tau}{2}$. The curvature of the line bundle associated to this canonical Spin^c structure is given in a local orthonormal frame $\{e_1, e_2, e_3 = \xi\}$ by

$$\Omega(e_1, e_2) = (\kappa - 4\tau^2) \text{ and } \Omega(e_i, e_j) = 0 \text{ if not.} \tag{9}$$

It is straightforward that $D\varphi = \frac{3}{2}\varphi$. Moreover, the scalar curvature of M is given by $2\kappa - 2\tau^2$. By definition of the canonical Spin^c structure, we have $|\Omega| = \kappa - 4\tau^2$. Finally, since $c_3 = 2$, we get $\frac{3}{8}(R - c_3|\Omega|) = \frac{9\tau^2}{4}$. Let now consider a domain of M bounded by a minimal surface. It remains to prove that φ satisfies the condition $P^m_{APS} = 0$. The restriction of φ to the boundary satisfies $D\varphi = H\varphi + \tau\gamma(\nu)\varphi$. Because the boundary is minimal, we have $D\varphi = \tau\gamma(\nu)\varphi$. Using the super-symmetry property $D(\gamma(\nu)\varphi) = -\gamma(\nu)D\varphi$, we have $D(\varphi + \gamma(\nu)\varphi) = \tau(\varphi + \gamma(\nu)\varphi)$. For $\tau < 0$, this implies that $P^m_{APS} = 0$. For Berger spheres with $\tau > 0$, we have to take the anti-canonical $\mathrm{Spin^c}$ structure that has a Killing spinor of opposite Killing constant. To summarize, we proved that every domain of the Berger sphere bounded by a minimal surface is an example of the limiting case of Inequality (3) for the condition mAPS. Such domains exist because we know examples of compact minimal surfaces embedded into the Berger spheres (For example, the equator of the Berger spheres and the minimal Clifford tori). In [14], Torralbo constructed a family of of minimal onduloïdes and some of them are embedded.

The case of MIT bag condition Under the MIT bag condition, the spectrum of the Dirac operator is an unbounded sequence of complex number with nonnegative imaginary part. Equality in (3) cannot hold. Following the ideas of Raulot [13], we will derive an optimal inequality for the eigenvalues of the Dirac operator for the boundary condition P_{MIT}.

Lemma 3.2. *Let μ b a complex number and ψ a non trivial* $\mathrm{Spin^c}$ *Ω-Killing spinor field of Killing constant μ, i.e., for any $X \in \Gamma(TM)$,*

$$\begin{cases} \nabla_X\psi = \mu X \cdot \psi, \\ i\Omega \cdot \psi = \frac{c_n}{2}|\Omega|\psi. \end{cases}$$

Then, μ is real number or an imaginary number and ψ has no zeros.

Proof: The fact that ψ has no zeros is well known (see [4]). The Schrödinger-Lichnerowicz formula applied for the spinor ψ, gives

$$D^2\psi = n^2\mu^2\psi = \nabla^*\nabla\psi + \frac{1}{4}R\psi - \frac{1}{4}c_n|\Omega|\psi = n\mu^2\psi + \frac{1}{4}R\psi - \frac{1}{4}c_n|\Omega|\psi.$$

Since ψ has no zeros, we deduce that $n(n-1)\mu^2 = \frac{1}{4}(R - c_n|\Omega|)$. Thus μ^2 is real and hence μ is real or pure imaginary. □

Proof of Theorem 1.2: Proceeding as in [13], we obtain a from the Spin^c Reilly formula, for an eigenspinor φ:

$$\int_M \left(\frac{n-1}{n}|\lambda - niH_0|^2 - \frac{R}{4} + \frac{c_n}{4}|\Omega| - n(n-1)H_0^2\right)|\varphi|^2 dv_g \geqslant 0, \tag{10}$$

where H_0 is the infimum of H on ∂M, with equality if and only if φ is a Killing spinor of Killing constant $-\frac{\lambda}{n}$, $\Omega \cdot \varphi = i\frac{c_n}{2}|\Omega|_g\varphi$, and H is constant (equals to H_0). From (10), we obtain immediately the desired lower bound. If equality holds in this lower bound, from Lemma 3.2, λ is either real or imaginary, but as an eigenvalue of the Dirac operator for the MIT boundary condition, λ has positive imaginary part. Hence, lambda is imaginary and φ is an imaginary Ω-Killing spinor. The fact that the boundary ∂M is umbilical is similar to the spin case (see [13]).

Examples: Riemannian manifolds with imaginary $\mathrm{Spin^c}$ Killing spinors of Killing number $i\mu$ have been classified in [5]. Such manifolds are: The hyperbolic space endowed with its unique Spin structure and the warped product with $\mathbb{R}$ of a Riemmanian $\mathrm{Spin^c}$ manifold carrying a parallel spinor, i.e, $(F^{n-1} \times \mathbb{R}, e^{4\mu t}h \oplus dt^2)$ where (F^{n-1}, h) is a complete $\mathrm{Spin^c}$ manifold with a parallel spinor field. As examples which are not Spin we can state $(\mathbb{C}P^2 \times \mathbb{R}, e^{4\mu t}g_{FS} \oplus dt^2)$ or $(F^{2m} \times \mathbb{R}, e^{4\mu t}h \oplus dt^2)$ where g_{FS} is the Fubini Study metric and (F^{2m}, h) is a Kähler manifold endowed with the canonical or the anti-canonical $\mathrm{Spin^c}$ structure. Totally umbilical embedded hypersurfaces of constant mean curvature in $(F^{2m} \times \mathbb{R}, e^{4\mu t}h \oplus dt^2)$ exist. For example, Montiel [10] proved that such a hypersurface is a leaf of the foliation $\mathcal{F}(\mathcal{X})$ (Here $\mathcal{X}$ denotes a non-trivial conformal closed vector field which exist for $(F^{2m}\times\mathbb{R}, e^{4\mu t}h\oplus dt^2)$) or a it is locally a Riemannian product $\mathbb{R}\times\mathcal{Q}^{2m-1}$ immersed into $\mathbb{R}^2\times\mathcal{Q}^{n-1}$ as $\gamma\times I_{\mathcal{Q}^{n-1}}$, where γ is a line in $\mathbb{R}^2$.

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