Homogenization and two-scale convergence

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Abstract. Following an idea of G. Nguetseng, the author defines a notion of "two-scale" convergence, which is aimed at a better description of sequences of oscillating functions. Bounded sequences in $L^2(\Omega)$ are proven to be relatively compact with respect to this new type of convergence. A corrector-type theorem (i.e., which permits, in some cases, replacing a sequence by its "two-scale" limit, up to a strongly convergent remainder in $L^2(\Omega)$) is also established. These results are especially useful for the homogenization of partial differential equations with periodically oscillating coefficients. In particular, a new method for proving the convergence of homogenization processes is proposed, which is an alternative to the so-called energy method of Tartar. The power and simplicity of the two-scale convergence method is demonstrated on several examples, including the homogenization of both linear and nonlinear second-order elliptic equations.

Key words. homogenization, two-scale convergence, periodic

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Introduction. This paper is devoted to the homogenization of partial differential equations with periodically oscillating coefficients. This type of equation models various physical problems arising in media with a periodic structure. Quite often the size of the period is small compared to the size of a sample of the medium, and, denoting their ratio by ε, an asymptotic analysis, as $\varepsilon \to 0$, is required: namely, starting from a microscopic description of a problem, we seek a macroscopic, or averaged, description. From a mathematical point of view, we have a family of partial differential operators L_ε (with coefficients oscillating with period ε), and a family of solutions u_ε which, for a given domain Ω and source term f, satisfy

$$(0.1) \qquad L_\varepsilon u_\varepsilon = f \quad \text{in } \Omega,$$

complemented by appropriate boundary conditions. Assuming that the sequence u_ε converges, in some sense, to a limit u, we look for a so-called homogenized operator $\bar{L}$ such that u is a solution of

$$(0.2) \qquad \bar{L}u = f \quad \text{in } \Omega.$$

Passing from (0.1) to (0.2) is the homogenization process. (There is a vast body of literature on that topic; see [10], [40] for an introduction, and additional references.) Although homogenization is not restricted to the case of periodically oscillating operators (cf. the Γ-convergence of DeGiorgi [16], [17], the H-convergence of Tartar [42], [34], or the G-convergence of Spagnolo [41], [49]), we restrict our attention to that particular case. This allows the use of the well-known two-scale asymptotic expansion method [7], [10], [27], [40] in order to find the precise form of the homogenized operator $\bar{L}$. The key to that method is to postulate the following ansatz for u_ε:

$$(0.3) \qquad u_\varepsilon(x) = u_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right) + \cdots,$$

where each term $u_i(x, y)$ is periodic in y. Then, inserting (0.3) in (0.1) and identifying powers of ε leads to a cascade of equations for each term u_i. In general, averaging with respect to y that for u_0 gives (0.2), and the precise form of $\bar{L}$ is computed with the help of a so-called cell equation in the unit period (see [10], [40] for details). This method is very simple and powerful, but unfortunately is formal since, a priori, the ansatz (0.3) does not hold true. Thus, the two-scale asymptotic expansion method is used only to guess the form of the homogenized operator $\bar{L}$, and other arguments are needed to prove the convergence of the sequence u_ε to u. To this end, the more general and powerful method is the so-called energy method of Tartar [42]. Loosely speaking, it amounts to multiplying equation (0.1) by special test functions (built with the solutions of the cell equation), and passing to the limit as $\varepsilon \to 0$. Although products of weakly convergent sequences are involved, we can actually pass to the limit thanks to some "compensated compactness" phenomenon due to the particular choice of test functions.

Despite its frequent success in the homogenization of many different types of equations, this way of proceeding is not entirely satisfactory. It involves two different steps, the formal derivation of the cell and homogenized equation, and the energy method, which have very little in common. In some cases, it is difficult to work out the energy method (the construction of adequate test functions could be especially tricky). The energy method does not take full advantage of the periodic structure of the problem (in particular, it uses very little information gained with the two-scale asymptotic expansion). The latter point is not surprising since the energy method was not conceived by Tartar for periodic problems, but rather in the more general (and more difficult) context of H-convergence. Thus, there is room for a more efficient homogenization method, dedicated to partial differential equations with periodically oscillating coefficients. The purpose of the present paper is to provide such a method that we call two-scale convergence method.

This new method relies on the following theorem, which was first proved by Nguetseng [36].

THEOREM 0.1. *Let u_ε be a bounded sequence in $L^2(\Omega)$ (Ω being an open set of $\mathbb{R}^N$). There exists a subsequence, still denoted by u_ε, and a function $u_0(x, y) \in L^2(\Omega \times Y)$ ($Y = (0 ; 1)^N$ is the unit cube) such that*

$$(0.4) \qquad \lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x) \psi\left(x, \frac{x}{\varepsilon}\right) dx = \int_\Omega \int_Y u_0(x, y) \psi(x, y) \, dx \, dy$$

for any smooth function $\psi(x, y)$, which is Y-periodic in y. Such a sequence u_ε is said to two-scale converge to $u_0(x, y)$.

We provide a simple proof of Theorem 0.1 along with a new corrector result.

THEOREM 0.2. *Let u_ε be a sequence that two-scale converges to $u_0(x, y)$. Then, u_ε weakly converges in $L^2(\Omega)$ to $u(x) = \int_Y u_0(x, y) \, dy$, and we have*

$$(0.5) \qquad \lim_{\varepsilon \to 0} \|u_\varepsilon\|_{L^2(\Omega)} \geqq \|u_0\|_{L^2(\Omega \times Y)} \geqq \|u\|_{L^2(\Omega)}.$$

Furthermore, if equality is achieved in the left part of (0.5), namely,

$$(0.6) \qquad \lim_{\varepsilon \to 0} \|u_\varepsilon\|_{L^2(\Omega)} = \|u_0\|_{L^2(\Omega \times Y)},$$

and if $u_0(x, y)$ is smooth, then we have

$$(0.7) \qquad \lim_{\varepsilon \to 0} \left\| u_\varepsilon(x) - u_0\left(x, \frac{x}{\varepsilon}\right) \right\|_{L^2(\Omega)} = 0.$$

Loosely speaking, Theorem 0.1 is a rigorous justification of the first term in the ansatz (0.3), while Theorem 0.2 gives the condition of a strong convergence to zero of the difference between u_ε and its ansatz. We are now equipped to explain the two-scale convergence method. We multiply equation (0.1) by a test function of the type $\psi(x, x/\varepsilon)$, where $\psi(x, y)$ is a smooth function, Y-periodic in y. After some integration by parts, we pass to the two-scale limit with the help of Theorem 0.1. In the limit, we read off a variational formulation for $u_0(x, y)$. The corresponding partial differential equation is called the two-scale homogenized problem. It is usually of the same type as the original problem (0.1), but it involves two variables x and y. Thus, averaging with respect to y leads to the homogenized problem (0.2). Eventually, so-called corrector results (i.e., strong or pointwise convergences) can be obtained by the application of Theorem 0.2.

We emphasize that the two-scale convergence method is self-contained, i.e., in a single process we find the homogenized equation and we prove convergence. This is in contrast with the former "usual" homogenization process (as described above) which is divided in two steps: first, find the homogenized and cell equations by means of asymptotic expansions; second, prove convergence with the energy method. Another interesting feature of the two-scale convergence method is the introduction of the two-scale homogenized problem. It turns out that it is a well-posed system of equations, which are a combination of the usual homogenized and cell equations. Indeed, if it is expected that the periodic oscillations in the operator L_ε generate only the same type of oscillations in the solution u_ε, the sequence u_ε is completely characterized by its two-scale limit $u_0(x, y)$. Thus, starting from a well-posed problem for u_ε, we should obtain in the limit a well-posed problem of the same type for u_0. However, this is not always the case for the usual macroscopic homogenized equation (the solution of which is $u(x) = \int_Y u_0(x, y)\, dy$). When averaging the two-scale homogenized problem with respect to y, its "nice" form can disappear, and, rather, we could obtain integro-differential terms (corresponding to memory effects), nonlocal terms, or nonexplicit equations. There are many such examples in the literature (see [5], [29], [32], [46], where "classical" methods are used, and [2], [3], [37], where two-scale convergence is applied). In these cases, the two-scale homogenized problem explains and simplifies the complicated form of the macroscopic limit equation, thanks to the additional microscopic variable y, which plays the role of a hidden variable.

Since Theorem 0.1 proves the existence of the first term in the ansatz (0.3), the two-scale convergence method appears as the mathematically rigorous version of the, intuitive and formal, two-scale asymptotic expansion method [7], [10], [27], [40]. The key of the success for such a method is to consider only periodic homogenization problems. This amounts to restricting the class of possible oscillations of the solutions to purely periodic ones. Working with the relatively small class of periodic oscillations allows us to obtain the representation formula (0.4) for weak limits of solutions. For general types of oscillations, a result like (0.4) seems to be out of reach (the main obstacle being how to choose the test functions). On the other hand, periodic homogenization can be cast into the framework of quasi-periodic, or almost-periodic (in the sense of Besicovitch) homogenization (see, e.g., [28], [38]), since periodic functions are a very special subclass of quasi-, or almost-, periodic functions. In this case, test functions can also be written $\psi(x, x/\varepsilon)$, where $\psi(x, y)$ is quasi-, or almost-, periodic in y. However, we do not know if Theorem 0.1 can be generalized to such test functions or if a new convergence method can thus be obtained.

The paper is organized as follows. Section 1 is devoted to the proof of Theorems 0.1 and 0.2, and other related results. In § 2, we show precisely how the two-scale

convergence method works on the homogenization of linear second-order elliptic equations (this is the favorite model problem in homogenization; see, e.g., Chapter 1 in [10]). We do this in a fixed domain Ω, but also in a periodically perforated domain Ω_ε (a porous medium), obtained by removing from Ω infinitely many small holes of size ε (their number is of order ε^{-N}), which support a Neumann boundary condition. Two-scale convergence is particularly well adapted to the latter case, and we recover previous results (see [13], [1], [4]) without using any extension techniques. Section 3 generalizes § 2 to the nonlinear case. In the periodic setting, we give a new proof of the Γ-convergence of convex energies (see [31], [16], [17]), and we revisit the homogenization of monotone operators [42]. On the contrary of §§ 2 and 3, § 4 deals with an example of homogenization where typical two-scale phenomena appear. We consider a linear elliptic second-order equation with periodic coefficients taking only two values 1 and ε^2. It models a diffusion process in a medium made of two highly heterogeneous materials. It turns out that the limit diffusion process is of a very special type: the usual homogenized problem is not an explicit partial differential equation. Finally, § 5 is devoted to the proof of a technical lemma used in § 1; more generally, we investigate under which regularity assumptions on a Y-periodic function $\psi(x, y)$ the following convergence holds true:

$$(0.8) \qquad \lim_{\varepsilon \to 0} \int_\Omega \left| \psi\left(x, \frac{x}{\varepsilon}\right) \right| dx = \int_\Omega \int_Y |\psi(x, y)| \, dx \, dy.$$

It is easily seen that continuous functions satisfy (0.8). We prove that (0.8) still holds true for functions of $L^1[\Omega; C_\#(Y)]$ or $L^1_\#[Y; C(\bar{\Omega})]$, which are continuous in only one variable, x or y. However, we cannot decrease the regularity of $\psi(x, y)$ too much. Indeed, we construct a counterexample to (0.8) for a function $\psi(x, y)$ of $C[\Omega; L^1_\#(Y)]$, which is not continuous in x for any value of y, but merely continuous in x in the "$L^1(Y)$-mean."

1. Two-scale convergence. Let us begin this section with a few notations. Throughout this paper Ω is an open set of $\mathbb{R}^N (N \geqq 1)$, and $Y = [0; 1]^N$ is the closed unit cube. As usual, $L^2(\Omega)$ is the Sobolev space of real-valued functions that are measurable and square summable in Ω with respect to the Lebesgue measure. We denote by $C^\infty_\#(Y)$ the space of infinitely differentiable functions in $\mathbb{R}^N$ that are periodic of period Y. Then, $L^2_\#(Y)$ (respectively, $H^1_\#(Y)$) is the completion for the norm of $L^2(Y)$ (respectively, $H^1(Y)$) of $C^\infty_\#(Y)$. Remark that $L^2_\#(Y)$ actually coincides with the space of functions in $L^2(Y)$ extended by Y-periodicity to the whole of $\mathbb{R}^N$.

Let us consider a sequence of functions u_ε in $L^2(\Omega)$ (ε is a sequence of strictly positive numbers which goes to zero). Following the lead of Nguetseng [36], we introduce the following.

DEFINITION 1.1. A sequence of functions u_ε in $L^2(\Omega)$ is said to *two-scale converge* to a limit $u_0(x, y)$ belonging to $L^2(\Omega \times Y)$ if, for any function $\psi(x, y)$ in $D[\Omega; C^\infty_\#(Y)]$, we have

$$(1.1) \qquad \lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x) \psi\left(x, \frac{x}{\varepsilon}\right) dx = \int_\Omega \int_Y u_0(x, y) \psi(x, y) \, dx \, dy.$$

This new notion of "two-scale convergence" makes sense because of the next compactness theorem.

THEOREM 1.2. *From each bounded sequence u_ε in $L^2(\Omega)$, we can extract a subsequence, and there exists a limit $u_0(x, y) \in L^2(\Omega \times Y)$ such that this subsequence two-scale converges to u_0.*

To establish Theorem 1.2, we need the following lemma, the proof of which may be found in § 5.

LEMMA 1.3. *Let $\psi(x, y)$ be a function in $L^2[\Omega; C_\#(Y)]$, i.e., measurable and square summable in $x \in \Omega$, with values in the Banach space of continuous functions, Y-periodic in y. Then, for any positive value of ε, $\psi(x, x/\varepsilon)$ is a measurable function on Ω, and we have*

$$(1.2) \qquad \left\| \psi\left(x, \frac{x}{\varepsilon}\right) \right\|_{L^2(\Omega)} \leqq \|\psi(x, y)\|_{L^2[\Omega;\, C_\#(Y)]} \equiv \left[\int_\Omega \operatorname*{Sup}_{y \in Y} |\psi(x, y)|^2 \, dx \right]^{1/2}$$

and

$$(1.3) \qquad \lim_{\varepsilon \to 0} \int_\Omega \psi\left(x, \frac{x}{\varepsilon}\right)^2 \, dx = \int_\Omega \int_Y \psi(x, y)^2 \, dx \, dy.$$

DEFINITION 1.4. A function $\psi(x, y)$, Y-periodic in y, and satisfying (1.3), is called an "admissible" test function.

It is well known (and easy to prove) that a continuous function $\psi(x, y)$ on $\Omega \times Y$, Y-periodic in y, satisfies (1.3). However, the situation is not so clear if the regularity of ψ is weakened: in particular, the measurability of $\psi(x, x/\varepsilon)$ is not obvious. To our knowledge, the minimal regularity hypothesis (if any) making of $\psi(x, y)$ an "admissible" test function is not known. In order that the right-hand side of (1.3) makes sense, $\psi(x, y)$ must at least belong to $L^2(\Omega \times Y)$ (in addition to being Y-periodic in y). But, as we shall see in § 5, this is not enough for (1.3) to hold (a counterexample is provided in Proposition 5.8). Loosely speaking, $\psi(x, y)$ turns out to be an "admissible" test function if it is continuous in one of its arguments (as is the case when ψ belongs to $L^2[\Omega; C_\#(Y)]$). For more details, see § 5, which is devoted to the proof of Lemma 1.3 and to the investigation of other regularity assumptions making of ψ an "admissible" test function.

Proof of Theorem 1.2. Let u_ε be a bounded sequence in $L^2(\Omega)$: there exists a positive constant C such that

$$\|u_\varepsilon\|_{L^2(\Omega)} \leqq C.$$

For any function $\psi(x, y) \in L^2[\Omega; C_\#(Y)]$, according to Lemma 1.3, $\psi(x, x/\varepsilon)$ belongs to $L^2(\Omega)$, and the Schwarz inequality yields

$$(1.4) \qquad \left| \int_\Omega u_\varepsilon(x) \psi\left(x, \frac{x}{\varepsilon}\right) dx \right| \leqq C \left\| \psi\left(x, \frac{x}{\varepsilon}\right) \right\|_{L^2(\Omega)} \leqq C \|\psi(x, y)\|_{L^2[\Omega; C_\#(Y)]}.$$

Thus, for fixed ε, the left-hand side of (1.4) turns out to be a bounded linear form on $L^2[\Omega; C_\#(Y)]$. The dual space of $L^2[\Omega; C_\#(Y)]$ can be identified with $L^2[\Omega; M_\#(Y)]$, where $M_\#(Y)$ is the space of Y-periodic Radon measures on Y. By virtue of the Riesz representation theorem, there exists a unique function $\mu_\varepsilon \in L^2[\Omega; M_\#(Y)]$ such that

$$(1.5) \qquad \langle \mu_\varepsilon, \psi \rangle = \int_\Omega u_\varepsilon(x) \psi\left(x, \frac{x}{\varepsilon}\right) dx,$$

where the brackets in the left-hand side of (1.5) denotes the duality product between $L^2[\Omega; C_\#(Y)]$ and its dual. Furthermore, in view of (1.4), the sequence μ_ε is bounded in $L^2[\Omega; M_\#(Y)]$. Since the space $L^2[\Omega; C_\#(Y)]$ is separable (i.e., contains a dense countable family), from any bounded sequence of its dual we can extract a subsequence that converges for the weak* topology. Thus, there exists $\mu_0 \in L^2[\Omega; M_\#(Y)]$ such that, up to a subsequence, and for any $\psi \in L^2[\Omega; C_\#(Y)]$,

$$(1.6) \qquad \langle \mu_\varepsilon, \psi \rangle \to \langle \mu_0, \psi \rangle.$$

By combining (1.5) and (1.6) we obtain, up to a subsequence, and for any $\psi \in L^2[\Omega; C_\#(Y)]$,

$$(1.7) \qquad \lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x)\psi\left(x, \frac{x}{\varepsilon}\right) dx = \langle \mu_0, \psi \rangle.$$

From Lemma 1.3 we know that

$$(1.8) \qquad \lim_{\varepsilon \to 0} \left\| \psi\left(x, \frac{x}{\varepsilon}\right) \right\|_{L^2(\Omega)} = \| \psi(x, y) \|_{L^2(\Omega \times Y)}.$$

Now, passing to the limit in the first two terms of (1.4) with the help of (1.7) and (1.8), we deduce

$$|\langle \mu_0, \psi \rangle| \leq C \|\psi\|_{L^2(\Omega \times Y)}.$$

By density of $L^2[\Omega; C_\#(Y)]$ in $L^2(\Omega \times Y)$, and by the Riesz representation theorem, μ_0 is identified with a function $u_0 \in L^2(\Omega \times Y)$, i.e.,

$$(1.9) \qquad \langle \mu_0, \psi \rangle = \int_\Omega \int_Y u_0(x, y)\psi(x, y) \, dx \, dy.$$

Equalities (1.7) and (1.9) are the desired result.   □

Remark 1.5. In the proof of Theorem 1.2, we considered test functions $\psi(x, y)$ in $L^2[\Omega; C_\#(Y)]$. Other choices of space of test functions are actually possible. For example, in the case where Ω is bounded, we could have replaced $L^2[\Omega; C_\#(Y)]$ by $C[\bar{\Omega}; C_\#(Y)]$, or by $L^2_\#[Y; C(\bar{\Omega})]$. The main ingredients of the proof would not be affected by this change. All these spaces have in common that they are separable Banach spaces, which is the required property in order to extract a weakly $*$ convergent subsequence from any bounded sequence in their dual. In any case the two-scale limit $u_0(x, y)$ is always the same, whatever the chosen space of test functions (see Remark 1.11).

Before developing further the theory, let us give a few examples of two-scale limits.

(∗) For any smooth function $a(x, y)$, being Y-periodic in y, the associated sequence $a_\varepsilon(x) = a(x, x/\varepsilon)$ two-scale converges to $a(x, y)$.

(∗∗) Any sequence u_ε that converges strongly in $L^2(\Omega)$ to a limit $u(x)$, two-scale converges to the same limit $u(x)$.

(∗∗∗) Any sequence u_ε that admits an asymptotic expansion of the type $u_\varepsilon(x) = u_0(x, x/\varepsilon) + \varepsilon u_1(x, x/\varepsilon) + \varepsilon^2 u_2(x, x/\varepsilon) + \cdots$, where the functions $u_i(x, y)$ are smooth and Y-periodic in y, two-scale converges to the first term of the expansion, namely, $u_0(x, y)$.

In view of the third example we already have a flavour of the main interest of two-scale convergence: even if the above asymptotic expansion does not hold (or is unknown), it is permitted to rigorously justify the existence of its first term $u_0(x, y)$. This is very helpful in homogenization theory, where such asymptotic expansions are frequently used in a heuristical way (see [10], [40]). This remark is the key of our two-scale convergence method, as explained in §§ 2, 3, and 4.

The next proposition establishes a link between two-scale and weak L^2-convergences.

PROPOSITION 1.6. *Let u_ε be a sequence of functions in $L^2(\Omega)$, which two-scale converges to a limit $u_0(x, y) \in L^2(\Omega \times Y)$. Then u_ε converges also to $u(x) = \int_Y u_0(x, y) \, dy$ in $L^2(\Omega)$ weakly. Furthermore, we have*

$$(1.10) \qquad \lim_{\varepsilon \to 0} \|u_\varepsilon\|_{L^2(\Omega)} \geq \|u_0\|_{L^2(\Omega \times Y)} \geq \|u\|_{L^2(\Omega)}.$$

Proof. By taking test functions $\psi(x)$, which depends only on x, in (1.1), we immediately obtain that u_ε weakly converges to $u(x) = \int_Y u_0(x, y) \, dy$ in $L^2(\Omega)$. To obtain (1.10), for $\psi(x, y) \in L^2[\Omega; C_\#(Y)]$, we compute

$$\int_\Omega \left[u_\varepsilon(x) - \psi\left(x, \frac{x}{\varepsilon}\right) \right]^2 dx = \int_\Omega u_\varepsilon(x)^2 \, dx + \int_\Omega \psi\left(x, \frac{x}{\varepsilon}\right)^2 dx$$

$$- 2 \int_\Omega u_\varepsilon(x) \psi\left(x, \frac{x}{\varepsilon}\right) dx \geqq 0.$$

Passing to the limit as $\varepsilon \to 0$ yields

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x)^2 \, dx \geqq 2 \int_\Omega \int_Y u_0(x, y) \psi(x, y) \, dx \, dy - \int_\Omega \int_Y \psi(x, y)^2 \, dx \, dy.$$

Then, using a sequence of smooth functions that converges strongly to u_0 in $L^2(\Omega \times Y)$ leads to

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x)^2 \, dx \geqq \int_\Omega \int_Y u_0(x, y)^2 \, dx \, dy.$$

On the other hand, the Cauchy–Schwarz inequality in Y gives the other inequality in (1.10). $\quad\square$

Remark 1.7. From Proposition 1.6, we see that, for a given bounded sequence in $L^2(\Omega)$, there is more information in its two-scale limit u_0 than in its weak L^2 limit u: u_0 contains some knowledge on the periodic oscillations of u_ε, while u is just the average (with respect to y) of u_0. However, let us emphasize that the two-scale limit captures only the oscillations that are in resonance with those of the test functions $\psi(x, x/\varepsilon)$. Contrary to the example (*) above, the sequence defined by $b_\varepsilon(x) = a(x, x/\varepsilon^2)$ (where $a(x, y)$ is a smooth function, Y-periodic in y) has the same two-scale limit and weak L^2 limit, namely, $\int_Y a(x, y) \, dy$. (This is a consequence of the difference of orders in the speed of oscillations for b_ε and the test functions $\psi(x, x/\varepsilon)$.) In this example, no oscillations are captured because the two-scale limit depends only on the variable x. Remark also here that the independence of the two-scale limit on the "fast" variable y does not imply strong convergence of the sequence in $L^2(\Omega)$.

We claim that there is more information in the two-scale limit of a sequence than in its weak L^2 limit. But does this supplementary knowledge yield some kind of strong convergence? This question is precisely answered by the following theorem.

THEOREM 1.8. *Let u_ε be a sequence of functions in $L^2(\Omega)$ that two-scale converges to a limit $u_0(x, y) \in L^2(\Omega \times Y)$. Assume that*

$$(1.11) \qquad\qquad \lim_{\varepsilon \to 0} \| u_\varepsilon \|_{L^2(\Omega)} = \| u_0 \|_{L^2(\Omega \times Y)}.$$

Then, for any sequence v_ε that two-scale converges to a limit $v_0(x, y) \in L^2(\Omega \times Y)$, we have

$$(1.12) \qquad\qquad u_\varepsilon(x) v_\varepsilon(x) \rightharpoonup \int_Y u_0(x, y) v_0(x, y) \, dy \quad \text{in } D'(\Omega).$$

Furthermore, if $u_0(x, y)$ belongs to $L^2[\Omega; C_\#(Y)]$, we have

$$(1.13) \qquad\qquad \lim_{\varepsilon \to 0} \left\| u_\varepsilon(x) - u_0\left(x, \frac{x}{\varepsilon}\right) \right\|_{L^2(\Omega)} = 0.$$

Remark 1.9. The condition (1.11) can be interpreted as "u_0 contains all the oscillations of the sequence u_ε." Indeed, (1.11) always takes place for a sequence $\psi(x, x/\varepsilon)$, with $\psi(x, y) \in L^2[\Omega; C_\#(Y)]$ or, more generally, being an "admissible" test

function in the sense of Definition 1.4. The result (1.12) can be defined as a *strong* two-scale convergence for the sequence u_ε; remarkably, it allows to pass to the limit in some product of two weak convergences in $L^2(\Omega)$.

Remark 1.10. As already pointed out before, for a given ε, the function $u_0(x, x/\varepsilon)$ need not be measurable in Ω, if $u_0(x, y)$ merely belongs to $L^2(\Omega \times Y)$. Thus, in order for (1.13) to make sense, some regularity on u_0 is required; more precisely, we restrict ourselves to functions $u_0(x, y)$ in $L^2[\Omega; C_\#(Y)]$ (more generally, $u_0(x, y)$ could be any "admissible" test function; see § 5 for details). However, we could wonder if all two-scale limits automatically are "admissible" test functions. Unfortunately, this is not true, and Lemma 1.13 below shows that any function in $L^2(\Omega \times Y)$ is attained as a two-scale limit. In view of the counterexample of Proposition 5.8, it is clear that, in general, a function of $L^2(\Omega \times Y)$ is not "admissible" in the sense of Definition 1.4. Thus, we cannot avoid an assumption on the regularity of u_0 in order to state (1.13).

Finally, we claim that, in the vocabulary of homogenization, (1.13) is a corrector-type result. Indeed, the sequence u_ε is approximated by its two-scale limit $u_0(x, x/\varepsilon)$ up to a strongly convergent reminder in $L^2(\Omega)$. Thus, the weak L^2-convergence of u_ε to its weak limit u is improved by (1.13), and the precise corrector is $u_0(x, x/\varepsilon) - u(x)$.

Proof of Theorem 1.8. Let $\psi_n(x, y)$ be a sequence of smooth functions in $L^2[\Omega; C_\#(Y)]$ that converges strongly to $u_0(x, y)$ in $L^2(\Omega \times Y)$. By definition of two-scale convergence for u_ε, and using Lemma 1.3 and assumption (1.11), we obtain

$$(1.14) \quad \lim_{\varepsilon \to 0} \int_\Omega \left[u_\varepsilon(x) - \psi_n\left(x, \frac{x}{\varepsilon}\right) \right]^2 dx = \int_\Omega \int_Y [u_0(x, y) - \psi_n(x, y)]^2 \, dx \, dy.$$

Passing to the limit as n goes to infinity, (1.14) yields

$$(1.15) \quad \lim_{n \to \infty} \lim_{\varepsilon \to 0} \int_\Omega \left[u_\varepsilon(x) - \psi_n\left(x, \frac{x}{\varepsilon}\right) \right]^2 dx = 0.$$

Let v_ε be a sequence that two-scale converges to a limit $v_0(x, y)$. For any $\phi(x) \in D(\Omega)$, we have

$$\int_\Omega \phi(x) u_\varepsilon(x) v_\varepsilon(x) \, dx = \int_\Omega \phi(x) \psi_n\left(x, \frac{x}{\varepsilon}\right) v_\varepsilon(x) \, dx$$

$$+ \int_\Omega \phi(x) \left[u_\varepsilon(x) - \psi_n\left(x, \frac{x}{\varepsilon}\right) \right] v_\varepsilon(x) \, dx.$$

Passing to the limit as ε goes to zero (and having in mind that v_ε is a bounded sequence in $L^2(\Omega)$) yields

$$\left| \lim_{\varepsilon \to 0} \int_\Omega \phi(x) u_\varepsilon(x) v_\varepsilon(x) \, dx - \int_\Omega \int_Y \phi(x) \psi_n(x, y) v_0(x, y) \, dx \, dy \right|$$

$$\leq C \lim_{\varepsilon \to 0} \left\| u_\varepsilon(x) - \psi_n\left(x, \frac{x}{\varepsilon}\right) \right\|_{L^2(\Omega)}.$$

Next, passing to the limit when n goes to infinity and using (1.15) leads to (1.12), i.e.,

$$\lim_{\varepsilon \to 0} \int_\Omega \phi(x) u_\varepsilon(x) v_\varepsilon(x) \, dx = \int_\Omega \int_Y \phi(x) u_0(x, y) v_0(x, y) \, dx \, dy.$$

Furthermore, if $u_0(x, y)$ is smooth, say $u_0 \in L^2[\Omega; C_\#(Y)]$, then (1.14) applies directly with u_0 instead of ψ_n, and it is nothing but (1.13). $\square$

Remark 1.11. As a consequence of Theorem 1.8, we can enlarge the class of test functions $\psi(x, y)$ used in the definition of two-scale convergence. In Definition 1.1, a

sequence u_ε two-scale converges to a limit u_0 if

$$(1.16) \qquad \lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x) \psi\left(x, \frac{x}{\varepsilon}\right) dx = \int_\Omega \int_Y u_0(x, y) \psi(x, y) \, dx \, dy$$

for any smooth test function ψ, namely, for $\psi(x, y) \in D[\Omega; C_\#^\infty(Y)]$. The class of test functions has already been considerably enlarged since the compactness Theorem 1.2 is proved for any $\psi(x, y) \in L^2[\Omega; C_\#(Y)]$. In view of Theorem 1.8, the validity of (1.16) is extended to all "admissible" test functions ψ in the sense of Definition 1.4. Indeed, an admissible test function satisfies hypothesis (1.11) in Theorem 1.8, and thus the sequence $\psi(x, x/\varepsilon)$ two-scale converges strongly to $\psi(x, y)$. Retrospectively, the choice of the space $L^2[\Omega; C_\#(Y)]$ in the proof of Theorem 1.2 appears to be purely technical: other choices would have led to the same two-scale limit.

Remark 1.12. Let us conclude this section by some bibliographical comments. As already said, the notion of two-scale convergence and the proof of the compactness Theorem 1.2 go back to Nguetseng [36]. Here we present a new proof of Theorem 1.2, which is simpler than the original one (note in passing that our proof has some similarities with that of Ball [8] for the existence of Young measures). Proposition 1.6 and Theorem 1.8 (concerning corrector results) are new. Recently, a generalization of two-scale convergence to Young measures has been introduced by E [19] in order to handle homogenization of nonlinear hyperbolic conservation laws (see Remark 3.8). Various authors have also developed ideas similar to two-scale convergence: Arbogast, Douglas, and Hornung [6] defined a so-called dilation operator for homogenization problems in porous media, while Mascarenhas [32] introduced a kind of two-scale Γ-convergence in the study of some memory effects in homogenization. All these works can be embedded in the general setting of two-scale convergence.

Now that the basic tools of the two-scale convergence method have been established, we give a few complementary results before explaining how it can be applied to the homogenization of partial differential equations with periodically oscillating coefficients. We first prove that two-scale limits have no extra regularity, as announced in Remark 1.10.

LEMMA 1.13. *Any function* $u_0(x, y)$ *in* $L^2(\Omega \times Y)$ *is attained as a two-scale limit.*

Proof. For any function $u_0(x, y) \in L^2(\Omega \times Y)$, we shall construct a bounded sequence u_ε in $L^2(\Omega)$ that two-scale converges to u_0. Let $u_n(x, y)$ be a sequence of smooth, Y-periodic in y functions that converge strongly to u_0 in $L^2(\Omega \times Y)$. Let $[\psi_k(x, y)]_{1 \leq k \leq \infty}$ be a dense family of smooth, Y-periodic in y functions in $L^2(\Omega \times Y)$, normalized such that $\|\psi_k\|_{L^2(\Omega \times Y)} = 1$. Obviously, for fixed n, the sequence $u_n(x, x/\varepsilon)$ two-scale converges to $u_n(x, y)$, i.e., for any $\delta > 0$, and for any smooth $\psi(x, y)$, there exists $\varepsilon_0(n, \delta, \psi) > 0$ such that $\varepsilon < \varepsilon_0$ implies

$$\left| \int_\Omega u_n\left(x, \frac{x}{\varepsilon}\right) \psi\left(x, \frac{x}{\varepsilon}\right) dx - \int_\Omega \int_Y u_n(x, y) \psi(x, y) \, dx \, dy \right| \leq \delta.$$

Now, we extract a diagonal sequence; namely, fixing $\delta_n = \|u_n - u_0\|_{L^2(\Omega \times Y)}$, there exists a sequence of positive numbers $\varepsilon(n)$, which goes to zero as $n \to \infty$ such that

$$\left| \int_\Omega u_n\left(x, \frac{x}{\varepsilon(n)}\right)^2 dx - \int_\Omega \int_Y u_n(x, y)^2 \, dx \, dy \right| \leq \delta_n$$

$$(1.17) \quad \left| \int_\Omega u_n\left(x, \frac{x}{\varepsilon(n)}\right) \psi_k\left(x, \frac{x}{\varepsilon(n)}\right) dx - \int_\Omega \int_Y u_n(x, y) \psi_k(x, y) \, dx \, dy \right| \leq \delta_n$$

$$\text{for } 1 \leq k \leq n.$$

Defining the diagonal sequence $u_{\varepsilon(n)}(x) \equiv u_n(x, x/\varepsilon(n))$, and recalling that δ_n is a sequence of positive numbers that goes to zero, it is clear from the first line of (1.17) that the sequence $u_{\varepsilon(n)}$ is bounded in $L^2(\Omega)$. By density of the family $[\psi_k(x, y)]_{1 \le k \le \infty}$ in $L^2(\Omega \times Y)$, the second line implies that $u_{\varepsilon(n)}$ two-scale converges to u_0. $\square$

So far we have only considered bounded sequences in $L^2(\Omega)$. The next proposition investigates some cases where we have additional bounds on sequences of derivatives.

PROPOSITION 1.14.

(i) *Let u_ε be a bounded sequence in $H^1(\Omega)$ that converges weakly to a limit u in $H^1(\Omega)$. Then, u_ε two-scale converges to $u(x)$, and there exists a function $u_1(x, y)$ in $L^2[\Omega; H^1_\#(Y)/\mathbb{R}]$ such that, up to a subsequence, ∇u_ε two-scale converges to $\nabla_x u(x) + \nabla_y u_1(x, y)$.*

(ii) *Let u_ε and $\varepsilon \nabla u_\varepsilon$ be two bounded sequences in $L^2(\Omega)$. Then, there exists a function $u_0(x, y)$ in $L^2[\Omega; H^1_\#(Y)]$ such that, up to a subsequence, u_ε and $\varepsilon \nabla u_\varepsilon$ two-scale converge to $u_0(x, y)$ and to $\nabla_y u_0(x, y)$, respectively.*

(iii) *Let u_ε be a divergence-free bounded sequence in $[L^2(\Omega)]^N$, which two-scale converges to $u_0(x, y)$ in $[L^2(\Omega \times Y)]^N$. Then, the two-scale limit satisfies $\operatorname{div}_y u_0(x, y) = 0$ and $\int_Y \operatorname{div}_x u_0(x, y)\, dy = 0$.*

Proof.

(i) Since u_ε (respectively, ∇u_ε) is bounded in $L^2(\Omega)$ (respectively, $[L^2(\Omega)]^N$), up to a subsequence, it two-scale converges to a limit $u_0(x, y) \in L^2(\Omega \times Y)$ (respectively, $\chi_0(x, y) \in [L^2(\Omega \times Y)]^N$). Thus for any $\psi(x, y) \in D[\Omega; C^\infty_\#(Y)]$ and any $\Psi(x, y) \in D[\Omega; C^\infty_\#(Y)]^N$, we have

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x) \psi\left(x, \frac{x}{\varepsilon}\right) dx = \int_\Omega \int_Y u_0(x, y) \psi(x, y)\, dx\, dy,$$

(1.18)

$$\lim_{\varepsilon \to 0} \int_\Omega \nabla u_\varepsilon(x) \cdot \Psi\left(x, \frac{x}{\varepsilon}\right) dx = \int_\Omega \int_Y \chi_0(x, y) \cdot \Psi(x, y)\, dx\, dy.$$

By integration by parts, we have

$$\varepsilon \int_\Omega \nabla u_\varepsilon(x) \cdot \Psi\left(x, \frac{x}{\varepsilon}\right) dx = -\int_\Omega u_\varepsilon(x) \left[\operatorname{div}_y \Psi\left(x, \frac{x}{\varepsilon}\right) + \varepsilon \operatorname{div}_x \Psi\left(x, \frac{x}{\varepsilon}\right) \right] dx.$$

Passing to the limit in both terms with the help of (1.18) leads to

$$0 = -\int_\Omega \int_Y u_0(x, y) \operatorname{div}_y \Psi(x, y)\, dx\, dy.$$

This implies that $u_0(x, y)$ does not depend on y. Since the average of u_0 is u, we deduce that for any subsequence the two-scale limit reduces to the weak L^2 limit u. Thus, the entire sequence u_ε two-scale converges to $u(x)$. Next, in (1.18) we choose a function Ψ such that $\operatorname{div}_y \Psi(x, y) = 0$. Integrating by parts we obtain

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x) \operatorname{div}_x \Psi\left(x, \frac{x}{\varepsilon}\right) dx = -\int_\Omega \int_Y \chi_0(x, y) \cdot \Psi(x, y)\, dx\, dy$$

$$= \int_\Omega \int_Y u(x) \operatorname{div}_x \Psi(x, y)\, dx\, dy.$$

Thus, for any function $\Psi(x, y) \in D[\Omega; C_\#^\infty(Y)]^N$ with $\text{div}_y \Psi(x, y) = 0$, we have

$$(1.19) \qquad \int_\Omega \int_Y [\chi_0(x, y) - \nabla u(x)] \cdot \Psi(x, y) \, dx \, dy = 0.$$

Recall that the orthogonal of divergence-free functions are exactly the gradients (see, if necessary, [43] or [47]). This well-known result can be very easily proved in the present context by means of Fourier analysis in Y. Thus, we deduce from (1.19) that there exists a unique function $u_1(x, y)$ in $L^2[\Omega; H_\#^1(Y)/\mathbb{R}]$ such that

$$\chi_0(x, y) = \nabla u(x) + \nabla_y u_1(x, y).$$

(ii) Since u_ε (respectively, $\varepsilon \nabla u_\varepsilon$) is bounded in $L^2(\Omega)$ (respectively, $[L^2(\Omega)]^N$), up to a subsequence, it two-scale converges to a limit $u_0(x, y) \in L^2(\Omega \times Y)$ (respectively, $\chi_0(x, y) \in [L^2(\Omega \times Y)]^N$). Thus for any $\psi(x, y) \in D[\Omega; C_\#^\infty(Y)]$ and any $\Psi(x, y) \in D[\Omega; C_\#^\infty(Y)]^N$, we have

$$(1.20) \qquad
\begin{aligned}
\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x) \psi\left(x, \frac{x}{\varepsilon}\right) dx &= \int_\Omega \int_Y u_0(x, y) \psi(x, y) \, dx \, dy, \\
\lim_{\varepsilon \to 0} \int_\Omega \varepsilon \nabla u_\varepsilon(x) \cdot \Psi\left(x, \frac{x}{\varepsilon}\right) dx &= \int_\Omega \int_Y \chi_0(x, y) \cdot \Psi(x, y) \, dx \, dy.
\end{aligned}$$

Integrating by parts in (1.20), we obtain

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x) \left[\text{div}_y \Psi\left(x, \frac{x}{\varepsilon}\right) + \varepsilon \, \text{div}_x \Psi\left(x, \frac{x}{\varepsilon}\right) \right] dx = - \int_\Omega \int_Y \chi_0(x, y) \cdot \Psi(x, y) \, dx \, dy$$

$$= \int_\Omega \int_Y u_0(x, y) \, \text{div}_y \Psi(x, y) \, dx \, dy.$$

Disintegrating by parts leads to $\chi_0(x, y) = \nabla_y u_0(x, y)$.

The proof of part (iii) is similar to the previous ones, and is left to the reader.   □

Two-scale convergence is not limited to bounded sequences in $L^2(\Omega)$. Our main result, Theorem 1.2, is easily generalized to bounded sequences in $L^p(\Omega)$, with $1 < p \leq +\infty$. Remark that the case $p = +\infty$ is included, while $p = 1$ is excluded (this is similar to what happens for weak convergence).

COROLLARY 1.15. *Let u_ε be a bounded sequence in $L^p(\Omega)$, with $1 < p \leq +\infty$. There exists a function $u_0(x, y)$ in $L^p(\Omega \times Y)$ such that, up to a subsequence, u_ε two-scale converges to u_0, i.e., for any function $\psi(x, y) \in D[\Omega; C_\#^\infty(Y)]$, we have*

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x) \psi\left(x, \frac{x}{\varepsilon}\right) dx = \int_\Omega \int_Y u_0(x, y) \psi(x, y) \, dx \, dy.$$

(*The proof is exactly the same as that of Theorem 1.2.*)

Of course, two-scale convergence is also easily generalized to n-scale convergence, with n any finite integer greater than two. This is a very helpful tool for what is called reiterated homogenization (see [10, Chap. 1, § 8]).

COROLLARY 1.16. *Let u_ε be a bounded sequence in $L^2(\Omega)$. There exists a function $u_0(x, y_1, \cdots, y_{n-1})$ in $L^2(\Omega \times Y^{n-1})$ such that, up to a subsequence, u_ε n-scale converges to u_0, i.e., for any function $\psi(x, y_1, \cdots, y_{n-1}) \in D[\Omega; C_\#^\infty(Y^{n-1})]$, we have*

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x) \psi\left(x, \frac{x}{\varepsilon}, \cdots, \frac{x}{\varepsilon^{n-1}}\right) dx$$

$$= \int_\Omega \int_{Y^{n-1}} u_0(x, y_1, \cdots, y_{n-1}) \psi(x, y_1, \cdots, y_{n-1}) \, dx \, dy_1 \cdots dy_{n-1}.$$

Remark 1.17. In the present paper, the test functions $\psi(x, y)$ are always assumed to be Y-periodic in y. Other choices for the period are possible. For a same sequence u_ε different two-scale limits can arise according to the period chosen for the test functions $y \to \psi(x, y)$, but they are related by a straightforward change of variables.

2. Homogenization of linear second-order elliptic equations. In this section we show how two-scale convergence can be used for the homogenization of linear second-order elliptic equations with periodically oscillating coefficients. We first revisit this favorite model problem of homogenization (see, e.g., [10, Chap. 1, § 6] in a fixed domain Ω, and later on we consider the case of perforated domains Ω_ε (see [13]). Besides recovering previous well-known results from a new point of view, we establish a new form of the limit problem, that we call the two-scale homogenized problem, and which is simply a combination of the usual homogenized problem and the cell problem (see [10], [40] for an introduction to the topic).

Let Ω be a bounded open set of $\mathbb{R}^N$. Let f be a given function in $L^2(\Omega)$. We consider the following linear second-order elliptic equation

$$(2.1) \qquad -\mathrm{div}\left(A\left(x, \frac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) = f \quad \text{in } \Omega,$$

$$u_\varepsilon = 0 \quad \text{on } \partial\Omega,$$

where $A(x, y)$ is a matrix defined on $\Omega \times Y$, Y-periodic in y, such that there exists two positive constants $0 < \alpha \leq \beta$ satisfying

$$(2.2) \qquad \alpha|\xi|^2 \leq \sum_{i,j=1}^{N} A_{ij}(x, y)\xi_i\xi_j \leq \beta|\xi|^2 \quad \text{for any } \xi \in \mathbb{R}^N.$$

Assumption (2.2) implies that the matrix $A(x, y)$ belongs to $[L^\infty(\Omega \times Y)]^{N^2}$, but it doesn't ensure that the function $x \to A(x, x/\varepsilon)$ is measurable, nor that it converges to its average $\int_Y A(x, y)\, dy$ in any suitable topology (see the counterexample of Proposition 5.8). Thus, we also require that $A_{ij}(x, y)$ is an "admissible" test function in the sense of Definition 1.4, namely, $A_{ij}(x, x/\varepsilon)$ is measurable and satisfies

$$(2.3) \qquad \lim_{\varepsilon \to 0} \int_\Omega A_{ij}\left(x, \frac{x}{\varepsilon}\right)^2 dx = \int_\Omega \int_Y A_{ij}(x, y)^2\, dx\, dy.$$

Assumption (2.3) is the weakest possible, but is rather vague. More precise, but also more restrictive, assumptions include, e.g., $A(x, y) \in L^\infty[\Omega; C_\#(Y)]^{N^2}$, $A(x, y) \in L_\#^\infty[Y; C(\bar{\Omega})]^{N^2}$, or $A(x, y) \in C[\Omega; L_\#^\infty(Y)]^{N^2}$ (the latter is the usual assumption in [10]). Under assumptions (2.2), (2.3), equation (2.1) admits a unique solution u_ε in $H_0^1(\Omega)$, which satisfies the a priori estimate

$$(2.4) \qquad \|u_\varepsilon\|_{H_0^1(\Omega)} \leq C\|f\|_{L^2(\Omega)},$$

where C is a positive constant that depends only on Ω and α, and not on ε. Thus, there exists $u \in H_0^1(\Omega)$ such that, up to a subsequence, u_ε converges weakly to u in $H_0^1(\Omega)$. The homogenization of (2.1) amounts to find a "homogenized" equation that admits the limit u as its unique solution.

Let us briefly recall the usual process of homogenization. In a first step, two-scale asymptotic expansions are used in order to obtain formally the homogenized equation (see, e.g., [10], [40]). In a second step, the convergence of the sequence u_ε to the solution u of the homogenized equation is proved (usually by means of the so-called energy method of Tartar [42]).

The results of the first (heuristic) step are summarized in the following.

DEFINITION 2.1. The homogenized problem is defined as

$$-\operatorname{div}[A^*(x)\nabla u(x)] = f \quad \text{in } \Omega,$$

(2.5)

$$u = 0 \quad \text{on } \partial\Omega,$$

where the entries of the matrix A^* are given by

(2.6) $$A_{ij}^*(x) = \int_Y A(x,y)[\nabla_y w_i(x,y) + e_i] \cdot [\nabla_y w_j(x,y) + e_j] \, dy$$

and, for $1 \leqq i \leqq N$, w_i is the solution of the so-called cell problem

$$-\operatorname{div}_y[A(x,y)[\nabla_y w_i(x,y) + e_i]] = 0 \quad \text{in } Y,$$

(2.7)

$$y \to w_i(x,y) \quad Y\text{-periodic.}$$

As a result of the second step, we have the following theorem [10, Chap. I, Thm. 6.1].

THEOREM 2.2. *The sequence u_ε of solutions of (2.1) converges weakly in $H_0^1(\Omega)$ to the unique solution u of (2.5).*

We are going to recover this last result with the help of two-scale convergence, but we also propose an alternative formulation of the limit problem by introducing the *two-scale homogenized problem*, which is a combination of the usual homogenized equation (2.5) and of the cell equation (2.7).

THEOREM 2.3. *The sequence u_ε of solutions of (2.1) converges weakly to $u(x)$ in $H_0^1(\Omega)$, and the sequence ∇u_ε two-scale converges to $\nabla u(x) + \nabla_y u_1(x,y)$, where (u, u_1) is the unique solution in $H_0^1(\Omega) \times L^2[\Omega; H_\#^1(Y)/\mathbb{R}]$ of the following two-scale homogenized system*:

$$-\operatorname{div}_y[A(x,y)[\nabla u(x) + \nabla_y u_1(x,y)]] = 0 \quad \text{in } \Omega \times Y,$$

(2.8) $$-\operatorname{div}_x\left[\int_Y A(x,y)[\nabla u(x) + \nabla_y u_1(x,y)] \, dy\right] = f \quad \text{in } \Omega,$$

$$u(x) = 0 \quad \text{on } \partial\Omega,$$

$$y \to u_1(x,y) \quad Y\text{-periodic.}$$

Furthermore, (2.8) is equivalent to the usual homogenized and cell equations (2.5)–(2.7) through the relation

(2.9) $$u_1(x,y) = \sum_{i=1}^N \frac{\partial u}{\partial x_i}(x) w_i(x,y).$$

Remark 2.4. The two-scale homogenized problem (2.8) is a system of two equations, two unknowns (u and u_1), where the two space variables x and y (i.e., the macroscopic and microscopic scales) are mixed. Although (2.8) seems to be complicated, it is a well-posed system of equations (cf. its variationial formulation (2.11) below), which is easily shown to have a unique solution. Remark that, here, the two equations of (2.8) can be decoupled in (2.5)–(2.7) (homogenized and cell equations) which are also two well-posed problems. However, we emphasize that this situation is very peculiar to the simple second-order elliptic equation (2.1). For many other types of problems, this decoupling is not possible, or leads to very complicated forms of the homogenized equation, including integro-differential operators and nonexplicit equations. Thus, the homogenized equation does not always belong to a class for which an existence and uniqueness theory is easily available, as opposed to the two-scale homogenized system, which is, in most cases, of the same type as the original problem, but with twice the variables (x and y) and unknowns (u and u_1). The supplementary, microscopic, variable and unknown play the role of "hidden" variables in the

vocabulary of mechanics (as remarked by Sanchez-Palencia [40]). Although their presence doubles the size of the limit problem, it greatly simplifies its structure (which could be useful for numerical purposes, too), while eliminating them introduces "strange" effects (like memory or nonlocal effects) in the usual homogenized problem. In short, both formulations ("usual" or two-scale) of the homogenized problem have their pros and cons, and none should be eliminated without second thoughts. Particularly striking examples of the above discussion may be found in § 4, in [2] (a convection-diffusion problem), or in [3] (unsteady Stokes flows in porous media).

Remark 2.5. As stated earlier, the two-scale homogenized problem (2.8) is equivalent to the homogenized system (2.5) and the cell problem (2.7), which are obtained by two-scale asymptotic expansions. This equivalence holds without any assumptions on the symmetry of the matrix A. Recall that, if A is not symmetric, the test functions used in the energy method are not the solutions of (2.7), but that of the dual cell problem (i.e., (2.7), where A is replaced by its transpose $'A$).

Proof of Theorem 2.3. Thanks to the a priori estimate (2.4), there exists a limit u such that, up to a subsequence, u_ε converges weakly to u in $H_0^1(\Omega)$. As a consequence of Proposition 1.14, there exists $u_1(x, y) \in L^2[\Omega; H_\#^1(Y)/\mathbb{R}]$ such that, up to another subsequence, ∇u_ε two-scale converges to $\nabla_x u(x) + \nabla_y u_1(x, y)$. In view of these limits, u_ε is expected to behave as $u(x) + \varepsilon u_1(x, x/\varepsilon)$. This suggests multiplying (2.1) by a test function $\phi(x) + \varepsilon \phi_1(x, x/\varepsilon)$, with $\phi(x) \in D(\Omega)$ and $\phi_1(x, y) \in D[\Omega; C_\#^\infty(Y)]$. This yields

$$(2.10) \qquad \begin{aligned} &\int_\Omega A\left(x, \frac{x}{\varepsilon}\right) \nabla u_\varepsilon \left[\nabla \phi(x) + \nabla_y \phi_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon \nabla_x \phi_1\left(x, \frac{x}{\varepsilon}\right)\right] dx \\ &= \int_\Omega f(x) \left[\phi(x) + \varepsilon \phi_1\left(x, \frac{x}{\varepsilon}\right)\right] dx. \end{aligned}$$

If the matrix $A(x, y)$ is smooth, then the function $'A(x, x/\varepsilon)[\nabla \phi(x) + \nabla_y \phi_1(x, x/\varepsilon)]$ can be considered as a test function in Theorem 1.2, and we pass to the two-scale limit in (2.10). Even if $A(x, y)$ is not smooth, at least, by assumption (2.3), the function $'A(x, x/\varepsilon)[\nabla \phi(x) + \nabla_y \phi_1(x, x/\varepsilon)]$ two-scale converges *strongly* to its limit $'A(x, y)[\nabla \phi(x) + \nabla_y \phi_1(x, y)]$ (i.e., condition (1.11) is satisfied in Theorem 1.8). Thus, using Theorem 1.8, we can still pass to the two-scale limit in (2.10):

$$(2.11) \qquad \begin{aligned} &\int_\Omega \int_Y A(x, y)[\nabla u(x) + \nabla_y u_1(x, y)] \cdot [\nabla \phi(x) + \nabla_y \phi_1(x, y)] \, dx \, dy \\ &= \int_\Omega f(x) \phi(x) \, dx. \end{aligned}$$

By density, (2.11) holds true for any (ϕ, ϕ_1) in $H_0^1(\Omega) \times L^2[\Omega; H_\#^1(Y)/\mathbb{R}]$. An easy integration by parts shows that (2.11) is a variational formulation associated to (2.8). Endowing the Hilbert space $H_0^1(\Omega) \times L^2[\Omega; H_\#^1(Y)/\mathbb{R}]$ with the norm $\|\nabla u(x)\|_{L^2(\Omega)} + \|\nabla_y u_1(x, y)\|_{L^2(\Omega \times Y)}$, we check the conditions of the Lax–Milgram lemma in (2.11). Let us focus on the coercivity in $H_0^1(\Omega) \times L^2[\Omega; H_\#^1(Y)/\mathbb{R}]$ of the bilinear form defined by the left-hand side of (2.11):

$$\int_\Omega \int_Y A(x, y)[\nabla \phi(x) + \nabla_y \phi_1(x, y)] \cdot [\nabla \phi(x) + \nabla_y \phi_1(x, y)] \, dx \, dy$$

$$\geqq \alpha \int_\Omega \int_Y |\nabla \phi(x) + \nabla_y \phi_1(x, y)|^2 \, dx \, dy$$

$$= \alpha \int_\Omega |\nabla \phi(x)|^2 \, dx + \alpha \int_\Omega \int_Y |\nabla_y \phi_1(x, y)|^2 \, dx \, dy.$$

Thus, by application of the Lax–Milgram lemma, there exists a unique solution of the two-scale homogenized problem (2.8). Consequently, the entire sequences u_ε and ∇u_ε converge to $u(x)$ and $\nabla u(x) + \nabla_y u_1(x, y)$. At this point, we could content ourselves with (2.8) as a homogenized problem, since its variational formulation (2.11) appears very naturally by application of two-scale convergence. However, it is usually preferable, from a physical or numerical point of view, to eliminate the microscopic variable y (one doesn't want to solve the small scale structure). This is an easy algebra exercise (left to the reader) to average (2.8) with respect to y, and to obtain the equivalent system (2.5)–(2.7), along with formula (2.6) for the homogenized matrix A^*.   □

Corrector results are easily obtained with the two-scale convergence method. The next theorem rigorously justifies the two first terms in the usual asymptotic expansion of the solution u_ε (see [10]).

THEOREM 2.6. *Assume that $\nabla_y u_1(x, y)$ is an "admissible" test function in the sense of Definition 1.4. Then, the sequence $[\nabla u_\varepsilon(x) - \nabla u(x) - \nabla_y u_1(x, x/\varepsilon)]$ converges strongly to zero in $[L^2(\Omega)]^N$. In particular, if $u_1, \nabla_x u_1$, and $\nabla_y u_1$ are "admissible," then we have*

$$\left[u_\varepsilon(x) - u(x) - \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) \right] \to 0 \quad \text{in } H^1(\Omega) \text{ strongly.}$$

Proof. Let us first remark that the assumption on u_1, being an "admissible" test function, is satisfied as soon as the matrix A is smooth, say $A(x, y) \in C[\Omega; L^\infty_\#(Y)]^{N^2}$, by standard regularity results for the solutions $w_i(x, y)$ of the cell problem (2.7).

Now, using this assumption, we can write

$$\int_\Omega A\left(x, \frac{x}{\varepsilon}\right) \left[\nabla u_\varepsilon(x) - \nabla u(x) - \nabla_y u_1\left(x, \frac{x}{\varepsilon}\right) \right]^2 dx$$

$$(2.12) \qquad = \int_\Omega f(x) u_\varepsilon(x)\, dx + \int_\Omega A\left(x, \frac{x}{\varepsilon}\right) \left[\nabla u(x) + \nabla_y u_1\left(x, \frac{x}{\varepsilon}\right) \right]^2 dx$$

$$- \int_\Omega (A + {}^t A)\left(x, \frac{x}{\varepsilon}\right) \nabla u_\varepsilon(x) \cdot \left[\nabla u(x) + \nabla_y u_1\left(x, \frac{x}{\varepsilon}\right) \right] dx.$$

Using the coercivity condition (2.2) and passing to the two-scale limit in the right-hand side of (2.12) yields

$$\alpha \lim_{\varepsilon \to 0} \left\| \nabla u_\varepsilon(x) - \nabla u(x) - \nabla_y u_1\left(x, \frac{x}{\varepsilon}\right) \right\|^2_{L^2(\Omega)}$$

$$(2.13)$$

$$\leqq \int_\Omega f(x) u(x)\, dx - \int_\Omega \int_Y A(x, y)[\nabla u(x) + \nabla_y u_1(x, y)]^2\, dx\, dy.$$

In view of (2.8), the right-hand side of (2.13) is equal to zero, which is the desired result.   □

Two-scale convergence can also handle homogenization problems in perforated domains, without requiring any extension lemmas or similar technical ingredients. Let us define a sequence Ω_ε of periodically perforated subdomains of a bounded open set Ω in $\mathbb{R}^N$. The period of Ω_ε is εY^*, where Y^* is a subset of the unit cube $Y = (0; 1)^N$, which is called the solid or material part (by opposition to the hole, or void part, $Y - Y^*$). We assume that the material domain E^*, obtained by Y-periodicity from Y^*, is a smooth connected open set in $\mathbb{R}^N$ (remark that no assumptions are made on

the void domain $\mathbb{R}^N - E^*$; thus, the holes $Y - Y^*$ may be connected or isolated). Denoting by $\chi(y)$ the characteristic function of E^* (a Y-periodic function), Ω_ε is defined as

$$(2.14) \qquad \Omega_\varepsilon = \left\{ x \in \Omega \,\Big/\, \chi\left(\frac{x}{\varepsilon}\right) = 1 \right\}.$$

We consider a linear second-order elliptic equation in Ω_ε,

$$-\mathrm{div}\left(A\left(x, \frac{x}{\varepsilon}\right) \nabla u_\varepsilon \right) + u_\varepsilon = f \quad \text{in } \Omega_\varepsilon,$$

$$(2.15) \qquad \frac{\partial u_\varepsilon}{\partial v_{A_\varepsilon}} = \left[A\left(x, \frac{x}{\varepsilon}\right) \nabla u_\varepsilon \right] \cdot n = 0 \quad \text{on } \partial\Omega_\varepsilon - \partial\Omega,$$

$$u_\varepsilon = 0 \quad \text{on } \partial\Omega \cap \partial\Omega_\varepsilon,$$

where the matrix A satisfies the same assumptions (2.2), (2.3) as before. From (2.15), we easily deduce the a priori estimates

$$(2.16) \qquad \| u_\varepsilon \|_{L^2(\Omega_\varepsilon)} \leq C \quad \text{and} \quad \| \nabla u_\varepsilon \|_{L^2(\Omega_\varepsilon)} \leq C,$$

where C is a constant which does not depend on ε. The main difficulty in homogenization in perforated domains is to establish that the sequence u_ε admits a limit u in $H^1(\Omega)$. From (2.16) we cannot extract a convergent subsequence by weak compactness in a given Sobolev space, since each u_ε is defined in a different space $H^1(\Omega_\varepsilon)$, which varies with ε.

Nevertheless, this problem has first been solved by Cioranescu and Saint Jean Paulin [13] in the case of domains perforated with isolated holes (i.e., $Y - Y^*$ is strictly included in Y), while the general case is treated in [1] and [4]. The main result of these three papers is the following theorem.

THEOREM 2.7. *The sequence u_ε of solutions of* (2.15) *"converges" to a limit u, which is the unique solution in $H_0^1(\Omega)$ of the homogenized problem*

$$-\mathrm{div}\,[A^* \nabla u] + \theta u = \theta f \quad in \ \Omega,$$

$$(2.17)$$

$$u = 0 \quad on \ \partial\Omega,$$

where θ is the volume fraction of material (i.e., $\theta = \int_Y \chi(y)\, dy = |Y^*|$), and the entries of the matrix A^* are given by

$$(2.18) \qquad A_{ij}^*(x) = \int_{Y^*} A(x, y)[\nabla_y w_i(x, y) + e_i] \cdot [\nabla_y w_j(x, y) + e_j]\, dy,$$

and, for $1 \leq i \leq N$, w_i is the solution of the cell problem

$$-\mathrm{div}_y\, (A(x, y)[\nabla_y w_i(x, y) + e_i]) = 0 \quad in \ Y^*,$$

$$(2.19) \qquad A(x, y)[\nabla_y w_i(x, y) + e_i] \cdot n = 0 \quad on \ \partial Y^* - \partial Y,$$

$$y \to w_i(x, y) \quad Y\text{-periodic}.$$

Remark 2.8. The convergence of the sequence u_ε is intentionally very "vague" in Theorem 2.7. In view of the a priori estimates (2.16), there is no clear notion of convergence for u_ε, which is defined on a varying set Ω_ε. In the literature this difficulty has been overcome in two different ways. In [13] and [1], an extension of u_ε to the whole domain Ω is constructed, and this extension is proved to converge weakly in $H^1(\Omega)$ to the homogenized limit u. In [4], no sophisticated extensions are used, but

a version of the Rellich theorem in perforated domains is established (loosely speaking, the embedding of $H^1(\Omega_\varepsilon)$ in $L^2(\Omega_\varepsilon)$ is compact, uniformly in ε), which allows us to prove that u_ε converges to u in the sense that $\|u_\varepsilon - u\|_{L^2(\Omega)_\varepsilon}$ goes to zero. All these references use classical methods of homogenization (the energy method of Tartar in [13] and [4], and the Γ-convergence of De Giorgi in [1]).

·In the next theorem we recover the results of Theorem 2.7, using two-scale convergence. As in [4], we do not use any sophisticated extensions (apart from the trivial extension by zero in the holes $\Omega - \Omega_\varepsilon$), and we give a new interpretation of the "vague" convergence mentioned above.

THEOREM 2.9. *Denote by* $\tilde{}$ *the extension by zero in the domain* $\Omega - \Omega_\varepsilon$. *The sequences* $\tilde{u}_\varepsilon$ *and* $\tilde{\nabla} u_\varepsilon$ *two-scale converge to* $u(x)\chi(y)$ *and* $\chi(y)[\nabla u(x) + \nabla_y u_1(x, y)]$, *respectively, where* (u, u_1) *is the unique solution in* $H_0^1(\Omega) \times L^2[\Omega; H_\#^1(Y^*)/\mathbb{R}]$ *of the following two-scale homogenized system*;

$$-\mathrm{div}_y\, (A(x, y)[\nabla u(x) + \nabla_y u_1(x, y)]) = 0 \quad in\ \Omega \times Y^*,$$

$$-\mathrm{div}_x\left[\int_{Y^*} A(x, y)[\nabla u(x) + \nabla_y u_1(x, y)]\, dy\right] + \theta u(x) = \theta f(x) \quad in\ \Omega,$$

(2.20)
$$u(x) = 0 \quad on\ \partial\Omega,$$

$$y \to u_1(x, y) \quad Y\text{-periodic}$$

$$(A(x, y)[\nabla u(x) + \nabla_y u_1(x, y)]) \cdot n = 0 \quad on\ \partial Y^* - \partial Y.$$

Furthermore, (2.20) *is equivalent to the usual homogenized and cell equations* (2.17)–(2.19) *through the relation*

(2.21)
$$u_1(x, y) = \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}(x) w_i(x, y).$$

Proof. In view of (2.16), the two sequences $\tilde{u}_\varepsilon$ and $\tilde{\nabla} u_\varepsilon$ are bounded in $L^2(\Omega)$, and by application of Theorem 1.2 they two-scale converge, up to a subsequence, to $u_0(x, y)$ and $\xi_0(x, y)$, respectively. Since, by definition, $\tilde{u}_\varepsilon$ and $\tilde{\nabla} u_\varepsilon$ are equal to zero in $\Omega - \Omega_\varepsilon$, their two-scale limit $u_0(x, y)$ and $\xi_0(x, y)$ are also equal to zero if $y \in Y - Y^*$. In order to find the precise form of u_0 and ξ_0 in $\Omega \times Y^*$, we argue as in Proposition 1.14(i). Let $\psi(x, y) \in D[\Omega; C_\#^\infty(Y)]$ and $\Psi(x, y) \in D[\Omega; C_\#^\infty(Y)]^N$ be two functions, equal to zero if $y \in Y - Y^*$ (hence, they belong to $D(\Omega_\varepsilon)$ and $[D(\Omega_\varepsilon)]^N$). We have

(2.22)
$$\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} u_\varepsilon(x)\psi\left(x, \frac{x}{\varepsilon}\right)dx = \int_\Omega \int_{Y^*} u_0(x, y)\psi(x, y)\, dx\, dy,$$

$$\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} \nabla u_\varepsilon(x) \cdot \Psi\left(x, \frac{x}{\varepsilon}\right)dx = \int_\Omega \int_{Y^*} \xi_0(x, y) \cdot \Psi(x, y)\, dx\, dy.$$

By integration by parts, we obtain

$$\varepsilon \int_{\Omega_\varepsilon} \nabla u_\varepsilon(x) \cdot \Psi\left(x, \frac{x}{\varepsilon}\right)dx = -\int_{\Omega_\varepsilon} u_\varepsilon(x)\left[\mathrm{div}_y\, \Psi\left(x, \frac{x}{\varepsilon}\right) + \varepsilon\, \mathrm{div}_x\, \Psi\left(x, \frac{x}{\varepsilon}\right)\right]dx.$$

Passing to the limit in both terms with the help of (2.22) leads to

$$0 = -\int_\Omega \int_{Y^*} u_0(x, y)\, \mathrm{div}_y\, \Psi(x, y)\, dx\, dy.$$

This implies that $u_0(x, y)$ does not depend on y in Y^*, i.e., there exists $u(x) \in L^2(\Omega)$ such that

$$u_0(x, y) = u(x)\chi(y).$$

Now, we add to the previous assumptions on $\Psi(x, y)$ the condition $\operatorname{div}_y \Psi(x, y) = 0$. Integrating by parts in Ω_ε gives

$$(2.23) \qquad \int_{\Omega_\varepsilon} \nabla u_\varepsilon(x) \cdot \Psi\left(x, \frac{x}{\varepsilon}\right) dx = -\int_{\Omega_\varepsilon} u_\varepsilon(x) \operatorname{div}_x \Psi\left(x, \frac{x}{\varepsilon}\right) dx.$$

Passing to the two-scale limit yields

$$(2.24) \qquad \int_\Omega \int_{Y^*} \xi_0(x, y) \cdot \Psi(x, y) \, dx \, dy = -\int_\Omega \int_{Y^*} u(x) \operatorname{div}_x \Psi(x, y) \, dx \, dy.$$

By using Lemma 2.10 below, the right-hand side of (2.24) becomes $\int_\Omega u(x) \operatorname{div}_x \theta(x) \, dx$, while the left-hand side is a linear continuous form in $\theta(x) \in [L^2(\Omega)]^N$. This implies that $u(x) \in H_0^1(\Omega)$. Then, integrating by parts in (2.24) shows that, for any function $\Psi(x, y) \in L^2[\Omega; L_\#^2(Y^*)]^N$ with $\operatorname{div}_y \Psi(x, y) = 0$ and $\Psi(x, y) \cdot n_y = 0$ on $\partial Y^* - \partial Y$, we have

$$(2.25) \qquad \int_\Omega \int_{Y^*} [\xi_0(x, y) - \nabla u(x)] \cdot \Psi(x, y) \, dx \, dy = 0.$$

Since the orthogonal of divergence-free functions is exactly the gradients, we deduce from (2.25) that there exists a function $u_1(x, y)$ in $L^2[\Omega; H_\#^1(Y^*)/\mathbb{R}]$ such that $\xi_0(x, y) = \chi(y)[\nabla u(x) + \nabla_y u_1(x, y)]$.

We are now in the position of finding the homogenized equations satisfied by u and u_1. Let us multiply the original equation (2.15) by the test function $\phi(x) + \varepsilon \phi_1(x, x/\varepsilon)$, where $\phi \in D(\Omega)$ and $\phi_1 \in D[\Omega; C_\#^\infty(Y)]$. Integrating by parts and passing to the two-scale limit yields

$$(2.26) \qquad \int_\Omega \int_{Y^*} A(x, y)[\nabla u(x) + \nabla_y u_1(x, y)] \cdot [\nabla \phi(x) + \nabla_y \phi_1(x, y)] \, dx \, dy + \theta \int_\Omega u\phi \, dx$$
$$= \theta \int_\Omega f\phi \, dx.$$

By density, (2.26) holds true for any (ϕ, ϕ_1) in $H_0^1(\Omega) \times L^2[\Omega; H_\#^1(Y^*)/\mathbb{R}]$. An easy integration by parts shows that (2.26) is a variational formulation associated to (2.20). It remains to prove existence and uniqueness in (2.26), and, as in Theorem 2.3, the main point is to show the coercivity of the left-hand side of (2.26). Indeed, it is an easy exercise (left to the reader) to check that $\|\nabla u(x) + \nabla_y u_1(x, y)\|_{L^2(\Omega \times Y^*)}$ is a norm for the Hilbert space $H_0^1(\Omega) \times L^2[\Omega; H_\#^1(Y^*)/\mathbb{R}]$. Remark, however, that this result relies heavily on the assumption on Y^* (namely, the Y-periodic set E^*, with period Y^*, is connected), and even fails if Y^* is strictly included in the unit cell Y. Remark also that here, to the contrary of the situation in Theorem 2.3, the above norm is not equal to $\|\nabla u(x)\|_{L^2(\Omega)} + \|\nabla_y u_1(x, y)\|_{L^2(\Omega \times Y^*)}$. $\square$

LEMMA 2.10. *For any function* $\theta(x) \in [L^2(\Omega)]^N$ *there exists* $\Psi(x, y) \in L^2[\Omega; H_\#^1(Y^*)]^N$ *such that*

$$\operatorname{div}_y \Psi(x, y) = 0 \quad \textit{in } Y^*,$$

$$\Psi(x, y) = 0 \quad \textit{on } \partial Y^* - \partial Y,$$

$$(2.27) \qquad \int_{Y^*} \Psi(x, y) \, dy = \theta(x),$$

$$\|\Psi(x, y)\|_{L^2(\Omega; H_\#^1(Y^*))^N} \leq C \|\theta(x)\|_{L^2(\Omega)^N}.$$

Proof. For $1 \leq i \leq N$, consider the following Stokes problem:

$$\nabla p_i - \Delta v_i = e_i \quad \text{in } Y^*,$$

$$\text{div } v_i = 0 \quad \text{in } Y^*,$$

$$v_i = 0 \quad \text{on } \partial Y^* - \partial Y,$$

$$p_i, v_i \quad Y\text{-periodic},$$

which admits a unique, nonzero solution (p_i, v_i) in $[L^2_\#(Y^*)/\mathbb{R}] \times [H^1_\#(Y^*)]^N$ since we have assumed that E^* (the Y-periodic set obtained from Y^*) is smooth and connected. Denote by A the constant, symmetric, positive definite matrix $(\int_{Y^*} \nabla v_i \cdot \nabla v_j)_{1 \leq i,j \leq N}$. Then, for any $\theta(x) \in [L^2(\Omega)]^N$, the function Ψ defined by

$$\Psi(x, y) = \sum_{i=1}^N \langle A^{-1}\theta(x), e_i \rangle v_i(y)$$

is easily seen to satisfy all the properties (2.27) since $\int_{Y^*} \nabla v_i \cdot \nabla v_j = \int_{Y^*} v_i \cdot e_j$. $\quad\square$

3. Homogenization of nonlinear operators. In this section we show how two-scale convergence can handle nonlinear homogenization problems. Again, we revisit two well-known model problems in nonlinear homogenization: first, the Γ-convergence of oscillating convex integral functionals, and second, the H-convergence (also known as G-convergence) of oscillating monotone operators. We begin this section by recovering some previous results of De Giorgi, and Marcellini [31], concerning Γ-convergence of convex functionals. Then we recover other results of Tartar [42], about H-convergence of monotone operators, and finally we conclude by giving a few references where generalizations of the two-scale convergence method are applied to the homogenization of nonlinear hyperbolic conservation laws, and nonlinear equations admitting viscosity solutions (see Remark 3.8).

Let Ω be a bounded open set in $\mathbb{R}^N$ and $f(x)$ a given function on Ω. We consider a family of functionals

$$(3.1) \qquad I_\varepsilon(v) = \int_\Omega \left[W\left(\frac{x}{\varepsilon}, \nabla v(x)\right) - f(x)v(x) \right] dx,$$

where $v(x)$ is a vector-valued function from Ω into $\mathbb{R}^n$, and the scalar energy $W(y, \lambda)$ satisfies, for some $p > 1$,

(3.2)

    (i)   for any λ, the function $y \to W(y, \lambda)$ is measurable and Y-periodic,

    (ii)   a.e. in y, the function $\lambda \to W(y, \lambda)$ is strictly convex and C^1 in $\mathbb{R}^{nN}$,

    (iii) $0 \leq c|\lambda|^p \leq W(y, \lambda) \leq C[1 + |\lambda|^p]$ a.e. in y, with $0 < c < C$,

    (iv) $\left| \dfrac{\partial W}{\partial \lambda}(y, \lambda) \right| \leq C[1 + |\lambda|^{p-1}]$ a.e. in y.

(Actually, assumption (iv) is easily seen to be a consequence of (ii) and (iii), as remarked by Francfort [24].) We also assume that $f(x) \in [L^{p'}(\Omega)]^n$ with $(1/p) + (1/p') = 1$. Since $W(y, \lambda)$ is convex in λ, for fixed ε, there exists a unique $u_\varepsilon(x) \in [W_0^{1,p}(\Omega)]^n$ that achieved the minimum of the functional $I_\varepsilon(v)$ on $[W_0^{1,p}(\Omega)]^n$, i.e.,

$$(3.3) \qquad I_\varepsilon(u_\varepsilon) = \mathop{\text{Inf}}_{v \in [W_0^{1,p}(\Omega)]^n} \int_\Omega \left[W\left(\frac{x}{\varepsilon}, \nabla v(x)\right) - f(x)v(x) \right] dx.$$

The homogenization of the functionals $I_\varepsilon(v)$ amounts to finding an "homogenized" functional $\bar{I}(v)$ such that the sequence of minimizers u_ε converges to a limit u, which is precisely the minimizer of $\bar{I}(v)$. This problem has been solved by Marcellini [31]. His result is the following.

THEOREM 3.1. *There exist a functional $\bar{I}$ and a function u such that*

$$u_\varepsilon \rightharpoonup u \quad \text{weakly in } [W_0^{1,p}(\Omega)]^n,$$

(3.4) $$I_\varepsilon(u_\varepsilon) \to \bar{I}(u),$$

$$\bar{I}(u) = \underset{v \in [W_0^{1,p}(\Omega)]^n}{\text{Inf}} \bar{I}(v).$$

Furthermore, $\bar{I}$ is given by

(3.5) $$\bar{I}(v) = \int_\Omega [\bar{W}(\nabla v(x)) - f(x)v(x)]\, dx,$$

where the energy $\bar{W}$ is defined by

(3.6) $$\bar{W}(\lambda) = \underset{w \in [W_\#^{1,p}(Y)]^n}{\text{Inf}} \int_Y W(y, \lambda + \nabla w(y))\, dy.$$

Remark 3.2. By definition, $\bar{I}$ is the homogenized functional, and the sequence I_ε is said to Γ-converge to $\bar{I}$. (For more details about the Γ-convergence of De Giorgi, see [16], [17].) In addition, it is easy to see that the energy $\bar{W}$ is also convex and C^1, and satisfies the same growth conditions as W. We emphasize that Theorem 3.1 is restricted to convex energies; the situation is completely different in the nonconvex case (see [12], [33]).

We are going to recover Theorem 3.1 using two-scale convergence, and without any tools form the theory of Γ-convergence.

THEOREM 3.3. *There exists a function $u(x)$ such that the sequence u_ε of solutions of* (3.3) *converges weakly to u in $[W_0^{1,p}(\Omega)]^n$. There also exists a function $u_1(x, y) \in L^p[\Omega; W_\#^{1,p}(Y)]^n$ such that the sequence ∇u_ε two-scale converges to $\nabla_x u(x) + \nabla_y u_1(x, y)$. Furthermore, the homogenized energy is also characterized as*

(3.7) $$\bar{I}(u) = I(u, u_1) = \underset{\substack{v \in [W_0^{1,p}(\Omega)]^n \\ v_1 \in L^p[\Omega; W_\#^{1,p}(Y)/\mathbb{R}]^n}}{\text{Inf}} I(v, v_1),$$

where $I(v, v_1)$ is the two-scale homogenized functional defined by

(3.8) $$I(v, v_1) = \int_\Omega \int_Y [W[y, \nabla v(x) + \nabla_y v_1(x, y)] - f(x)v(x)]\, dx\, dy.$$

Remark 3.4. Theorem 3.3 furnishes a new characterization of the homogenized problem, which turns out to be a double minimization over two different spaces of functions of two variables x and y. In the quadratic case, this characterization was also proposed by Lions (see his "averaging principle" in the calculus of variations [30, § 5, Chap. 1]). Theorem 3.1 is easily deduced from Theorem 3.3 by averaging the two-scale homogenized functional $I(v, v_1)$ with respect to y to recover the usual homogenized functional $\bar{I}(v)$. The difference between $\bar{I}(v)$ and $I(v, v_1)$ corresponds exactly to the difference in the linear case between the usual and two-scale homogenized problems (see Remark 2.4).

Proof of Theorem 3.3. In view of the growth condition (3.2)(iii) for the energy W, the sequence of minimizers u_ε is bounded in $[W_0^{1,p}(\Omega)]^n$. Thus, there exists a function u such that, up to a subsequence, u_ε converges weakly to u in $[W_0^{1,p}(\Omega)]^n$.

Applying Proposition 1.14 and Corollary 1.15, there also exists a function $u_1(x, y) \in L^p[\Omega; W^{1,p}_\#(Y)/\mathbb{R}]^n$ such that, up to another subsequence, ∇u_ε two-scale converges to $\nabla_x u(x) + \nabla_y u_1(x, y)$.

In a first step we give a lower bound for $I_\varepsilon(u_\varepsilon)$. Since $W(\cdot, \lambda)$ is convex and differentiable, we have

$$(3.9) \qquad W(\cdot, \lambda) \geqq W(\cdot, \mu) + \left\langle \frac{\partial W}{\partial \lambda}(\cdot, \mu), \lambda - \mu \right\rangle.$$

By specifying (3.9), we obtain

$$(3.10) \quad W\left[\frac{x}{\varepsilon}, \nabla u_\varepsilon(x)\right] \geqq W\left[\frac{x}{\varepsilon}, \mu\left(x, \frac{x}{\varepsilon}\right)\right] + \left\langle \frac{\partial W}{\partial \lambda}\left[\frac{x}{\varepsilon}, \mu\left(x, \frac{x}{\varepsilon}\right)\right], \nabla u_\varepsilon(x) - \mu\left(x, \frac{x}{\varepsilon}\right) \right\rangle.$$

For a smooth function $\mu(x, y) \in D[\Omega; C^\infty_\#(Y)]^n$, we can integrate (3.10) on Ω, and then pass to the two-scale limit in the right-hand side. This leads to

$$\lim_{\varepsilon \to 0} I_\varepsilon[u_\varepsilon(x)] \geqq \int_\Omega \int_Y \left(W[y, \mu(x, y)] - f(x)u(x) \right) dx \, dy$$

$$(3.11)$$

$$+ \int_\Omega \int_Y \left\langle \frac{\partial W}{\partial \lambda}[y, \mu(x, y)], \nabla_x u(x) + \nabla_y u_1(x, y) - \mu(x, y) \right\rangle dx \, dy.$$

Now, we apply (3.11) to a sequence of smooth functions $\mu(x, y)$, Y-periodic in y, which converges to $\nabla_x u(x) + \nabla_y u_1(x, y)$ strongly in $[L^p(\Omega \times Y)]^{nN}$. In view of the growth conditions (3.2)(iii) and (iv) on W and $\partial W/\partial \lambda$, we can pass to the limit in (3.11) and obtain

$$(3.12) \qquad \lim_{\varepsilon \to 0} I_\varepsilon[u_\varepsilon(x)] \geqq \int_\Omega \int_Y [W[y, \nabla_x u(x) + \nabla_y u_1(x, y)] - f(x)u(x)] \, dx \, dy$$

$$= I(u, u_1).$$

Now, in a second step we establish an upper bound for $I_\varepsilon(u_\varepsilon)$. For $\phi(x) \in [D(\Omega)]^n$ and $\phi_1(x, y) \in D[\Omega; C^\infty_\#(Y)]^n$, since u_ε is the minimizer, we have

$$(3.13) \qquad I_\varepsilon[u_\varepsilon(x)] \leqq I_\varepsilon\left[\phi(x) + \varepsilon\phi_1\left(x, \frac{x}{\varepsilon}\right)\right].$$

Passing to the two-scale limit in the right-hand side of (3.13) yields

$$(3.14) \qquad \lim_{\varepsilon \to 0} I_\varepsilon[u_\varepsilon(x)] \leqq \int_\Omega \int_Y [W[y, \nabla_x \phi(x) + \nabla_y \phi_1(x, y)] - f(x)\phi(x)] \, dx \, dy$$

$$= I(\phi, \phi_1).$$

The functional $I(\phi, \phi_1)$ is called the two-scale homogenized functional. By density, we deduce from (3.14) that

$$(3.15) \qquad \lim_{\varepsilon \to 0} I_\varepsilon[u_\varepsilon(x)] \leqq \inf_{\substack{v \in [W^{1,p}_0(\Omega)]^n \\ v_1 \in L^p[\Omega; W^{1,p}_\#(Y)/\mathbb{R}]^n}} I(v, v_1).$$

Combining (3.12) and (3.15) yields

$$(3.16) \qquad \lim_{\varepsilon \to 0} I_\varepsilon[u_\varepsilon(x)] = I(u, u_1) = \inf_{\substack{v \in [W^{1,p}_0(\Omega)]^n \\ v_1 \in L^p[\Omega; W^{1,p}_\#(Y)/\mathbb{R}]^n}} I(v, v_1).$$

Since $W(\cdot, \lambda)$ is a strictly convex energy, there exists a unique minimizer (u, u_1) of (3.16). Thus, the entire sequence u_ε converges weakly to u in $[W_0^{1,p}(\Omega)]^n$, and the entire sequence ∇u_ε two-scale converges to $\nabla_x u(x) + \nabla_y u_1(x, y)$.   □

So far, we have considered minimization problems. Instead, we could have solved the corresponding nonlinear Euler equations, satisfied by the minimizers. More generally, we could consider nonlinear second-order elliptic equations, which may not correspond to any energy minimization. Indeed, we are going to generalize Theorem 3.3 to the case of monotone operators, thus recovering previous results of Tartar [42]. Define an operator $a(y, \lambda)$ from $Y \times \mathbb{R}^{nN}$ in $\mathbb{R}^{nN}$ as follows:

(3.17)

    (i)   for any λ, the function $y \to a(y, \lambda)$ is measurable and Y-periodic,

    (ii)   a.e. in y, the function $\lambda \to a(y, \lambda)$ is continuous,

    (iii) $0 \leq c|\lambda|^p \leq a(y, \lambda) \cdot \lambda$, for $0 < c$, and $p > 1$,

    (iv) $|a(y, \lambda)| \leq C[1 + |\lambda|^{p-1}]$, for $0 < C$.

Furthermore, the operator a is strictly monotone, i.e.,

$$(3.18) \qquad [a(y, \lambda) - a(y, \mu)] \cdot (\lambda - \mu) > 0 \quad \text{for any } \lambda \neq \mu.$$

For $f(x) \in [L^{p'}(\Omega)]^n$ (with $(1/p) + (1/p') = 1$), we consider the equation

$$(3.19) \qquad -\text{div}\, a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right) = f \quad \text{in } \Omega,$$

$$u_\varepsilon = 0 \quad \text{on } \partial\Omega,$$

which admits a unique solution u_ε in $[W_0^{1,p}(\Omega)]^n$.

THEOREM 3.5. *The sequence u_ε of solutions of* (3.19) *converges weakly to a function* $u(x)$ *in* $[W_0^{1,p}(\Omega)]^n$, *and the sequence ∇u_ε two-scale converges to $\nabla_x u(x) + \nabla_y u_1(x, y)$, where (u, u_1) is the unique solution in* $[W_0^{1,p}(\Omega)]^n \times L^p[\Omega; W_\#^{1,p}(Y)/\mathbb{R}]^n$ *of the homogenized problem*

$$(3.20) \qquad \begin{aligned} -\text{div}_x \left[\int_Y a[y, \nabla u(x) + \nabla_y u_1(x, y)]\, dy \right] &= f \quad \text{in } \Omega \\ -\text{div}_y\, a[y, \nabla u(x) + \nabla_y u_1(x, y)] &= 0 \quad \text{in } Y \\ u &= 0 \quad \text{on } \partial\Omega \\ y \to u_1(x, y) \quad &Y\text{-periodic.} \end{aligned}$$

Proof. From the growth conditions (3.17), we easily obtain a priori estimates on u_ε, which is bounded in $[W_0^{1,p}(\Omega)]^n$, and $g_\varepsilon = a(x/\varepsilon, \nabla u_\varepsilon)$, which is bounded in $[L^{p'}(\Omega)]^{nN}$. Thus, up to a subsequence, u_ε converges weakly to a limit u in $[W_0^{1,p}(\Omega)]^n$, while ∇u_ε and g_ε two-scale converge to $\nabla u(x) + \nabla_y u_1(x, y)$ and $g_0(x, y)$, respectively. Since $f + \text{div}\, g_\varepsilon = 0$, arguing as in Proposition 1.14, it is not difficult to check that the two-scale limit g_0 satisfies

$$\text{div}_y\, g_0(x, y) = 0$$

$$(3.21) \qquad f(x) + \text{div}_x \left[\int_Y g_0(x, y)\, dy \right] = 0.$$

The problem is to identify g_0 in terms of a, u, and u_1. To this end, for any positive number t, and any functions ϕ, $\phi_1 \in D[\Omega; C_\#^\infty(Y)]^n$, we introduce a test function defined by

$$\mu_\varepsilon(x) = \nabla \left[u(x) + \varepsilon \phi_1\left(x, \frac{x}{\varepsilon}\right) \right] + t\phi\left(x, \frac{x}{\varepsilon}\right),$$

which two-scale converges to a limit $\mu_0(x, y) = \nabla u(x) + \nabla_y \phi_1(x, y) + t\phi(x, y)$. The monotonicity property (3.18) yields

$$\int_\Omega \left[g_\varepsilon - a\left(\frac{x}{\varepsilon}, \mu_\varepsilon\right) \right] \cdot (\nabla u_\varepsilon - \mu_\varepsilon) \, dx \geqq 0,$$

or, equivalently,

$$(3.22) \quad \int_\Omega \left[-\operatorname{div} g_\varepsilon \cdot u_\varepsilon - a\left(\frac{x}{\varepsilon}, \mu_\varepsilon\right) \cdot \nabla u_\varepsilon - g_\varepsilon \cdot \mu_\varepsilon + a\left(\frac{x}{\varepsilon}, \mu_\varepsilon\right) \cdot \mu_\varepsilon \right] dx \geqq 0.$$

Using (3.19) in the first term of (3.22), and passing to the two-scale limit in all the other terms leads to

$$(3.23) \quad \int_\Omega \int_Y \left[f \cdot u - a(y, \mu_0) \cdot [\nabla u(x) + \nabla_y u_1(x, y)] - g_0 \cdot \mu_0 + a(y, \mu_0) \cdot \mu_0 \right] dx \, dy \geqq 0.$$

In view of the growth conditions (3.17) on the operator a, we can pass to the limit in (3.23) when considering a sequence of functions $\phi_1(x, y)$ that converges strongly to $u_1(x, y)$ in $[L^p(\Omega; W_\#^{1,p}(Y))]^n$. Thus, replacing μ_0 by $\nabla u(x) + \nabla_y u_1(x, y) + t\phi(x, y)$ and integrating by parts, (3.23) becomes

$$(3.24) \quad \begin{aligned} &\int_\Omega \left[f(x) + \operatorname{div}\left(\int_Y g_0(x, y) \, dy \right) \right] \cdot u(x) \, dx + \int_\Omega \int_Y \operatorname{div}_y g_0(x, y) \cdot u_1(x, y) \, dx \, dy \\ &+ \int_\Omega \int_Y \left[a[y, \nabla u(x) + \nabla_y u_1(x, y) + t\phi(x, y)] - g_0(x, y) \right] t\phi(x, y) \, dx \, dy \geqq 0. \end{aligned}$$

Thanks to (3.21), the first two terms of (3.24) are equal to zero. Then, dividing by $t > 0$, and passing to the limit, as t goes to zero, gives for any function $\phi(x, y)$,

$$\int_\Omega \int_Y \left[a[y, \nabla u(x) + \nabla_y u_1(x, y)] - g_0(x, y) \right] \phi(x, y) \, dx \, dy \geqq 0.$$

Thus, we conclude that $g_0(x, y) = a[y, \nabla u(x) + \nabla_y u_1(x, y)]$. Combined with (3.21) it implies that (u, u_1) is a solution of the homogenized system (3.20). Since the operator a is strictly monotone, system (3.20) has a unique solution, and the entire sequence u_ε converges.   □

In the case $p = 2$, and under the further assumption that the operator a is uniformly monotone, i.e., there exists a positive constant c such that

$$(3.25) \qquad [a(y, \lambda) - a(y, \mu)] \cdot (\lambda - \mu) \geqq c|\lambda - \mu|^2 \quad \text{for any } \lambda, \mu,$$

we obtain a corrector result similar to Theorem 2.6 in the linear case.

THEOREM 3.6. *Assume that the function $u_1(x, y)$ is smooth. Then, the sequence $u_\varepsilon(x) - u(x) - \varepsilon u_1(x, x/\varepsilon)$ converges strongly to zero in $H^1(\Omega)$.*

Remark 3.7. Corrector results for monotone operators in the general framework of H-convergence have been obtained by Murat [35] (see also [15] in the periodic case). By lack of smoothness for $\nabla u(x)$, the corrector in [35] is not explicit. Here, on the contrary, the corrector is explicitly given as $\nabla_y u_1(x, x/\varepsilon)$. However, we still have to assume that $u_1(x, y)$ is smooth in order to state Theorem 3.6 (more precisely, $\nabla_y u_1(x, x/\varepsilon)$ is required to be, at least, an admissible test function in the sense of Definition 1.4).

Proof of Theorem 3.6. Since $u_1(x, y)$ is assumed to be smooth, we consider the function

$$\mu_\varepsilon(x) = \nabla \left[u(x) + \varepsilon u_1 \left(x, \frac{x}{\varepsilon} \right) \right],$$

which two-scale converges to $\mu_0(x, y) = \nabla u(x) + \nabla_y u_1(x, y)$. The monotonicity property (3.25) yields

$$(3.26) \qquad \int_\Omega \left[g_\varepsilon - a \left(\frac{x}{\varepsilon}, \mu_\varepsilon \right) \right] \cdot (\nabla u_\varepsilon - \mu_\varepsilon) \, dx \geqq c \int_\Omega |\nabla u_\varepsilon - \mu_\varepsilon|^2 \, dx.$$

As in the proof of Theorem 3.5, the left-hand side of (3.26) goes to zero, which implies that the sequence $\nabla[u_\varepsilon(x) - u(x) - \varepsilon u_1(x, x/\varepsilon)]$ converges to zero in $[L^2(\Omega)]^N$. □

Remark 3.8. In the literature, homogenization has also been applied to other types of nonlinear equations. A first example is given by certain fully nonlinear, first- or second-order, partial differential equation, which fall within the scope of the theory of viscosity solutions (see the review paper of Crandall, Ishii, and Lions [14]). The key point of the viscosity solutions theory is that it provides a maximum principle that permits comparison between solutions. Based on this fact is the so-called "perturbed test function" method of Evans [22], [23], which provides very elegant proof of convergence for the homogenization of such equations. A perturbed test function is a function of the type $\phi(x) + \varepsilon^i \phi_1(x, x/\varepsilon)$ ($i = 1, 2$, depending on the order of the equation), which is, thus very similar to that of the two-scale convergence method. Indeed, the perturbed test function method appears, a posteriori, as the ad hoc version of two-scale convergence in the context of viscosity solutions of nonlinear equations.

A second example is nonlinear hyperbolic conservation laws. To handle homogenization of such equations, E [19] introduced so-called two-scale Young measures, which are a combination of the usual Young measures (introduced for PDEs by Tartar [45]) with two-scale convergence. Combined with DiPerna's method for reducing measure-valued solutions of conservation laws to Dirac masses [18], it allows us to rigorously homogenize nonlinear transport equations, and nonlinear hyperbolic equations with oscillating forcing terms [19], [20]. In the case of linear hyperbolic equations, two-scale convergence has also been applied by Amirat, Hamdache, and Ziani [5] and Hou and Xin [26].

4. Homogenization of a diffusion process in highly heterogeneous media. In § 2 we studied the homogenization of a second-order elliptic equation with varying coefficients $A(x, x/\varepsilon)$. This can be regarded as a stationary diffusion process in a medium made of two materials, if $A(x, x/\varepsilon)$ takes only two different values (of the same order of magnitude). The present section is also devoted to the homogenization of a diffusion process, but the main novelty with respect to § 2 is the high heterogeneity of the two materials: namely, ε being the microscale, the ratio of their diffusion coefficients is taken of order ε^2 (this precise scaling corresponds to an equipartition of the energy in both materials, see Remark 4.9). As we shall see, it changes completely the form of the homogenized problem, which is genuinely of "two-scale" type (see 4.6)). In particular, the elimination of the microscale in the homogenized system does not yield a partial differential equation (see (4.9)).

Let us turn to a brief description of the geometry of the heterogeneous medium. We consider two materials, periodically distributed in a domain Ω (a bounded open set in $\mathbb{R}^N$), with period εY (ε is a small positive number, and $Y = (0 ; 1)^N$ is the unit cube). The unit period Y is divided in two complementary parts Y_1 and Y_2, which are occupied by material 1 and material 2, respectively. Let $\chi_1(y)$ (respectively, $\chi_2(y)$)

be the characteristic function of Y_1 (respectively, Y_2), extended by Y-periodicity to the whole $\mathbb{R}^N$. They satisfy

$$\chi_1(y) + \chi_2(y) = 1 \quad \text{in } Y.$$

The domain Ω is thus divided in two subdomains Ω_ε^1 and Ω_ε^2 (occupied by materials 1 and 2, respectively), which are defined by

$$\Omega_\varepsilon^1 = \left\{ x \in \Omega / \chi_1\left(\frac{x}{\varepsilon}\right) = 1 \right\} \quad \text{and} \quad \Omega_\varepsilon^2 = \left\{ x \in \Omega / \chi_2\left(\frac{x}{\varepsilon}\right) = 1 \right\}.$$

We make the fundamental assumption that, in the heterogeneous domain Ω, material 1 is the "matrix," while material 2 can be either "inclusions" or another matrix (like interconnected fibers). More precisely, denoting by E_1 the subset of $\mathbb{R}^N$ obtained by Y-periodicity from Y_1, we assume that E_1 is smooth and connected. On the contrary, no such assumptions are made on E_2 (the Y-periodic set built with Y_2).

Let μ_1 and μ_2 be two positive constants. We define the varying diffusion coefficient μ_ε of the heterogeneous medium Ω by

$$(4.1) \qquad \mu_\varepsilon(x) = \mu_1 \chi_1\left(\frac{x}{\varepsilon}\right) + \varepsilon^2 \mu_2 \chi_2\left(\frac{x}{\varepsilon}\right).$$

For a given source term f and positive constant α, we consider the following diffusion process for a scalar u_ε

$$(4.2) \qquad \begin{aligned} -\text{div}\,[\mu_\varepsilon \nabla u_\varepsilon] + \alpha u_\varepsilon &= f \quad \text{in } \Omega, \\ u_\varepsilon &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

We implicitly assume in (4.2) the usual transmission condition at the interface of the two materials, namely, u_ε and $\mu_\varepsilon \partial u_\varepsilon / \partial n$ are continuous through $\partial\Omega_\varepsilon^1 \cap \partial\Omega_\varepsilon^2$.

Remark 4.1. We emphasize the particular scaling of the coefficients defined in (4.2): the order of magnitude of μ_ε is 1 in material 1 (the "matrix"), and ε^2 in material 2 (the "inclusions" or the "fibers"). This explains why such a medium is called "highly" heterogeneous. (For a motivation of the precise scaling, see Remark 4.9 below.) Problem (4.2) is a simplified version of a system studied by Arbogast, Douglas, and Hornung [6], which models single phase flow in fractured porous media. Its homogenization leads to the so-called double porosity model. In their context, u_ε is the fluid pressure, and μ_ε is the permeability that is much larger in the network of fractures Ω_ε^1 than in the porous rocks Ω_ε^2. Problem (4.2) can also be interpreted as the heat equation. Then, u_ε is the temperature, and μ_ε is the thermal diffusion. (Thus, material 1 is a good conductor, while material 2 is a poor one.) Under additional assumptions on the geometry and the regularity of the source term, problem (4.2) has been studied by Panasenko [39] with the help of the maximum principle (that we do not use here).

Assuming $f \in L^2(\Omega)$, it is well known that there exists a unique solution of (4.2) in $H_0^1(\Omega)$. Multiplying (4.2) by u_ε and integrating by parts leads to

$$(4.3) \qquad \int_\Omega \mu_\varepsilon |\nabla u_\varepsilon|^2 + \alpha \int_\Omega u_\varepsilon^2 = \int_\Omega f u_\varepsilon.$$

Then, if α is strictly positive, the solution u_ε is easily seen to satisfy the a priori estimates

$$(4.4) \qquad \begin{aligned} \|u_\varepsilon\|_{L^2(\Omega)} &\leq C, \\ \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon^1)} &\leq C, \\ \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon^2)} &\leq \frac{C}{\varepsilon}, \end{aligned}$$

where C is a positive constant which does not depend on ε.

Remark 4.2. The a priori estimates (4.4) are easily deduced from (4.3) when $\alpha > 0$. Actually they hold true even when $\alpha = 0$, but with a new ingredient, namely, a Poincaré-type inequality. Under the additional assumption that Y_1 is connected in Y, there exists a constant C, which does not depend on ε, and such that, for any $v \in H_0^1(\Omega)$,

$$(4.5) \qquad \|v\|_{L^2(\Omega)} \leq C[\|\nabla v\|_{L^2(\Omega_\varepsilon^1)} + \varepsilon \|\nabla v\|_{L^2(\Omega_\varepsilon^2)}].$$

Obviously the Poincaré-type inequality (4.5), applied to u_ε, implies (4.4) even for $\alpha = 0$. The proof of (4.5) is rather technical and out of the scope of the present paper. The interested reader is referred to Lemma 3.4 in [4] for a similar proof. Thus, this is only for simplicity that a zero-order term has been introduced in (4.2).

Before stating the main result of the present section, let us define the Hilbert space $H_{0\#}^1(Y_2)$ made of functions of $H_\#^1(Y_2)$, which vanishes on the interface $\partial Y_1 \cap \partial Y_2$.

THEOREM 4.3. *The sequence* u_ε *of solutions of* (4.2) *two-scale converges to a limit* $u(x) + \chi_2(y)v(x, y)$, *where* (u, v) *is the unique solution in* $H_0^1(\Omega) \times L^2[\Omega; H_{0\#}^1(Y_2)]$ *of the homogenized problem*

$$-\mu_1 \operatorname{div}_x [A^* \nabla_x u(x)] + \alpha u(x) = f(x) - \alpha \int_{Y_2} v(x, y)\, dy \quad in\ \Omega,$$

$$-\mu_2 \Delta_{yy} v(x, y) + \alpha v(x, y) = f(x) - \alpha u(x) \quad in\ Y_2,$$

$$(4.6) \qquad\qquad u = 0 \quad on\ \partial\Omega,$$

$$v(x, y) = 0 \quad on\ \partial Y_1 \cap \partial Y_2,$$

$$y \to v(x, y) \quad Y\text{-}periodic,$$

where the entries of the constant matrix A^* *are given by*

$$(4.7) \qquad A_{ij}^* = \int_{Y_1} [\nabla_y w_i(y) + e_i] \cdot [\nabla_y w_j(y) + e_j]\, dy,$$

and, for $1 \leq i \leq N$, $w_i(y)$ *is the solution of the cell problem*

$$-\operatorname{div}_y [\nabla_y w_i + e_i] = 0 \quad in\ Y_1,$$

$$[\nabla_y w_i + e_i] \cdot n = 0 \quad on\ \partial Y_1 \cap \partial Y_2,$$

$$y \to w_i(y) \quad Y\text{-}periodic.$$

Thanks to a separation of variables, the homogenized system (4.6) can be simplified. Denoting by $U(x)$ the weak limit in $L^2(\Omega)$ of the sequence u_ε, we obtain an equation for U. (Let us note in passing that $U(x)$ is not equal to $u(x)$, but rather to $u(x) + \int_{Y_2} v(x, y)\, dy$.)

PROPOSITION 4.4. *Let* $w(y)$ *be the unique solution in* $H_{0\#}^1(Y_2)$ *of*

$$-\mu_2 \Delta_{yy} w(y) + \alpha w(y) = 1 \quad in\ Y_2,$$

$$w(y) = 0 \quad on\ \partial Y_1 \cap \partial Y_2,$$

$$y \to w(y) \quad Y\text{-}periodic.$$

Then, $v(x, y) = w(y)[f(x) - \alpha u(x)]$, *and* $u(x)$ *is the unique solution in* $H_0^1(\Omega)$ *of*

$$-\mu_1 \operatorname{div}_x [A^* \nabla_x u(x)] + \alpha \left(1 - \alpha \int_{Y_2} w(y)\, dy\right) u(x) = \left(1 - \alpha \int_{Y_2} w(y)\, dy\right) f(x) \quad in\ \Omega$$

$$(4.8)$$

$$u = 0 \quad on\ \partial\Omega.$$

Denoting by L^{-1} the solution operator of (4.8) *from* $H^{-1}(\Omega)$ *to* $H_0^1(\Omega)$ (*i.e.,* $u(x) = L^{-1}f(x)$), $U(x)$ *can be written as*

$$(4.9) \qquad U(x) = L^{-1}f(x) + \left[\int_{Y_2} w(y) \, dy \right] f(x).$$

Remark 4.5. In view of (4.9), $U(x)$ is the solution of a very special diffusion process for which no simple partial differential equation can be found. Of course, if the source term $f(x)$ is smooth, we can apply the operator L to (4.9) and obtain the equation

$$(4.10) \qquad L[U(x)] = f(x) + \left[\int_{Y_2} w(y) \, dy \right] L[f(x)].$$

But (4.10) is only formal, since, a priori, the solution $U(x)$ does not satisfy the required Dirichlet boundary condition. Thus, it seems preferable to write $U(x)$ as the sum of two terms, which are solutions of a more standard problem (4.6). The homogenized problem (4.6) is a system of two coupled equations, one "macroscopic" (in Ω) and the other one "microscopic" (in Y_2): $u(x)$ is the contribution coming from material 1 in Ω_ε^1, and $v(x, y)$ is the additional contribution from material 2 in Ω_ε^2. This is definitely a "two-scale" phenomenon, since in the limit as $\varepsilon \to 0$ (4.6) keeps track of the two different materials on two different scales. This phenomenon allowed Arbogast, Douglas, and Hornung [6] to recover the so-called double porosity model in porous media flows.

The two-scale convergence of u_ε towards $u(x) + \chi_2(y)v(x, y)$ can be improved with the following corrector result.

PROPOSITION 4.6. *Assume that* $v(x, y)$ *is smooth* (*namely, that it is an admissible test function in the sense of Definition* 1.4). *Then we have*

$$(4.11) \qquad \left[u_\varepsilon(x) - u(x) - \chi_2\left(\frac{x}{\varepsilon}\right) v\left(x, \frac{x}{\varepsilon}\right) \right] \to 0 \quad \text{in } L^2(\Omega) \text{ strongly.}$$

For the proof of Theorem 4.3 we need the following.

LEMMA 4.7. *There exist functions* $u(x) \in H_0^1(\Omega)$, $v(x, y) \in L^2[\Omega; H_{0\#}^1(Y_2)]$, *and* $u_1(x, y) \in L^2[\Omega; H_\#^1(Y_1)/\mathbb{R}]$ *such that, up to a subsequence,*

$$(4.12) \qquad \begin{pmatrix} u_\varepsilon \\ \chi_1(x/\varepsilon)\nabla u_\varepsilon \\ \varepsilon\chi_2(x/\varepsilon)\nabla u_\varepsilon \end{pmatrix} \text{ two-scale converge to } \begin{pmatrix} u(x) + \chi_2(y)v(x, y) \\ \chi_1(y)[\nabla u(x) + \nabla_y u_1(x, y)] \\ \chi_2(y)\nabla_y v(x, y) \end{pmatrix}.$$

Proof. In view of the a priori estimates (4.4), the three sequences in (4.12) admit two-scale limits. Arguing as in Theorem 2.9, it is easily seen that there exist $u(x) \in H_0^1(\Omega)$ and $u_1(x, y) \in L^2[\Omega; H_\#^1(Y_1)/\mathbb{R}]$ such that $\chi_1(x/\varepsilon)u_\varepsilon$ and $\chi_1(x/\varepsilon)\nabla u_\varepsilon$ two-scale converge to $\chi_1(y)u(x)$ and $\chi_1(y)[\nabla u(x) + \nabla_y u_1(x, y)]$. On the other hand, it follows from Proposition 1.14 that there exists a function $u_0(x, y) \in L^2[\Omega; H_\#^1(Y_2)]$ such that $\chi_2(x/\varepsilon)u_\varepsilon$ and $\varepsilon\chi_2(x/\varepsilon)\nabla u_\varepsilon$ two-scale converge to $\chi_2(y)u_0(x, y)$ and $\chi_2(y)\nabla_y u_0(x, y)$. It remains to find the relationship between $u(x)$ and $u_0(x, y)$.

Consider the sequence $\varepsilon\nabla u_\varepsilon$ in the whole domain Ω. For any function $\phi(x, y) \in D[\Omega; C_\#^\infty(Y)]^N$, we know from the above results that

$$(4.13) \qquad \lim_{\varepsilon \to 0} \int_\Omega \varepsilon\nabla u_\varepsilon(x) \cdot \phi\left(x, \frac{x}{\varepsilon}\right) dx = \int_\Omega \int_Y \chi_2(y)\nabla_y u_0(x, y) \cdot \phi(x, y) \, dx \, dy.$$

By integration by parts, the left-hand side of (4.13) is also equal to

$$\lim_{\varepsilon \to 0} - \int_{\Omega} u_{\varepsilon}(x) \left[\mathrm{div}_y \, \phi\left(x, \frac{x}{\varepsilon}\right) + \varepsilon \, \mathrm{div}_x \, \phi\left(x, \frac{x}{\varepsilon}\right) \right] dx$$

$$= - \int_{\Omega} \int_Y [\chi_1(y) u(x) + \chi_2(y) u_0(x, y)] \, \mathrm{div}_y \, \phi(x, y) \, dx \, dy.$$

By equality between the two limits, we obtain that $u_0(x, y) = u(x)$ on $\partial Y_1 \cap \partial Y_2$. Thus, there exists $v(x, y) \in L^2[\Omega; H^1_{0\#}(Y_2)]$ such that $u_0(x, y) = u(x) + v(x, y)$.   □

Proof of Theorem 4.3. In view of the two-scale limit of the sequence u_ε, we multiply (4.2) by a test function of the form $\phi(x) + \varepsilon \phi_1(x, x/\varepsilon) + \psi(x, x/\varepsilon)$, where $\phi(x) \in D(\Omega)$, $\phi_1(x, y) \in D[\Omega; C^\infty_\#(Y)]$, and $\psi(x, y) \in D[\Omega; C^\infty_\#(Y)]$ with $\psi(x, y) = 0$ for $y \in Y_1$. Integrating by parts and passing to the two-scale limit yields

$$\int_{\Omega} \int_{Y_1} \mu_1 [\nabla u(x) + \nabla_y u_1(x, y)] \cdot [\nabla \phi(x) + \nabla_y \phi_1(x, y)] \, dx \, dy$$

$$+ \int_{\Omega} \int_{Y_2} \mu_2 \nabla_y v(x, y) \cdot \nabla_y \psi(x, y) \, dx \, dy$$

(4.14)

$$+ \alpha \int_{\Omega} \int_Y [u(x) + \chi_2(y) v(x, y)] \cdot [\phi(x) + \chi_2(y) \psi(x, y)] \, dx \, dy$$

$$= \int_{\Omega} \int_Y f(x) [\phi(x) + \chi_2(y) \psi(x, y)] \, dx \, dy.$$

By density (4.14) holds true for any $(\phi, \phi_1, \psi) \in H^1_0(\Omega) \times L^2[\Omega; H^1_\#(Y_1)/\mathbb{R}] \times L^2[\Omega; H^1_{0\#}(Y_2)]$. Its left-hand side is easily seen to be coercive on the above functional space; thus (4.14) admits a unique solution (u, u_1, v). Another integration by parts shows that (4.14) is a variational formulation of the following two-scale homogenized system for $u, u_1,$ and v:

$$-\mu_1 \, \mathrm{div}_x \left[\int_{Y_1} [\nabla_x u(x) + \nabla_y u_1(x, y)] \, dy \right] + \alpha u(x) = f(x) - \alpha \int_{Y_2} v(x, y) \, dy \quad \text{in } \Omega,$$

$$-\mathrm{div}_y [\nabla_x u(x) + \nabla_y u_1(x, y)] = 0 \quad \text{in } Y_1,$$

$$-\mu_2 \Delta_{yy} v(x, y) + \alpha v(x, y) = f(x) - \alpha u(x) \quad \text{in } Y_2,$$

(4.15)
$$u = 0 \quad \text{on } \partial\Omega,$$

$$[\nabla_x u(x) + \nabla_y u_1(x, y)] \cdot n_y = 0 \quad \text{on } \partial Y_1 \cap \partial Y_2,$$

$$y \to u_1(x, y) \quad Y\text{-periodic},$$

$$v(x, y) = 0 \quad \text{on } \partial Y_1 \cap \partial Y_2,$$

$$y \to v(x, y) \quad Y\text{-periodic}.$$

In (4.15), the equation in u_1 can be decoupled from the two other ones, as we did in Theorem 2.9. Then, introducing the matrix A^* defined in (4.7), (4.8), the elimination of u_1 leads to system (4.6).   □

Proof of Proposition 4.6. Recall the energy equation (4.3):

(4.3)
$$\int_{\Omega} \mu_\varepsilon |\nabla u_\varepsilon|^2 + \alpha \int_{\Omega} u_\varepsilon^2 = \int_{\Omega} f u_\varepsilon.$$

Passing to the limit in the right-hand side of (4.3), and using the variational formulation (4.14) yields

$$\lim_{\varepsilon \to 0} \int_\Omega \mu_\varepsilon |\nabla u_\varepsilon|^2 + \alpha \int_\Omega u_\varepsilon^2$$

$$(4.16) \qquad = \int_\Omega \int_{Y_1} \mu_1 |\nabla u(x) + \nabla_y u_1(x, y)|^2 \, dx \, dy$$

$$+ \int_\Omega \int_{Y_2} \mu_2 |\nabla_y v(x, y)|^2 \, dx \, dy + \alpha \int_\Omega \int_Y [u(x) + \chi_2(y)v(x, y)]^2 \, dx \, dy.$$

By application of Proposition 1.6, the limit of each term in the left-hand side of (4.16) is larger than the corresponding two-scale limit in the right-hand side. Thus equality holds for each contribution. In particular, if $\alpha > 0$, we have

$$(4.17) \qquad \lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon^2 = \int_\Omega \int_Y [u(x) + \chi_2(y)v(x, y)]^2 \, dx \, dy.$$

In view of (4.17) and Theorem 1.8, we obtain the desired result (4.11). The result holds true also for $\alpha = 0$: first we obtain a corrector result for the gradients $\chi_1(x/\varepsilon)\nabla u_\varepsilon$ and $\varepsilon\chi_2(x/\varepsilon)\nabla u_\varepsilon$, second we use again the Poincaré-type inequality (4.5) to deduce (4.11). □

Remark 4.8. Similarly to the scalar equation (4.2), we could consider a Stokes problem in a domain filled with two fluids having a highly heterogeneous viscosity μ_ε (still defined by (4.1))

$$(4.18) \qquad \begin{aligned} \nabla p_\varepsilon - \mathrm{div}\,[\mu_\varepsilon \nabla u_\varepsilon] &= f \quad \text{in } \Omega, \\ \mathrm{div}\, u_\varepsilon &= 0 \quad \text{in } \Omega, \\ u_\varepsilon &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

with the usual transmission condition at the interface: u_ε and $p_\varepsilon n - \mu_\varepsilon \partial u_\varepsilon/\partial n$ are continuous through $\partial\Omega_\varepsilon^1 \cap \partial\Omega_\varepsilon^2$ (u_ε and p_ε are the velocity and pressure of the fluids). Assuming that Y_2 is a "bubble" strictly included in the period Y, (4.18) can be regarded as a model for bubbly fluids, where the viscosity is much smaller in the bubble than in the surrounding fluid. Because of its simplicity, this model is very academic since the size, the shape, and the periodic arrangement of the bubbles are kept fixed. Nevertheless, in view of Theorem 4.3, the homogenization of (4.18) could be interesting to derive averaged equations for bubbly fluids. Unfortunately, it turns out that the homogenized system can be drastically simplified in the Stokes case. Drawing upon the ideas of [44], Theorem 4.3 can be generalized to the Stokes equation (4.18), and a homogenized system similar to (4.6) is obtained:

$$(4.19) \qquad \begin{aligned} \nabla p(x) - \mu_1 \,\mathrm{div}_x\,[A^* \nabla_x u(x)] &= f(x) \quad \text{in } \Omega, \\ \mathrm{div}\, u(x) &= 0 \quad \text{in } \Omega, \\ \nabla_y q(x, y) - \mu_2 \Delta_{yy} v(x, y) &= f(x) - \nabla p(x) \quad \text{in } Y_2, \\ \mathrm{div}_y\, v(x, y) &= 0 \quad \text{in } Y_2, \\ u &= 0 \quad \text{on } \partial\Omega, \\ v(x, y) &= 0 \quad \text{on } \partial Y_2, \end{aligned}$$

where A^* is a given positive fourth-order tensor. Since we assumed that the bubble Y_2 does not touch the faces of Y, they are no periodic boundary condition for $q(x, y)$ and $v(x, y)$. Thus, the unique solution of (4.19) satisfies $v = 0$ and $q = y \cdot [f(x) - \nabla p(x)]$. As the weak limit in $[L^2(\Omega)]^N$ of the sequence u_ε is $u(x) + \int_{Y_2} v(x, y) \, dy$, it coincides with $u(x)$. Thus the homogenized problem can be reduced to the Stokes equation for $u(x)$. In other words, there are no contributions from the bubbles in the limit, and thus no interesting phenomena due to the bubbles appear in the homogenized Stokes equations.

Remark 4.9. We have chosen a very special scaling of the diffusion coefficients in (4.2): the order of magnitude of μ_ε is 1 in material 1, and ε^2 in material 2. Indeed, we could more generally consider a scaling ε^k in material 2, with k any positive real number. Let us motivate our choice of the scaling $k = 2$, and to make things easier, we assume that there is no zero-order term in (4.2), i.e., $\alpha = 0$. Then, it turns out that the value $k = 2$ is the only one (apart from zero) that insures a balance between the energies in material 1 and 2 is the only one (apart from zero) that insures a balance between the energies in material 1 and 2, i.e., as ε goes to zero, both terms $\int_{\Omega_\varepsilon^1} \mu_\varepsilon |\nabla u_\varepsilon|^2$ and $\int_{\Omega_\varepsilon^2} \mu_\varepsilon |\nabla u_\varepsilon|^2$ have the same order of magnitude. Thus, for $k = 2$, the limit problem will exhibit a coupling between material 1 and 2. On the contrary, for $k < 2$ the energy is much larger in material 1 than in 2, and in the limit no contributions from material 2 remains (material 2 behaves as a perfect conductor on the microscopic level). For $k > 2$ the energy is much smaller in material 1 than in 2, and in the limit no contributions from material 1 remains (actually, material 2 is a very poor conductor on the microscopic level, but since the source term is of order one its energy goes to infinity).

In other words, our scaling is the only one which makes of material 1 (respectively, 2) a good conductor on the macroscopic (respectively, microscopic) level, yielding an asymptotic (as ε goes to zero) equipartition of the energies stored in materials 1 and 2.

5. On convergence results for periodically oscillating functions. This section is devoted to the proof of Lemma 1.3, and more generally to the convergence of periodically oscillating functions $\psi(x, x/\varepsilon)$. Although in §1 the convergence of the sequence $\psi(x, x/\varepsilon)$ was studied in $L^2(\Omega)$, for the sake of clarity we recast Lemma 1.3 and Definition 1.4 in the framework of $L^1(\Omega)$. More precisely, we consider functions of two variables $\psi(x, y)$ ($x \in \Omega$ open set in $\mathbb{R}^N$, $y \in Y$ the unit cube of $\mathbb{R}^N$), periodic of period Y in y, and we investigate the weak convergence of the sequence $\psi(x, x/\varepsilon)$ in $L^1(\Omega)$, as $\varepsilon \to 0$. Recall the analogue of Definition 1.4 obtained by replacing L^2 by L^1.

DEFINITION 5.1. A function $\psi(x, y) \in L^1(\Omega \times Y)$, Y-periodic in y, is called an "admissible" test function if and only if

$$(5.1) \qquad \lim_{\varepsilon \to 0} \int_\Omega \left| \psi\left(x, \frac{x}{\varepsilon}\right) \right| dx = \int_\Omega \int_Y |\psi(x, y)| \, dx \, dy.$$

The purpose of this section is to investigate under which assumptions a function $\psi(x, y)$ is admissible in the sense of Definition 5.1. It is easily seen that continuous functions on $\Omega \times Y$ are admissible. However, when less smoothness is assumed on $\psi(x, y)$, the verification of (5.1) is not obvious (first of all, the measurability of $\psi(x, x/\varepsilon)$ is not always clear). In the sequel we propose several regularity assumptions for $\psi(x, y)$ to be admissible (see Lemma 5.2, Corollary 5.4, and Lemma 5.5). They all involve the continuity of ψ in, at least, one of the variables x or y. We emphasize that it is definitely not a necessary condition for (5.1). However, to our knowledge this is the only way to obtain, in general, the measurability of $\psi(x, x/\varepsilon)$, by asserting that $\psi(x, y)$ is a

Caratheodory-type function (for a precise definition, see, e.g., Definition 1.2 of Chapter VIII in [21]). We also emphasize that this question of measurability is not purely technical and futile, but is very much linked to possible counterexamples to (5.1). We actually exhibit a counterexample to (5.1), which clearly indicates that the regularity of $\psi(x, y)$ cannot be decreased too much, even if $\psi(x, x/\varepsilon)$ is measurable (see Proposition 5.8).

Our first and main result is the L^1 equivalent of Lemma 1.3, which we recall for the reader's convenience.

LEMMA 5.2. *Let* $\psi(x, y) \in L^1[\Omega; C_\#(Y)]$. *Then, for any positive value of* ε, $\psi(x, x/\varepsilon)$ *is a measurable function on* Ω *such that*

$$(5.2) \qquad \left\| \psi\left(x, \frac{x}{\varepsilon}\right) \right\|_{L^1(\Omega)} \leqq \|\psi(x, y)\|_{L^1[\Omega; C_\#(Y)]},$$

and $\psi(x, y)$ *is an "admissible" test function, i.e., satisfies* (5.1).

By definition, $L^1[\Omega; C_\#(Y)]$ is the space of functions, measurable and summable in $x \in \Omega$, with values in the Banach space of continuous functions, Y-periodic in y. More precisely, $L^1[\Omega; C_\#(Y)]$ is a space of *classes* of functions (two functions belong to the same class if they are equal almost everywhere in Ω); however, for simplicity we shall not distinguish a class or any of its representatives. The above definition of $L^1[\Omega; C_\#(Y)]$ is not very explicit, but we also have the following characterization, which implies, in particular, that any function of $L^1[\Omega; C_\#(Y)]$ is of Caratheodory type, i.e., satisfies (i) and (ii).

LEMMA 5.3. *A function* $\psi(x, y)$ *belongs to* $L^1[\Omega; C_\#(Y)]$ *if and only if there exists a subset* E *(independent of* y) *of measure zero in* Ω *such that*

(i) *For any* $x \in \Omega - E$, *the function* $y \to \psi(x, y)$ *is continuous and* Y-*periodic;*

(ii) *For any* $y \in Y$, *the function* $x \to \psi(x, y)$ *is measurable on* Ω;

(iii) *The function* $x \to \mathrm{Sup}_{y \in Y} |\psi(x, y)|$ *has a finite* $L^1(\Omega)$-*norm.*

Proof. We simply sketch the proof that relies on the equivalence between strong and weak measurability for functions with values in a separable Banach space. Recall the following result of functional analysis (see [11, Prop. 10, Chap. IV.5], or Petti's theorem [48, Chap. V]): let $f(x)$ be a function defined on Ω with values in a separable Banach space E, and let ϕ_n be a weak * dense, countable, family of functions in the unit ball of the dual E' of E; the function f is measurable if and only if all the real-valued functions $x \to \langle \phi_n(x), f(x) \rangle_{E', E}$ are measurable.

Applying this result with $E = C_\#(Y)$, and ϕ_n the family of Dirac masses at rational points of Y, yields the result. □

Proof of Lemma 5.2. From Lemma 5.3, we know that $\psi(x, y)$ is a Caratheodory-type function, and this establishes the measurability of $\psi(x, x/\varepsilon)$. Then, inequality (5.2) is a consequence of the definition of the norm $\|\psi(x, y)\|_{L^1[\Omega; C_\#(Y)]} \equiv \int_\Omega \mathrm{Sup}_{y \in Y} |\psi(x, y)| \, dx$. Let us check that $\psi(x, y)$ satisfies (5.1).

For any integer n, we introduce a paving of the unit cube Y made of n^N small cubes Y_i of size n^{-1}. The main properties of this paving are

$$(5.3) \qquad Y = \bigcup_{i=1}^{n^N} Y_i, \quad |Y_i| = \frac{1}{n^N}, \quad |Y_i \cap Y_j| = 0 \quad \text{if } i \neq j.$$

Let $\chi_i(y)$ be the characteristic function of the set Y_i extended by Y-periodicity to $\mathbb{R}^N$, and let y_i be a point in Y_i. We approximate any function $\psi(x, y)$ in $L^1[\Omega; C_\#(Y)]$ by

a step function in y defined by

$$(5.4) \qquad \psi_n(x, y) = \sum_{i=1}^{n^N} \psi(x, y_i) \chi_i(y).$$

We first prove (5.1) for ψ_n, and then show that passing to the limit as n goes to infinity yields the result for ψ. Thanks to Lemma 5.3 the function $x \to \psi(x, y_i)$ belongs to $L^1(\Omega)$, while $\chi_i(x/\varepsilon)$ is in $L^\infty(\Omega)$. Due to the periodicity of χ_i, a well-known result on oscillating functions leads to

$$(5.5) \qquad \lim_{\varepsilon \to 0} \int_\Omega \psi(x, y_i) \chi_i\left(\frac{x}{\varepsilon}\right) dx = \int_\Omega \psi(x, y_i) \, dx \, |Y_i|.$$

Summing equalities (5.5) for $i \in [1, \cdots, n^N]$ leads to (5.1) for ψ_n.

It remains to pass to the limit in n. Let us first prove that ψ_n converges to ψ in the strong topology of $L^1[\Omega; C_\#(Y)]$. Define

$$(5.6) \qquad \delta_n(x) = \operatorname*{Sup}_{y \in Y} |\psi_n(x, y) - \psi(x, y)|.$$

The function $y \to [\psi_n(x, y) - \psi(x, y)]$ is piecewise continuous in Y almost everywhere in x. Thus, in (5.6) the supremum over $y \in Y$ can be replaced by the supremum over $y \in Y \cap \mathbb{Q}$. This implies that δ_n, being the supremum of a countable family of measurable functions, is measurable, too (see if necessary [11, Chap. IV.5, Thm. 2]). On the other hand, as a result of the continuity in y of ψ, we have

$$\lim_{n \to +\infty} \delta_n(x) = 0 \quad \text{a.e. in } \Omega.$$

Furthermore,

$$0 \leq \delta_n(x) \leq 2 \operatorname*{Sup}_{y \in Y} |\psi(x, y)| \in L^1(\Omega).$$

By application of the Lebesgue theorem of dominated convergence, the sequence δ_n strongly converges to zero in $L^1(\Omega)$. Thus ψ_n strongly converges to ψ in $L^1[\Omega; C_\#(Y)]$.

Let us estimate the difference

$$\left| \int_\Omega \psi\left(x, \frac{x}{\varepsilon}\right) dx - \int_\Omega \int_Y \psi(x, y) \, dx \, dy \right| \leq \left| \int_\Omega \psi\left(x, \frac{x}{\varepsilon}\right) dx - \int_\Omega \psi_n\left(x, \frac{x}{\varepsilon}\right) dx \right|$$

$$(5.7) \qquad\qquad + \left| \int_\Omega \psi_n\left(x, \frac{x}{\varepsilon}\right) dx - \int_\Omega \int_Y \psi_n(x, y) \, dx \, dy \right|$$

$$\qquad\qquad + \left| \int_\Omega \int_Y \psi(x, y) \, dx \, dy - \int_\Omega \int_Y \psi_n(x, y) \, dx \, dy \right|.$$

The first term in the right-hand side of (5.7) is bounded by

$$\int_\Omega \left| \psi\left(x, \frac{x}{\varepsilon}\right) - \psi_n\left(x, \frac{x}{\varepsilon}\right) \right| dx \leq \int_\Omega \operatorname*{Sup}_{y \in Y} |\psi(x, y) - \psi_n(x, y)| \, dx = \|\psi_n - \psi\|_{L^1[\Omega; C_\#(Y)]}.$$

For fixed n we pass to the limit in (5.7) as $\varepsilon \to 0$:

$$(5.8) \qquad \lim_{\varepsilon \to 0} \left| \int_\Omega \psi\left(x, \frac{x}{\varepsilon}\right) dx - \int_\Omega \int_Y \psi(x, y) \, dx \, dy \right| \leq 2\|\psi_n - \psi\|_{L^1[\Omega; C_\#(Y)]}.$$

Then, we pass to the limit in (5.8) as $n \to \infty$, and we obtain (5.1). □

Reversing the role of x and y (namely, assuming continuity in x and measurability in y), the same proof as that of Lemma 5.2 works also for the following corollary.

COROLLARY 5.4. *Assume that Ω is a bounded open set (its closure $\bar{\Omega}$ is thus compact). Let $\psi(y, x)$ be a function in $L^1_\#[Y; C(\bar{\Omega})]$, i.e., measurable, summable, and Y-periodic in y, with values in the Banach space of continuous functions in $\bar{\Omega}$. Then, for any positive value of ε, $\psi(x/\varepsilon, x)$ is a measurable function on Ω such that*

$$(5.9) \qquad \left\| \psi\left(\frac{x}{\varepsilon}, x\right) \right\|_{L^1(\Omega)} \leqq C(\Omega) \| \psi(y, x) \|_{L^1_\#[Y; C(\bar{\Omega})]},$$

and $\psi(y, x)$ is an "admissible" test function, i.e., $\lim_{\varepsilon \to 0} \int_\Omega |\psi(x/\varepsilon, x)| \, dx = \int_\Omega \int_Y |\psi(y, x)| \, dx \, dy$.

In the literature (see, e.g., [10]) the favorite assumption on $\psi(x, y)$, ensuring it is an admissible test function, is $\psi(x, y) \in C_c[\Omega; L^\infty_\#(Y)]$ (i.e., continuous with compact support in Ω, with values in the Banach space of measurable, essentially bounded, and Y-periodic functions in Y). The next two lemmas are concerned with this situation.

LEMMA 5.5. *Let $\psi(x, y)$ be a function such that there exist a subset $E \subset Y$, of measure zero, independent of x, and a compact subset $K \subset \Omega$ independent of y, satisfying*

(i) *For any $y \in Y - E$, the function $x \to \psi(x, y)$ is continuous, with compact support K;*

(ii) *For any $x \in \Omega$, the function $y \to \psi(x, y)$ is Y-periodic and measurable on Y;*

(iii) *The function $x \to \psi(x, y)$ is continuous on K, uniformly with respect to $y \in Y - E$.*

Then, for any positive value of ε, $\psi(x, x/\varepsilon)$ is a measurable function on Ω, and $\psi(x, y)$ is an admissible test function in the sense of Definition 5.1, i.e., satisfies

$$(5.1) \qquad \lim_{\varepsilon \to 0} \int_\Omega \left| \psi\left(x, \frac{x}{\varepsilon}\right) \right| \, dx = \int_\Omega \int_Y |\psi(x, y)| \, dx \, dy.$$

Before proving Lemma 5.5, let us remark that any function satisfying (i)–(iii) obviously belongs to $C_c[\Omega; L^\infty_\#(Y)]$. The converse is more subtle. Indeed, since $\psi(x, y)$ is an element of $C_c[\Omega; L^\infty_\#(Y)]$, for each $x \in \Omega$, its value $y \to \psi(x, y)$ is a class of functions in $L^\infty_\#(Y)$: picking up a representative for each x and collecting them gives a "representative" of $\psi(x, y)$ in $C_c[\Omega; L^\infty_\#(Y)]$.

LEMMA 5.6. *Let $\psi(x, y)$ be a function in $C_c[\Omega; L^\infty_\#(Y)]$. Then, there exists a "representative" of $\psi(x, y)$ for which properties (i)–(iii) in Lemma 5.5 hold.*

Proof. Let $\psi(x, y) \in C_c[\Omega; L^\infty_\#(Y)]$. By definition, for any value of $x \in \Omega$, the function $y \to \psi(x, y)$ is measurable on Y, Y-periodic, and there exists a subset $E(x)$ of measure zero in Y such that $\psi(x, y)$ is bounded on $Y - E(x)$. The continuity of $x \to \psi(x, y)$ from Ω in $L^\infty_\#(Y)$ is equivalent to

$$(5.10) \qquad \lim_{\eta \to 0} \operatorname{Sup}_{y \in Y - [E(x) \cup E(x+\eta)]} |\psi(x+\eta, y) - \psi(x, y)| = 0 \quad \text{for any } x \in \Omega.$$

We emphasize that, a priori, the exceptional set $E(x)$, where the function $y \to \psi(x, y)$ is not defined, depends on x. Nevertheless, thanks to the continuity of $\psi(x, y)$ with respect to the x variable, we are going to exhibit a "representative" of $\psi(x, y)$ for which $E(x)$ is included in a fixed set E of measure zero.

Let $K \subset \Omega$ be the compact support of $x \to \psi(x, y)$. Let $(K_i)_{i=1}^n$ be a sequence of partitions of K (i.e., $\cup_{i=1}^n K_i = K$ and $|K_i \cap K_j| = 0$ if $i \neq j$) such that $\lim_{n \to +\infty} \operatorname{Sup}_{1 \leq i \leq n} \operatorname{diam}(K_i) = 0$. Let $\chi_i(x)$ be the characteristic function of K_i, and x_i a point in K_i. Define the step function $\psi_n(x, y)$ by

$$\psi_n(x, y) = \sum_{i=1}^n \psi(x_i, y) \chi_i(x).$$

By definition of the partitions $(K_i)_{i=1}^n$, and continuity of $x \to \psi(x, y)$ from Ω in $L^\infty_\#(Y)$, we have

$$(5.11) \qquad \lim_{n \to +\infty} \operatorname{Sup}_{x \in K} \| \psi(x, y) - \psi_n(x, y) \|_{L^\infty_\#(Y)}.$$

In view of its definition, $\psi_n(x, y)$ is defined and bounded on $\Omega \times (Y - E_n)$, where E_n is a set of measure zero that does not depend on x. Then, the set $E = \cup_{n=1}^\infty E_n$ is also of measure zero and does not depend on x. From (5.11) it is easily deduced that $\psi_n(x, y)$ converges pointwise in $\Omega \times (Y - E)$ to a limit $\tilde\psi(x, y)$ that is continuous in $x \in \Omega$ uniformly with respect to $y \in Y - E$. As announced, $\tilde\psi$ is a "representative" of $\psi(x, y)$, which has the desired properties (i)–(iii). $\square$

Proof of Lemma 5.5. Properties (i) and (ii) imply that $\psi(x, y)$ is a Caratheodory-type function, and thus $\psi(x, x/\varepsilon)$ is measurable on Ω. Using the approximating sequence of step functions $\psi_n(x, y)$ introduced in the proof of Lemma 5.6, and arguing as in Lemma 5.2, leads to (5.1) for ψ. $\square$

In the three previous results, the function $\psi(x, y)$ is assumed to be continuous in, at least, one variable x or y. Of course, it is not a necessary assumption that ψ be an "admissible" test function. For example, if a separation of variables holds, namely, ψ is the product of two functions, each depending on only one variable, we have the following well-known result (for a proof, see, e.g., [9]).

LEMMA 5.7. *Assume that Ω is a bounded open set. Let $\phi_1(x) \in L^p(\Omega)$ and $\phi_2(y) \in L^{p'}_\#(Y)$ with $(1/p) + (1/p') = 1$ and $1 \leq p \leq +\infty$. (In case $p = 1$ and $p' = +\infty$, the set Ω can be unbounded.) Then, for any positive value of ε, $\phi_1(x)\phi_2(x/\varepsilon)$ is a measurable function on Ω, and $\phi_1(x)\phi_2(y)$ is an "admissible" test function in the sense of Definition 5.1.*

In general the regularity of ψ cannot be weakened too much: even if $\psi(x, x/\varepsilon)$ is measurable, the function $\psi(x, y)$ may be not "admissible" in the sense of Definition 5.1. Following an idea of Gérard and Murat [25], we are able to construct a counter-example to (5.1) with $\psi(x, y) \in C[\bar\Omega; L^1_\#(Y)]$.

PROPOSITION 5.8. *Let $\bar\Omega = Y = [0; 1]$. There exists $v(x, y) \in C([0, 1]; L^1_\#[0, 1])$, which is not an "admissible" test function, namely,*

$$(5.12) \qquad \lim_{n \to +\infty} \int_0^1 |v(x, nx)| \, dx \neq \int_0^1 \int_0^1 |v(x, y)| \, dx \, dy.$$

Remark 5.9. In general, a function $\psi(x, y) \in C[\bar\Omega; L^1_\#(Y)]$ is not of Caratheodory type, i.e., is not continuous in x almost everywhere in Y. Thus, the measurability of $\psi(x, x/\varepsilon)$ is usually not guaranteed.

Proof of Proposition 5.8. Let us fix $\bar\Omega = Y = [0; 1]$. In the square $[0; 1]^2$, we are going to construct an increasing sequence of measurable subset E_n, which converges to a set E. The desired function $v(x, y)$ will be defined as the characteristic function of E extended by $[0; 1]$-periodicity in y.

For each integer n, we consider the n lines defined in the plane by

$$y = nx - p \quad \text{with } p \in \{0, 1, 2, \cdots, n-1\}.$$

Then, we define the set D_n made of all the points (x, y) in $[0; 1]^2$ that are at a distance less than αn^{-3} of one of the lines $y = nx - p$ for $p = 0, 1, \cdots, n-1$ (the distance is the usual Euclidean distance, and α is a small strictly positive number). The set D_n is made of n strips of width $2\alpha n^{-3}$ and length of order 1. Next, we define the measurable set $E_n = \cup_{p=1}^n D_p$. The sequence E_n is increasing in $[0; 1]^2$, and thus converges to a

measurable limit set E. We have a bound on its measure

$$(5.13) \qquad |E| \leq \sum_{n=1}^{\infty} |D_n| \leq 4\alpha \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Let $v(x, y)$ be the characteristic function of E extended by $[0; 1]$-periodicity in y. For sufficiently small α, we deduce from (5.13) that $\int_{[0;1]^2} v(x, y) \, dx \, dy = |E| < 1$. Meanwhile, we obviously have $v(x, nx) = 1$ for $x \in [0; 1]$. Thus, the sequence $\int_{[0;1]} v(x, nx) \, dx$ cannot converge to the average of v. To complete the proof it remains to show that $v(x, y)$ belongs to $C([0, 1]; L^1_{\#}[0, 1])$, i.e., for any $x \in [0; 1]$,

$$(5.14) \qquad \lim_{\varepsilon \to 0} \int_{[0;1]} |v(x + \varepsilon, y) - v(x, y)| \, dy = 0.$$

(By definition of E, $v(x, y)$ is measurable in $[0; 1]^2$ and is easily seen to be also measurable, at fixed x, in y.) Let $E(x)$ (respectively, $D_n(x)$) be the section of E (respectively, D_n) at fixed abcissa x, i.e.,

$$E(x) = \{ y \in [0; 1] / (x, y) \in E \},$$

$$D_n(x) = \{ y \in [0; 1] / (x, y) \in D_n \}.$$

Then

$$\int_{[0;1]} |v(x + \varepsilon, y) - v(x, y)| \, dy = |E(x) \cap ([0; 1] - E(x + \varepsilon))|$$

$$+ |E(x + \varepsilon) \cap ([0; 1] - E(x))|.$$

Since $E(x) = \bigcup_{n=1}^{\infty} D_n(x)$, we have

$$|E(x) \cap ([0; 1] - E(x + \varepsilon))| \leq \sum_{n=1}^{\infty} |D_n(x) \cap ([0; 1] - D_n(x + \varepsilon))|.$$

It is easily seen that $|D_n(x) \cap ([0; 1] - D_n(x + \varepsilon))|$ is constant when x varies in $[0; 1]$. Thus

$$(5.15) \qquad \int_{[0;1]} |v(x + \varepsilon, y) - v(x, y)| \, dy \leq 2 \sum_{n=1}^{\infty} |D_n(x) \cap ([0; 1] - D_n(x + \varepsilon))|.$$

Let us fix $\varepsilon > 0$. Recall that D_n is made of n strips of width $2\alpha n^{-3}$. Denote by l_n (respectively, L_n) the length of the intersection of one strip with the x-axis (respectively, y-axis). It is easily seen that l_n is of order n^{-3}, while L_n is of order n^{-2}. Both points (x, y) and $(x + \varepsilon, y)$ lie in the same strip of D_n if n is smaller than $\varepsilon^{-1/3}$. This suggests to cut the sum in (5.15) in two parts, the first one being

$$(5.16) \qquad \sum_{n=\varepsilon^{-1/3}}^{\infty} |D_n(x) \cap ([0; 1] - D_n(x + \varepsilon))| \leq \sum_{n=\varepsilon^{-1/3}}^{\infty} |D_n(x)|,$$

while the second one is

$$(5.17) \qquad \sum_{n=1}^{\varepsilon^{-1/3}} |D_n(x) \cap ([0; 1] - D_n(x + \varepsilon))|.$$

Since $|D_n(x)| = L_n$ is of order n^{-2}, (5.16) is bounded by

$$\sum_{n=\varepsilon^{-1/3}}^{\infty} |D_n(x)| \leq C \sum_{n=\varepsilon^{-1/3}}^{\infty} \frac{1}{n^2} \leq C\varepsilon^{1/3}.$$

On the other hand, an easy calculation shows that, for any value of n, $|D_n(x) \cap ([0; 1] - D_n(x + \varepsilon))|$ is bounded by $C\varepsilon n$. Thus, (5.17) is bounded by

$$\sum_{n=1}^{\varepsilon^{-1/3}} |D_n(x) \cap ([0; 1] - D_n(x + \varepsilon))| \leq C \sum_{n=1}^{\varepsilon^{-1/3}} \varepsilon n \leq C\varepsilon^{1/3}.$$

This leads to

$$\int_{[0;1]} |v(x + \varepsilon, y) - v(x, y)| \, dy \leq C\varepsilon^{1/3},$$

where C is a constant independent of ε. Letting $\varepsilon \to 0$ yields (5.14).   □

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