ON THE $L^p$–THEORY OF ANISOTROPIC SINGULAR PERTURBATIONS OF ELLIPTIC PROBLEMS

CHOKRI OGABI

Academie de Grenoble, 38300 France

Abstract. In this article we give an extension of the $L^2$–theory of anisotropic singular perturbations for elliptic problems. We study a linear and some non-linear problems involving $L^p$ data ($1 < p < 2$). Convergences in pseudo Sobolev spaces are proved for weak and entropy solutions, and rate of convergence is given in cylindrical domains.

1. Introduction

1.1. Preliminaries. In this article we shall give an extension of the $L^2$–theory of the asymptotic behavior of elliptic, anisotropic singular perturbations problems. This kind of singular perturbations has been introduced by M. Chipot [6]. From the physical point of view, these problems can modelize diffusion phenomena when the diffusion coefficients in certain directions are going toward zero. The $L^2$ theory of the asymptotic behavior of these problems has been studied by M. Chipot and many co-authors. First of all, let us begin by a brief discussion on the uniqueness of the weak solution ( by weak a solution we mean a solution in the sense of distributions) to the problem

$$\begin{cases}
-\text{div}(A \nabla u) = f \\
u = 0 \quad \text{on } \partial \Omega
\end{cases}$$

(1)

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$ is a bounded Lipschitz domain, we suppose that $f \in L^p(\Omega)$ ($1 < p < 2$). The diffusion matrix $A = (a_{ij})$ is supposed to be bounded and satisfies the ellipticity assumption on $\Omega$ ( see assumptions (2) and (3) in subsection 1.2). It is well known that (1) has at least a weak solution in $W^{1,p}_0(\Omega)$. Moreover, if $A$ is symmetric and continuous and $\partial \Omega \in C^2$ [2] then (1) has a unique solution in $W^{1,p}_0(\Omega)$. If $A$ is discontinuous the uniqueness assertion is false, in [15] Serrin has given a counterexample when $N \geq 3$. However, if $N = 2$ and if $\partial \Omega$ is sufficiently smooth and without any continuity assumption on $A$, (1) has a unique weak solution in $W^{1,p}_0(\Omega)$. The proof is based on the Meyers regularity theorem (see for instance [13]). To treat this pathology, Benilin, Boccardo, Gallouet, and al have introduced the concept of the entropy solution [4] for problems involving $L^1$ data (or more generally a Radon measure).

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For every \( k > 0 \) we define the function \( T_k : \mathbb{R} \to \mathbb{R} \) by
\[
T_k(s) = \begin{cases} 
  s & |s| \leq k \\
  k \text{sgn}(x) & |s| \geq k
\end{cases}
\]
And we define the space \( T^{1,2}_0 \) introduced in [4].
\[
T^{1,2}_0(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable such that for any } k > 0 \text{ there exists } \phi_n \in H^1_0(\Omega) : \phi_n \to T_k(u) \text{ a.e in } \Omega \right. \\
\left. \text{ and } \left( \nabla \phi_n \right)_{n \in \mathbb{N}} \text{ is bounded in } L^2(\Omega) \right\}
\]
This definition of \( T^{1,2}_0 \) is equivalent to the original one given in [4]. In fact, this is a characterization of this space [4]. Now, more generally, for \( f \in L^1(\Omega) \) we have the following definition of entropy solution [4].

**Definition 1.** A function \( u \in T^{1,2}_0(\Omega) \) is said to be an entropy solution to (1) if
\[
\int_{\Omega} A \nabla u \cdot \nabla T_k(u - \varphi) dx \leq \int_{\Omega} f T_k(u - \varphi) dx, \varphi \in \mathcal{D}(\Omega), k > 0
\]

We refer the reader to [4] for more details about the sense of this formulation. The main results of [4] show that (1) has a unique entropy solution which is also a weak solution of (1) moreover since \( \Omega \) is bounded then this solution belongs to
\[
\bigcap_{1 \leq r < \frac{N}{N-1}} W^{1,r}_0(\Omega).
\]

1.2. Description of the problem and functional setting. Throughout this article we will suppose that \( f \in L^p(\Omega), 1 < p < 2 \), (we can suppose that \( f \notin L^2(\Omega) \)). We give a description of the linear problem (some nonlinear problems will be studied later). Consider the following singular perturbations problem
\[
\begin{cases} 
  -\text{div}(A \nabla u_e) = f & \\
  u_e = 0 & \text{ on } \partial \Omega
\end{cases},
\]
where \( \Omega \) is a bounded Lipschitz domain of \( \mathbb{R}^N \). Let \( q \in \mathbb{N}^*, N - q \geq 2 \). We denote by \( x = (x_1, \ldots, x_N) = (X_1, X_2) \in \mathbb{R}^q \times \mathbb{R}^{N-q} \) i.e. we split the coordinates into two parts. With this notation we set
\[
\nabla = \left( \partial_{x_1}, \ldots, \partial_{x_N} \right)^T = \left( \nabla_{X_1}, \nabla_{X_2} \right),
\]
where
\[
\nabla_{X_1} = \left( \partial_{x_1}, \ldots, \partial_{x_q} \right)^T \text{ and } \nabla_{X_2} = \left( \partial_{x_{q+1}}, \ldots, \partial_{x_N} \right)^T.
\]
Let \( A = (a_{ij}(x)) \) be a \( N \times N \) matrix which satisfies the ellipticity assumption
\[
\exists \lambda > 0 : A \xi \cdot \xi \geq \lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^N \text{ for a.e } x \in \Omega,
\]
and
\[
a_{ij}(x) \in L^\infty(\Omega), \forall i, j = 1, 2, \ldots, N,
\]
We have decomposed \( A \) into four blocks
\[
A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},
\]
where $A_{11}$, $A_{22}$ are respectively $q \times q$ and $(N-q) \times (N-q)$ matrices. For $0 < \epsilon \leq 1$ we have set

$$A_\epsilon = \begin{pmatrix} \epsilon^2 A_{11} & \epsilon A_{12} \\ \epsilon A_{21} & A_{22} \end{pmatrix}$$

We denote $\Omega_{X_1} = \{(X_1, X_2) \in \Omega \}$ and $\Omega^1 = P_1 \Omega$ where $P_1 : \mathbb{R}^N \to \mathbb{R}^p$ is the usual projector. We introduce the space

$$V_p = \left\{ u \in L^p(\Omega) \mid \nabla X_2 u \in L^p(\Omega), \text{ and for a.e } X_1 \in \Omega^1, u(X_1, \cdot) \in W^{1,p}_0(\Omega_{X_1}) \right\}$$

We equip $V_p$ with the norm

$$\|u\|_{V_p} = \left(\|u\|_{L^p(\Omega)}^p + \|\nabla X_2 u\|_{L^p(\Omega)}^p\right)^{\frac{1}{p}},$$

then one can show easily that $(V_p, \|\cdot\|_{V_p})$ is a separable reflexive Banach space.

The passage to the limit (formally) in (2) gives the limit problem

$$\begin{cases} - \text{div}_{X_2}(A_{22} \nabla X_2 u_0(X_1, \cdot)) = f(X_1, \cdot) \\ u_0(X_1, \cdot) = 0 \text{ on } \partial \Omega_{X_1} \\ X_1 \in \Omega^1 \end{cases} \quad (5)$$

The $L^2$-theory (when $f \in L^2$) of problem (2) has been treated in [8], convergence has been proved in $V_2$ and rate of convergence in the $L^2$-norm has been given. For the $L^2$–theory of several nonlinear problems we refer the reader to [9],[10],[14].

This article is mainly devoted to study the $L^p$–theory of the asymptotic behavior of linear and nonlinear singularly perturbed problems. In other words, we shall study the convergence $u_\epsilon \to u_0$ in $V_p$ (Notice that in [9], authors have treated some problems involving $L^p$ data where some others data of the equations depend on $p$, one can check easily that it is not the $L^p$ theory which we expose in this manuscript).

Let us briefly summarize the content of the paper:

- In section 2: We study the linear problem, we prove convergences for weak and entropy solutions.
- In section 3: We give the rate of convergence in a cylindrical domain when the data is independent of $X_1$.
- In section 4: We treat some nonlinear problems.

2. The Linear Problem

The main results in this section are the following

**Theorem 1.** Assume (3), (4) then there exists a sequence $(u_\epsilon)_{0 < \epsilon \leq 1} \subset W^{1,p}_0(\Omega)$ of weak solutions to (2) and $u_0 \in V_p$ such that $\epsilon \nabla X_1 u_\epsilon \to 0$ in $L^p(\Omega)$, $u_\epsilon \to u_0$ in $V_p$ where $u_0$ satisfies (5) for a.e $X_1 \in \Omega^1$.

**Corollary 1.** Assume (3), (4) then if $A$ is symmetric and continuous and $\partial \Omega \in C^2$, then there exists a unique $u_0 \in V_p$ such that $u_0(X_1, \cdot)$ is the unique solution to (5) in $W^{1,p}_0(\Omega_{X_1})$ for a.e $X_1$. Moreover the sequence $(u_\epsilon)_{0 < \epsilon \leq 1}$ of the unique solutions (in $W^{1,p}_0(\Omega)$) to (2) converges in $V_p$ to $u_0$ and $\epsilon \nabla X_1 u_\epsilon \to 0$ in $L^p(\Omega)$.

**Proof.** This corollary follows immediately from Theorem 1 and uniqueness of the solutions of (2) and (5) as mentioned in subsection 1.1 (Notice that $\partial \Omega_{X_1} \in C^2$).
Theorem 2. Assume (3), (4) then there exists a unique \( u_0 \in V_p \) such that \( u_0(X_1, \cdot) \) is the unique entropy solution of (5). Moreover, the sequence of the entropy solutions \( (u_\epsilon)_{0<\epsilon \leq 1} \) of (2) converges to \( u_0 \) in \( V_p \) and \( \epsilon \nabla X_1 u_\epsilon \to 0 \) in \( L^p(\Omega) \).

2.1. Weak convergence. Let us prove the following primary result

Theorem 3. Assume (3), (4) then there exists a sequence \( (u_{\epsilon_k})_{k \in \mathbb{N}} \subset W^{1,p}_0(\Omega) \) of weak solutions to (2) \( (\epsilon_k \to 0 \text{ as } k \to \infty) \) and \( u_0 \in V_p \) such that \( \nabla X_2 u_{\epsilon_k} \to \nabla X_2 u_0 \), \( \epsilon_k \nabla X_1 u_{\epsilon_k} \to 0 \), \( u_{\epsilon_k} \to u_0 \) in \( L^p(\Omega) \) weak-* and \( u_0 \) satisfies (5) for a.e \( X_1 \in \Omega \).

Proof. By density let \( (f_n)_{n \in \mathbb{N}} \subset L^2(\Omega) \) be a sequence such that \( f_n \to f \) in \( L^p(\Omega) \), we can suppose that \( \forall n \in \mathbb{N} : \|f_n\|_{L^p} \leq M, M \geq 0 \). Consider the regularized problem

\[
\begin{align*}
u^n_\epsilon & \in H^1_0(\Omega), \quad \int_\Omega A_\epsilon \nabla u^n_\epsilon \cdot \nabla \varphi dx = \int_\Omega f_n \varphi dx, \quad \varphi \in \mathcal{D}(\Omega) \tag{6} \\
\end{align*}
\]

Assumptions (2) and (3) shows that \( u^n_\epsilon \) exists and it is unique by the Lax-Milgram theorem. (Notice that \( u^n_\epsilon \) also belongs to \( W^{1,p}_0(\Omega) \)). We introduce the function

\[
\theta(t) = \int_0^t (1 + |s|)^{p-2} ds, \quad t \in \mathbb{R}
\]

This kind of function has been used in [3]. We have \( \theta'(t) = (1 + |t|)^{p-2} \leq 1 \) and \( \theta(0) = 0 \), therefore we have \( \theta(u) \in H^1_0(\Omega) \) for every \( u \in H^1_0(\Omega) \). Testing with \( \theta(u^n_\epsilon) \) in (6) and using the ellipticity assumption we deduce

\[
\begin{align*}
\lambda \epsilon^2 \int_\Omega (1 + |u^n_\epsilon|)^{p-2} |\nabla X_1 u^n_\epsilon|^2 dx & + \lambda \int_\Omega (1 + |u^n_\epsilon|)^{p-2} |\nabla X_2 u^n_\epsilon|^2 dx \\
& \leq \int_\Omega f_n \theta(u^n_\epsilon) dx \leq \frac{2}{p-1} \int_\Omega |f_n| (1 + |u^n_\epsilon|)^{p-1} dx,
\end{align*}
\]

where we have used \( |\theta(t)| \leq \frac{2(|t|^{p-1})(p-1)}{2} \). In the other hand, by Hölder’s inequality we have

\[
\int_\Omega |\nabla X_2 u^n_\epsilon|^p dx \leq \left( \int_\Omega (1 + |u^n_\epsilon|)^{p-2} |\nabla X_2 u^n_\epsilon|^2 dx \right)^{\frac{p}{2}} \left( \int_\Omega (1 + |u^n_\epsilon|)^p dx \right)^{1-\frac{p}{2}}
\]

From the two previous integral inequalities we deduce

\[
\int_\Omega |\nabla X_2 u^n_\epsilon|^p dx \leq \left( \frac{2}{\lambda(p-1)} \int_\Omega |f_n| (1 + |u_{\epsilon_k}|)^{p-1} dx \right)^{\frac{p}{2}} \times \left( \int_\Omega (1 + |u_{\epsilon_k}|)^p dx \right)^{1-\frac{p}{2}}
\]

By Hölder’s inequality we get

\[
\begin{align*}
\|\nabla X_2 u^n_\epsilon\|_{L^p(\Omega)} & \leq \left( \frac{2\|f_n\|_{L^p}}{\lambda(p-1)} \right)^{\frac{1}{2}} \left( \int_\Omega (1 + |u^n_\epsilon|)^p dx \right)^{\frac{1}{2}}.
\end{align*}
\]

Using Minkowski inequality we get

\[
\|\nabla X_2 u^n_\epsilon\|_{L^p(\Omega)} \leq C(1 + \|u^n_\epsilon\|_{L^p(\Omega)}),
\]

(7)
Thanks to Poincaré’s inequality \( \|u^n_e\|_{L^p(\Omega)} \leq C_\Omega \|\nabla X_1 u^n_e\|_{L^p(\Omega)} \) we obtain
\[
\|\nabla X_2 u^n_e\|_{L^p(\Omega)} \leq C' (1 + \|\nabla X_2 u^n_e\|_{L^p(\Omega)}),
\]
where the constant \( C' \) depends on \( p, \lambda, mes(\Omega), M \) and \( C_\Omega \). Whence, we deduce
\[
\|u^n_e\|_{L^p(\Omega)} \cdot \|\nabla X_2 u^n_e\|_{L^p(\Omega)} \leq C''
\]
Similarly we obtain
\[
\|\epsilon\nabla X_1 u^n_e\|_{L^p(\Omega)} \leq C''',
\]
where the constants \( C'', C''' \) are independent of \( n \) and \( \epsilon \), so
\[
\|u^n_e\|_{W^{1,p}(\Omega)} \leq \frac{\text{Const}}{\epsilon}
\]
Fix \( \epsilon \), since \( W^{1,p}(\Omega) \) is reflexive then (10) implies that there exists a subsequence \( (u^n_{e_k(\epsilon)})_{k \in \mathbb{N}} \) and \( u_e \in W^{1,p}_0(\Omega) \) such that \( u^n_{e_k(\epsilon)} \rightharpoonup u_e \in W^{1,p}_0(\Omega) \) (as \( l \to \infty \)) in \( W^{1,p}(\Omega) \)–weak. Now, passing to the limit in (6) as \( l \to \infty \) we deduce
\[
\int_{\Omega} A_2 \nabla u_e \cdot \nabla \varphi \, dx = \int_{\Omega} f \varphi \, dx \quad \varphi \in \mathcal{D}(\Omega)
\]
Whence \( u_e \) is a weak solution of (2) \( (u_e = 0 \text{ on } \partial \Omega \text{ in the trace sense of } W^{1,p} \text{-functions, indeed the trace operator is well defined since } \partial \Omega \text{ is Lipschitz}) \).

Now, from (8) and (9) we deduce
\[
\|u_e\|_{L^p(\Omega)} \leq \liminf_{l \to \infty} \left\|u^n_{e_k(\epsilon)}\right\|_{L^p(\Omega)} \leq C'
\]
and similarly we obtain
\[
\|\epsilon\nabla X_1 u_e\|_{L^p(\Omega)} \cdot \|\nabla X_2 u_e\|_{L^p(\Omega)} \leq C'
\]
Using reflexivity and continuity of the derivation operator on \( \mathcal{D}'(\Omega) \) one can extract a subsequence \( (u_{e_k})_{k \in \mathbb{N}} \) such that \( \nabla X_2 u_{e_k} \rightharpoonup \nabla X_2 u_0, \epsilon_k \nabla X_1 u_{e_k} \to 0, u_{e_k} \rightharpoonup u_0 \) in \( L^p(\Omega) \) – weak. Passing to the limit in (11) we get
\[
\int_{\Omega} A_{22} \nabla X_2 u_0 \cdot \nabla X_2 \varphi \, dx = \int_{\Omega} f \varphi \, dx \quad \varphi \in \mathcal{D}(\Omega)
\]
Now, we will prove that \( u_0 \in V_p \). Since \( \nabla X_2 u_{e_k} \rightharpoonup \nabla X_2 u_0 \) and \( u_{e_k} \rightharpoonup u_0 \) in \( L^p(\Omega) \) – weak then there exists a sequence \( (U_n)_{n \in \mathbb{N}} \subset \text{conv}(\{u_{e_k}\}_{k \in \mathbb{N}}) \) such that \( \nabla X_2 U_n \rightharpoonup \nabla X_2 u_0 \) in \( L^p(\Omega) \) – strong, where \( \text{conv}(\{u_{e_k}\}_{k \in \mathbb{N}}) \) is the convex hull of the set \( \{u_{e_k}\}_{k \in \mathbb{N}} \). Notice that we have \( U_n \in W^{1,p}_0(\Omega) \) then -up to a subsequence- we have \( U_n(X_{1, \cdot}) \in W^{1,p}_0(\Omega X_1) \), a.e \( X_1 \in \Omega^1 \). And we also have -up to a subsequence- \( \nabla X_2 U_n(X_{1, \cdot}) \rightharpoonup \nabla X_2 u_0(X_{1, \cdot}) \) in \( L^p(\Omega X_1) \) – strong a.e \( X_1 \in \Omega^1 \). Whence \( u_0(X_{1, \cdot}) \in W^{1,p}_0(\Omega X_1) \) for a.e \( X_1 \in \Omega^1 \), so \( u_0 \in V_p \).

Finally, we will prove that \( u_0 \) is a solution of (5). Let \( E \) be a Banach space, a family of vectors \( \{e_n\}_{n \in \mathbb{N}} \) in \( E \) is said to be a Banach basis or a Schauder basis of \( E \) if for every \( x \in E \) there exists a family of scalars \( (\alpha_n)_{n \in \mathbb{N}} \) such that \( x = \sum_{n=0}^{\infty} \alpha_n e_n \), where the series converges in the norm of \( E \). Notice that Schauder basis does not always exist. In [11] P. Enflo has constructed a separable reflexive Banach space without Schauder basis!. However, the Sobolev space \( W^{1, r}_0 (1 < r < \infty) \) has a Schauder basis whenever the boundary of the domain is sufficiently smooth [12]. Now, we are ready to finish the proof. Let \( (U_i \times V_i)_{i \in \mathbb{N}} \) be a countable covering of \( \Omega \)
such that $U_i \times V_i \subset \Omega$ where $U_i \subset \mathbb{R}^q, V_i \subset \mathbb{R}^{N-q}$ are two bounded open domains, where $\partial V_i$ is smooth ($V_i$ are Euclidian balls for example), such a covering always exists. Now, fix $\psi \in \mathcal{D}(V_i)$ then it follows from (12) that for every $\varphi \in \mathcal{D}(U_i)$ we have

$$
\int_{U_i} \varphi dX_1 \int_{V_i} A_{22} \nabla_{X_2} u_0 \cdot \nabla_{X_3} \psi dX_2 = \int_{U_i} \varphi dX_1 \int_{V_i} f \psi dX_2
$$

Whence for a.e $X_1 \in U_i$ we have

$$
\int_{V_i} A_{22}(X_1, \cdot) \nabla_{X_2} u_0(X_1, \cdot) \cdot \nabla_{X_3} \psi dX_2 = \int_{V_i} f(X_1, \cdot) \psi dX_2
$$

Notice that by density we can take $\psi \in W^{1,p'}_0(V_i)$ where $p'$ is the conjugate of $p$. Using the same techniques as in [8], where we use a Schauder basis of $W^{1,p'}_0(V_i)$ and a partition of the unity, one can easily obtain

$$
\int_{\Omega_{X_1}} A_{22}(X_1, \cdot) \nabla_{X_2} u_0(X_1, \cdot) \cdot \nabla_{X_3} \varphi dx = \int_{\Omega_{X_1}} f(X_1, \cdot) \varphi dx, \varphi \in \mathcal{D}(\Omega),
$$

for a.e $X_1 \in \Omega^1$. Finally, since $u_0(X_1, \cdot) \in W^{1,p}_0(\Omega_{X_1})$ (as proved above) then $u_0(X_1, \cdot)$ is a solution of (5) (Notice that $\Omega_{X_1}$ is also a Lipschitz domain so the trace operator is well defined).

2.2. Strong convergence. Theorem 1 will be proved in three steps. the proof is based on the use of the approximated problem (6). In the first step, we shall construct the solution of the limit problem

**Step 1**: Let $u^n_0 \in H^1_0(\Omega)$ be the unique solution to (6), existence and uniqueness of $u^n_0$ follows from assumptions (3), (4) as mentioned previously. One have the following

**Proposition 1.** Assume (3), (4) then there exists $(u^n_0)_{n \in \mathbb{N}} \subset V_2$ such that $cu^n_0 \to 0$ in $L^2(\Omega)$, $u^n_0 \rightharpoonup u_0^n$ in $V_2$ for every $n \in \mathbb{N}$, in particular the two convergences holds in $L^p(\Omega)$ and $V_p$ respectively. And $u_0^n$ is the unique weak solution in $V_2$ to the problem

$$
\begin{align*}
\begin{cases}
\text{div}_{X_2}(A_{22}(X_1, \cdot) \nabla_{X_2} u_0^n(X_1, \cdot)) = f_n(X_1, \cdot), & X_1 \in \Omega^1 \\
u_0^n(X_1, \cdot) = 0 & \text{on } \partial \Omega_{X_1}
\end{cases}
\end{align*}
$$

(13)

**Proof.** This result follows from the $L^2$-theory (Theorem 1 in [8]). The convergences in $V_p$ and $L^p(\Omega)$ follow from the continuous embedding $V_2 \hookrightarrow V_p, L^2(\Omega) \hookrightarrow L^p(\Omega)$ ($p < 2$).

Now, we construct $u_0$ the solution of the limit problem (5). Testing with $\varphi = \theta(u_0^n(X_1, \cdot))$ in the weak formulation of (13) ($\theta$ is the function introduced in subsection 2.1) and estimating like in the proof of Theorem 3 we obtain as in (7)

$$
\left\| \nabla_{X_2} u_0^n(X_1, \cdot) \right\|_{L^p(\Omega_{X_1})} \leq \left( \frac{\left\| f_n(X_1, \cdot) \right\|_{L^p(\Omega_{X_1})}}{\lambda(p-1)} \right)^{\frac{1}{p}} \times \left( \int_{\Omega_{X_1}} (1 + |u_0^n(X_1, \cdot)|^p) dX_2 \right)^{\frac{1}{p}}
$$

(14)
Integrating over $\Omega^1$ and using Cauchy-Schwarz’s inequality in the right hand side we get
\[
\|\nabla u_0^n\|_{L^p(\Omega)}^p \leq C \|f_n\|_{L^p(\Omega)}^p \left( \int_{\Omega} (1 + |u_0^n|)^p dx \right) \frac{1}{2} 
\]
and therefore
\[
\|\nabla u_0^n\|_{L^p(\Omega)}^2 \leq C'(1 + \|u_0^n\|_{L^p(\Omega)})
\]
Using Poincaré’s inequality $\|u_0^n\|_{L^p(\Omega)} \leq C_\Omega \|\nabla u_0^n\|_{L^p(\Omega)}$ (which holds since $u_0^n(X_1,\cdot) \in W^{1,p}(\Omega_X)$ a.e $X_1 \in \Omega^1$), one can obtain the estimate
\[
\|u_0^n\|_{L^p(\Omega)} \leq C'' \text{ for every } n \in \mathbb{N},
\]
where $C''$ is independent of $n$. Now, using the linearity of the problem and (13) with the test function $\theta(u_0^n(X_1,\cdot) - u_0^m(X_1,\cdot))$, $m, n \in \mathbb{N}$ one can obtain like in (14)
\[
\|\nabla (u_0^n(X_1,\cdot) - u_0^m(X_1,\cdot))\|_{L^p(\Omega_X)}^2 \leq \left( \frac{\|f_n(X_1,\cdot) - f_m(X_1,\cdot)\|_{L^p(\Omega_X)}}{\lambda (p - 1)} \right)^{\frac{1}{2}} \times \left( \int_{\Omega_X} (1 + |u_0^n(X_1,\cdot) - u_0^m(X_1,\cdot)|)^p dX_2 \right)^{\frac{1}{p}}
\]
integrating over $\Omega^1$ and using Cauchy-Schwarz and (15) yields
\[
\|\nabla (u_0^n - u_0^m)\|_{L^p(\Omega)} \leq C \|f_n - f_m\|_{L^p(\Omega)}^\frac{1}{2},
\]
where $C$ is independent of $m$ and $n$. The Poincaré’s inequality shows that
\[
\|u_0^n - u_0^m\|_{V_p} \leq C' \|f_n - f_m\|_{L^p(\Omega)}^\frac{1}{2}
\]
Since $(f_n)_{n \in \mathbb{N}}$ is a converging sequence in $L^p(\Omega)$ then this last inequality shows that $(u_0^n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $V_p$, consequently there exists $u_0 \in V_p$ such that $u_0^n \to u_0$ in $V_p$. Now, passing to the limit in (6) as $\epsilon \to 0$ we get
\[
\int_{\Omega} A_{22} \nabla u_0^n \cdot \nabla \varphi dX_2 = \int_{\Omega} f_\varphi dX_2, \quad \varphi \in \mathcal{D}(\Omega)
\]
Passing to the limit as $n \to \infty$ we deduce
\[
\int_{\Omega} A_{22} \nabla u_0 \cdot \nabla \varphi dX_2 = \int_{\Omega} f \varphi dX_2, \quad \varphi \in \mathcal{D}(\Omega)
\]
Then it follows as proved in Theorem 3 that $u_0$ satisfies (5). Whence we have proved the following

**Proposition 2.** Under assumption of Proposition 1 there exists $u_0 \in V_p$ solution to (5) such that $u_0^n \to u_0$ in $V_p$ where $(u_0^n)_{n \in \mathbb{N}}$ is the sequence given in Proposition 1.

**Step 2:** In this second step we will construct the sequence $(u_\epsilon)_{0 < \epsilon \leq 1}$ solutions of (2), one can prove the following
**Proposition 3.** There exists a sequence \((u^n_\epsilon)_{0<\epsilon<1} \subset W^{1,p}_0(\Omega)\) of weak solutions to (2) such that \(u^n_\epsilon \to u_\epsilon\) in \(W^{1,p}(\Omega)\) for every \(\epsilon\) fixed. Moreover, \(u^n_\epsilon \to u_\epsilon\) in \(V_p\) and \(\epsilon \nabla X_2 u^n_\epsilon \to \epsilon \nabla X_2 u_\epsilon\), uniformly in \(\epsilon\).

**Proof.** Using the linearity of (6) testing with \(\theta(u^n_\epsilon - u^m_\epsilon)\), \(m,n \in \mathbb{N}\) we obtain as in (7)

\[
\| \nabla X_2 u^n_\epsilon - u^m_\epsilon \|_{L^p(\Omega)} \leq \left( \frac{\|f_n - f_m\|_{L^p}}{\lambda(p-1)} \right)^\frac{1}{p} \left( \int_\Omega (1 + |u^n_\epsilon - u^m_\epsilon|^p) \right)^\frac{1}{p} 
\]

And (8) gives

\[
\| \nabla X_2(u^n_\epsilon - u^m_\epsilon) \|_{L^p(\Omega)} \leq C \|f_n - f_m\|_{L^p}^{\frac{7}{p}}
\]

where \(C\) is independent of \(\epsilon\) and \(n\), whence Poincaré’s inequality implies

\[
\|u^n_\epsilon - u^m_\epsilon\|_{V_p} \leq C' \|f_n - f_m\|_{L^p}^{\frac{7}{p}} \tag{16}
\]

Similarly we obtain

\[
\|\epsilon \nabla X_2(u^n_\epsilon - u^m_\epsilon)\|_{L^p(\Omega)} \leq C'' \|f_n - f_m\|_{L^p}^{\frac{7}{p}} \tag{17}
\]

its follows that

\[
\|u^n_\epsilon - u^m_\epsilon\|_{W^{1,p}(\Omega)} \leq C \|f_n - f_m\|_{L^p}^{\frac{1}{p}} \tag{18}
\]

The last inequality implies that for every \(\epsilon\) fixed \((u^n_\epsilon)_{n \in \mathbb{N}}\) is a Cauchy sequence in \(W^{1,p}_0(\Omega)\). Then there exists \(u_\epsilon \in W^{1,p}_0(\Omega)\) such that \(u^n_\epsilon \to u_\epsilon\) in \(W^{1,p}(\Omega)\), then the passage to the limit in (6) shows that \(u_\epsilon\) is a weak solution of (2). Finally (16) and (17) show that \(u^n_\epsilon \to u_\epsilon\) (resp \(\epsilon \nabla X_2 u^n_\epsilon \to \epsilon \nabla X_2 u_\epsilon\)) in \(V_p\) (resp in \(L^p(\Omega)\)) uniformly in \(\epsilon\).

**Step3:** Now, we are ready to conclude. Proposition 1, 2 and 3 combined with the triangular inequality show that \(u_\epsilon \to u_0\) in \(V_p\) and \(\epsilon \nabla X_2 u_\epsilon \to 0\) in \(L^p(\Omega)\), and the proof of Theorem 1 is finished.

### 2.3. Convergence of the entropy solutions

As mentioned in section 1 the entropy solution \(u_\epsilon\) of (2) exists and it is unique. We shall construct this entropy solution. Using the approximated problem (6), one has a \(W^{1,p}\)-strongly converging sequence \(u^n_\epsilon \to u_\epsilon \in W^{1,p}_0(\Omega)\) as shown in Proposition 3. We will show that \(u_\epsilon \in T^{1,2}_0(\Omega)\). Clearly we have \(T_k(u^n_\epsilon) \in H^1_0(\Omega)\) for every \(k > 0\). Now testing with \(T_k(u^n_\epsilon)\) in (6) we obtain

\[
\int_\Omega A_\epsilon \nabla u^n_\epsilon \cdot \nabla T_k(u^n_\epsilon)dx = \int_\Omega f_n T_k(u^n_\epsilon)dx
\]

Using the ellipticity assumption we get

\[
\int_\Omega |\nabla T_k(u^n_\epsilon)|^2 \leq \frac{Mk}{\lambda(1+\epsilon^2)} \tag{18}
\]

Fix \(\epsilon, k\), we have \(u^n_\epsilon \to u_\epsilon\) in \(L^p(\Omega)\) then there exists a subsequence \((u^{n_i}_\epsilon)_{i \in \mathbb{N}}\) such that \(u^{n_i}_\epsilon \to u_\epsilon\) a.e \(x \in \Omega\) and since \(T_k\) is bounded then it follows that \(T_k(u^{n_i}_\epsilon) \to T_k(u_\epsilon)\) a.e in \(\Omega\) and strongly in \(L^2(\Omega)\) whence \(u_\epsilon \in T^{1,2}_0(\Omega)\).

It follows by (18) that there exists a subsequence still labelled \(T_k(u^{n_i}_\epsilon)\) such that \(\nabla T_k(u^{n_i}_\epsilon) \to v_{\epsilon,k} \in L^2(\Omega)\). The continuity of \(\nabla\) on \(D'(\Omega)\) implies that \(v_{\epsilon,k} = \)
\[ \nabla T_k(u_n), \text{ whence } T_k(u_n^{m}) \to T_k(u_e) \text{ in } H^1(\Omega). \] Now, since \( T_k(u_n^{m}) \in H^1_0(\Omega) \) then we deduce that \( T_k(u_e) \in H^1_0(\Omega). \)

It follows [4] that

\[ \int_{\Omega} A_e \nabla u_e \cdot \nabla T_k(u_e - \varphi) \, dx \leq \int_{\Omega} f T_k(u_e - \varphi) \, dx \]

Whence \( u_e \) is the entropy solution of (2). Similarly the function \( u_0 \) (constructed in Proposition 2) is the entropy solution to (5) for a.e \( X_1 \) The uniqueness of \( u_0 \) in \( V_p \) follows from the uniqueness of the entropy solution of problem (5). Finally, the convergences given in Theorem 2 follows from Theorem 1.

**Remark 1.** Uniqueness of the entropy solutions implies that it does not depend on the choice of the approximated sequence \((f_n)_n\).

2.4. A regularity result for the entropy solution of the limit problem.
In this subsection we assume that \( \Omega = \omega_1 \times \omega_2 \) where \( \omega_1, \omega_2 \) are two bounded Lipschitz domains of \( \mathbb{R}^n, \mathbb{R}^{N-q} \) respectively. We introduce the space

\[ W_p = \{ u \in L^p(\Omega) \mid \nabla X_1 u \in L^p(\Omega) \} \]

We suppose the following

\[ f \in W_p \text{ and } A_{22}(x) = A_{22}(X_2) \text{ i.e } A_{22} \text{ is independent of } X_1 \] (19)

**Theorem 4.** Assume (3), (4), (19) then \( u_0 \in W^{1,p}(\Omega) \), where \( u_0 \) is the entropy solution of (5).

**Proof.** Let \((u_n^{0})\) the sequence constructed in subsection 2.2, we have \( u_n^{0} \to u_0 \) in \( V_p \), where \( u_0 \) is the entropy solution of (5) as mentioned in the above subsection.

Let \( \omega'_1 \subset \subset \omega_1 \) be an open subset, for \( 0 < h < d(\partial \omega_1, \omega'_1) \) and for \( X_1 \in \omega'_1 \) we set \( \tau_h u_0^n = u_0^n(X_1 + hab) \) where \( e_i = (0,..,1,..,0) \) then we have by (13)

\[ \int_{\omega_2} A_{22} \nabla X_1 (\tau_h u_0^n - u_0^n) \cdot \nabla X_1 \varphi \, dX_2 = \int_{\omega_2} (\tau_h f_n - f_n) \varphi \, dX_2, \quad \varphi \in \mathcal{D}(\omega_2) \]

where we have used \( A_{22}(x) = A_{22}(X_2) \).

We introduce the function \( \theta_\delta(t) = \int_0^t (\delta + |s|)^{p-2} \, ds, \quad \delta > 0, \ t \in \mathbb{R} \) we have

\[ 0 < \theta_\delta'(t) = (\delta + |t|)^{p-2} \leq \delta^{p-2} \text{ and } |\theta_\delta(t)| \leq \frac{2(\delta + |t|)^{p-1}}{p-1} \]

Testing with \( \varphi = \frac{1}{h} \theta_\delta(\frac{\tau_h u_0^n - u_0^n}{h}) \in H^1_0(\omega_2) \). To make the notations less heavy we set

\[ U = \frac{\tau_h u_0^n - u_0^n}{h}, \quad \frac{\tau_h f_n - f_n}{h} = F \]

Then we get

\[ \int_{\omega_2} \theta_\delta'(U) A_{22} \nabla X_1 U \cdot \nabla X_1 U dX_2 = \int_{\omega_2} F \theta_\delta(U) \, dX_2 \]

Using the ellipticity assumption for the left hand side and Hölder’s inequality for the right hand side of the previous inequality we deduce

\[ \lambda \int_{\omega_2} \theta_\delta(U) |\nabla X_1 U|^2 \, dX_2 \leq \frac{2}{p-1} ||F||_{L^p(\omega_2)} \left( \int_{\omega_2} (\delta + |U|)^p \, dX_2 \right)^{\frac{p-1}{p}} \]
Using Hölder’s inequality we derive
\[ \|\nabla X_2 U\|_{L^p(\omega_2)}^p \leq \left( \int_{\omega_2} \theta_\delta(U) |\nabla X_2 U|^2 \, dX_2 \right)^{\frac{p}{2}} \left( \int_{\omega_2} \theta_\delta(U) \, dX_2 \right)^{\frac{p-2}{2}} \]
\[ \leq \left( \frac{2}{\lambda(p-1)} \|F\|_{L^p(\omega_2)} \left( \int_{\omega_2} (\delta + |U|)^p \, dX_2 \right)^{\frac{p-2}{p}} \times \right. \]
\[ \left. \left( \int_{\omega_2} \theta_\delta(U) \, dX_2 \right)^{\frac{p-2}{2}} \right) \]

Then we deduce
\[ \|\nabla X_2 U\|_{L^p(\omega_2)}^2 \leq \frac{2}{\lambda(p-1)} \|F\|_{L^p(\omega_2)} \left( \int_{\omega_2} (\delta + |U|)^p \, dX_2 \right)^{\frac{1}{p}} \]
Now passing to the limit as \( \delta \to 0 \) using the Lebesgue theorem we deduce
\[ \|\nabla X_2 U\|_{L^p(\omega_2)}^2 \leq \frac{2}{\lambda(p-1)} \|F\|_{L^p(\omega_2)} \left( \int_{\omega_2} (|U|)^p \, dX_2 \right)^{\frac{1}{p}}, \]
and Poincaré’s inequality gives
\[ \|\nabla X_2 U\|_{L^p(\omega_2)} \leq \frac{2C_{\omega_2}}{\lambda(p-1)} \|F\|_{L^p(\omega_2)} \]
Now, integrating over \( \omega_1 \) yields
\[ \left\| \frac{\tau_h u_0^n - u_0^n}{h} \right\|_{L^p(\omega_1 \times \omega_2)} \leq \frac{2C_{\omega_2}}{\lambda(p-1)} \left\| \frac{(\tau_h f_n - f_n)}{h} \right\|_{L^p(\omega_1 \times \omega_2)} \]
Passing to the limit as \( n \to \infty \) using the invariance of the Lebesgue measure under translations we get
\[ \left\| \frac{\tau_h u_0 - u_0}{h} \right\|_{L^p(\omega_1 \times \omega_2)} \leq \frac{2C_{\omega_2}}{\lambda(p-1)} \left\| \frac{(\tau_h f - f)}{h} \right\|_{L^p(\omega_1 \times \omega_2)} \]
Whence, since \( f \in W_p \) then
\[ \left\| \frac{\tau_h u_0 - u_0}{h} \right\|_{L^p(\omega_1 \times \omega_2)} \leq C, \]
where \( C \) is independent of \( h \), therefore we have \( \nabla X_1 u_0 \in L^p(\Omega) \). Combining this with \( u_0 \in V_p \) we get the desired result. \( \square \)

3. The Rate of Convergence Theorem

In this section we suppose that \( \Omega = \omega_1 \times \omega_2 \) where \( \omega_1, \omega_2 \) are two bounded Lipschitz domains of \( \mathbb{R}^3 \) and \( \mathbb{R}^{N-q} \) respectively. We suppose that \( A_{12}, A_{22} \) and \( f \) depend on \( X_2 \) only i.e \( A_{12}(x) = A_{12}(X_2), A_{22}(x) = A_{22}(X_2) \) and \( f(x) = f(X_2) \in L^p(\omega_2) (1 < p < 2), f \notin L^2(\omega_2) \).

Let \( u_\epsilon, u_0 \) be the unique entropy solutions of (2), (5) respectively then under the above assumptions we have the following

**Theorem 5.** For every \( \omega_1' \subset \subset \omega_1 \) and \( m \in \mathbb{N}^* \) there exists \( C \geq 0 \) independent of \( \epsilon \) such that
\[ \|u_\epsilon - u_0\|_{W^{p}(\omega_1' \times \omega_2)} \leq C\epsilon^m \]
Proof. Let $u_\epsilon, u_0$ be the entropy solutions of (2), (5) respectively, we use the approximated sequence $(u^n_\epsilon)_{\epsilon,n}, (u^n_0)_n$ introduced in section 2. Subtracting (13) from (6) we obtain

$$
\int_\Omega A_\epsilon \nabla (u^n_\epsilon - u^n_0) \cdot \nabla \varphi dx = 0,
$$

where we have used that $u^n_0$ is independent of $X_1$ (since $f$ and $A_{22}$ are independent of $X_1$) and that $A_{12}$ is independent of $X_1$.

Let $\omega'_1 \subset \omega_1$ then there exists $\omega'_1 \subset \omega'_1 \subset \omega_1$. We introduce the function $\rho \in D(\omega)' \subset \omega'_1$ and $\rho = 1$ on $\omega'_1$( we can choose $0 \leq \rho \leq 1$) Testing with $\varphi = \rho^2 \theta'_\delta(u^n_\epsilon - u^n_0)$ we obtain (6) we deduce

$$
(\rho^2 \theta'_\delta(u^n_\epsilon - u^n_0) A_\epsilon \nabla (u^n_\epsilon - u^n_0) \cdot \nabla \rho dx
$$

$$
= - \int_\Omega \rho \theta'_\delta(u^n_\epsilon - u^n_0) A_\epsilon \nabla (u^n_\epsilon - u^n_0) \cdot \nabla \rho dx
$$

$$
= - \epsilon^2 \int_\Omega \rho \theta'_\delta(u^n_\epsilon - u^n_0) A_{11} \nabla X_1(u^n_\epsilon - u^n_0) \cdot \nabla X_1 \rho dx
$$

$$
- \epsilon \int_\Omega \rho \theta'_\delta(u^n_\epsilon - u^n_0) A_{12} \nabla X_2(u^n_\epsilon - u^n_0) \cdot \nabla X_2 \rho dx
$$

where we have used that $\rho$ is independent of $X_2$.

Using the ellipticity assumption for the left hand side and assumption (4) for the right hand side of previous equality we deduce

$$
\epsilon^2 \lambda \int_\Omega \theta'_\delta(u^n_\epsilon - u^n_0) |\rho \nabla X_1 (u^n_\epsilon - u^n_0)|^2 dx + \lambda \int_\Omega \theta'_\delta(u^n_\epsilon - u^n_0) |\rho \nabla X_2 (u^n_\epsilon - u^n_0)|^2 dx
$$

$$
\leq \epsilon^2 C \int_\Omega \rho |\theta'_\delta(u^n_\epsilon - u^n_0)| |\nabla X_1 (u^n_\epsilon - u^n_0)| dx
$$

$$
+ \epsilon C \int_\Omega \rho |\theta'_\delta(u^n_\epsilon - u^n_0)| |\nabla X_2 (u^n_\epsilon - u^n_0)| dx
$$

Where $C \geq 0$ depends on $A$ and $\rho$. Using Young's inequality $ab \leq \frac{a^2}{2c} + \frac{b^2}{2}$ for the two terms in the right hand side of the previous inequality we obtain

$$
\epsilon^2 \lambda \int_\Omega \theta'_\delta(u^n_\epsilon - u^n_0) |\rho \nabla X_1 (u^n_\epsilon - u^n_0)|^2 dx + \frac{\lambda}{2} \int_\Omega \theta'_\delta(u^n_\epsilon - u^n_0) |\rho \nabla X_2 (u^n_\epsilon - u^n_0)|^2 dx
$$

$$
\leq \epsilon^2 C' \int_\Omega |\theta'_\delta(u^n_\epsilon - u^n_0)|^2 |\theta'_\delta(u^n_\epsilon - u^n_0)|^{-1} dx
$$

Whence

$$
\epsilon^2 \lambda \int_\Omega \theta'_\delta(u^n_\epsilon - u^n_0) |\rho \nabla X_1 (u^n_\epsilon - u^n_0)|^2 dx + \frac{\lambda}{2} \int_\Omega \theta'_\delta(u^n_\epsilon - u^n_0) |\rho \nabla X_2 (u^n_\epsilon - u^n_0)|^2 dx
$$

$$
\leq \frac{4}{(p-1)^2} \epsilon^2 C' \int_\Omega |\delta + |u^n_\epsilon - u^n_0||^p dx
$$

where $C''$ is independent of $\epsilon$ and $n$. 
Now, using Hölder’s inequality and the previous inequality we deduce

\[
\epsilon^2 \frac{\lambda}{2} \left( \| \nabla X_1 (u^n - u_0^n) \|_{L^p(\Omega)} + \| \nabla X_2 (u^n - u_0^n) \|_{L^p(\Omega)} \right) \\
\leq \left[ \epsilon^2 \frac{\lambda}{2} \left( \int_{\Omega} \theta_\delta'(u^n - u_0^n) \| \nabla X_1 (u^n - u_0^n) \|_{L^p(\Omega)} dx \\
+ \frac{\lambda}{2} \left( \int_{\Omega} \theta_\delta'(u^n - u_0^n) \| \nabla X_2 (u^n - u_0^n) \|_{L^p(\Omega)} dx \right) \right) \times \\
\left( \int_{\omega_{1}^\varepsilon \times \omega_2} (\delta + |u^n - u_0^n|)^{\frac{2-p}{p}} \right) \\
\leq \frac{4C'}{(p-1)^2} \epsilon^2 \left( \int_{\omega_{1}^\varepsilon \times \omega_2} (\delta + |u^n - u_0^n|)^{\frac{2}{p}} dx \right)^{\frac{p}{2}} 
\]

Passing to the limit as \( \delta \to 0 \) using the Lebesgue theorem. Passing to the limit as \( n \to \infty \) we get

\[
\epsilon^2 \left( \| \nabla X_1 (u^n - u_0) \|_{L^p(\omega_{1}^\varepsilon \times \omega_2)} + \| \nabla X_2 (u^n - u_0) \|_{L^p(\omega_{1}^\varepsilon \times \omega_2)} \right) \\
\leq C'' \epsilon^2 \left( \| u^n - u_0 \|_{L^p(\omega_{1}^\varepsilon \times \omega_2)} \right) \\
\tag{20}
\]

Using Poincaré’s inequality

\[
\| (u^n - u_0) \|_{L^p(\omega_{1}^\varepsilon \times \omega_2)} \leq C_{\omega_2} \| \nabla X_2 (u^n - u_0) \|_{L^p(\omega_{1}^\varepsilon \times \omega_2)} ,
\]

we obtain

\[
\epsilon^2 \left( \| \nabla X_1 (u^n - u_0) \|_{L^p(\omega_{1}^\varepsilon \times \omega_2)}^2 + \| \nabla X_2 (u^n - u_0) \|_{L^p(\omega_{1}^\varepsilon \times \omega_2)}^2 \right) \\
\leq C'' \epsilon^2 \left( \| \nabla X_2 (u^n - u_0) \|_{L^p(\omega_{1}^\varepsilon \times \omega_2)}^2 \right)
\]

Let \( m \in \mathbb{N}^* \) then there exists \( \omega_{1}' \subset \subset \omega_{1}'' \subset \subset \omega_{1}^{(m+1)} \subset \subset \omega_1 \). Iterating the above inequality \( m \)-times we deduce

\[
\epsilon^2 \left( \| \nabla X_1 (u^n - u_0) \|_{L^p(\omega_{1}^\varepsilon \times \omega_2)}^2 + \| \nabla X_2 (u^n - u_0) \|_{L^p(\omega_{1}^\varepsilon \times \omega_2)}^2 \right) \\
\leq C'' \epsilon^2 \left( \| \nabla X_2 (u^n - u_0) \|_{L^p(\omega_{1}^\varepsilon \times \omega_2)}^2 \right)
\]

Now, from (20) (with \( \omega_{1}' \) and \( \omega_{1}'' \) replaced by \( \omega_{1}^{(m)} \) and \( \omega_{1}^{(m+1)} \) respectively) we deduce

\[
\epsilon^2 \left( \| \nabla X_1 (u^n - u_0) \|_{L^p(\omega_{1}^\varepsilon \times \omega_2)}^2 + \| \nabla X_2 (u^n - u_0) \|_{L^p(\omega_{1}^\varepsilon \times \omega_2)}^2 \right) \\
\leq C'' \epsilon^2 \left( \| \nabla X_2 (u^n - u_0) \|_{L^p(\omega_{1}^\varepsilon \times \omega_2)}^2 \right)
\]

Since \( u_n \to u_0 \) in \( L^p(\Omega) \) then \( \| u_n - u_0 \|_{L^p(\Omega)} \) is bounded and therefore we obtain

\[
\| u_n - u_0 \|_{W^p(\omega_{1}^\varepsilon \times \omega_2)} \leq C'' \epsilon^m
\]

And the proof of the theorem is finished.

Can one obtain a more better convergence rate? In fact, the anisotropic singular perturbation problem (2) can be seen as a problem in a cylinder becoming
Indeed the two problems can be connected to each other via a scaling $\epsilon = \frac{1}{\ell}$ (see [5] for more details). So let us consider the problem

\[
\begin{cases}
- \text{div}(\tilde{A}\nabla u_\ell) = f \\
u_\ell = 0 \quad \text{on } \partial \Omega_\ell
\end{cases}
\tag{21}
\]

where $\tilde{A} = (\tilde{a}_{ij})$ is a $N \times N$ matrix such that

\[
\tilde{a}_{ij} \in L^\infty(\mathbb{R}^q \times \omega_2)
\tag{22}
\]

\[
\exists \lambda > 0 : \tilde{A}\xi \cdot \xi \geq \lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^N \text{ for a.e } x \in \mathbb{R}^q \times \omega_2,
\tag{23}
\]

$\Omega_\ell = \ell \omega_1 \times \omega_2$ a bounded domain where $\omega_1$, $\omega_2$ are two bounded Lipschitz domain with $\omega_1$ convex and containing 0.

We assume that $f \in L^p(\omega_2)$ ($1 < p < 2$) and $\tilde{A}_{22}(x) = \tilde{A}_{22}(X_2)$, $\tilde{A}_{12}(x) = \tilde{A}_{12}(X_2)$.

We consider the limit problem

\[
\begin{cases}
- \text{div}(\tilde{A}_{22}\nabla_{X_2} u_\infty) = f \\
u_\infty = 0 \quad \text{on } \partial \omega_2
\end{cases}
\tag{24}
\]

Then under the above assumptions we have

**Theorem 6.** Let $u_\ell$, $u_\infty$ be the unique entropy solutions to (21) and (24) then for every $\alpha \in (0, 1)$ there exists $C \geq 0, c > 0$ independent of $\ell$ such that

\[
\|\nabla (u_\ell - u_\infty)\|_{W^{1,p}(\Omega_\ell)} \leq Ce^{-\alpha \ell}
\]

*Proof.* Let $u_\ell$, $u_\infty$ the unique entropy solutions to (21) and (24) respectively, and let $(u^n_\ell)$ and $(u^n_\infty)$ the approximation sequences (as in section 2). we have $u^n_\ell \to u_\ell$ in $W^{1,p}_{0}(\Omega_\ell)$ and $u^n_\infty \to u_\infty$ in $W^{1,p}_{0}(\omega_2)$. Subtracting the associated approximated problems to (21) and (24) and take the weak formulation we get

\[
\int_{\Omega_\ell} \tilde{A}\nabla(u^n_\ell - u^n_\infty) \nabla \varphi dx = 0, \varphi \in \mathcal{D}(\Omega)
\tag{25}
\]

Where we have used that $\tilde{A}_{22}$, $\tilde{A}_{12}$, $u^n_\infty$ are independent of $X_1$. Now we will use the iteration technique introduced in [7], let $0 < \ell_0 \leq \ell - 1$, and let $\rho \in D(\mathbb{R}^q)$ a bump function such that

\[
0 \leq \rho \leq 1, \rho = 1 \text{ on } \ell_0 \omega_1 \text{ and } \rho = 0 \text{ on } \mathbb{R}^q \setminus (\ell_0 + 1) \omega_1, |\nabla X_1 \rho| \leq c_0
\]

where $c_0$ is the universal constant (see [5]). Testing with $\rho^2 \theta_\delta(u^n_\ell - u^n_\infty) \in H^1_0(\Omega_\ell)$ in (25) we get

\[
\int_{\Omega_\ell} \rho^2 \theta_\delta(u^n_\ell - u^n_\infty) \tilde{A}\nabla(u^n_\ell - u^n_\infty) \cdot \nabla(u^n_\ell - u^n_\infty) dx
\]

\[
+ \int_{\Omega_\ell} \rho \theta_\delta(u^n_\ell - u^n_\infty) \tilde{A}\nabla(u^n_\ell - u^n_\infty) \cdot \nabla \rho dx = 0
\]

Using the ellipticity assumption (23)

\[
\int_{\Omega_\ell} \rho^2 \theta_\delta(u^n_\ell - u^n_\infty) |\nabla(u^n_\ell - u^n_\infty)|^2 dx
\]

\[
\leq 2 \int_{\Omega_\ell} \rho |\theta_\delta(u^n_\ell - u^n_\infty)| \left| \tilde{A}\nabla(u^n_\ell - u^n_\infty) \right| |\nabla \rho| dx
\]
Notice that $\nabla \rho = 0$ on $\Omega_{t_0}$, and $\Omega_{t_0} \subset \Omega_{t_0+1}$ (since $\omega_1$ is convex and containing 0). Then by the Cauchy-Schwaz inequality we get
\[
\int_{\Omega_t} \rho^2 \theta'_\delta(u^n_t - u^n_\infty) |\nabla(u^n_t - u^n_\infty)|^2 \, dx 
\leq 2c_0 C \int_{\Omega_{t_0+1} \setminus \Omega_{t_0}} \rho |\theta'_\delta(u^n_t - u^n_\infty)||\nabla(u^n_t - u^n_\infty)| \, dx 
\leq 2c_0 C \left( \int_{\Omega_t} \rho^2 \theta'_\delta(u^n_t - u^n_\infty) |\nabla(u^n_t - u^n_\infty)|^2 \, dx \right)^{\frac{1}{2}} \times 
\left( \int_{\Omega_{t_0+1} \setminus \Omega_{t_0}} |\theta'_\delta(u^n_t - u^n_\infty)|^2 \theta'_\delta(u^n_t - u^n_\infty)^{-1} \, dx \right)^{\frac{1}{2}}
\]
where we have used (22). Whence we get (since $\rho = 1$ on $\Omega_{t_0}$)
\[
\int_{\Omega_{t_0}} \theta'_\delta(u^n_t - u^n_\infty) |\nabla(u^n_t - u^n_\infty)|^2 \, dx \leq \int_{\Omega_t} \rho^2 \theta'_\delta(u^n_t - u^n_\infty) |\nabla(u^n_t - u^n_\infty)|^2 \, dx 
\leq \left( \frac{4c_0 C}{p - 1} \right)^2 \int_{\Omega_{t_0+1} \setminus \Omega_{t_0}} (\delta + |u^n_t - u^n_\infty|)^p \, dx
\]
From Hölder’s inequality it holds that
\[
\|\nabla(u^n_t - u^n_\infty)\|_{L^p(\Omega_{t_0})}^2 
\leq \left( \int_{\Omega_{t_0}} \theta'_\delta(u^n_t - u^n_\infty) |\nabla(u^n_t - u^n_\infty)|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega_{t_0}} (\delta + |u^n_t - u^n_\infty|)^p \, dx \right)^{\frac{2-p}{2}} 
\leq \left( \frac{4c_0 C}{p - 1} \right)^2 \left( \int_{\Omega_{t_0+1} \setminus \Omega_{t_0}} (\delta + |u^n_t - u^n_\infty|)^p \, dx \right)^{\frac{2-p}{2}} \left( \int_{\Omega_{t_0}} (\delta + |u^n_t - u^n_\infty|)^p \, dx \right)^{\frac{p}{2}}
\]
Passing to the limit as $\delta \to 0$ (using the Lebesgue theorem) we get
\[
\|\nabla(u^n_t - u^n_\infty)\|_{L^p(\Omega_{t_0})}^2 
\leq C_1 \left( \int_{\Omega_{t_0+1} \setminus \Omega_{t_0}} |u^n_t - u^n_\infty|^p \, dx \right) \times \left( \int_{\Omega_{t_0}} |u^n_t - u^n_\infty|^p \, dx \right)^{\frac{2-p}{p}}.
\]
where we have used $0 \leq \rho \leq 1$. Using Poincaré’s inequality
\[
\|\nabla(u^n_t - u^n_\infty)\|_{L^p(\Omega_{t_0})} \leq C_{\omega_2} \|\nabla(u^n_t - u^n_\infty)\|_{L^p(\Omega_{t_0})}
\]
we get
\[
\|\nabla(u^n_t - u^n_\infty)\|_{L^p(\Omega_{t_0})}^p \leq C_2 \|u^n_t - u^n_\infty\|_{L^p(\Omega_{t_0+1} \setminus \Omega_{t_0})}^p
\]
Using Poincaré’s inequality
\[
\|u^n_t - u^n_\infty\|_{L^p(\Omega_{t_0+1} \setminus \Omega_{t_0})} \leq C_{\omega_2} \|\nabla(u^n_t - u^n_\infty)\|_{L^p(\Omega_{t_0+1} \setminus \Omega_{t_0})}
\]
we get
\[
\|\nabla(u^n_t - u^n_\infty)\|_{L^p(\Omega_{t_0})}^p \leq C_3 \|\nabla(u^n_t - u^n_\infty)\|_{L^p(\Omega_{t_0+1} \setminus \Omega_{t_0})}^p
\]
Whence
\[ \| \nabla (u^n - u_\infty^n) \|_{L^p(\Omega_\ell)} \leq \frac{C_3}{C_3 + 1} \| \nabla (u^n_\ell - u_\infty^n) \|_{L^p(\Omega_{\ell+1})} \]

Let \( \alpha \in (0, 1) \), iterating this formula starting from \( \ell \) we get
\[ \| \nabla (u^n_\ell - u_\infty^n) \|_{L^p(\Omega_\ell)} \leq \left( \frac{C_3}{C_3 + 1} \right)^{|\alpha \ell|} \| \nabla (u^n_\ell - u_\infty^n) \|_{L^p(\Omega_{\alpha \ell + |1-\alpha| \ell})} \]

Whence
\[ \| \nabla (u^n_\ell - u_\infty^n) \|_{L^p(\Omega_\ell)} \leq c e^{-c' \ell} \| \nabla (u^n_\ell - u_\infty^n) \|_{L^p(\Omega_\ell)} \] (26)

where \( c, c' > 0 \) are independent of \( \ell \) and \( n \).

Now we have to estimate the right hand side of (26). Testing with \( \theta(u^n_\ell) \) in the approximated problem associated to (21) one can obtain as in subsection 2.1
\[ \| \nabla u^n_\ell \|_{L^p(\Omega_\ell)} \leq C \ell^{\frac{p}{2}} \] (27)

Similarly testing with \( \theta(u^n_\infty) \) in the approximated problem associated to (24), we get
\[ \| \nabla u^n_\infty \|_{L^p(\Omega_\ell)} \leq C' \ell^{\frac{p}{2}} \] (28)

Replace (28), (27) in (26) and passing to the limit as \( n \to \infty \) we obtain the desired result.

**Corollary 2.** Under the above assumptions then for every \( \alpha \in (0, 1) \) there exists \( C \geq 0, c > 0 \) independent of \( \ell \) such that
\[ \| u_\ell - u_0 \|_{W^{1, p}(\Omega_\ell \times \Omega_\infty)} \leq C e^{-c \ell} \]

where \( u_\ell, u_0 \) are the entropy solutions to (2) and (5) respectively

**Remark 2.** It is very difficult to prove the rate convergence theorem for general data. When \( f(x) = f_1(X_2) + f_2(x) \) with \( f_1 \in L^p(\Omega_2) \) and \( f_2 \in W_2 \) we only have the estimates
\[ \epsilon \| \nabla X_1(u_\epsilon - u_0) \|_{L^p(\omega_1 \times \omega_2)} + \| \nabla X_2(u_\epsilon - u_0) \|_{L^p(\omega_1 \times \omega_2)} + \| u_\epsilon - u_0 \|_{L^p(\omega_1 \times \omega_2)} \leq C \epsilon \]

This follows from the linearity of the equation, Theorem 5 and the \( L^2 \)-theory [8].

4. **Some Extensions to nonlinear problems and applications**

4.1. **A semilinear monotone problem.** We consider the semilinear problem
\[
\begin{cases}
- \text{div}(A_\epsilon \nabla u_\epsilon) = f + a(u_\epsilon) \\
u_\epsilon = 0 & \text{on } \partial \Omega
\end{cases}
\] (29)

Where the \( a : \mathbb{R} \to \mathbb{R} \) is a continuous nonincreasing function which satisfies the growth condition
\[ \forall x \in \mathbb{R} : |a(x)| \leq K(1 + |x|), \quad K \geq 0 \] (30)
and \( f \in L^p(\Omega) \) where \( 1 < p < 2 \), \( f \notin L^2(\Omega) \) and \( A \) is given as in Subsection 1.2. Clearly the Nemytskii operator \( u \to a(u) \) maps \( L^r(\Omega) \to L^r(\Omega) \) continuously for every \( 1 \leq r < \infty \). The passage to the limit (formally) gives the limit problem

\[
\begin{align*}
-\text{div}_X (A_{22}(X_1,\cdot) \nabla u_0(X_1,\cdot)) &= f(X_1,\cdot) + a(u_0(X_1,\cdot)) \\
\quad u_0(X_1,\cdot) &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

We can suppose that \( a(0) = 0 \). Indeed, in the general case the right hand side of (29) can be replaced by \( (a(0) + f) + b(x) \) where \( b(x) = a(x) - a(0) \). Clearly \( b \) is continuous nonincreasing and satisfies \( |b(x)| \leq (K + |a(0)|)(1 + |x|) \).

First of all, suppose that \( f \in L^2(\Omega) \), then we have the following

**Proposition 4.** Assume (3), (4) and \( a(0) = 0 \). Let \( u \) be the unique weak solution in \( H^1(\Omega) \) to (29) then \( \epsilon \nabla_X u \to 0 \) in \( L^2(\Omega) \) and \( u \to u_0 \) in \( V_2 \) where \( u_0 \) is the unique solution in \( V_2 \) to the limit problem (31).

**Proof.** Existence of \( u \) follows directly by a simple application of the Schauder fixed point theorem for example. The uniqueness follows from monotonicity of \( a \) and the Poincaré’s inequality.

Take \( u_e \) as a test function in (29) then one can obtain the estimates

\[
\epsilon \| \nabla_X u_e \|_{L^2(\Omega)} \cdot \| \nabla X_2 u_e \|_{L^2(\Omega)} \cdot \| u_e \|_{L^2(\Omega)} \leq C,
\]

where \( C \) is independent of \( \epsilon \), we have used that \( \int_\Omega a(u_e) u_e \, dx \leq 0 \) (thanks to monotonicity assumption and \( a(0) = 0 \)). And we also have (thanks to assumption (30))

\[
\| a(u_e) \|_{L^2(\Omega)} \leq K(|\Omega|^\frac{1}{2} + C)
\]

so there exists \( v \in L^2(\Omega) \), \( u_0 \in L^2(\Omega) \), \( \nabla_X u_0 \in L^2(\Omega) \) and a subsequence \( (u_{e_k})_{k \in \mathbb{N}} \) such that

\[
a(u_{e_k}) \to v, \epsilon_k \nabla_X u_{e_k} \to 0, \nabla X_2 u_{e_k} \to \nabla X_2 u_0, u_{e_k} \to u_0 \text{ in } L^2(\Omega)-\text{weak}
\]

Passing to the in the weak formulation of (29) we get

\[
\int_\Omega A_{22} \nabla X_2 u_0 \cdot \nabla X_2 \varphi \, dx = \int_\Omega f \varphi \, dx + \int_\Omega v \varphi \, dx, \varphi \in \mathcal{D}(\Omega)
\]

Take \( \varphi = u_{e_k} \) in the previous equality and passing to the limit we get

\[
\int_\Omega A_{22} \nabla X_2 u_0 \cdot \nabla X_2 u_0 \, dx = \int_\Omega f u_0 \, dx + \int_\Omega v u_0 \, dx
\]

(34)
Let us computing the quantity

\[ 0 \leq I_k = \int_{\Omega} A_{e_k} \left( \frac{\nabla X_1 u_{e_k}}{\nabla X_2 (u_{e_k} - u_0)} \right) \cdot \left( \frac{\nabla X_1 u_{e_k}}{\nabla X_2 (u_{e_k} - u_0)} \right) \, dx \]

\[ - \int_{\Omega} (a(u_{e_k}) - a(u_0))(u_{e_k} - u_0) \, dx \]

\[ = \int_{\Omega} f u_{e_k} \, dx - \epsilon \int_{\Omega} A_{12} \nabla X_2 u_0 \cdot \nabla X_1 u_{e_k} \, dx - \epsilon \int_{\Omega} A_{21} \nabla X_1 u_{e_k} \cdot \nabla X_2 u_0 \, dx \]

\[ - \int_{\Omega} A_{22} \nabla X_2 u_{e_k} \cdot \nabla X_2 u_0 \, dx - \int_{\Omega} A_{22} \nabla X_2 u_0 \cdot \nabla X_2 u_{e_k} \, dx \]

\[ + \int_{\Omega} f u_0 \, dx + \int_{\Omega} v u_0 \, dx + \int_{\Omega} a(u_0) u_{e_k} \, dx \]

\[ + \int_{\Omega} a(u_{e_k}) u_0 \, dx - \int_{\Omega} a(u_0) u_0 \, dx \]

(This quantity is positive thanks to the ellipticity and monotonicity assumptions).

Passing to the limit as \( k \to \infty \) using (32), (33), (34) we get

\[ \lim_{k} I_k = 0 \]

And finally The ellipticity assumption and Poincaré’s inequality show that

\[ \| \epsilon_k \nabla X_1 u_{e_k} \|_{L^2(\Omega)}, \| \nabla X_2 (u_{e_k} - u_0) \|_{L^2(\Omega)}, \| u_{e_k} - u_0 \|_{L^2(\Omega)} \to 0 \]  \hspace{1cm} (35)

Whence (33) becomes

\[ \int_{\Omega} A_{22} \nabla X_2 u_0 \cdot \nabla X_2 \varphi \, dx = \int_{\Omega} f \varphi \, dx + \int_{\Omega} a(u_0) \varphi \, dx, \varphi \in \mathcal{D}(\Omega) \]  \hspace{1cm} (36)

\[ \| \nabla X_2 (u_{e_k} - u_0) \|_{L^2(\Omega)} \to 0 \]

shows that \( u_0 \in V_2 \), and therefore

\[ \int_{\Omega \setminus X_1} A_{22} \nabla X_2 u_0 \cdot \nabla X_2 \varphi \, dx = \int_{\Omega \setminus X_1} f \varphi \, dx + \int_{\Omega \setminus X_1} a(u_0) \varphi \, dx, \varphi \in \mathcal{D}(\Omega \setminus X_1) \]

Hence \( u_0(X_1, \cdot) \) is a solution to (31). The uniqueness in \( H^1_0(X_1, \cdot) \) of the the solution of the limit problem (31) shows that \( u_0 \) is the unique function in \( V_2 \) which satisfies (36). Therefore the convergences (35) hold for the whole sequence \( (u_0)_{0 \leq k \leq 1} \).

Now, we are ready to give the main result of this subsection

**Theorem 7.** Suppose that \( f \in L^p(\Omega) \) where \( 1 < p < 2 \) (we can suppose that \( f \notin L^2(\Omega) \)) then there exists \( u_0 \in V_p \) such that \( u_0(X_1, \cdot) \) is the unique entropy solution to (31) and we have \( u_r = u_0 \) in \( V_p, \epsilon \nabla X_1 u_r \to 0 \) in \( L^p(\Omega) \), where \( u_r \) is the unique entropy solution to (29).

**Proof.** We only give a sketch of the proof. Existence and uniqueness of the entropy solutions to (29) and (31) follows from the general result proved in [4]. As in proof of Theorem 2 we shall construct the entropy solution \( u_r \). We consider the approximated problem

\[
\begin{align*}
- \text{div}(A_r \nabla u_r^n) &= f_n + a(u_r^n) \\
u_r^n &= 0 \quad \text{on } \partial \Omega
\end{align*}
\]
We follow the same arguments as in section 2, where we use the above proposition and the following
\[
\int_{\Omega} (a(u) - a(v))\theta(u - v) dx \leq 0
\]
Which holds for every \( u, v \in L^2(\Omega) \), in fact this follows from monotonicity of \( a \) and \( \theta \).

4.2. Nonlinear problem without monotonicity assumption. Suppose that \( \Omega = \Omega_1 \times \Omega_2 \) where \( \Omega_1, \Omega_2 \) and consider the following nonlinear problem
\[
\begin{cases}
- \text{div}(A(x)\nabla u_e) = f + B(u_e) \\
u_e = 0 \quad \text{on } \partial \Omega
\end{cases}
\tag{37}
\]
Where \( f \in L^p(\Omega), \, 1 < p < 2 \) and \( B : L^p(\Omega) \to L^p(\Omega) \) is a continuous nonlinear operator. We suppose that
\[
\exists M \geq 0, \forall u \in L^p(\Omega) : \|B(u)\|_{L^p} \leq M \tag{38}
\]

**Proposition 5.** Assume (3), (4), and (38) then:
1) There exists a sequence \((u_e)_{0 < \epsilon \leq 1} \subset W^{1,p}_0(\Omega)\) of an entropy solutions to (37) which are also weak solutions such that
\[
\epsilon \|\nabla X_1 u_e\|_{L^p(\Omega)} + \|\nabla X_2 u_e\|_{L^p(\Omega)} + \|u_e\|_{L^p(\Omega)} \leq C_0,
\]
where \( C_0 \geq 0 \) is independent of \( \epsilon \) (the constant \( C_0 \) depends only on \( \Omega, \lambda, f \) and \( M \)).
2) If \((u_e)_{0 < \epsilon \leq 1}\) is a sequence of entropy and weak solutions to (37) then we have the above estimates.

**Proof.** 1) The existence of \( u_e \) is based on the Schauder fixed point theorem, we define the mapping \( \Gamma : L^p(\Omega) \to L^p(\Omega) \) by
\[
v \in L^p(\Omega) \to \Gamma(v) = v_e \in W^{1,p}_0(\Omega)
\]
where \( v_e \) is the entropy solution of the linearized problem
\[
\begin{cases}
- \text{div}(A(x)\nabla v_e) = f + B(v) \\
v_e = 0 \quad \text{on } \partial \Omega
\end{cases}
\tag{39}
\]
Since the entropy solution is unique then \( \Gamma \) is well defined. We can prove easily (by using the approximation method) that \( \Gamma \) is continuous. As in subsection 2.1 we can obtain the estimates
\[
\epsilon \|\nabla X_1 u_e\|_{L^p(\Omega)} + \|\nabla X_2 u_e\|_{L^p(\Omega)} + \|u_e\|_{L^p(\Omega)} \leq C_0
\]
where \( C_0 \) is independent of \( \epsilon \) and \( v \) (thanks to (38))

Now, define the subset
\[
K = \left\{ u \in W^{1,p}_0(\Omega) : \epsilon \|\nabla X_1 u\|_{L^p(\Omega)} + \|\nabla X_2 u\|_{L^p(\Omega)}, \|u\|_{L^p(\Omega)} \leq C_0 \right\}
\]
The subset \( K \) is convex and compact in \( L^p(\Omega) \) thanks to the Sobolev compact embedding \( W^{1,p}_0(\Omega) \subset L^p(\Omega) \).

The subset \( K \) is stable under \( \Gamma \) (since \( C_0 \) is independent of \( v \) as mentioned above). Whence \( \Gamma \) admits at least a fixed point \( u_e \in K \), in other words \( u_e \) is a weak solution to (37) which is also an entropy solution, this last assertion follows from the definition of \( \Gamma \)
2) Let \((u_\epsilon)_{0<\epsilon<1}\) be a sequence of entropy and weak solutions to (37) \(u_\epsilon\) is the unique entropy solution to (39) with \(v\) replaced by \(u_\epsilon\) and therefore we obtain the desired estimates as proved above.

\[\square\]

**Remark 3.** In the general case the entropy solution \(u_\epsilon\) of (37) is not necessarily unique.

Now, assume that

\[f(x) = f(X_2), A_{22}(x) = A_{22}(X_2), A_{12}(x) = A_{12}(X_2)\]

(40)

And assume that for every \(E \subset W^p\) bounded in \(L^p(\Omega)\) we have

\[\overline{\operatorname{conv}}\{B(E)\} \subset W_2,\]

(41)

where \(\overline{\operatorname{conv}}\{B(E)\}\) is the closed convex-hull of \(B(E)\) in \(L^p(\Omega)\). Assumption (41) appears strange. We shall give later some concrete examples of operators which satisfy this assumption. Let us prove the following

**Theorem 8.** Assume (3), (4), (38), (40) and (41). Let \((u_\epsilon)_{0<\epsilon<1} \subset W^{1,p}_0(\Omega)\) be an entropy and weak solution to (37) then for every \(\epsilon\) there exists \(C_0\) independent of \(\epsilon\) such that

\[\forall \epsilon : \|u_\epsilon\|_{W^{1,p}(\Omega)} \leq C_0\]

**Proof.** The proof is similar the one given in our preprint [14]. Let \((\Omega_j)_{j\in\mathbb{N}}\) an open covering of \(\Omega\) such that \(\overline{\Omega_j} \subset \Omega_{j+1}\). We equip the space \(Z = W^{1,p}_{loc}(\Omega)\) with the topology generated by the family of seminorms \((p_j)_{j\in\mathbb{N}}\) defined by

\[p_j(u) = \|u_p\|_{W^{1,p}(\Omega_j)}\]

Equipped with this topology, \(Z\) is a separated locally convex topological vector space. We set \(Y = L^p(\Omega)\) equipped with its natural topology. We define the family of the linear continuous mappings

\[\Lambda_\epsilon : Y \to Z\]

deﬁned by: \(g \in Y\), \(\Lambda_\epsilon(g) = v_\epsilon\) where \(v_\epsilon\) is the unique entropy solution to

\[
\begin{cases}
-\operatorname{div}(A_\epsilon \nabla v_\epsilon) = g \\
v_\epsilon = 0 \quad \text{on } \partial \Omega
\end{cases}
\]

The continuity of \(\Lambda_\epsilon\) follows immediately if we observe \(\Lambda_\epsilon\) as a composition of \(\Lambda_\epsilon : Y \to Y\) and the canonical injection \(Y \to Z\).

Now, we denote \(Z_w\), \(Y_w\) the spaces \(Z\), \(Y\) equipped with the weak topology respectively. then \(\Lambda_\epsilon : Y_w \to Z_w\) is also continuous.

Consider the bounded (in \(Y\)) subset

\[E_0 = \left\{ u \in W_p \mid \|u\|_{L^p(\Omega)} \leq C_0 \right\},\]

where \(C_0\) is the constant introduced in Proposition 5. Consider the subset \(G = f + \overline{\operatorname{conv}}\{B(E_0)\}\) where the closure is taken in the \(L^p\)-topology. Thanks to assumption (41) and (38) \(G\) is closed convex and bounded in \(Y\). Now for every \(g \in G\) the orbit \(\{\Lambda_\epsilon g\}_\epsilon\) is bounded in \(Z\) thanks to Remark 2. And therefore \(\{\Lambda_\epsilon g\}_\epsilon\) is bounded in \(Z_w\).
Clearly the set $G$ is compact in $Y_w$. Then it follows by the Banach-Steinhaus theorem (applied on the quadruple $\Lambda_c$, $G$, $Y_w$, $Z_w$) that there exists a bounded subset $F$ in $Z_w$ such that

$$\forall \epsilon : \Lambda_c(G) \subset F.$$ 

The boundedness of $F$ in $Z_w$ implies its boundedness in $Z$. i.e. For every $j \in \mathbb{N}$ there exists $C_j \geq 0$ independent of $\epsilon$ such that

$$\forall \epsilon : p_j(\Lambda_c(G)) \leq C_j.$$ 

Let $u_\epsilon$ be an entropy and weak solution to (37) then we have $u_\epsilon \in E_0$ as proved in Proposition 5 then $\Lambda_c(f + B(u_\epsilon)) = u_\epsilon \in F$ for every $\epsilon$, therefore

$$\forall \epsilon : \|u_\epsilon\|_{W^{1,p}(\Omega_j)} \leq C_j.$$ 

Whence for every $\Omega' \subset \subset \Omega$ there exists $C_{\Omega'} \geq 0$ independent of $\epsilon$ such that

$$\forall \epsilon : \|u_\epsilon\|_{W^{1,p}(\Omega')} \leq C_{\Omega'}.$$

Now we are ready to prove the convergence theorem. Assume that

$$B : (L^p(\Omega), \tau_{L^p(\Omega)}) \to L^p(\Omega)$$

is continuous

(42)

where $(L^p(\Omega), \tau_{L^p(\Omega)})$ is the space $L^p(\Omega)$ equipped with the $L^p_{\text{loc}}(\Omega)$-topology. Notice that (42) implies that $B : L^p(\Omega) \to L^p(\Omega)$ is continuous. Then we have the following

**Theorem 9.** Under assumptions of Theorem 8, assume in addition (42), suppose that $\Omega$ is convex, then there exists $u_0 \in V_p$ and a sequence $(u_{\epsilon_k})_{k \in \mathbb{N}}$ of entropy and weak solution to (37) such that

$$\epsilon_k \nabla X_1 u_{\epsilon_k} \to 0, \nabla X_2 u_{\epsilon_k} \to \nabla X_2 u_0 \text{ in } L^p(\Omega) \text{ weak}$$

and $u_{\epsilon_k} \to u_0$ in $L^p_{\text{loc}}(\Omega) \text{ strong}$

Moreover $u_0$ satisfies in $D'(\omega_2)$ the equation

$$-\nabla X_3 (A_{22} \nabla X_2 u_0(X_1, \cdot)) = f + B(u_0)(X_1, \cdot)$$

for a.e $X_1 \in \omega_1$

**Proof.** The estimates given in Proposition 5 show that there exists $u_0 \in L^p(\Omega)$ and a sequence $(u_{\epsilon_k})_{k \in \mathbb{N}}$ solutions to (37) such that

$$\epsilon_k \nabla X_1 u_{\epsilon_k} \to 0, \nabla X_2 u_{\epsilon_k} \to \nabla X_2 u_0 \text{ and } u_{\epsilon_k} \to u_0 \text{ in } L^p(\Omega) \text{ weak}$$

(43)

As we have proved in Theorem 3 we have $u_0 \in V_p$. The particular difficulty is the passage to the limit in the nonlinear term. This assertion is guaranteed by Theorem 8. Indeed, since $\Omega$ is convex and Lipschitz then there an open covering $(\Omega_j)_{j \in \mathbb{N}}$, $\Omega_j \subset \Omega_{j+1}$ and $\overline{\Omega_j} \subset \Omega$ such that each $\Omega_j$ is a Lipschitz domain (Take an increasing sequence of number $0 < \beta_j < 1$ with $\lim \beta_j = 1$. Fix $x_0 \in \Omega$ and take $\Omega_j = \beta_j(\Omega - x_0) + x_0$, since $\Omega$ is convex then $\overline{\Omega_j} \subset \Omega$. The Lipschitz character is conserved since the multiplication by $\beta_j$ and translations are $C^\infty$ diffeomorphisms).

Theorem 8 shows that for every $j \in \mathbb{N}$ there exists $C_j \geq 0$ such that

$$\|u_\epsilon\|_{W^{1,p}(\Omega_j)} \leq C_{\Omega_j}$$
Since $\Omega_j$ is Lipschitz then the embedding $W^{1,p}(\Omega_j) \hookrightarrow L^p(\Omega_j)$ is compact \cite{1} and therefore for each $k$ there exists a subsequence $(u_{e,k})_k \subset L^p(\Omega_j)$ such that

$$u_{e,k}|_{\Omega_j} \to u_0|_{\Omega_j}$$

By the diagonal process one can construct a sequence $(u_{e,k})_k$ such that $u_{e,k} \to u_0$ in $L^p(\Omega_j)$ for every $j$, in other words we have

$$u_{e,k} \to u_0 \text{ in } L^p_{loc}(\Omega) \quad \text{strongly} \quad (44)$$

Now passing to the limit in the weak formulation of (37) we deduce

$$-\text{div}_{X_2}(A_{22}\nabla_{X_2}u_0(X_1,\cdot)) = f + B(u_0)(X_1,\cdot),$$

where we have used (43) for the passage to the limit in the left hand side. For the passage to the limit in the nonlinear term we have used (44) and assumption (42).

\textbf{Example 1.} We give a concrete example of application of the above abstract analysis. Let $\Omega = \omega_1 \times \omega_2$ be a Lispchitz convex domain of $\mathbb{R}^q \times \mathbb{R}^{N-q}$ and let $A$ be a bounded $(N-q) \times (N-q)$ matrix defined on $\omega_2$ which satisfies the ellipticity assumption. Let us consider the integro-differential problem

$$\begin{cases}
-\text{div}_{X_2}(A(X_2)\nabla_{X_2}u) = f(X_2) + \int_{\omega_1} h(X'_1,X_1,X_2)a(u(X'_1,X_2))dX'_1 \\
u(X_1,\cdot) = 0 \quad \text{on } \partial \omega_2
\end{cases} \quad (45)$$

where $h \in L^\infty(\omega_1 \times \Omega)$ and $f \in L^p(\omega_2)$, $1 < p < 2$, and $a$ is a continuous real bounded function.

This equation is based on the Neutron transport equation (see for instance \cite{10}).

A solution to (45) is a function $u \in V_p$. Which satisfies (45) in $\mathcal{D}'(\omega_2)$. suppose that

$$\nabla_{X_1}h(X'_1,X_1,X_2) \in L^\infty(\omega_1 \times \Omega)$$

Then we have

\textbf{Theorem 10.} Under the assumptions of this example, (45) has at least a solution in $V_p$ in the sense of $\mathcal{D}'(\omega_2)$ for a.e $X_1 \in \omega_1$

\textbf{Proof.} We introduce the singular perturbation problem

$$\begin{cases}
-\text{div}_{X}(A_{e}\nabla u_{e}) = f(X_2) + \int_{\omega_1} h(X'_1,X_1,X_2)a(u_{e}(X'_1,X_2))dX'_1 \\
u_{e} = 0 \quad \text{on } \partial \Omega
\end{cases} \quad \text{where}

A_e = \begin{pmatrix}
e^2I & 0 \\
0 & A
\end{pmatrix}$$

Clearly $A_e$ satisfies the ellipticity assumption and it is Clear that the operator

$$u \to \int_{\omega_1} h(X'_1,X_1,X_2)a(u(X'_1,X_2))dX'_1$$

satisfies assumption (38).
We can prove easily that the above operator satisfies assumption (42). Indeed, let $u_n \to u$ in $L^p_{\text{loc}}(\Omega)$ then there exists a subsequence $(u_{n_k})$ (constructed by the diagonal process) such that $u_{n_k} \to u$ a.e in $\Omega$. Since $a$ is bounded then it follows by the Lebesgue theorem that

$$
\int_{\omega_1} h(X'_1, X_1, X_2) a(u_{n_k}(X'_1, X_2)) dX'_1 \to \int_{\omega_1} h(X'_1, X_1, X_2) a(u(X'_1, X_2)) dX'_1,
$$
in $L^p(\Omega)$. Whence by a contradiction argument we get

$$
\int_{\omega_1} h(X'_1, X_1, X_2) a(u_{n_k}(X'_1, X_2)) dX'_1 \to \int_{\omega_1} h(X'_1, X_1, X_2) a(u(X'_1, X_2)) dX'_1,
$$
in $L^p(\Omega)$

We can prove similarly as in [14] that (41) holds, therefore the assertion of the theorem is a simple application of theorem 9.

**Remark 4.** Notice that the compacity of the operator given in the previous example is not sufficient to prove a such result as in the $L^2$ theory [10]. This shows the importance of assumption (41) wich holds for the above operator.

Does operator whose assumption (41) holds admit necessarily an integral representation as in (45)?

**Example 2.** We shall replace the integral by a general linear operator. Let us consider the following problem: Find $u \in V_p$ such that

$$
\begin{cases}
- \text{div}_{X_2}(A \nabla_{X_2} u) = f(X_2) + gP(ha(u)) \\
u(X_1, \cdot) = 0 \quad \text{on } \partial \omega_2
\end{cases}
$$

(46)

where $a$, $A$ and $f$ are defined as in Example 1.

We suppose that $g, h \in L^\infty(\Omega)$ with $\text{Supp}(h) \subset \Omega$ compact. Assume $\nabla_{X_2} g \in L^\infty(\Omega)$ and $P : L^p(\Omega) \to L^2(\omega_2)$ is a bounded linear operator.

When $P$ is not compact then the operator $u \to gP(ha(u))$ is not necessarily compact, if this is the case then this operator cannot admit an integral representation.

**Theorem 11.** Under the assumptions of this example there exists at least a solution $u \in V_p$ to (46) in the sense of $\mathcal{D}'(\omega_2)$ for a.e $X_1 \in \omega_1$.

**Proof.** Similarly, the proof is a simple application of theorem 9.

5. **Some Open questions**

**Problem 1.** Suppose that $\infty > p > 2$. Given $f \in L^p$ and consider (2), since $f \in L^2$ then $u_e \to u_0$ in $V_2$. Assume that $\Omega$ and $A$ are sufficiently regular. Can one prove that $u_e \to u_0$ in $V_p$?

**Problem 2.** What happens when $f \in L^1$? As mentioned in the introduction there exists a unique entropy solution to (2) which belongs to $\bigcap_{1 \leq r < \frac{N}{N-1}} W^{1,r}_0(\Omega)$. Can one prove that $u_e \to u_0$ in $V_r$ for some $1 \leq r < \frac{N}{N-1}$? Can one prove at least weak convergence in $L^r$ for some $1 < r < \frac{N}{N-1}$ as given in Theorem 4?
References


E-mail address: chokri.ogabi@ag-grenoble.fr