On the zeros of a class of generalised Dirichlet series-XVIII (a few remarks on littlewood’s theorem and Totchmarsh points)

R Balasubramanian, K Ramachandra, A Sankaranarayanan

To cite this version:

HAL Id: hal-01109311
https://hal.archives-ouvertes.fr/hal-01109311
Submitted on 26 Jan 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
ON THE ZEROS OF A CLASS OF GENERALISED DIRICHLET SERIES-XVIII
(A FEW REMARKS ON LITTLEWOOD’S THEOREM AND TITCHMARSH POINTS)

BY

R. BALASUBRAMANIAN, K. RAMACHANDRA and A. SANKARANARAYANAN

(DEDICATED TO PROFESSOR K. CHANDRASEKHARAN
ON HIS SEVENTY-FIFTH BIRTHDAY)

§ 1. INTRODUCTION AND NOTATION. This paper was necessitated because we observed that the proofs of the results of the paper XVI[7] can be simplified and that the results therein can at the same time be generalised. (In § 1, § 2 and § 3 we prove six theorems in all. For an attractive application of these see Theorem 11 of § 4). We write \( s = \sigma + it \) as usual.

We begin by stating a generalisation of Theorem 9.15 (A) (on page 230 of [9]). We need some definitions. (We fix two positive constants \( a \) and \( b \) with \( a < b \) throughout). The parameter \( T \) will be assumed to exceed a large positive constant.

GENERALISED DIRICHLET SERIES (GDS). Let \( \{\lambda_n\} \) be a sequence of real numbers with \( a < \lambda_1 < \lambda_2 < \cdots, \lambda_1 < b \) and \( a \leq \lambda_{n+1} - \lambda_n \leq b \) for \( n \geq 1 \). Let \( \{A_n\} \) be any sequence of complex numbers such that \( A_1 \neq 0 \)
and

\[ Z(s) = \sum_{n=1}^{\infty} A_n \lambda_n^{-s}. \]

(1)

converges for some complex \( s = s_0 \). Then \( Z(s) \) is called a generalised Dirichlet series (GDS). We remark that if \( Z(s) \) converges at \( s = s_0 \), then it is absolutely convergent at \( s = s_0 + 2 \). Note that a GDS is different from zero if real part of \( s(Re s) \) exceeds a certain constant. In fact as \( Re s \to \infty \), \( |Z(s)| \) tends to a non-zero constant. A GDS is said to be a normalised generalised Dirichlet series (NGDS) if \( \sum_{n \leq x} |A_n| ^2 \ll x^{1+\varepsilon} \) for every \( \varepsilon > 0 \). A GDS is said to be a Dirichlet series if \( \{\alpha_n\} \) is a subsequence of the sequence of natural numbers.

\{\alpha_n\} TRANSFORMATION OF AN NGDS. Let \( Z(s) \) be an NGDS. We consider only such sequences \( \{\alpha_n\} \) of real numbers which satisfy

\[ \sum_{n \leq x} |A_n\alpha_n| ^2 \ll x^{1+\varepsilon} \] for every \( \varepsilon > 0 \). \( F(s) = \sum_{n=1}^{\infty} A_n(\lambda_n + \alpha_n)^{-s} \) is said to be an \( \{\alpha_n\} \) transformation of \( Z(s) \) if \( F(s) \) is a GDS. Note that

\[ F(s) = D(s) + Z(s) \]

where \( D(s) = \sum_{n=1}^{\infty} A_n((\lambda_n + \alpha_n)^{-s} - \lambda_n^{-s}) \).

(2)

and that \( D(s) \) is analytic in \( \sigma > 0 \). Moreover we have

LEMA 1. For \( \sigma > 0 \) and every \( \varepsilon > 0 \) we have \( D(s) = O_{\sigma}(|t| + 2)^2 \) and also

\[ \frac{1}{T} \int_{T}^{2T} |D(\sigma + it)|^2 \, dt \ll_{\varepsilon} \max(T^{2(\frac{1}{2} - \sigma) + \varepsilon}, T^\varepsilon). \]

(3)

PROOF. See Theorems 7 and 7' of XV[8].

THEOREM 1 (J.E. LITTLEWOOD). Let \( Z(s) \) be a GDS which can be continued analytically in \( (\sigma \geq \frac{1}{2} - \delta_0, T - \log T \leq t \leq 2T + \log T) \), where \( \delta_0(> 0) \) is a constant, and there \( \log(\max |Z(s)| + 100) \) is \( \ll \log T \). Let \( Z(s) \to 1 \) as \( Re \, s \to \infty \). For \( \alpha \geq \frac{1}{2} \) let \( N(\alpha, T) \) denote the number of zeros of \( Z(s) \) in \( (\sigma \geq \alpha, T \leq t \leq 2T) \). Then (for \( \sigma_0 > \frac{1}{2} \)) we have

\[ 2\pi \int_{\sigma_0}^{\infty} N(\sigma, T) d\sigma = \int_{T}^{2T} \log |Z(\sigma_0 + it)| \, dt + O(\log T). \]

(4)
and hence
\[ N\left(\frac{1}{2} + 2\delta, T\right) \leq (g\delta)^{-1}T \log \left( \frac{1}{T} \int_T^{2T} \left| Z\left(\frac{1}{2} + \delta + it\right) \right|^g dt \right) + O(\log T) \]
(5)
holds uniformly for all real positive constants \( g \) and \( \delta \). If \( \delta \) is any fixed constant we may take \( \delta_0 = 0 \) and then replace \( O(\log T) \) by \( O_T(\log T) \).

**REMARK.** This theorem is essentially due to J.E. Littlewood, since the special case \( g = 2 \) and \( Z(s) = \zeta(s) \) (due to J.E. Littlewood) is dealt with on pages 229 and 230 of [9]. The general case stated as Theorem 1 above follows by a trivial generalisation of Littlewood's method. If we do not assume \( Z(s) \to 1 \) as \( \text{Re} \ s \to \infty \), we have to replace \( O(\log T) \) by \( O(T) \) in (4) and (5). This does not matter for our purposes.

§ 2. A COROLLARY TO THEOREM 1.

**THEOREM 2.** Let \( r \geq 1 \) be any integer constant and let \( \varphi_1(s), \varphi_2(s), \ldots, \varphi_r(s) \) be \( r \) Dirichlet series each of which is continuably analytically in \( (\sigma \geq \frac{1}{2} - \delta_0, T - \log T \leq t \leq 2T + \log T) \) and there \( \log(\max | \varphi_j(s) | + 100) \ll \log T \).

Suppose further that
\[ \max_j \left( \frac{1}{T} \int_{T-\log T}^{2T+\log T} \left| \varphi_j\left(\frac{1}{2} + it\right) \right|^2 dt \right) \ll \varepsilon T^\varepsilon \]
(6)
holds for every \( \varepsilon > 0 \). Let \( P(X_1, \ldots, X_r) \) be any fixed polynomial (with complex coefficients) such that when we put \( X_j = \varphi_j(s)(i = 1, 2, \ldots, r), P = P(s) = P(X_1, \ldots, X_r) \) is a normalised Dirichlet series. Let \( F(s) \) be any \( \{\alpha_n\} \) transformation of \( P(s) \). Then the function \( N(\sigma, T) \) defined (as before) for \( F(s) \) satisfies
\[ N(\sigma, T) \ll \sigma T \quad (\sigma > \frac{1}{2}). \]
(7)

**REMARK.** We define the degree of a monomial \( X_1^{d_1} \cdots X_r^{d_r} \) to be \( d_1 + \cdots + d_r \) and the degree of \( P(X_1, \ldots, X_r) \) to be the maximum of \( d_1 + \cdots + d_r \) taken over all monomials occurring in \( P(X_1, \ldots, X_r) \). If the degree of \( P \) is 1 then we can allow each \( \varphi_j(s) \) to be a GDS. Then \( P \) has to be an NGDS.
PROOF. The proof follows from the fact that (6) implies
\[
\max_j \left( \frac{1}{T} \int_T^{2T} | \varphi_j(\sigma + it) |^2 \, dt \right) \ll \sigma \quad (\sigma > \frac{1}{2}),
\]
and that for a suitable small constant \( g > 0 \) we have
\[
| F(s) |^g \ll | D(s) |^2 + | \varphi_1(s) |^2 + \cdots + | \varphi_r(s) |^2 + 1. \tag{9}
\]
Note that in view of Lemma 1 it is not hard to deduce that
\[
\frac{1}{T} \int_T^{2T} | D(\sigma + it) |^2 \, dt \ll \sigma \quad (\sigma > \frac{1}{2}). \tag{10}
\]
From these facts Theorem 2 follows from Theorem 1.

§ 3. TITCHMARSH POINTS. Let \( F(s) \) be a GDS continuable analytically in \((\sigma \geq \beta, T - \log T \leq t \leq 2T + \log T)\) and there \( \log(\max | F(s) | + 100) \ll \log T. \) A point \( s_0 = \sigma_0 + it_0 \) in \((\sigma \geq \beta + \delta_1, T \leq t \leq 2T)\), where \( \delta_1 > 0 \) is a constant, is said to be a Titchmarsh point with the lower bound \( T^\ell \) for \( | F(s) | \) if \( \ell(> 0) \) is bounded below independent of \( T \) and \( t_0. \)

THEOREM 3. If \( s_0 = \sigma_0 + it_0 \) (with \( F(s) \) as above) is a Titchmarsh point of \( F(s) \), then the region \((\sigma \geq \beta, | t - t_0 | \leq \delta_2 \) where \( \delta_2(> 0) \) is any small constant, contains \( \gg \log T \) zeros of \( F(s) \).

PROOF. For the proof of this theorem due to R. Balasubramanian and K. Ramachandra see Theorem 3 of [III]1. It should be mentioned that this theorem is not too-trivial a generalisation of Theorem 9.14 (on page 227 of [9]) due to E.C. Titchmarsh.

WELL-SPACED POINTS. The points \( s^{(q)} = \sigma_q + it_q \) \((q = 1, 2, \cdots)\) in the complex plane are said to be well-spaced if \( | s^{(q)} - s^{(q')} | \) is bounded below for all pairs \((q, q')\) with \( q \neq q' \).

THEOREM 4. If there are \( N_0 \) well-spaced Titchmarsh points for \( F(s) \) \((F(s) \) as in Theorem 3), then \( F(s) \) has \( \gg N_0 \log T \) zeros in \((\sigma \geq \beta, T \leq t \leq 2T)\).

PROOF. The proof follows from the fact that \( | F(s) | \) tends to a non-zero
THEOREM 5. Let $\beta(< \frac{1}{2})$ be a constant and $r \geq 1$ any integer constant and $\varphi_1(s), \ldots, \varphi_r(s)$ be $r$ Dirichlet series each of which is continu­able analytically in $(\sigma \geq \beta, T - \log T \leq t \leq 2T + \log T)$ and there 
$\log \max(|\varphi_j(s)| + 100)$ is $\ll \log T$. Suppose further that for $j = 1, 2, \ldots, r$ and $\sigma \geq \beta$, we have

$$\frac{1}{T} \int_{T-\log T}^{2T+\log T} |\varphi_j(\sigma + it)|^2 \, dt \ll_{\varepsilon} \max\left(T^{2m_j\left(\frac{1}{2}-\sigma\right)+\varepsilon}, T^\varepsilon\right)$$

(11)

where $m_j > 0$ are constants. Let $\mu > 0$ be a constant. Put $X_0 = T^{\mu(\frac{1}{2}-\sigma)-\varepsilon}$. Let $X_0^d_0 X_1^d_1 \cdots X_r^d_r$ ($d_j \geq 0$ integers, $j = 0, 1, 2, \ldots, r$) be any fixed mono­mial in $X_0, X_1, \ldots, X_r$. Let the weighted $\mu$–degree $d(\mu)$ of the monomial be 
defined as $\mu d_0 + m_1 d_1 + \cdots + m_r d_r$. Put $Q_0(s) = X_0^{d_0}(\varphi_1(s))^{d_1} \cdots (\varphi_r(s))^{d_r}$. Then given any well-spaced set of points $\{s_q\}$ with $s_q = \sigma + it_q (q = 1, 2, \ldots ; \sigma \geq \beta + \delta_3, T \leq t_q \leq 2T)$ where $\delta_3 > 0$ is a small constant we have

$$|Q_0(\sigma + it_q)| \ll \max(T^{d(\mu)(\frac{1}{2}-\sigma)+\varepsilon}, T^\varepsilon),$$

(12)

except for $O(T^{1-\varepsilon})$ values of $q$.

REMARK. If $\sum_{j=1}^r d_j = 1$, then we can allow $\varphi_j(s)(j = 1, 2, \ldots, r)$ to be 
GDS.

PROOF. We use the fact that the value $|\varphi_j(s_q)|$ of $\varphi_j(s)$ is majorised by the 
mean value over a disc (with $s_q$ as centre and $\varepsilon$ as radius) of $|\varphi_j(s)|$. We 
choose a small radius and sum over all the discs taking $s_q$ to be $\sigma + it_q$. We 
obtain

$$\frac{1}{T} \sum_{q} |\varphi_j(s_q)|^2 \ll \max\left(T^{2m_j\left(\frac{1}{2}-\sigma\right)+\varepsilon}, T^\varepsilon\right).$$

Hence $|\varphi_j(s_q)| > \max(T^{m_j\left(\frac{1}{2}-\sigma\right)+\varepsilon}, T^{\varepsilon})$ is possible for at most $O(T^{1-\varepsilon})$ 
values of $q$. We next sum over all $j$ and obtain the result.

THEOREM 6. Let $\beta(< \frac{1}{2})$ be a constant and let $\varphi_0(s)$ be a Dirichlet se­ries continu­able analytically in $(\sigma \geq \beta, T - \log T \leq t \leq 2T + \log T)$ and there
log max(|φ₀(s)| + 100) ≪ log T. Suppose that it has \( \gg T \text{resp.} T(\log \log T)^{-1} \) well-spaced Titchmarsh points \( \{\sigma + it\} \) (where \( \sigma \) is any constant with \( \beta < \sigma < \frac{1}{2} \)) with the lower bound \( T^{\mu (\frac{1}{2} - \sigma)^{-\varepsilon}} \) where \( \mu(>0) \) is a constant. Let \( Q(X₀, X₁, \ldots, X_r) \) be a fixed polynomial (with complex coefficients) such that for some positive integer \( M \), the maximum of \( d(p) \) (defined in Theorem 5) taken over all the monomials \( X₀^{d₀} X₁^{d₁} \cdots X_r^{d_r} \) occurring in \( Q(X₀, X₁, \ldots, X_r) \) is less than \( M\mu \). Put \( Q(s) = Q(φ₀(s), φ₁(s), \ldots, φ_r(s)) \) (where \( φ_j(s) \) \( j = 1, 2, \ldots, r \) are as in Theorem 5). Assume that \( (φ₀(s))^M - Q(s) \) is an NGDS, and let \( F(s) \) be its \( \{αₙ\} \) transformation. Then \( F(s) \) has \( \gg T \log T \text{resp.} T(\log T)(\log \log T)^{-1} \) zeros in \( (\sigma - \varepsilon, \sigma + t) \).

**Proof.** Follows from \( |(φ₀(s))^M - Q(s)| \geq |φ₀(s)|^M (1 - |Q(s)| - |φ₀(s)|^{-M}) \).

**Remark.** Note that none of the functions \( φ₀(s), (φ₀(s))^M \) and \( Q(s) \) need be normalised Dirichlet series. If \( M = 1 \), then \( Q(s) \) does not involve \( φ₀(s) \) (which can now be taken to be a GDS). If \( M = 1 \) and \( Q(X₀, X₁, \ldots, X_r) \) (now independent of \( X₀ \)) is linear in \( X₁, \ldots, X_r \) (i.e. \( \sum d_j \leq 1 \) for every monomial \( X₁^{d₁} \cdots X_r^{d_r} \) occurring in \( Q(X₀, X₁, \ldots, X_r) \) and equality holds for at least one monomial) then all of \( φ₁(s), \ldots, φ_r(s) \) can be taken to be GDS. Also \( Q(X₀, X₁, \ldots, X_r) \) can be a constant.

§ 4. SOME APPLICATIONS OF THEOREMS 2 TO 6. Theorems 2 to 6 are only easy formalisms. These would be completely uninteresting without examples. Finding examples is a difficult task. For example we do not know how to prove the expected result \( N(σ, T) ≪ T(σ > \frac{1}{2}) \) for the abelian \( L \)-series of an algebraic number field. However we have a somewhat general theorem namely.

**Theorem 7.** Let \( \{λₙ\}(n = 1, 2, \ldots) \) be a sequence of real numbers as in the definition of GDS. Let \( |\sum aₙ| ≤ B(x), \sum |aₙ|^2 ≤ xB(x) \) and \( \sum m ≤ x \sum n ≤ m |aₙ|^2 ≤ xB(x) \), where \( B(x) \) depends on \( x \). If \( B(x) ≪ x^ε \) (for every
\( \epsilon > 0 \) then \( Z(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s} \) converges uniformly over compact subsets of \( \sigma > 0 \) and hence is analytic there. We have

\[
N\left(\frac{1}{2} + \delta, T\right) \ll_{\delta} T \quad (\delta > 0).
\]

(13)

If further \( \log B(x) \ll \log \log x \) then we have

\[
N\left(\frac{1}{2} + \delta, T\right) \ll T \log(\delta^{-1})
\]

(14)

uniformly for \( 0 < \delta \leq \frac{1}{2} \).

REMARK. Results like

\[
\frac{1}{T} \int_{T-\log T}^{2T+\log T} |Z\left(\frac{1}{2} + it\right)|^2 \, dt \ll T^{1-\epsilon}
\]

(15)

for every \( \epsilon > 0 \) and more general and powerful results have been proved in paper V[6]. Results like (15) imply (13) and (14). If \( \{Z(s)\} \) is any finite set of Dirichlet series each subject to (15) we can apply Theorem 2.

We now turn to series of the type

\[
\sum_{n=1}^{\infty} a_n b_n e^{2\pi i n \theta} \lambda_n^{-s} \quad (\theta \text{ is a real constant}),
\]

(16)

their analytic continuations and their Titchmarsh points. Investigations dealing with such series were carried out in a series of papers by R. Balasubramanian and K. Ramachandra (see [I][1], [IV][2], V[6], VI[3], XIV[4] and also the paper [8] by K. Ramachandra and A. Sankaranarayanan). The paper XIV[4] is nearly final. In paper XIV[4] the condition \( a_n = O(1) \) is assumed. This can be relaxed to \( \sum_{n < x} |a_n|^2 = O(x) \). This last mentioned condition on \( a_n \) will be assumed in the rest of this paper.

Lest we get lost in generalities we state two special cases first.

THEOREM 8. Let \( \theta_0 (0 < \theta_0 < \frac{1}{2}) \) be a constant and let \( \{a_n\} \) be a sequence of complex numbers satisfying the inequality

\[
|\sum_{m=1}^{N} a_m - N| \leq (\frac{1}{2} - \theta_0)^{-1} \text{ for } N \geq 1.
\]
On the zeros of a class-XVIII

\[ N = 1, 2, 3, \ldots \text{. Also for } n = 1, 2, 3, \ldots \text{, let } \alpha_n \text{ be real and } | \alpha_n | \leq C(\theta_0) \text{ where } C(\theta_0) \text{ is a certain (small) constant depending only on } \theta_0. \text{ Then the number of zeros of the function} \\
\sum_{n=1}^{\infty} a_n(n + \alpha_n)^{-s} = \zeta(s) + \sum_{n=1}^{\infty} (a_n(n + \alpha_n)^{-s} - n^{-s}) \]

in the rectangle (| \sigma - \frac{1}{2} | \leq \delta, T \leq t \leq 2T) \text{ is } \geq C(\theta_0, \delta) T \log T \text{ where } C(\theta_0, \delta) \text{ is a positive constant depending only on } \theta_0 \text{ and } \delta, \text{ and } T \geq T_0(\theta_0, \delta) \text{ a large positive constant.} 

**PROOF.** Theorem 10 (below) gives \( T \) well spaced Titchmarsh points on every line segment \((\sigma = \frac{1}{2} - \delta, T \leq t \leq 2T)\) with the lower bound \( T^\delta \), while actually (14) gives

\[ N \left( \frac{1}{2} + \frac{C \log \log T}{\log T}, T \right) \ll C^{-1} T \log T \]

for every fixed \( C(>0) \). (It is not hard to prove the required mean-square upper bound for the function). 

**THEOREM 9.** In the above theorem we can relax the condition on \( a_n \) to

\[ | \sum_{m=1}^{N} a_m - N | \leq (\frac{1}{2} - \theta_0)^{-1} N^{\theta_0} \text{ and } \sum_{n \leq x} | a_n |^2 \leq (\frac{1}{2} - \theta_0)^{-1} x. \]

Then the lower bound for the number of zeros in \((\sigma \geq \frac{1}{2} - \delta, T \leq t \leq 2T)\) (\( \delta \) being any constant with \( \frac{1}{2} - \delta > \theta_0 \)) is \( \geq C(\theta_0, \delta) T(\log T)(\log \log T)^{-1} \). But only when \( \sum_{n \leq x} a_n = x + O_e(x^e) \) we can prove that \( N(\frac{1}{2} + \delta, T) \ll \delta T. \) Also if

\[ \sum_{n \leq x} a_n = x + O((\log x)^{C_1}) \text{ (} C_1 > 0 \text{ being a constant we can prove} \]

\[ N \left( \frac{1}{2} + \frac{C(\log \log T)^2}{\log T}, T \right) \ll C^{-1} T(\log T)(\log \log T)^{-1} \]

for every fixed \( C > 0. \)

**PROOF.** Theorem 10 (below) gives \( T(\log \log T)^{-1} \) well-spaced Titchmarsh points on every line segment \((\sigma = \frac{1}{2} - \delta, T \leq t \leq 2T)\) with the lower
THEOREM 10. (i) Let \( \{\lambda_n\} \) be as in the definition of GDS. This sequence will be further restricted by the condition (vii) or (viii) as the case may be. \( \theta \) will denote a real constant.

Let \( f(x) \) and \( g(x) \) be positive real valued functions defined in \( x \geq 0 \) satisfying

(ii) \( f(x)x^n \) is monotonic increasing and \( f(x)x^{-\eta} \) is monotonic decreasing for every fixed \( \eta > 0 \) and all \( x \geq x_0(\eta) \).

(iii) \( \lim_{x \to \infty} (g(x)x^{-1}) = 1 \).

(iv) For all \( x \geq 0 \), \( g'(x) \) lies between two positive constants and \( (g'(x))^2 - g(x)g''(x) \) lies between two positive constants (it being assumed that \( g(x) \) is twice continuously differentiable for \( x \geq x_0 \)).

Let \( \{a_n\} \) and \( \{b_n\} \) be two sequences of complex numbers having the following properties.

(v) \( |b_n| (f(n))^{-1} \) lies between two positive constants (for all integers \( n \geq n_0 \)) and \( \left( \sum_{n \leq x} |a_n|^2 \right)x^{-1} \) does not exceed a positive constant for all \( x \geq 1 \).

(vi) For all \( X \geq 1 \), \( \sum_{X \leq n \leq 2X} |b_{n+1} - b_n| \ll f(X) \).

We next assume that \( \{a_n\} \) and \( \{b_n\} \) satisfy at least one of the two following conditions.

(vii) **MONOTONICITY CONDITION.** There exists an arithmetic progression \( A \) (of integers) such that

\[
\lim_{x \to \infty} \left( x^{-1} \sum_{n \leq x} a_n \right) = h \quad (h \neq 0),
\]

where the accent denotes the restriction of \( n \) to \( A \). Also for every positive constant \( \nu \) we have that \( |b_n| \lambda_n^{-\nu} \) is monotonic decreasing for all \( n(\geq n_0) \) in \( A \).

(viii) **REAL PART CONDITION.** There exists an arithmetic pro-
gression $A$ (of integers) such that

$$\lim \inf_{x \to \infty} \left( \frac{1}{x} \sum_{x \leq \lambda_n \leq 2x, \Re a_n > 0} Re a_n \right) > 0$$

and

$$\lim_{x \to \infty} \left( \frac{1}{x} \sum_{x \leq \lambda_n \leq 2x, \Re a_n < 0} Re a_n \right) = 0$$

where the accent denotes the restriction of $n$ to $A$.

(ix) Finally we set $\lambda_n = g(n)$ and let $\{\alpha_n\}$ be a sequence of real numbers such that $|\alpha_n|$ does not exceed a small positive constant (depending on other constants). We suppose that the GDS

$$F(s) = \sum_{n=1}^{\infty} a_n b_n e^{2\pi i \theta (\lambda_n + \alpha_n)^{-s}}$$

can be continued analytically in $(\sigma \geq \frac{1}{2} - \delta, T - \log T \leq t \leq 2T + \log T)$ and there $\log \max(|F(s)| + 100) \ll \log T$.

Then on every line segment $(\sigma = \frac{1}{2} - \delta_4, T \leq t \leq 2T)$ ($\delta_4$ being a constant with $0 < \delta_4 \leq \delta$) there are $\gg T (\log \log T)^{-1}$ well-spaced Titchmarsh points with the lower bound $\gg T^{\delta_4} f(T)$. If further we have

$$\frac{1}{T} \int_{T}^{2T} |F(\frac{1}{2} - \delta_4 + it)|^2 dt \ll T^{2\delta_4} (f(T))^2$$

for every constant $\delta_4$ (with $0 < \delta_4 \leq \delta$), then the number of well-spaced Titchmarsh points on the line segment $(\sigma = \frac{1}{2} - \delta_4, T \leq t \leq 2T)$ (with the lower bound $\gg T^{\delta_4} f(T)$) is $\gg T$.

**Remark.** This theorem is proved by R. Balasubramanian and K. Ramachandra in this form in the paper XIV$^{[a]}$ except that we have now to use $\sum_{n \leq x} |a_n|^2 \ll x$ in place of $a_n = O(1)$ and also except that we have to involve $\theta$. Lemmas necessary (see Lemma 6 of IV$^{[2]}$) for these generalities and also the method have been developed in previous papers mentioned before by R. Balasubramanian and K. Ramachandra.
Finally we would like to mention paper XVI of this series of papers. Here we assume a functional equation of a very general type for a GDS and prove that a large class of \( \{ \alpha_n \} \) transformations of it have \( \gg T \) well-spaced Titchmarsh points on every line segment \( (\sigma = \frac{1}{2} - \delta, T \leq t \leq 2T) \) with a lower bound of the type \( \gg T^{m\delta-\varepsilon} \) where \( m > 0 \) is a real constant and \( \varepsilon(>0) \) is an arbitrary constant (for example for the zeta function of a ray class in an algebraic number field of degree \( m \). If \( m \geq 2 \) we can allow \( \sum_{n \leq x} |\alpha_n|^2 \ll x^{1+\varepsilon} \) in place of \( \sum_{n \leq x} |A_n\alpha_n|^2 \ll x^{1+\varepsilon} \), see the definition in § 1 for the meaning of \( A_n \). Note that if \( A_n = O_x(n^\varepsilon) \) then the condition on \( \alpha_n \) is simply \( \sum_{n \leq x} |\alpha_n|^2 \ll x^{1+\varepsilon} \). These results are very general. But out of these GDS only in very special cases (but still a somewhat large class of GDS) we can prove that

\[
\frac{1}{T} \int_T^{2T} |F(\frac{1}{2} + \delta + it)|^2 \, dt \ll_{\varepsilon, \delta} T^{\varepsilon}
\]

for all \( \delta > 0 \) and \( \varepsilon > 0 \). Some examples (not already covered by Theorem 7) are (i) zeta function of any ray class of a quadratic field (ii) zeta function of a positive definite quadratic form \( Q(X_1, \ldots, X_\ell) \) (in \( \ell \geq 2 \) variables and with integer coefficients) namely \( \sum_{n=1}^\infty \left( a_n n^{-\frac{\ell}{2}+1} \right) n^{-s} \), where \( a_n \) is the number of \( \ell \)-tuples \( (m_1, \ldots, m_\ell) \) of integers with \( Q(m_1, \ldots, m_\ell) = n \). In this case \( m = 1 \) and the lower bound is \( \gg T^\delta \) (resp. \( \gg T^{2\delta-\varepsilon} \)) according as \( \ell > 2 \) or \( \ell = 2 \) see [8].

Instead of enumerating all the applications of this theory we state a beautiful theorem (namely, Theorem 11 below). Many other theorems can be deduced in a similar manner by the interested readers from the results of papers mentioned above and the results of § 1, § 2 and § 3, (see also the post-script at the end of this paper).

**THEOREM 11.** Let \( F \) denote the class of Dirichlet series of the form

\[
\zeta(s) + \sum_{n=1}^\infty a_n n^{-s}
\]

with complex number sequence \( \{ a_n \} \) satisfying \( \sum_{n \leq x} a_n = O(1) \). Let \( \varphi_j = \varphi_j(s) (j = 0, 1, 2, \ldots, r) \) be any \( r + 1 \) Dirichlet series (may
On the zeros of a class-XVIII

not be distinct) of the class $\mathcal{F}$. Let $P(X_0, X_1, \ldots, X_r)$ be any fixed polynomial (with complex coefficients) of degree $d$ (being the maximum of $d_0 + d_1 + \cdots + d_r$ taken over all monomials $X_0^{d_0} X_1^{d_1} \cdots X_r^{d_r}$ occurring in $P(X_0, \ldots, X_r)$). Let $Q$ be defined by

$$Q = (\varphi_0(s))^{d+1} - P(\varphi_0, \varphi_1, \ldots, \varphi_r) = \sum_{n=1}^{\infty} B_n n^{-\sigma}, (\sigma > 1).$$

Then first we have $B_n \neq 0$ for at least one $n$ (also $Q$ is analytic in $\sigma > 0$, $t \geq 1$). Next put

$$F(s) = \sum B_n ((n + \alpha_n)^{-\sigma} - n^{-\sigma}) + Q$$

where $\{\alpha_n\}$ is any sequence of real numbers with $|\alpha_n| \leq \frac{1}{3}$. Then in $(\sigma \geq \frac{1}{2} - \delta, T \leq t \leq 2T), F(s)$ has $\gg T \log T$ zeros and in $(\sigma \geq \frac{1}{2} + \frac{C \log \log T}{\log T}, T \leq t \leq 2T)$ only $\ll C^{-1} T \log T$ zeros ($C \geq 1$ being any constant).

REMARK 1. If $d \geq 1$ we can allow $\sum_{n \leq x} |\alpha_n|^{1/2} \ll x^{1+\varepsilon}$ in place of $|\alpha_n| \leq \frac{1}{3}$. But then we have to stipulate that $F(s)$ should be a GDS.

REMARK 2. That $B_n \neq 0$ for at least one $n$ of course follows since $Q$ has a pole of order $(d + 1)$ at $s = 1$. But then we mention that the conclusion of Theorem 11 are valid for $\varphi_0(s) = (1 - 2^{1-s})\zeta(s)$ and $\varphi_j(s) = \sum_{n=1}^{\infty} a_n^{(j)} n^{-s} (j = 1, 2, \ldots, r)$ where $\max \left| \sum_{n \leq x} a_n^{(j)} \right| = O(1)$. 

23
REFERENCES


POST-SCRIPT

1. In view of Theorem 2, it is important to find Dirichlet series whose satisfy (6). This will enable us to prove $N\left(\frac{1}{2} + \delta, T\right) \ll \delta T (\delta > 0)$ for larger and larger class of GDS.

2. In view of Theorem 4, it is important to find $N_0$ (as large as possible) well-spaced Titchmarsh points with the lower bound $\geq T^{k-\epsilon}$ (for some $k > 0$ and every $\epsilon > 0$) on the line segment $(\sigma = \frac{1}{2} - \delta, T \leq t \leq 2T)$ for a large class of Dirichlet series. In this direction we have Balasubramanian-Ramachandra functions given by Theorem 10 (Theorems 8 and 9 are special cases of these functions). Also we have the $\ell$th derivative ($\ell \geq 0$ integer) of a class of GDS which satisfy a very general functional equation (see equation (5) of XVII). The case $\ell = 0$ is treated in XVII and it is proved that $N_0 \gg T$. We can cover all integers $\ell$ as follows. We make use of the following lemma.

**LEMMA 2.** Let $h(x)$ be an $n$-times continuously differentiable function defined in $a_0 \leq x \leq a_0 + nd$, where $a_0 > 0, d > 0$ are constants and $n$ is any fixed integer $\geq 1$. Then

$$\sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} h(a_0 + rd) = \int_0^d \cdots \int_0^d h(a_0 + u_1 + \cdots + u_n) du_1 du_2 \cdots du_n.$$ 

**PROOF.** Follows by trivial induction.

We apply this lemma to $h(\sigma) = h(\sigma, t) = \chi(\sigma + it)$ of equation (5) of XVII and obtain $|\chi^{(\ell)}(s_0)| \gg T^{k\left(\frac{1}{2} - \sigma\right)} (\log T)^\ell$ for any fixed $t(T \leq t \leq 2T)$ and a suitable $s_0 = \sigma_0 + it$ (with $\sigma_0$ at a distance of $O((\log T)^{-1})$ from any arbitrarily given $\sigma$). At the same time for all $s$ and $\ell$ we have (by Cauchy's theorem), $|\chi^{(\ell)}(s)| \ll T^{k\left(\frac{1}{2} - \sigma\right)} (\log T)^\ell$.

Next we apply local convexity (see for example the references [PS-1] and [PS-2] below, see especially Theorem 6-C of [PS-2] for a correction in [PS-1]) to the zeta-function like analytic function $Z^{(\ell)}(s)(\chi(s))^{-1}(\log T)^{-\ell}$ to prove that the integral of its absolute value taken over $|t - t_0| \leq C(\epsilon)$ on $\sigma = \frac{1}{2} + \delta$ exceeds $t_0^{-\epsilon} (T \leq t_0 \leq 2T)$, where $C(\epsilon)$ depends only on $\epsilon$. From this it follows that for $Z^{(\ell)}(s)$ we have $N_0 \gg T$ the lower bound
is $\geq T^{k\delta - \epsilon}$.

3. Next given (arbitrarily) $N_0$ well-spaced points on $(\sigma = \frac{1}{2} - \delta, T \leq t \leq 2T)$ we can sometimes obtain a subset (of these points) of cardinality $\gg N_0$ ($= T$, sometimes $T'(\log \log T)^{-1}$) Titchmarsh points for a class of Dirichlet series or GDS. But this class of Dirichlet series is a very restricted one. Let $Z(s)$ be a Dirichlet series (see equation (5) of X\textsuperscript{V}[3]) which have

(a) Euler product for $Z_1(s)$.

(b) Functional equation with $1 \leq k \leq 2$.

(c) Mean-square on the critical line (see equation (6) of the present paper) $\sigma = \frac{1}{2}$.

(We have to mention that (c) follows from (b))

(d) $|\chi^{(\ell)}(s)| \leq t^{\delta/2}(\log t)^\ell$ for all integers $\ell \geq 0$.

From these we can deduce.

**Lemma 3.** Let $\{t_j\}(T \leq t_j \leq 2T)$ be a well-spaced set of points with cardinality $\gg T$. Then out of these points we can select a subset of points $t'_j$ (with cardinality $\gg T$) satisfying

$$|Z_1(\frac{1}{2} + \delta + it'_j)| \gg 1 \quad \text{and} \quad |Z_1^{(\ell)}(\frac{1}{2} + \delta + it'_j)| \ll 1$$

($\ell = 1, 2, \cdots, \ell_0$) where $\ell_0 \geq 1$ is any integer.

**Proof.** This lemma is contained implicitly in the proof of Theorem 1 of [PS-3].

From these we can formulate a general principle.

**General Principle.** In Theorem 11 we can replace $(\varphi_0(s))^{d+1}$ by $Q_1 \equiv (F_1(s))^{M_1}(\varphi^{(\ell_1)}(s))^{M_2}$ with integers $\ell_1 \geq 0, M_1 \geq 0, M_2 \geq 0, M_1 + M_2 \geq 1$, where $F_1(s)$ is a power product (with non-negative integral exponents) of derivatives of functions like $Z(s)$ satisfying (a),(b),(c) and (d) above and $\varphi(s)$ is either a Balasubramanian-Ramachandra function or a function which has a functional equation such as (5) of XV\textsuperscript{[3]}. In place of
On the zeros of a class-XVIII

$P(\varphi_0, \varphi_1, \cdots, \varphi_r)$ of Theorem 11, we can have a suitable modification say $Q_2$ such that $Q_1 - Q_2$ has $\gg T$ (resp. $T(\log \log T)^{-1}$) well-spaced Titchmarsh points on $\sigma = \frac{1}{2} - \delta$. Accordingly we have lower bounds for the number of zeros of $Q_1 - Q_2$ in $(\sigma \geq \frac{1}{2} - 2\delta, T \leq t \leq 2T)$ (and upper bounds for $N(\frac{1}{2} + \delta, T)$ only sometimes). We can say similar things about the $\{\alpha_n\}$ transformations of $Q_1 - Q_2$.

REFERENCES ADDED TO POST-SCRIPT


ADDRESSES OF AUTHORS

1. PROFESSOR R. BALASUBRAMANIAN
   MATSCIENCE
   THARAMANI P.O.
   MADRAS 600 0113, INDIA
   e-mail: BALU@IMSC.ERNET.IN
2. PROFESSOR K. RAMACHANDRA  
SCHOOL OF MATHEMATICS  
TATA INSTITUTE OF FUNDAMENTAL RESEARCH  
HOMI BHABHA ROAD  
BOMBAY 400 005  
e-mail : KRAM@TIFRVAX.TIFR.RES.IN

3. DR. A. SANKARANARAYANAN  
SCHOOL OF MATHEMATICS  
TATA INSTITUTE OF FUNDAMENTAL RESEARCH  
HOMI BHABHA ROAD  
BOMBAY 400 005  
e-mail : SANK@TIFRVAX.TIFR.RES.IN

MANUSCRIPT COMPLETED ON 11th OCTOBER 1995.