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Ciprian A. Tudor, Yimin Xiao. Sample Paths of the Solution to the Fractional-colored Stochastic Heat Equation. 2015. hal-01108721v2

HAL Id: hal-01108721

https://hal.archives-ouvertes.fr/hal-01108721v2

Preprint submitted on 27 Jan 2015

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Sample Paths of the Solution to the Fractional-colored Stochastic Heat Equation

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January 27, 2015

Abstract

Let $u = \{u(t,x), t \in [0,T], x \in \mathbb{R}^d\}$ be the solution to the linear stochastic heat equation driven by a fractional noise in time with correlated spatial structure. We study various path properties of the process u with respect to the time and space variable, respectively. In particular, we derive their exact uniform and local moduli of continuity and Chung-type laws of the iterated logarithm.

2010 AMS Classification Numbers: 60H55, 60H07, 91G70.

Key words: Stochastic heat equation, fractional Brownian motion, Gaussian noise, path regularity, law of the iterated logarithm.

1 Introduction

Stochastic analysis of fractional Brownian motion (fBm) naturally led to the study of stochastic partial differential equations (SPDEs) driven by this Gaussian process. The motivation comes from wide applications of fBm. We refer, among others, to [13], [21], [26],

*Supported by the CNCS grant PN-II-ID-PCCE-2011-2-0015 (Romania).
†Research of Y. Xiao is partially supported by NSF grants DMS-1307470 and DMS-1309856.

[30] and [34] for theoretical studies of SPDEs driven by fBm. To list only a few examples of applications of fractional noises in various areas, we mention [15] for biophysics, [5] for financial time series, [11] for electrical engineering, and [7] for physics.

The purpose of our present paper is to study fine properties of the solution to the stochastic heat equation driven by a Gaussian noise which is fractional in time and colored in space. Our work continues, in part, the line of research which concerns SPDEs driven by the fBm but at the same time it follows the research line initiated by Dalang [8] which treats equations with white noise in time and non trivial correlation in space. More precisely, we consider a stochastic linear equation driven by a Gaussian noise that behaves as a fractional Brownian motion with respect to its time variable and has a correlated spatial structure. A necessary and sufficient condition for the existence of the solution has been given in [3] and other results has been given in [35], [27], [1], [2] among others.

To briefly describe the context, consider the stochastic heat equation with additive noise

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2}\Delta u + \dot{W}, \quad t \in [0,T],\ x \in \mathbb{R}^d, \\ u(0,x) &= 0, \quad x \in \mathbb{R}^d, \end{aligned} \tag{1}$$

where $T > 0$ is a constant and $\dot{W}$ is usually referred to as a *space-time white noise*. It is well-known (see for example the now classical paper [8]) that (1) admits a unique mild solution if and only if $d = 1$. This mild solution is defined as

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) W(ds, dy), \qquad t \in [0,T],\ x \in \mathbb{R}. \tag{2}$$

In the above, $W = \{W(t,A); t \geq 0,\ A \in \mathcal{B}_b(\mathbb{R}^d)\}$ is a zero-mean Gaussian process with covariance given by

$$\mathbb{E}\left(W(t,A)W(s,B)\right) = (s \wedge t)\lambda(A \cap B), \tag{3}$$

where λ denotes the Lebesque measure and $\mathcal{B}_b(\mathbb{R}^d)$ is the collection of all bounded Borel subsets of $\mathbb{R}^d$. The integral in (2) is a Wiener integral with respect to the Gaussian process W and G is the Green kernel of the heat equation given by

$$G(t,x) = \begin{cases} (2\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{2t}\right) & \text{if } t > 0, x \in \mathbb{R}^d, \\ 0 & \text{if } t \leq 0, x \in \mathbb{R}^d. \end{cases} \tag{4}$$

Consequently the mild solution $\{u(t,x), t \in [0,T], x \in \mathbb{R}\}$ is a centered two-parameter Gaussian process (also called a Gaussian random field). It is well-known (see e.g. [33] or [35]) that the solution defined by the (2) is well-defined if and only if $d = 1$ and in this case the covariance of the solution (2), when $x \in \mathbb{R}$ is fixed, satisfies

$$\mathbb{E}\left(u(t,x)u(s,x)\right) = \frac{1}{\sqrt{2\pi}}\left(\sqrt{t+s} - \sqrt{|t-s|}\right), \text{ for every } s,t \in [0,T]. \tag{5}$$

This establishes an interesting connection between the law of the solution (2) and the so-called *bifractional Brownian motion* introduced by [14]. Recall that, given constants $H \in (0,1)$ and $K \in (0,1]$, the bifractional Brownian motion $(B_t^{H,K})_{t\in[0,T]}$ is a centered Gaussian process with covariance

$$R^{H,K}(t,s) := R(t,s) = \frac{1}{2^K}\left(\left(t^{2H}+s^{2H}\right)^K - |t-s|^{2HK}\right), \qquad s,t \in [0,T]. \tag{6}$$

Relation (5) implies that, when $x \in \mathbb{R}$ is fixed, the Gaussian process $\{u(t,x), t \in [0,T]\}$ is a bifractional Brownian motion with parameters $H = K = \frac{1}{2}$ multiplied by the constant $2^{-K}\frac{1}{\sqrt{2\pi}}$. Therefore, many sample path properties of the solution to the heat equation driven by space time white-noise follow from [31, 16, 31, 36].

In order to avoid the restriction to the dimension $d = 1$, several authors considered other types of noises, usually more regular than the time-space white noise. An approach is to consider the *white-colored noise*, meaning a Gaussian process $W = \{W(t,A); t \geq 0,\ A \in \mathcal{B}_b(\mathbb{R}^d)\}$ with covariance with zero mean and covariance

$$\mathbb{E}\left(W(t,A)W(s,B)\right) = (t \wedge s)\int_A\int_B f(z-z')dzdz'. \tag{7}$$

Here the kernel f is the Fourier transform of a tempered non-negative measure μ on $\mathbb{R}^d$, i.e. μ is a non-negative measure which satisfies:

$$\int_{\mathbb{R}^d}\left(\frac{1}{1+|\xi|^2}\right)^l \mu(d\xi) < \infty \quad \text{for some } l > 0. \tag{8}$$

Recall that the Fourier transform f of μ is defined through

$$\int_{\mathbb{R}^d} f(x)\varphi(x)dx = \int_{\mathbb{R}^d}\mathcal{F}\varphi(\xi)\mu(d\xi), \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^d),$$

where $\mathcal{S}(\mathbb{R}^d)$ denotes the Schwarz space on $\mathbb{R}^d$. A useful connection between the kernel f and its associated measure μ is the Parseval inequality: for any $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\varphi(x)f(x-y)\psi(y)dxdy = (2\pi)^{-d}\int_{\mathbb{R}^d}\mathcal{F}\varphi(\xi)\overline{\mathcal{F}\psi(\xi)}\mu(d\xi). \tag{9}$$

In this work, we will include the following two following basic examples.

Example 1 *The Riesz kernel of order α:*

$$f(x) = R_\alpha(x) := \gamma_{\alpha,d}|x|^{-d+\alpha}, \tag{10}$$

where $0 < \alpha < d$, $\gamma_{\alpha,d} = 2^{d-\alpha}\pi^{d/2}\Gamma((d-\alpha)/2)/\Gamma(\alpha/2)$. In this case, $\mu(d\xi) = |\xi|^{-\alpha}d\xi$.

Example 2 *The Bessel kernel of order α:*

$$f(x) = B_\alpha(x) := \gamma'_\alpha \int_0^\infty w^{(\alpha-d)/2-1} e^{-w} e^{-|x|^2/(4w)} dw,$$

where $\alpha > 0$, $\gamma'_\alpha = (4\pi)^{\alpha/2}\Gamma(\alpha/2)$. In this case, $\mu(d\xi) = (1+|\xi|^2)^{-\alpha/2} d\xi$.

For any function $g \in L^1(\mathbb{R}^d)$ we denote by $\mathcal{F}g$ the Fourier transform of g, i.e.,

$$(\mathcal{F}g)(\xi) = \int_{\mathbb{R}^d} e^{-i\langle \xi, x\rangle} g(x) dx, \qquad \xi \in \mathbb{R}^d,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^d$. Recall that for any $t \in \mathbb{R}$ and $x \in \mathbb{R}^d$, the Fourier transform of the fundamental solution (4) is given by

$$\mathcal{F}G(t, x - \cdot)(\xi) = \exp\left(i\langle x, \xi\rangle - \frac{t|\xi|^2}{2} \right) \mathbf{1}_{\{t>0\}}(\xi), \quad \xi \in \mathbb{R}^d. \tag{11}$$

Dalang [8] proved that the stochastic heat equation with white-colored noise admits a unique solution if and only if

$$\int_{\mathbb{R}^d} \frac{1}{1+|\xi|^2} \mu(d\xi) < \infty. \tag{12}$$

Obviously, this condition allows x to be in higher dimensional space. For example in the case of the Riesz kernel, the stochastic heat equation with white-colored noise admits an unique solution if and only if $d < 2 + \alpha$.

Under (12), the solution of (1) with white-colored noise can still be written as in (2). One can compute the covariance of the solution with respect to the time variable (see e.g. [28]). For fixed $x \in \mathbb{R}^d$, and for every $s \le t$ it follows from Parseval's identity and (11)

$$\mathbb{E}\big(u(t,x)u(s,x)\big) = (2\pi)^{-d} \int_0^s du \int_{\mathbb{R}^d} \mu(d\xi) e^{-\frac{1}{2}(t-u)|\xi|^2} e^{-\frac{1}{2}(s-u)|\xi|^2}. \tag{13}$$

In the case when f is the Riesz kernel (i.e., $\mu(d\xi) = |\xi|^{-\alpha} d\xi$, we get

$$\mathbb{E}\big(u(t,x)u(s,x)\big) = (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^\alpha} e^{-\frac{1}{2}|\xi|^2} \frac{1}{1-\frac{d-\alpha}{2}} \left((t+s)^{1-\frac{d-\alpha}{2}} - (t-s)^{1-\frac{d-\alpha}{2}} \right).$$

In this case, the solution coincides, modulo a constant, with a bifractional Brownian motion with parameters $H = \frac{1}{2}$ and $K = 1 - \frac{d-\alpha}{2}$. Thus the sample path properties of this process can be deduced from [16, 31, 36].

Our purpose in this paper is to study the linear heat equation driven by a fractional-colored Gaussian noise (see section 2 for details). When the structure of the noise with respect to the time variable changes and the white noise is replaced by a fractional noise, the solution does not coincide with a bifractional Brownian motion anymore (see [6] or

[35]). Some new methods are needed for analyzing the path properties of the solution with respect to the time and to the space variables. By appealing to general methods for Gaussian processes and fields (cf. e.g., [20, 39, 40]), we establish the exact uniform and local moduli of continuity and Chung type laws of the iterated logarithm for the solution of (1) with a fractional-colored Gaussian noise.

Throughout this paper, we will use C to denote unspecified positive finite constants which may be different in each appearance. More specific constants are numbered as $c_1, c_2, \ldots$.

2 The solution to the stochastic heat equation with fractional-colored noise

We consider a Gaussian field $\{W^H(t, A), t \in [0, T], A \in \mathcal{B}_b(\mathbb{R}^d)\}$ with covariance

$$\mathbb{E}\left(W^H(t, A)W^H(s, B)\right) = R_H(t, s) \int_A \int_B f(z - z')dzdz', \tag{14}$$

where $R_H(t, s) := \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H})$ is the covariance of the fractional Brownian motion with index H and f is the Fourier transform of a tempered measure μ. This noise is usually called *fractional-colored* noise. We will assume throughout the paper that the Hurst parameter H is contained in the interval $(\frac{1}{2}, 1)$.

Consider the linear stochastic heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + \dot{W}^H, \quad t \in [0, T], x \in \mathbb{R}^d \tag{15}$$

with vanishing initial condition, where $\{W^H(t, x), t \in [0, T], x \in \mathbb{R}^d\}$ is a centered Gaussian noise with covariance (14). In the following we collect some known facts:

- A necessary and sufficient condition for the existence of the mild solution to the fractional-colored heat equation (15) has been given in [3] (see also [28]). Namely, (15) has a solution $\{u(t, x), t \geq 0, x \in \mathbb{R}^d\}$ that satisfies

$$\sup_{t\in[0,T],x\in\mathbb{R}^d} \mathbb{E}\left(u(t, x)^2\right) < +\infty$$

  if and only if

$$\int_{\mathbb{R}^d} \left(\frac{1}{1 + |\xi|^2}\right)^{2H} \mu(d\xi) < \infty. \tag{16}$$

- When (16) holds, the solution to (15) can be written in the mild form as

$$u(t, x) = \int_0^t \int_{\mathbb{R}^d} G(t - u, x - y)W^H(du, dy), \quad t \in [0, T], \ x \in \mathbb{R}^d. \tag{17}$$

It follows from [3] or [28] that, for $x \in \mathbb{R}^d$ fixed, the covariance function of the Gaussian process $\{u(t,x), t \in [0,T]\}$ can be written as

$$\mathbb{E}\left(u(t,x)u(s,x)\right) = \alpha_H (2\pi)^{-d} \int_0^t \int_0^s \frac{dudv}{|u-v|^{2-2H}} \int_{\mathbb{R}^d} \mu(d\xi) e^{-\frac{(t-u)|\xi|^2}{2}} e^{-\frac{(s-v)|\xi|^2}{2}} \quad (18)$$

with $\alpha_H = H(2H-1)$. In the particular case where the spatial covariance is given by the Riesz kernel, the process $t \mapsto u(t,x)$ is self-similar with parameter $H - \frac{d-\alpha}{4}$. However, for $H \in (\frac{1}{2}, 1)$, $\{u(t,x), t \geq 0\}$ is no longer a bifractional Brownian motion.

For $0 < \alpha < d$, the notation

$$\mu(d\xi) \asymp |\xi|^{-\alpha} d\xi, \quad (19)$$

means that for every non-negative function h there exist two positive and finite constants C and C', which may depend on h, such that

$$C' \int_{\mathbb{R}^d} h(\xi)|\xi|^{-\alpha} d\xi \leq \int_{\mathbb{R}^d} h(\xi)\mu(d\xi) \leq C \int_{\mathbb{R}^d} h(\xi)|\xi|^{-\alpha} d\xi. \quad (20)$$

It has been proven in [35] that, under condition (19), there exist two strictly positive constants c_1, c_2 such that for all $t, s \in [0,1]$ and for all $x \in \mathbb{R}^d$,

$$c_1 |t-s|^{2H - \frac{d-\alpha}{2}} \leq \mathbb{E}\left(|u(t,x) - u(s,x)|^2\right) \leq c_2 |t-s|^{2H - \frac{d-\alpha}{2}}. \quad (21)$$

Remark 1 *The Riesz kernel obviously satisfies (19). The Bessel kernel satisfies (19) and the constants in* (20) *are* $C = 1$ *and* $C' > 0$ *depending on* h.

Under Condition (19), the condition (16) is equivalent to

$$d < 4H + \alpha. \quad (22)$$

In the sequel (e.g., Theorem 1 below), we will also use the notation $f(x) \asymp g(x)$ which means that there exists two positive and finite constants C and C' such that $C' \leq f(x)/g(x) \leq C$ for all x in the domain of f and g. In the rest of the paper, we will assume that (19) is satisfied.

2.1 Sharp regularity in time

For any fixed $x \in \mathbb{R}^d$, to further study the regularity properties of the sample function $t \mapsto u(t,x)$, we decompose the solution $\{u(t,x), t \geq 0\}$ in (17) into the sum of a Gaussian process $U = \{U(t), t \geq 0\}$, which has stationary increments, and another Gaussian process whose sample functions are continuously differentiable on $(0, \infty)$. In particular, when the spatial covariance of noise W^H is given by the Riesz kernel (10), then $U = \{U(t), t \geq 0\}$ becomes fractional Brownian motion with Hurst parameter $\gamma = H - \frac{d-\alpha}{4}$. We will apply this decomposition to obtain the regularity in time of the solution $\{u(t,x), t \geq 0\}$.

Motivated by [24] (see also [25, 37] for related results), we introduce the pinned string process in time $\{U(t), t \geq 0\}$ defined by

$$
\begin{aligned}
U(t) \;&=\; \int_{-\infty}^{0} \int_{\mathbb{R}^d} \big(G(t-u, x-y) - G(-u, x-y)\big) W^H(du, dy) \\
&\qquad\qquad + \int_0^t \int_{\mathbb{R}^d} G(t-u, x-u) W^H(du, du).
\end{aligned}
$$

Note that $U(0) = 0$ and $U(t)$ can be expressed as

$$
U(t) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \big(G((t-u)_+, x-y) - G((-u)_+, x-y)\big) W^H(du, dy). \tag{23}
$$

In the above, $a_+ = \max(a, 0)$. The following theorem shows that $\{U(t), t \geq 0\}$ has stationary increments and identifies its spectral measure. This information is useful for appleying the methods in [20, 39, 40] to study sample path properties of $\{U(t), t \geq 0\}$.

Theorem 1 *The Gaussian process $\{U(t), t \geq 0\}$ given by (23) has stationary increments and its spectral density is given by*

$$
f_U(\tau) = \frac{2(2\pi)^{-d} \alpha_H}{|\tau|^{2H-1}} \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{\tau^2 + \frac{|\xi|^4}{4}}. \tag{24}
$$

Moreover, under condition (19), we have

$$
f_U(\tau) \asymp \frac{1}{|\tau|^{2H - \frac{d-\alpha}{2} + 1}}. \tag{25}
$$

Proof : For every $0 \leq s < t$, by Parseval's identity (9) and relation (11), we can write

$$
\begin{aligned}
&\mathbb{E}\left[(U(t) - U(s))^2\right] \\
&= \mathbb{E}\left(\int_{\mathbb{R}} \int_{\mathbb{R}^d} \left(G((t-u)_+, x-y) - G((s-u)_+, x-y)\right) W^H(du, dy)\right)^2 \\
&= (2\pi)^{-d} \alpha_H \int_{\mathbb{R}^d} \mu(d\xi) \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{du dv}{|u-v|^{2-2H}} \left(e^{-\frac{1}{2}(t-u)|\xi|^2} \mathbf{1}_{\{t>u\}} - e^{-\frac{1}{2}(s-u)|\xi|^2} \mathbf{1}_{\{s>u\}}\right) \\
&\qquad\qquad \times \left(e^{-\frac{1}{2}(t-v)|\xi|^2} \mathbf{1}_{\{t>v\}} - e^{-\frac{1}{2}(s-v)|\xi|^2} \mathbf{1}_{\{s>v\}}\right).
\end{aligned}
$$

Let $\varphi(u) = e^{-\frac{1}{2}(t-u)|\xi|^2} \mathbf{1}_{\{t>u\}} - e^{-\frac{1}{2}(s-u)|\xi|^2} \mathbf{1}_{\{s>u\}}$. Then its Fourier transform is

$$
\mathcal{F}\varphi(\tau) = \big(e^{-it\tau} - e^{-is\tau}\big) \frac{1}{i\tau + \frac{1}{2}|\xi|^2}. \tag{26}
$$

By using again Parseval's relation (9) in dimension $d = 1$ and (26), we get

$$\begin{aligned}\mathbb{E}\left[(U(t)-U(s))^2\right] &= (2\pi)^{-d}\alpha_H \int_{\mathbb{R}^d}\mu(d\xi)\int_{\mathbb{R}}\frac{d\tau}{|\tau|^{2H-1}}\left|\mathcal{F}\varphi(\tau)\right|^2\\ &= (2\pi)^{-d}\alpha_H \int_{\mathbb{R}^d}\mu(d\xi)\int_{\mathbb{R}}\frac{d\tau}{|\tau|^{2H-1}}\frac{2\left[1-\cos((t-s)\tau)\right]}{\tau^2+\frac{|\xi|^4}{4}}\\ &= 2(2\pi)^{-d}\alpha_H \int_{\mathbb{R}}\left[1-\cos((t-s)\tau)\right]\frac{d\tau}{|\tau|^{2H-1}}\int_{\mathbb{R}^d}\frac{\mu(d\xi)}{\tau^2+\frac{|\xi|^4}{4}},\end{aligned}$$

where the last step follows from Fubini's theorem and the convergence of the last integral in $\mu(d\xi)$ is guaranteed by relation (16). It follows that the Gaussian process $\{U(t), t \geq 0\}$ has stationary increments and its spectral density is given by (24).

Under condition (19), we have

$$f_U(\tau) \asymp \frac{1}{|\tau|^{2H-1}}\int_{\mathbb{R}^d}\frac{d\xi}{|\xi|^\alpha\left(\tau^2+\frac{|\xi|^4}{2}\right)} \asymp \frac{1}{|\tau|^{2H-\frac{d-\alpha}{2}+1}}.$$

This finishes the proof of (25) and thus the conclusion of Theorem 1 is obtained. ■

When the spatial covariance of noise W^H is given by the Riesz kernel (10), we have

$$\begin{aligned}f_U(\tau) &= \frac{\alpha_H 2^{2H-1}\Gamma(H-\frac{1}{2})}{(2\pi)^{d+\frac{1}{2}}\Gamma(1-H)\,|\tau|^{2H-1}}\int_{\mathbb{R}^d}\frac{d\xi}{|\xi|^\alpha\left(\tau^2+\frac{|\xi|^4}{4}\right)}\\ &= \frac{\alpha_H 2^{2H-1}\Gamma(H-\frac{1}{2})}{(2\pi)^{d+\frac{1}{2}}\Gamma(1-H)\,|\tau|^{2H-\frac{d-\alpha}{2}+1}}\int_{\mathbb{R}^d}\frac{d\eta}{|\eta|^\alpha\left(1+\frac{|\eta|^4}{4}\right)}.\end{aligned}$$

Therefore, in the Riesz kernel case, $\{U(t), t \geq 0\}$ is, up to a constant, a fractional Brownian motion B^γ with Hurst parameter $\gamma = H - \frac{d-\alpha}{4}$.

Recall that the spectral density of the fBm B^γ with Hurst index $\gamma \in (0,1)$ is given by $f_\gamma(\lambda) = c_\gamma|\lambda|^{-(1+2\gamma)}$ with $c_\gamma = \frac{\sin(\pi\gamma)\Gamma(1+2\gamma)}{2\pi}$.) Hence, we have the following corollary.

Corollary 1 *Let $U = \{U(t), t \geq 0\}$ be the Gaussian process defined by (23) such that the spatial covariance of W^H is given by the Riesz kernel (10).Then U coincides in distribution with $C_0 B^\gamma$ with $\gamma = H - \frac{d-\alpha}{4}$ and*

$$C_0^2 = \frac{(2\pi)^{-d+\frac{1}{2}}\alpha_H 2^{2H-1}\Gamma(H-\frac{1}{2})}{\sin\left(\pi(d-\frac{H-\alpha}{4})\right)\Gamma(1+2H-\frac{d-\alpha}{2})\Gamma(1-H)}\int_{\mathbb{R}^d}\frac{d\eta}{|\eta|^\alpha\left(1+\frac{|\eta|^4}{4}\right)}.$$

Now for every $t \geq 0$ we have the following decomposition

$$u(t,x) = U(t) - Y(t),$$

where

$$Y(t) = \int_{-\infty}^{0} \int_{\mathbb{R}^d} \left(G(t-u, x-y) - G(-u, x-y)\right) W^H(du, dy).$$

The following theorem shows that the sample function of $\{Y(t), t \geq 0\}$ is smooth, which is useful for studying the regularity properties of the solution process $\{u(t,x), t \geq 0\}$ in the time variable.

Theorem 2 *Let $x \in \mathbb{R}^d$ be fixed and let $[a, b] \subset (0, \infty)$. Then for any $k \geq 1$ there is a modification of $\{Y(t), t \geq 0\}$ such that its sample function is almost surely continuously differentiable on $[a, b]$ of order k.*

Proof : The method of proof is similar to those of [12, Proposition 3.1] and [41, Theorem 4.8], but is more complicated in our fractional-colored noise case.

The mean square derivative of Y at $t \in (0, \infty)$ can be expressed as

$$Y'(t) = \int_{-\infty}^{0} \int_{\mathbb{R}^d} G'(t-u, x-y)\, W^H(du, dy),$$

where $G' := \partial G/\partial t$. This can be verified by checking the covariance functions. For every $s, t \in (0, \infty)$ with $s \leq t$, similarly to the proof of Theorem 1, we derive

$$\begin{aligned}
\mathbb{E}\big(|Y'(t) - Y'(s)|^2\big) &= \mathbb{E}\left(\int_{-\infty}^{0} \int_{\mathbb{R}^d} \big(G'(t-u, x-y) - G'(s-u, x-y)\big)\, W^H(du, dy)\right)^2 \\
&= \frac{\alpha_H}{4(2\pi)^d} \int_{\mathbb{R}^d} |\xi|^{4-\alpha} d\xi \int_0^\infty \int_0^\infty \frac{du dv}{|u-v|^{2-2H}} \\
&\quad \times \left(e^{-\frac{1}{2}(t+u)|\xi|^2} - e^{-\frac{1}{2}(s+u)|\xi|^2}\right)\left(e^{-\frac{1}{2}(t+v)|\xi|^2} - e^{-\frac{1}{2}(s+v)|\xi|^2}\right).
\end{aligned}$$

In the above, we have used the fact that the Fourier transform of the function $y \mapsto G'(t+u, y)$ is

$$\frac{\partial}{\partial t}\Big(\mathcal{F}G(t+u, \cdot)(\xi)\Big) = -\frac{1}{2}|\xi|^2 e^{-\frac{1}{2}(t+u)|\xi|^2}.$$

Denote by $\mathcal{F}_{0,\infty}$ the restricted Fourier transform of $f \in L^1(0, \infty)$ defined by $(\mathcal{F}_{0,\infty} f)(\tau) = \int_0^\infty e^{-ix\tau} f(x) dx$, $\tau \in \mathbb{R}$. By applying the Parseval relation (9) for the restricted transform (see Lemma A1 in [3]), we see that for all $s, t \in [a, b]$ with $s < t$,

$$\begin{aligned}
\mathbb{E}\big(|Y'(t) - Y'(s)|^2\big) &= C \int_{\mathbb{R}^d} |\xi|^{4-\alpha} d\xi \int_{\mathbb{R}} \frac{d\tau}{|\tau|^{2H-1}} \left|\mathcal{F}_{0,\infty}\left(e^{-\frac{1}{2}(t+\cdot)|\xi|^2} - e^{-\frac{1}{2}(s+\cdot)|\xi|^2}\right)(\tau)\right|^2 \\
&= C \int_{\mathbb{R}^d} |\xi|^{4-\alpha} d\xi \int_{\mathbb{R}} \frac{d\tau}{|\tau|^{2H-1}} \left|e^{-\frac{1}{2}t|\xi|^2} - e^{-\frac{1}{2}s|\xi|^2}\right|^2 \frac{1}{\tau^2 + \frac{|\xi|^4}{4}} \\
&= C \int_{\mathbb{R}^d} |\xi|^{4-\alpha-4H} e^{-s|\xi|^2} \left|1 - e^{-\frac{1}{2}(t-s)|\xi|^2}\right|^2 d\xi \int_{\mathbb{R}} \frac{d\tau}{(|\tau|^2 + \frac{1}{4})|\tau|^{2H-1}} \\
&\leq C|t-s|^2 \int_{\mathbb{R}^d} |\xi|^{8-\alpha-4H} e^{-a|\xi|^2}\, d\xi.
\end{aligned}$$

Hence, as in [12, 41], by using Kolmogorov's continuity theorem, we can find a modification of Y such that $Y(t)$ is continuously differentiable on $[a,b]$. Iterating this argument yields the conclusion of Theorem 2. ■

Remark 2 *In [25], the authors obtained a similar decomposition for the solution to the linear heat equation with white noise in time and Riesz covariance in space in dimension $d = 1$. Our Theorem 1 shows that the time behavior of the process u is very similar to the behavior of the bifractional Brownian motion (see the main result in [16]).*

By applying Theorems 1, 2 and the results on uniform and local moduli of continuity for Gaussian processes (see e.g. [20, Chapter 7] or [22]), we derive the following regularity results on the solution process $\{u(t,x), t \geq 0\}$, when $x \in \mathbb{R}^d$ is fixed. For simplicity, we avoid the point $t = 0$.

Proposition 1 *Let $x \in \mathbb{R}^d$ be fixed. Then for any $0 < a < b < \infty$, we have*

$$\lim_{\varepsilon \to 0} \sup_{s,t \in [a,b], |s-t| \leq \varepsilon} \frac{|u(t,x) - u(s,x)|}{|s-t|^\gamma \sqrt{\log(1/(t-s)}} = c_3 \qquad a.s.,$$

where $0 < c_3 < \infty$ is a constant that may depend on a, b, γ and x. Or

$$\overline{\lim_{\varepsilon \to 0}} \sup_{s,t \in [a,b], |s-t| \leq \varepsilon} \frac{|u(t,x) - u(s,x)|}{\varepsilon^\gamma \sqrt{\log(1/\varepsilon)}} = c_3 \qquad a.s.$$

Here and in Propositions 2 and 3, $\gamma = H - \frac{d-\alpha}{4}$.

Proposition 2 *Let $x \in \mathbb{R}^d$ be fixed. Then for any $t_0 > 0$ we have*

$$\overline{\lim_{\varepsilon \to 0}} \frac{\sup_{|t-t_0| \leq \varepsilon} |u(t,x) - u(t_0,x)|}{\varepsilon^\gamma \sqrt{\log \log(1/\varepsilon)}} = c_4 \quad a.s.,$$

where $0 < c_4 < \infty$ is a constant.

By applying Theorems 1, 2 and the Chung-type law of iterated logarithm in [23], we derive immediately

Proposition 3 *Let $x \in \mathbb{R}^d$ be fixed. Then for any $t_0 > 0$ we have*

$$\underline{\lim}_{\varepsilon \to 0} \frac{\sup_{|t-t_0| \leq \varepsilon} |u(t,x) - u(t_0,x)|}{(\varepsilon / \log \log(1/\varepsilon))^\gamma} = c_5 \qquad a.s.,$$

where $0 < c_5 < \infty$ is a constant depending on the small ball probability of the Gaussian process U in Theorem 1.

Further properties on the local times and fractal behavior of the solution process $\{u(t,x), t \geq 0\}$, when $x \in \mathbb{R}^d$ is fixed, can be derived from [38, 39, 40].

2.2 Sharp regularity in space

In this section we fix $t > 0$ and analyze the space regularity of the solution $\{u(t,x), x \in \mathbb{R}^d\}$. We start with the following result.

Theorem 3 *For each $t > 0$, the Gaussian random field $\{u(t,x), x \in \mathbb{R}^d\}$ is stationary with spectral measure*

$$\Delta(d\xi) = \alpha_H (2\pi)^{-d} \int_0^t \int_0^t \frac{dudv}{|u-v|^{2-2H}} e^{-\frac{(u+v)|\xi|^2}{2}} \mu(d\xi).$$

Proof : It follows from the Fourier transform of the Green kernel in (11) and Parseval's identity (9) that the covariance function of $\{u(t,x), x \in \mathbb{R}^d\}$ is

$$\begin{aligned} \mathbb{E}\big(u(t,x)u(t,y)\big) &= \frac{\alpha_H}{(2\pi)^d} \int_0^t \int_0^t \frac{dudv}{|u-v|^{2-2H}} \int_{\mathbb{R}^d} e^{i\langle x-y,\xi\rangle} e^{-\frac{1}{2}(t-u)|\xi|^2} e^{-\frac{1}{2}(t-v)|\xi|^2} \mu(d\xi) \\ &= \frac{\alpha_H}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\langle x-y,\xi\rangle} \left(\int_0^t \int_0^t \frac{dudv}{|u-v|^{2-2H}} e^{-\frac{1}{2}(u+v)|\xi|^2} \right) \mu(d\xi). \end{aligned}$$

The conclusion of Theorem 3 follows. ■

It has been shown in [4, Proposition 4.3] that there exist two strictly positive constants $c_{1,H}, c_{2,H}$ such that

$$\begin{aligned} c_{1,H}(t^{2H} \wedge 1) \left(\frac{1}{1+|\xi|^2} \right)^{2H} &\leq \int_0^t \int_0^t \frac{dudv}{|u-v|^{2-2H}} e^{-\frac{(u+v)|\xi|^2}{2}} \\ &\leq c_{2,H}(t^{2H}+1) \left(\frac{1}{1+|\xi|^2} \right)^{2H}. \end{aligned} \tag{27}$$

This and Condition (19) imply that the spectral measure $\Delta(d\xi)$ is comparable with an absolutely continuous measure with a density function that is comparable to $|\xi|^{-(\alpha+4H)}$ for all $\xi \in \mathbb{R}^d$ with $|\xi| \geq 1$. As shown in [29, 19, 39, 40], this information is very useful for studying regularity and other sample path properties of the Gaussian random field $\{u(t,x), x \in \mathbb{R}^d\}$. In the following we show some consequences.

We start with the following estimate on $\mathbb{E}\big(|u(t,x) - u(t,y)|^2\big)$. To this end, let $\beta = \min\{1, 2H - \frac{d-\alpha}{2}\}$, and let

$$\rho = \begin{cases} 1 & \text{if } \beta = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4 *Assume that (19) and (22) hold. For any $M > 0$ and $t > 0$, there exist positive and finite constants c_6, c_7 such that for any $x, y \in [-M, M]^d$,*

$$c_6 |x-y|^{2\beta} \Big(\log \frac{1}{|x-y|} \Big)^{\rho} \leq \mathbb{E}\big(|u(t,x) - u(t,y)|^2 \big) \leq c_7 |x-y|^{2\beta} \Big(\log \frac{1}{|x-y|} \Big)^{\rho}. \tag{28}$$

Proof : Take $x, y \in [-M, M]^d$ and let $z := x - y \in \mathbb{R}^d$. Using again (11) and Parseval's identity, we can write

$$
\begin{aligned}
&\mathbb{E}\big(\,|u(t,x+z)-u(t,x)|^2\,\big)\\
&= \alpha_H (2\pi)^{-d} \int_0^t \int_0^t \frac{dudv}{|u-v|^{2-2H}} \int_{\mathbb{R}^d} \left|e^{-i\langle \xi, z\rangle} - 1\right|^2 e^{-\frac{u|\xi|^2}{2}}\, e^{-\frac{v|\xi|^2}{2}} \mu(d\xi)\\
&\asymp \int_{\mathbb{R}^d} (1-\cos\langle \xi, z\rangle) \frac{d\xi}{|\xi|^\alpha} \int_0^t \int_0^t \frac{dudv}{|u-v|^{2-2H}}\, e^{-\frac{(u+v)|\xi|^2}{2}}.
\end{aligned}
$$

Let us first prove the lower bound in (28). Using (27), with $c_{1,H,t}$ a generic strictly positive constant depending on t, H (that may change from line to line) and the lower bound in (22), we derive

$$
\begin{aligned}
\mathbb{E}\,|u(t,x+z)-u(t,x)|^2 \;&\geq\; c_{1,H,t} \int_{|\xi|\geq 1} \frac{d\xi}{|\xi|^\alpha} \left(\frac{1}{1+|\xi|^2}\right)^{2H} (1-\cos\langle \xi, z\rangle)\\
&\geq\; c_{1,H,t} \int_{|\xi|\geq 1} \frac{d\xi}{|\xi|^{\alpha+4H}} (1-\cos\langle \xi, z\rangle).
\end{aligned}
$$

By making the change of variables using spherical coordinates, we have

$$
\mathbb{E}\,|u(t,x+z)-u(t,x)|^2 \geq c_{1,H,t}|z|^{-d+\alpha+4H} \int_{\mathbb{S}^{d-1}} \int_{|z|}^\infty r^{d-1-\alpha-4H}(1-\cos(r\langle \theta, \theta_z\rangle)dr\sigma(d\theta), \tag{29}
$$

where $\theta_z = \frac{z}{|z|}$ and $\sigma(d\theta)$ is the uniform measure on the unit sphere $\mathbb{S}^{d-1}$.

Next we distinguish three cases: (i) $2H - \frac{d-\alpha}{2} > 1$, (ii) $2H - \frac{d-\alpha}{2} = 1$ and (iii) $2H - \frac{d-\alpha}{2} < 1$.

In case (i) and (ii), we observe that for $|z|$ small,

$$
\begin{aligned}
\int_{|z|}^\infty r^{d-1-\alpha-4H}(1-\cos(r\langle \theta, \theta_z\rangle)dr \;&\geq\; \frac{1}{2}\int_{|z|}^1 r^{d-1-\alpha-4H}(r\langle \theta, \theta_z\rangle)^2 dr\\
&\geq\; C|z|^{d+2-\alpha-4H}\Big(\log\frac{1}{|z|}\Big)^\rho \langle \theta, \theta_z\rangle^2,
\end{aligned}
$$

where the extra factor $\log\frac{1}{|z|}$ appears in case (ii). Plugging this into (29) gives the desired lower bound.

In case (iii), the integrand $r \mapsto r^{d-1-\alpha-4H}(1-\cos(r\langle \theta, \theta_z\rangle)$ is integrable at both 0 and infinity. The fact that $x, y \in [-M, M]^d$ ensures the integral has a positive lower bound. This gives the lower bound in (28).

Now we verify the upper bound in (28). Similarly to the above, the right-hand side of (27) and (22) imply

$$
\begin{aligned}
&\mathbb{E}\big(\,|u(t,x+z)-u(t,x)|^2\,\big)\\
&\leq c_{2,H,t}|z|^{-d+\alpha+4H} \int_{\mathbb{S}^{d-1}} \int_0^\infty r^{d-1-\alpha}\left(\frac{1}{|z|^2+r^2}\right)^{2H} (1-\cos(r\langle \theta, \theta_z\rangle)dr\sigma(d\theta).
\end{aligned}
$$

Again, by distinguish three cases: (i) $2H-\frac{d-\alpha}{2}>1$, (ii) $2H-\frac{d-\alpha}{2}=1$ and (iii) $2H-\frac{d-\alpha}{2}<1$, we can verify that the upper bound in (28). Since this is elementary, we omit the details. ■

Theorem 4 suggests that the sample function $x\mapsto u(t,x)$ is rough (or fractal) when $2H-\frac{d-\alpha}{2}\le 1$, and is differentiable when $2H-\frac{d-\alpha}{2}>1$. This is indeed the case as shown by the following theorem.

Theorem 5 *Assume that (19) and (22) hold and $t>0$ is fixed. If $2H-\frac{d-\alpha}{2}>1$, then $\{u(t,x), x\in\mathbb{R}^d\}$ has a modification (still denoted by the same notation) such that almost surely the sample function $x\mapsto u(t,x)$ is continuously differentiable on $\mathbb{R}^d$. Moreover, for any $M>0$, there exists a positive positive random variable K with all moments such that for every $j=1,\ldots,d$, the partial derivative $\frac{\partial}{\partial x_j}u(t,x)$ has the following modulus of continuity on $[-M,M]^d$:*

$$\sup_{x,y\in[-M,M]^d,|x-y|\le\varepsilon}\Big|\frac{\partial}{\partial x_j}u(t,x)-\frac{\partial}{\partial y_j}u(t,y)\Big|\le K\varepsilon^{2H-\frac{d-\alpha}{2}-1}\sqrt{\log\frac{1}{\varepsilon}}. \tag{30}$$

Proof : The method of proof is similar to that of Theorem 2 above or [41, Theorem 4.8]. By applying Theorem 3, we can show that the mean square partial derivative $\frac{\partial}{\partial x_j}u(t,x)$ exists and

$$\begin{aligned}\mathbb{E}\bigg(\Big|\frac{\partial}{\partial x_j}u(t,x)-\frac{\partial}{\partial y_j}u(t,y)\Big|^2\bigg) &= \int_{\mathbb{R}^d}\xi_j^2\big|e^{i\langle\xi,x\rangle}-e^{i\langle\xi,y\rangle}\big|^2\Delta(d\xi)\\ &\asymp \int_{\mathbb{R}^d}\xi_j^2\big(1-\cos\langle\xi,x-y\rangle\big)\Big(\frac{1}{1+|\xi|^2}\Big)^{2H}\frac{d\xi}{|\xi|^\alpha}\\ &\le C\,|x-y|^{4H+\alpha-d-2}.\end{aligned}$$

In the above, we have used the fact that $0<4H+\alpha-d-2<2$, so the last inequality can be derived as in the case of fractional Brownian motion. Finally the desired result follows from the general results on modulus of continuity of Gaussian random fields (cf. [20, Chapter 7] or [40, Section 4]). ■

Finally, we consider the non-smooth case. For simplicity, we assume $2H-\frac{d-\alpha}{2}<1$. The case $2H-\frac{d-\alpha}{2}=1$ is more subtle and will require significant extra work.

By combining Theorem 3 and relation (27) with the results in [29, Section 8] (see also [19, 39]), we obtain the following useful lemma.

Lemma 1 *Suppose $2H-\frac{d-\alpha}{2}<1$. Then ,for every fixed $t>0$, the Gaussian field $\{u(t,x), x\in\mathbb{R}^d\}$ is strongly locally nondeterministic. Namely, for every $M>0$, there exists a constant $c_8>0$ (depending on t and M) such that for every $n\ge 1$ and for every $x,y_1,...,y_n\in[-M,M]^d$,*

$$\mathrm{Var}\left(u(t,x)|u(t,y_1),\ldots,u(t,y_n)\right)\ge c_8\min_{0\le j\le n}\{|x-y_j|^{4H+\alpha-d}\},$$

where $y_0=0$.

Because of this, the Gaussian random field $\{u(t,x), x \in \mathbb{R}^d\}$ shares many local properties with fractional Brownian motion $B^\beta = \{B^\beta(x), x \in \mathbb{R}^d\}$ with $\beta = 2H - \frac{d-\alpha}{2}$. We give some examples.

Proposition 4 *(Uniform and local moduli of continuity) Suppose* $2H - \frac{d-\alpha}{2} < 1$. *Let* $t > 0$ *and* $M > 0$ *be fixed. Then*

(a) almost surely,

$$\lim_{\varepsilon\to 0} \frac{\max_{x\in[-M,M]^d, |h|\le\varepsilon} |u(t,x+h) - u(t,x)|}{\varepsilon^\beta \sqrt{\log(1/\varepsilon)}} = c_9.$$

(b) For $x_0 \in \mathbb{R}^d$,

$$\overline{\lim_{\varepsilon\to 0}} \frac{\max_{|h|\le\varepsilon} |u(t,x_0+h) - u(t,x_0)|}{\varepsilon^\beta \sqrt{\log\log(1/\varepsilon)}} = c_{10}.$$

In the above, $\beta = 2H - \frac{d-\alpha}{2}$ *and* $0 < c_9, c_{10} < \infty$ *are constants.*

Proof : The conclusion (a) follows from Lemma 1 and [22, Theorem 4.1], while (b) follows from [22, Theorem 5.1]. ■

Proposition 5 *(Chung's LIL) Suppose* $2H - \frac{d-\alpha}{2} < 1$. *Then for every* $t > 0$ *and* $x_0 \in \mathbb{R}^d$, *we have*

$$\underline{\lim}_{\varepsilon\to 0} \frac{\max_{|h|\le\varepsilon} |u(t,x_0+h) - u(t,x_0)|}{\varepsilon^\beta / (\log\log(1/\varepsilon))^\beta} = c_{10},$$

where $\beta = 2H - \frac{d-\alpha}{2}$ *and* $0 < c_{10} < \infty$ *is a constant.*

Proof : It follows from Lemma 1 and Li and Shao [17]. See also [18, Theorem 3.2] for related results. ■

We conclude with the following remarks.

Remark 3 *(i) Propositions 4 and 5 show that, when* $\beta = 2H - \frac{d-\alpha}{2} < 1$ *and* $t > 0$ *is fixed, the solution process* $\{u(t,x), x \in \mathbb{R}^d\}$ *behaves locally like a fractional Brownian motion* $\{B^\beta(x), x \in \mathbb{R}^d\}$. *This is not surprising. Indeed, similarly to Theorem 2 and [12, Proposition 3.1], one can show that* $\{u(t,x), x \in \mathbb{R}^d\}$ *is a smooth perturbation of* $\{B^\beta(x), x \in \mathbb{R}^d\}$.

(ii) Further properties on the local times and fractal behavior of the solution process $\{u(t,x), x \in \mathbb{R}^d\}$, *when* $t > 0$ *is fixed, can be derived from [38, 39, 40]. It is also possible to investigate sample path properties of the Gaussian random field* $\{u(t,x), t \ge 0, x \in \mathbb{R}^d\}$ *in both time and space variables. The methods developed in [9] will be useful for this purpose. We will study these problems in a subsequent paper.*

Acknowledgment. Yimin Xiao thanks Professor Raluca M. Balan for stimulating discussions.

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