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▶ To cite this version:

Aymen Braghtha. Local monomialization conjecture of a singular foliation of Darboux type. 2015. hal-01108702

HAL Id: hal-01108702 https://hal.science/hal-01108702

Preprint submitted on 23 Jan 2015

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Local monomialization conjecture of a singular foliation of Darboux type

Aymen Braghtha

January 23, 2015

Abstract

After the nice result introduced by Belotto in [1] concerning the local monomialization of a singular foliation given by n first integrals, this work is a continuation in the same spirit. In this paper, we introduce a important conjecture about local monomialization of a singular foliation of Darboux type (see section 1). This conjecture can be used to study pseudo-abelian integrals [2,4].

1 Introduction

Let M be an analytic manifold of dimension n+2. Given a families of first integrals of Darboux type H_{ϵ}

$$H_{\epsilon}(x,y) = H(x,y,\epsilon_1,\dots,\epsilon_n) = \prod_{i=1}^k P^{a_i}(x,y,\epsilon_1,\dots,\epsilon_n), \quad a_i > 0.$$
 (1)

Let F be the foliation of codimension one in M with coordinates $(x, y, \epsilon_1, \dots, \epsilon_n)$ which is given by the analytic one form ω

$$\omega = \frac{H_x}{\phi} dx + \frac{H_y}{\phi} dy + \sum_{i=1}^n \frac{H_{\epsilon_i}}{\phi} d\epsilon_i = 0, \tag{2}$$

where $H_x = \frac{\partial H}{\partial x}$, $H_y = \frac{\partial H}{\partial y}$, $H_{\epsilon_i} = \frac{\partial H}{\partial \epsilon_i}$ and $\phi = \prod_{i=1}^k P_i^{a_i-1}(x, y, \epsilon_1, \dots, \epsilon_n)$ (integrating factor). Let $F_i, i = 1, \dots, n$ are foliations of codimension one in M with coordinates $(x, y, \epsilon_1, \dots, \epsilon_n)$ which are given by the one forms ω_i

$$\omega_i = d\epsilon_i = 0, \quad i = 1, \dots, n. \tag{3}$$

Let $\mathcal{F} = (F, F_1, \dots, F_n)$ be the result foliation of dimension one in M where its leaves are given by the transversal intersection of leaves of F, F_1, \ldots, F_n . Otherwise speaking, the singular foliation \mathcal{F} is given by

$$\Omega = \omega \wedge \omega_1 \wedge \ldots \wedge \omega_n \tag{4}$$

$$=Q_1(x,y,\epsilon_1,\ldots,\epsilon_n)dx\wedge d\epsilon_1\wedge\ldots\wedge d\epsilon_n+Q_2(x,y,\epsilon_1,\ldots,\epsilon_n)dy\wedge d\epsilon_1\wedge\ldots\wedge d\epsilon_n=0,$$
 (5)

where $Q_1 = \frac{H_x}{\phi}$, $Q_2 = \frac{H_y}{\phi}$ are polynomials.

We shall say that Ω is a foliation of Darboux type with first integrals $(H, \epsilon_1, \dots, \epsilon_n)$.

Example. Let $H_{\epsilon}(x,y) = H(x,y,\epsilon) = (x-\epsilon)^{a_1}(x-y)^{a_2}(x+y)^{a_3}$ be a the first integral of Darboux type. The foliation F of codimension one in three dimensional space M with coordinates (x, y, ϵ) is given by the one form

$$\omega = (a_1(x-y)(x+y) + a_2(x-\epsilon)(x+y) + a_3(x-\epsilon)(x-y))dx - (a_2(x-\epsilon)(x+y) - a_3(x-\epsilon)(x-y))dy - a_1(x-y)(x+y)d\epsilon = 0$$

and the foliation F_1 of codimension one in M is given by the one form

$$\omega_1 = d\epsilon = 0.$$

The result foliation $\mathcal{F} = (F, F_1)$ is given by the two-form

$$\Omega = \omega \wedge \omega_1 = (a_1(x-y)(x+y) + a_2(x-\epsilon)(x+y) + a_3(x-\epsilon)(x-y))dx \wedge d\epsilon - (a_2(x-\epsilon)(x+y) - a_3(x-\epsilon)(x-y))dy \wedge d\epsilon = 0$$

Observe that the foliation $\mathcal{F} = (F, F_1)$ has a complicated singularity at the origin $(0, 0, 0) \subset D_0 = \{\epsilon = 0\}$.

Conjecture. There exist sequences of local blowings-up such that the total transform of the foliation $\mathcal{F}:\omega\wedge\omega_1\wedge\ldots\wedge\omega_n=0$ has locally n+1 monomial first integrals $(z^{\gamma_0},z^{\gamma_1}...,z^{\gamma_n})$ where $z^{\gamma_i}=z_1^{\gamma_{i,1}}\cdots z_{n+2}^{\gamma_{i,n+2}}$ and the exponents matrix

$$m(a_1, \dots, a_k) = \begin{pmatrix} \gamma_0^1 & \dots & \gamma_0^{n+2} \\ \gamma_1^1 & \dots & \gamma_1^{n+2} \\ \vdots & \vdots & \vdots \\ \gamma_n^1 & \dots & \gamma_n^{n+2} \end{pmatrix}$$

HAS A MAXIMAL RANK.

2 Blowing-up of the foliation \mathcal{F}

In this section, we introduce the fundamental idea to prove the conjecture which is based in first step on Hironaka's reduction of singularities [3]. Let $D_0 = \{\epsilon_1 = \epsilon_2 = \ldots = \epsilon_n = 0\}$ be a initial exceptional divisor.

Theorem 1. There exist a morphism Φ such that the pull-back foliation $\widetilde{\Phi}^*\mathcal{F} = \widetilde{\mathcal{F}}$ is given locally in neighborhood U_1 of the divisor $\widetilde{\Phi}^*(D_0)$ with coordinates $z = (z_1, \ldots, z_{n+2})$ by the following system

$$\begin{cases}
\widetilde{H} = z^{\gamma_0} \cdot \Delta_0, \\
\widetilde{\epsilon}_1 = z^{\gamma_1} \cdot \Delta_1, \\
\vdots \\
\widetilde{\epsilon}_n = z^{\gamma_n} \cdot \Delta_n,
\end{cases}$$
(6)

where Δ_i , i = 0, ..., n are a units.

Proof. (1) In first step, we monomialize the principal ideal $I_1 = \langle P_1 \rangle$, Hironaka theorem's guarantee the existence oft a sequence of blow-ups $\widetilde{\Phi}_1 = \widetilde{\Phi}_{n_1}^1 \circ \widetilde{\Phi}_{n_1-1}^1 \circ \ldots \circ \widetilde{\Phi}_1^1$ with initial center $C_0 \subset D_0$ (which is possibly a submanifold of M) such that

$$(\widetilde{\Phi}_1^* P_1)^{a_1} = \delta_1 \prod_{i=1}^{n+2} z_i^{a_1 \widetilde{\beta}_i^1}, \quad \delta_1(0) \neq 0.$$

(2) In the second step, we consider the principal ideal $I_2 = \langle \widetilde{\Phi}_1^* P_2 \rangle$ and by Hironaka theorem's there exist a sequence of blow-ups $\widetilde{\Phi}_2 = \Phi_{n_2}^2 \circ \Phi_{n_2-1}^2 \circ \ldots \circ \Phi_1^2$ such that

$$(\widetilde{\Phi}_{2}^{*} \circ \widetilde{\Phi}_{1}^{*} P_{2})^{a_{2}} = \delta_{2} \prod_{i=1}^{n+2} z_{i}^{a_{2} \widetilde{\beta}_{i}^{2}}, \quad \delta_{2}(0) \neq 0.$$

In the k-th step there exist a sequence of blow-ups $\widetilde{\Phi}_k = \Phi^k_{n_k} \circ \Phi^k_{n_k-1} \circ \ldots \circ \Phi^k_1$ such that the principal ideal $I_k = <\widetilde{\Phi}^*_{k-1} \circ \widetilde{\Phi}^*_{k-2} \circ \ldots \circ \widetilde{\Phi}^*_1 P_k > \text{has a normal crossings i.e.}$

$$(\widetilde{\Phi}_{k-1}^* \circ \widetilde{\Phi}_{k-2}^* \circ \dots \circ \widetilde{\Phi}_1^* P_k)^{a_k} = \delta_k \prod_{i=1}^{n+2} z_i^{a_k \widetilde{\beta}_i^k}, \quad \delta_k(0) \neq 0.$$

Finally, after desingularisation of each polynomial P_i of the first integral $H = \prod_{i=1}^k P_i^{a_i}$, the equations $z_1 = 0, \dots, z_{n+2} = 0$ are corresponding the irreducibles components of the exceptional divisor. For this

raison after desingularisation of $\widetilde{\Phi}_{i-1}^* \circ \widetilde{\Phi}_{i-2}^* \circ \dots \circ \widetilde{\Phi}_1^* P_i$, the polynomial $\widetilde{\Phi}_i^* \circ \widetilde{\Phi}_{i-1}^* \circ \widetilde{\Phi}_{i-2}^* \circ \dots \circ \widetilde{\Phi}_1^* P_{i-1}$ has a normal crossings. So locally we have

$$\begin{cases} \widetilde{H} = z^{\gamma_0}.\Delta_0, \\ \widetilde{\epsilon}_1 = z^{\gamma_1}.\Delta_1, \\ \vdots \\ \widetilde{\epsilon}_n = z^{\gamma_n}.\Delta_n, \end{cases}$$

where
$$z = (z_1, \dots, z_{n+2}), \gamma_0 = \sum_{i=1}^k a_i \beta_i, \beta_i = (\beta_i^1, \dots, \beta_i^{n+2}), \gamma_i = (\gamma_i^1, \dots, \gamma_i^{n+2}).$$

To complete the proof its necessairy to eliminate the units $\Delta_0, \Delta_1, \ldots, \Delta_n$ in the system (6). Now we define the resonant locus of the foliation $(z^{\gamma_0}.\Delta_0, z^{\gamma_1}.\Delta_1, \ldots, z^{\gamma_n}.\Delta_n)$

$$\mathcal{R} := \{ a = (a_1, \dots, a_k) : \gamma_0 \land \sum_{j=1}^n \gamma_j = 0 \}.$$

To prove the conjecture, we distinguish two cases

- generic case $a \notin \mathcal{R}$.
- nongeneric case $a \in \mathcal{R}$.

3 Some examples in dimension three

To more understand the problem, we see some examples in dimension three.

Example 1: Let \mathcal{F} be the local foliation which is obtained by after k blow-ups. The foliation \mathcal{F} is given by the following system

$$\begin{cases} H_a = x^{a_1} y^{a_2} (1+z) \\ f = xy \end{cases}$$
 (7)

In this example we have $\gamma_0: a_1\beta_1 + a_2\beta_2$ where $\beta_1 = (1,0,0), \beta_2 = (0,1,0), \gamma_1 = (1,1,0)$ and $\mathcal{R} = \{a = (a_1,a_2): \gamma_0 \wedge \gamma_1 = 0\}$. Our goal is to kill the unit 1+z in the first integral H_a without modifying the second monomial f in the sense to preserve its monomiality structure. For this raison, we distinguish two different cases:

(a) The generic case $a_1 \neq -a_2 \Leftrightarrow a \notin \mathcal{R}$: We take the change of variables $\tilde{x} = x(1+z)^{\frac{1}{a_1+a_2}}$, $\tilde{y} = y(1+z)^{\frac{1}{a_2+a_1}}$ and $\tilde{z} = z$. Then, we obtain the following system

$$\begin{cases}
H_a = \tilde{x}^{a_1} \tilde{y}^{a_2} \\
f = \tilde{x}\tilde{y}
\end{cases}$$
(8)

Question: How to calculate the generator vector field of the monomial foliation (7)?

Let us assume that the foliation \mathcal{F} is generated locally by the vector field $X(\tilde{x}, \tilde{y}, z) = \alpha_1 \tilde{x} \frac{\partial}{\partial \tilde{x}} + \alpha_2 \tilde{y} \frac{\partial}{\partial \tilde{y}} + \alpha_3 z \frac{\partial}{\partial z}$ which satisfies

$$X(H) = X(\tilde{x}^{a_1}\tilde{y}^{a_2}) = 0, \qquad X(f) = X(\tilde{x}\tilde{y}) = 0$$

To determine the vector $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ we use the two following relations

$$<\alpha, \gamma_0>=0$$
 (i.e. $X(H)=X(\tilde{x}^{a_1}\tilde{y}^{a_2})=0), <\alpha, \gamma_1>=0$ (i.e. $X(f)=X(\tilde{x}\tilde{y})=0).$

where <,> is scalar product in \mathbb{C}^3 . Finally, the vector $(\alpha_1,\alpha_2,\alpha_3) \in \{e_3\}$ and then

$$\mathcal{F} = \{z \frac{\partial}{\partial z}\}$$

Now, we express the vector field X in the original coordinates (x, y, z). If we write $X(x, y, z) = Ax \frac{\partial}{\partial x} + By \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}$, to determine A, B we use the fact that

$$X(xy) = 0 \iff A = -B$$

and so $X(x,y,z)=A(x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y})+z\frac{\partial}{\partial z}$ on the other hand we have

$$Ax = X(x) = X(\tilde{x}(1+z)^{\frac{1}{a_1+a_2}}) = z\frac{\partial}{\partial z}(\tilde{x}(1+z)^{\frac{1}{a_1+a_2}}) = \tilde{x}(1+z)^{\frac{1}{a_1+a_2}-1}\frac{z}{a_1+a_2}$$

Finally, we obtain

$$X(x,y,z) = \frac{1}{a_1 + a_2} \frac{z}{1 + z} \left(x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}\right) + z \frac{\partial}{\partial z} \Rightarrow Y(x,y,z) = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} + (a_1 + a_2)(1 + z) \frac{\partial}{\partial z}$$

Remark 1. In dimension three, if we consider the foliation \mathcal{F} which is given locally by

$$\begin{cases} f_1 = x^a y^b z^c \\ f_2 = x^{\tilde{a}} y^{\tilde{b}} z^{\tilde{c}} \end{cases}$$

where $rank \begin{pmatrix} a & b & c \\ \tilde{a} & \tilde{b} & \tilde{c} \end{pmatrix} = 2$. The generator vector field X of the form

$$X(x,y,z) = \hat{a}x\frac{\partial}{\partial x} + \hat{b}y\frac{\partial}{\partial y} + \hat{c}z\frac{\partial}{\partial z},$$

where

$$<(\hat{a},\hat{b},\hat{c}),(a,b,c)>=0, \quad and \quad <(\hat{a},\hat{b},\hat{c}),(\tilde{a},\tilde{b},\tilde{c})>=0.$$

In our example we observe that in the neighborhood of the leaf $\{z=0\}$ the vector field

$$Y \simeq x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} + (a_1 + a_2)z \frac{\partial}{\partial z}$$

is linearizable and consequently Y is transversal to the leaf $\{z=0\}$.

(b) The problem suppose where $a_1 = -a_2$ i.e $a \in \mathcal{R} = \{a = (a_1, a_2) : \gamma_0 \wedge \gamma_1 = 0\}$. In this case near the leaf $\{z = 0\}$, we have

$$Y \simeq x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}.$$

It is clear that the condition of transversality of Y and the leaf $\{z=0\}$ is not satisfieted.

Example 2: let \mathcal{F} be the local foliation which is given (after a sequence of blow-ups) by

$$\begin{cases} H = x^{a_1}y^{a_2}z^{a_3}(1 + g(x, y, z)) \\ f = xyz. \end{cases}$$

The foliation \mathcal{F} is given also

$$\begin{cases} \frac{H}{f^{a_1}} = y^{a_2 - a_1} z^{a_3 - a_1} (1 + g(x, y, z)) \\ f = x y z. \end{cases}$$

If $a = (a_1, a_2, a_3) \notin \mathcal{R} = \{a : (a_1\beta_1 + a_2\beta_2 + a_3\beta_3) \land (1, 1, 1) = 0\}$ (resonant locus), we can take the following variables change $x = \tilde{x}, y = \tilde{y}(1 + g(x, y, z))^{\frac{1}{a_2 - a_3}}$ and $z = \tilde{z}(1 + g(x, y, z))^{\frac{1}{a_3 - a_2}}$ and in this case the local foliation \mathcal{F} is generated by the vector field

$$X(\tilde{x}, \tilde{y}, \tilde{z}) = (a_2 - a_3)\tilde{x}\frac{\partial}{\partial \tilde{x}} + (a_3 - a_1)\tilde{y}\frac{\partial}{\partial \tilde{y}} + (a_1 - a_2)\tilde{z}\frac{\partial}{\partial \tilde{z}}$$

let us express the vector field X in the original coordinates (x, y, z), so we have

$$X(x) = X(\tilde{x}) = (a_2 - a_3)x \frac{\partial}{\partial x}$$

$$X(\tilde{y}(1+g(x,y,z)^{\frac{1}{a_2-a_3}})) = (a_3-a_1)y + \frac{1}{a_2-a_3}y\frac{X(g(x,y,z))}{1+g(x,y,z)}$$

and

$$X(\tilde{z}(1+g(x,y,z)^{\frac{1}{a_3-a_2}})) = (a_1 - a_2)z + \frac{1}{a_3 - a_2}z \frac{X(g(x,y,z))}{1 + g(x,y,z)}.$$

Finally the vector field X of the form

$$X(x,y,z) = (a_2 - a_3)x\frac{\partial}{\partial x} + (a_3 - a_1)y\frac{\partial}{\partial y} + (a_1 - a_2)z\frac{\partial}{\partial z} + \frac{1}{a_2 - a_3}\frac{X(g(x,y,z))}{1 + g(x,y,z)}(y\frac{\partial}{\partial x} - z\frac{\partial}{\partial z}).$$

Proposition 1. For $a \notin \mathcal{R}$, there exist a local diffeomorphism $\phi : z = (z_1, \dots, z_{n+2}) \mapsto \tilde{z} = (\tilde{z}_1, \dots, \tilde{z}_{n+2})$ such that the foliation \mathcal{F} is given locally by $(\tilde{z}^{\gamma_0}, \dots, \tilde{z}^{\gamma_{n+2}})$

Proof. We just make a convenable change of variables.

Open question. To complete the proof of Conjecture we must solve the nongeneric case $a = (a_1, \ldots, a_k) \in \mathcal{R}$ because in this case the rank of exponent matrix

$$m(a_1, \dots, a_k) = \begin{pmatrix} \gamma_0^1 & \dots & \gamma_0^{n+2} \\ \gamma_1^1 & \dots & \gamma_1^{n+2} \\ \vdots & \vdots & \vdots \\ \gamma_n^1 & \dots & \gamma_n^{n+2} \end{pmatrix}$$

is not maximal.

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