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Error estimates for a numerical approximation to the compressible barotropic Navier-Stokes equations

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Abstract

We present here a general method based on the investigation of the relative energy of the system, that provides an unconditional error estimate for the approximate solution of the barotropic Navier Stokes equations obtained by time and space discretization. We use this methodology to derive an error estimate for a specific DG/finite element scheme for which the convergence was proved in [26].

Keywords: Compressible fluids, Navier-Stokes equations, Relative energy, Error estimates, Finite element methods, Finite volume methods

AMS classification 35Q30, 65N12, 65N30, 76N10, 76N15, 76M10, 76M12

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1 - Introduction

The aim of this paper is to derive an error estimate for approximate solutions of the compressible barotropic Navier-Stokes equations obtained by a discretisation scheme. These equations are posed on the time-space domain $Q_T = (0, T) \times \Omega$, where Ω is a bounded polyhedral domain of \mathbb{R}^d , $d = 2, 3$ and $T > 0$, and read:

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0, \quad (1.1a)$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - \mu \Delta \mathbf{u} - (\mu + \lambda) \nabla \operatorname{div} \mathbf{u} + \nabla_x p(\varrho) = \mathbf{0}, \quad (1.1b)$$

supplemented with the initial conditions

$$\varrho(0, x) = \varrho_0(x), \quad \varrho \mathbf{u}(0, x) = \varrho_0 \mathbf{u}_0, \quad (1.2)$$

where ϱ_0 and \mathbf{u}_0 are given functions from Ω to \mathbb{R}_+ and \mathbb{R}^d respectively, and boundary conditions

$$\mathbf{u}|_{(0, T) \times \partial \Omega} = \mathbf{0}. \quad (1.3)$$

In the above equations, the unknown functions are the scalar density field $\varrho(t, x) \geq 0$ and vector velocity field $\mathbf{u} = (u_1, \dots, u_d)(t, x)$, where $t \in (0, T)$ denotes the time and $x \in \Omega$ is the space variable. The viscosity coefficients μ and λ are such that

$$\mu > 0, \quad \lambda + \frac{2}{d} \mu \geq 0. \quad (1.4)$$

The pressure p is given by an equation of state, that is a function of density which satisfies

$$p \in C([0, \infty)) \cap C^1(0, \infty), \quad p(0) = 0, \quad p'(\varrho) > 0. \quad (1.5)$$

In addition to (1.5), in the error analysis, we shall need to prescribe the asymptotic behavior of the pressure at large densities

$$\lim_{\varrho \rightarrow \infty} \frac{p'(\varrho)}{\varrho^{\gamma-1}} = p_\infty > 0 \quad \text{with some } \gamma \geq 1; \quad (1.6)$$

furthermore, if $\gamma < 2$ in (1.6), we need the additional condition (for small densities):

$$\liminf_{\varrho \rightarrow 0} \frac{p'(\varrho)}{\varrho^{\alpha+1}} = p_0 > 0 \quad \text{with some } \alpha \leq 0. \quad (1.7)$$

In particular, for $d = 3$, our error estimate is valid in the case of the equation of state $p(\varrho) = \varrho^\gamma$ (isotropic gas) with $\gamma \geq \frac{3}{2}$ (see Remark 3.1 for the simpler case $d = 2$).

The main underlying idea of this paper is to derive the error estimates for approximate solutions of problem (1.1)–(1.3) obtained by time and space discretization by using the discrete version of the *relative energy method* introduced on the continuous level in [12, 13, 15]. In spite of the fact that the relative energy method looks at the first glance pretty much similar to the widely used *relative entropy method* (and both approaches translate the same thermodynamic stability conditions), they are very different in appearance and formulation and may provide different results. The notions of relative entropy and relative entropy inequality were first introduced by Dafermos [7] in the context of systems of conservation laws and in particular for the compressible Euler equations. The relative energy functional was suggested and successfully used for the investigation of the stability of weak solutions to the equations of viscous compressible and heat conducting fluids in [13]. In contrast with the relative

entropy of Dafermos, for the viscous and heat conducting fluids, the relative energy approach is able to provide the structural stability of weak solutions, while the relative entropy approach fails in this case.

Both functionals coincide (modulo a change of variables) in the case of (viscous) compressible flows in the barotropic regime. The relative energy functional and the intrinsic version of the relative energy inequality have been recently employed to obtain several stability results for the weak solutions to these equations, including the weak strong uniqueness principle, see [12, 15]. Note that particular versions of the relative entropy inequality with particular specific test functions had been previously derived in the context of low Mach number limits, see e.g. [30, 32].

The discrete version of the Dafermos relative entropy was employed in the non viscous case to derive an error estimate for the numerical approximation to a hyperbolic system of conservation laws and, in particular, to the compressible Euler equations [4]. In this latter paper, the authors assume an L^∞ bound for the discrete solution, which is uniform with respect to the size of the space and time discretization (usually called stability hypothesis), that is not provided by the discrete equations. The same method with the same severe hypotheses have been used in [33] to treat the compressible Navier-Stokes equations. The error analysis in the present paper relies on the theoretical background introduced in [12] and yields an *unconditional result*; in particular, *we do not need any assumed bound* on the solution to get the error estimate.

The mathematical analysis of numerical schemes for the discretization of the steady and/or non steady compressible Navier-Stokes and/or compressible Stokes equations has been the object of some recent works. The convergence of the discrete solutions to the weak solutions of the compressible stationary Stokes was shown for a finite volume– non conforming P1 finite element [10, 11, 18] and for the wellknown MAC scheme which was introduced in [21] and is widely used in computational fluid dynamics. The unsteady Stokes problem was also discretized by some other discretization schemes on a reformulation of the problem, which were proven to be convergent [23–25]. The unsteady barotropic Navier-Stokes equations was recently investigated in [26] in the case $\gamma > 3$ (there is a real difficulty in the realistic case $\gamma \leq 3$ arising from the treatment of the non linear convective term). However, in these works, the rate of convergence is not provided; in fact, to the best of our knowledge, no error analysis has yet been performed for any of the numerical schemes that have been designed for the compressible Navier-Stokes equations, in spite of its great importance for the numerical analysis of the equations and for the mathematical simulations of compressible fluid flows. We present here a general technique to obtain an error analysis and apply it to one of the available numerical schemes. To the best of our knowledge, this is the first result of this type in the mathematical literature on the subject.

To achieve the goal, we systematically use the relative energy method on the discrete level. From this point of view, this paper is as valuable for the introduced methodology as for the result itself. Here, we apply the method to the scheme of [26]. In spite of the fact that this latter scheme is not used in practice (see e.g. [27] for a related schemes used in industrial codes), we begin the error analysis with the scheme [26] because of its readily available convergence proof. In fact, we aim to use this approach to investigate the numerical errors of less academic numerical schemes, such as the finite volume – non conforming P1 finite element [17, 19, 20, 27] or the MAC scheme [1, 22].

The paper is organized as follows. After recalling the fundamental setting of the problem and the relative energy inequality in the continuous case in Section 2, we proceed in Section 3 to the discretization: we introduce the discrete functional spaces and the definition of the numerical scheme, and state the main result of the paper, that is the error estimate formulated in Theorem 3.1. The remaining sections are devoted to the proof of Theorem 3.1:

- In Section 4 we recall the existence theorem for the numerical scheme (Lemma 3.1) and derive

estimates provided by the scheme.

- In Section 5, we derive the discrete intrinsic version of the relative energy inequality for the solutions of the numerical scheme (see Theorem 5.1).
- The relative energy inequality is transformed to a more convenient form in Section 6, see Lemma 6.1.
- Finally, in Section 7, we investigate the form of the discrete relative energy inequality with the test function being a strong solution to the original problem. This investigation is formulated in Lemma 7.1 and finally leads to a Gronwall type estimate formulated in Lemma 8.1. The latter yields the error estimates and finishes the proof of the main result.

Fundamental properties of the discrete functional spaces needed throughout the paper are reported in Appendix (Section 9). Some of them (especially those referring to the L^p setting, $p \neq 2$ that are not currently available in the mathematical literature) are proved. Section 9 is therefore of the independent interest.

2 - The continuous problem

The aim of this section is to recall some fundamental notions and results. We begin by the definition of weak solutions to problem (1.1)–(1.3).

Definition 2.1 (Weak solutions). *Let $\varrho_0 : \Omega \rightarrow [0, +\infty)$ and $\mathbf{u}_0 : \Omega \rightarrow \mathbb{R}^d$ with finite energy $E_0 = \int_{\Omega} (\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + H(\varrho_0)) dx$ and finite mass $0 < M_0 = \int_{\Omega} \varrho_0 dx$. We shall say that the pair (ϱ, \mathbf{u}) is a weak solution to the problem (1.1)–(1.3) emanating from the initial data $(\varrho_0, \mathbf{u}_0)$ if:*

(a) $\varrho \in L^\infty(0, T; L^1(\Omega))$, $\varrho \geq 0$ a.e. in $(0, T)$, and $\mathbf{u} \in L^2(0, T; W_0^{1,2}(\Omega))$.

(b) $\varrho \in C_{\text{weak}}([0, T]; L^1(\Omega))$, and the continuity equation (1.1a) is satisfied in the following weak sense

$$\int_{\Omega} \varrho \varphi dx \Big|_0^\tau = \int_0^\tau \int_{\Omega} \left(\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi \right) dx dt, \quad \forall \tau \in [0, T], \quad \forall \varphi \in C_c^\infty([0, T] \times \bar{\Omega}). \quad (2.1)$$

(c) $\varrho \mathbf{u} \in C_{\text{weak}}([0, T]; L^1(\Omega))$, and the momentum equation (1.1b) is satisfied in the weak sense,

$$\begin{aligned} \int_{\Omega} \varrho \mathbf{u} \cdot \varphi dx \Big|_0^\tau &= \int_0^\tau \int_{\Omega} \left(\varrho \mathbf{u} \cdot \partial_t \varphi + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla \varphi + p(\varrho) \operatorname{div} \varphi \right) dx dt \\ &\quad - \int_0^\tau \int_{\Omega} \left(\mu \nabla \mathbf{u} : \nabla_x \varphi + (\mu + \lambda) \operatorname{div} \mathbf{u} \operatorname{div} \varphi \right) dx dt, \quad \forall \tau \in [0, T], \quad \forall \varphi \in C_c^\infty([0, T] \times \Omega; \mathbb{R}^3). \end{aligned} \quad (2.2)$$

(d) The following energy inequality is satisfied

$$\int_{\Omega} \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + H(\varrho) \right) dx \Big|_0^\tau + \int_0^\tau \int_{\Omega} \left(\mu |\nabla \mathbf{u}|^2 + (\mu + \lambda) |\operatorname{div} \mathbf{u}|^2 \right) dx dt \leq 0, \quad \text{for a.a. } \tau \in (0, T), \quad (2.3)$$

$$\text{with } \varrho H'(\varrho) - H(\varrho) = p(\varrho), \quad (\text{for example } H(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2} dz). \quad (2.4)$$

Here and hereafter the symbol $\int_{\Omega} g dx \Big|_0^\tau$ is meant for $\int_{\Omega} g(\tau, x) dx - \int_{\Omega} g_0(x) dx$.

In the above definition, we tacitly assume that all the integrals in the formulas (2.1)–(2.3) are defined and we recall that $C_{\text{weak}}([0, T]; L^1(\Omega))$ is the space of functions of $L^\infty([0, T]; L^1(\Omega))$ which are continuous for the weak topology.

Note that the existence of weak solutions emanating from the finite energy initial data is well-known on bounded Lipschitz domains under assumptions (1.5) and (1.6) provided $\gamma > d/(d-1)$, see Lions [28] for "large" values of γ , Feireisl and coauthors [14] for $\gamma > d/(d-1)$.

Let us now introduce the notion of relative energy. We first introduce the function

$$\begin{aligned} E : [0, \infty) \times (0, \infty) &\rightarrow \mathbb{R}, \\ (\varrho, r) &\mapsto E(\varrho|r) = H(\varrho) - H'(r)(\varrho - r) - H(r), \end{aligned} \quad (2.5)$$

where H is defined by (2.4). Due to the monotonicity hypothesis in (1.5), H is strictly convex on $[0, \infty)$, and therefore

$$E(\varrho|r) \geq 0 \quad \text{and} \quad E(\varrho|r) = 0 \Leftrightarrow \varrho = r.$$

In order to measure a "distance" between a weak solution (ϱ, \mathbf{u}) of the compressible Navier-Stokes system and any other state (r, \mathbf{U}) of the fluid, we introduce the relative energy functional, defined by

$$\mathcal{E}(\varrho, \mathbf{u}|r, \mathbf{U}) = \int_{\Omega} \left(\frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + E(\varrho|r) \right) dx. \quad (2.6)$$

It was proved recently in [12] that, provided assumption (1.5) holds, any weak solution satisfies the following so-called relative energy inequality

$$\begin{aligned} &\mathcal{E}(\varrho, \mathbf{u}|r, \mathbf{U})(\tau) - \mathcal{E}(\varrho, \mathbf{u}|r, \mathbf{U})(0) + \int_0^\tau \int_{\Omega} \left(\mu |\nabla(\mathbf{u} - \mathbf{U})|^2 + (\mu + \lambda) |\operatorname{div}(\mathbf{u} - \mathbf{U})|^2 \right) dx dt \\ &\leq \int_0^\tau \int_{\Omega} \left(\mu \nabla \mathbf{U} : \nabla(\mathbf{U} - \mathbf{u}) + (\mu + \lambda) \operatorname{div} \mathbf{U} \operatorname{div}(\mathbf{U} - \mathbf{u}) \right) dx dt \\ &\quad + \int_0^\tau \int_{\Omega} \varrho \partial_t \mathbf{U} \cdot (\mathbf{U} - \mathbf{u}) dx dt + \int_0^\tau \int_{\Omega} \varrho \mathbf{u} \cdot \nabla \mathbf{U} \cdot (\mathbf{U} - \mathbf{u}) dx dt \\ &\quad - \int_0^\tau \int_{\Omega} p(\varrho) \operatorname{div} \mathbf{U} dx dt + \int_0^\tau \int_{\Omega} (r - \varrho) \partial_t H'(r) dx dt - \int_0^\tau \int_{\Omega} \varrho \nabla H'(r) \cdot \mathbf{u} dx dt \end{aligned} \quad (2.7)$$

for a.a. $\tau \in (0, T)$, and for any pair of test functions

$$r \in C^1([0, T] \times \bar{\Omega}), \quad r > 0, \quad \mathbf{U} \in C^1([0, T] \times \bar{\Omega}; \mathbb{R}^3), \quad \mathbf{U}|_{\partial\Omega} = 0.$$

The stability of strong solutions in the class of weak solutions is stated in the following proposition.

Proposition 2.1 (Estimate on the relative energy). *Let Ω be a Lipschitz domain. Assume that the viscosity coefficients satisfy assumptions (1.4), that the pressure p is a twice continuously differentiable function on $(0, \infty)$ satisfying (1.5) and (1.6), and that (ϱ, \mathbf{u}) is a weak solution to problem (1.1)–(1.3) emanating from initial data $(\varrho_0 \geq 0, \mathbf{u}_0)$, with finite energy E_0 and finite mass $M_0 > 0$. Let (r, \mathbf{U}) in the class*

$$\begin{cases} r \in C^1([0, T] \times \bar{\Omega}), \quad 0 < r = \min_{(t,x) \in \bar{Q}_T} r(t,x) \leq r(t,x) \leq \bar{r} = \max_{(t,x) \in \bar{Q}_T} r(t,x), \\ \mathbf{U} \in C^1([0, T] \times \bar{\Omega}; \mathbb{R}^3), \quad \mathbf{U}|_{(0,T) \times \partial\Omega} = 0 \end{cases} \quad (2.8)$$

be a (strong) solution of problem (1.1) emanating from the initial data (r_0, \mathbf{U}_0) . Then there exists

$$c = c(T, \Omega, M_0, E_0, \underline{r}, \bar{r}, \|p'\|_{C^1([\underline{r}, \bar{r}])}, \|(\nabla r, \partial_t r, \mathbf{U}, \nabla \mathbf{U}, \partial_t \mathbf{U})\|_{L^\infty(Q_T; \mathbb{R}^{19})}) > 0$$

such that for almost all $t \in (0, T)$,

$$\mathcal{E}(\varrho, \mathbf{u}|r, \mathbf{U})(t) \leq c \mathcal{E}(\varrho_0, \mathbf{u}_0|r_0, \mathbf{U}_0). \quad (2.9)$$

This estimate (implying among others the weak-strong uniqueness) was proved in [12] (see also [15]) for pressure laws (1.6) with $\gamma > d/(d-1)$. It remains valid under weaker hypothesis on the pressure, such as (1.6) with $\gamma \geq 1$; this can be proved using ideas introduced in [2] and [29].

3 - The numerical scheme

3.1 Partition of the domain

We suppose that Ω is a bounded domain of \mathbb{R}^d , polygonal if $d = 2$ and polyhedral if $d = 3$. Let \mathcal{T} be a decomposition of the domain Ω in tetrahedra, which we call hereafter a triangulation of Ω , regardless of the space dimension. By $\mathcal{E}(K)$, we denote the set of the edges ($d = 2$) or faces ($d = 3$) σ of the element $K \in \mathcal{T}$ called hereafter faces, regardless of the dimension. The set of all faces of the mesh is denoted by \mathcal{E} ; the set of faces included in the boundary $\partial\Omega$ of Ω is denoted by \mathcal{E}_{ext} and the set of internal faces (i.e. $\mathcal{E} \setminus \mathcal{E}_{\text{ext}}$) is denoted by \mathcal{E}_{int} . The triangulation \mathcal{T} is assumed to be regular in the usual sense of the finite element literature (see e.g. [5]), and in particular, \mathcal{T} satisfies the following properties:

- $\bar{\Omega} = \cup_{K \in \mathcal{T}} \bar{K}$;
- if $(K, L) \in \mathcal{T}^2$, then $\bar{K} \cap \bar{L} = \emptyset$ or $\bar{K} \cap \bar{L}$ is a vertex or $\bar{K} \cap \bar{L}$ is a common face of K and L ; in the latter case it is denoted by $K|L$.

For each internal face of the mesh $\sigma = K|L$, $\mathbf{n}_{\sigma,K}$ stands for the normal vector of σ , oriented from K to L (so that $\mathbf{n}_{\sigma,K} = -\mathbf{n}_{\sigma,L}$). We denote by $|K|$ and $|\sigma|$ the (d and $d - 1$ dimensional) Lebesgue measure of the tetrahedron K and of the face σ respectively, and by h_K and h_σ the diameter of K and σ respectively. We measure the regularity of the mesh thanks to the parameter θ defined by

$$\theta = \inf \left(\left\{ \frac{\xi_K}{h_K}, K \in \mathcal{T} \right\} \cup \left\{ \frac{h_K}{h_L}, \frac{h_L}{h_K}, K|L \in \mathcal{E}_{\text{int}} \right\} \right) \quad (3.1)$$

where ξ_K stands for the diameter of the largest ball included in K . Last but not least we denote by h the maximal size of the mesh,

$$h = \max(\max_{K \in \mathcal{T}} h_K, \max_{\sigma \in \mathcal{E}} h_\sigma). \quad (3.2)$$

The triangulation \mathcal{T} is said to be regular if it satisfies

$$\theta \geq \theta_0 > 0.$$

For a regular triangulation, there exists $c_i = c_i(\theta_0, K) > 0$ such that

$$c_1 h_K \leq h_\sigma \leq c_2 h_K, \quad c_1 |K| \leq |\sigma| h_\sigma \leq c_2 |\sigma| h_K \leq c_3 |K| \quad (3.3)$$

for any $\sigma \in \mathcal{E}(K)$.

3.2 Discrete functional spaces

Let \mathcal{T} be a mesh of Ω . We denote by $L_h(\Omega)$ the space of piecewise constant functions on the cells of the mesh; the space $L_h(\Omega)$ is the approximation space for the pressure and density. For $1 \leq p < \infty$, the mapping

$$q \mapsto \|q\|_{L_h^p(\Omega)} = \|q\|_{L^p(\Omega)} = \left(\sum_{K \in \mathcal{T}} |K| |q_K|^p \right)^{1/p}$$

is a norm on $L_h(\Omega)$. We also introduce spaces of non-negative and positive functions:

$$L_h^+(\Omega) = \{q \in L_h(\Omega), q_K \geq 0, \forall K \in \mathcal{T}\}, \quad L_h^{++}(\Omega) = \{q \in L_h(\Omega), q_K > 0, \forall K \in \mathcal{T}\}.$$

The approximation space for the velocity field is the space $\mathbf{W}_h(\Omega) = (V_h(\Omega))^d$, where $V_h(\Omega)$ is the non conforming piecewise linear finite element space [6, 8] defined by:

$$V_h(\Omega) = \{v \in L^2(\Omega), \forall K \in \mathcal{T}, v|_K \in \mathbb{P}_1(K), \\ \forall \sigma \in \mathcal{E}_{\text{int}}, \sigma = K|L, \int_\sigma v|_K \, dS = \int_\sigma v|_L \, dS, \quad \forall \sigma \in \mathcal{E}_{\text{ext}}, \int_\sigma v \, dS = 0\}, \quad (3.4)$$

where $\mathbb{P}_1(K)$ denotes the space of affine functions on K and dS the integration with respect to the $(d-1)$ -dimensional Lebesgue measure on the face σ . Each element $v \in V_h(\Omega)$ can be written in the form

$$v = \sum_{\sigma \in \mathcal{E}_{\text{int}}} v_\sigma \varphi_\sigma, \quad v_\sigma \in \mathbb{R}, \quad (3.5)$$

where the set $\{\varphi_\sigma\}_{\sigma \in \mathcal{E}_{\text{int}}} \subset V_h(\Omega)$ is the classical basis determined by

$$\forall (\sigma, \sigma') \in \mathcal{E}_{\text{int}}^2, \int_{\sigma'} \varphi_\sigma dS = \delta_{\sigma, \sigma'}, \quad \forall \sigma' \in \mathcal{E}_{\text{ext}}, \int_{\sigma'} \varphi_\sigma dS = 0. \quad (3.6)$$

Notice that $V_h(\Omega)$ approximates the functions with zero traces in the sense that for all elements in $V_h(\Omega)$, $v_\sigma = 0$ provided $\sigma \in \mathcal{E}_{\text{ext}}$. Since only the continuity of the integral over each face of the mesh is imposed, the functions in $V_h(\Omega)$ may be discontinuous through each face; the discretization is thus nonconforming in $W^{1,p}(\Omega; \mathbb{R}^d)$, $1 \leq p \leq \infty$. Finally, we notice that for any $1 \leq p < \infty$ the expression

$$|v|_{V_h^p(\Omega)} = \left(\sum_{K \in \mathcal{T}} \|\nabla v\|_{L^p(K; \mathbb{R}^d)}^p \right)^{1/p}$$

is a norm on $V_h(\Omega)$ and we denote by $V_h^p(\Omega)$ the space $V_h(\Omega)$ endowed with this norm.

We finish this section by introducing some notations. For a function v in $L^1(\Omega)$, we set

$$v_K = \frac{1}{|K|} \int_K v dx \text{ for } K \in \mathcal{T} \text{ and } \hat{v} = \sum_{K \in \mathcal{T}} v_K 1_K, \quad (3.7)$$

so that $\hat{v} \in L_h(\Omega)$. If $v \in W_0^{1,p}(\Omega)$, we set

$$v_\sigma = \frac{1}{|\sigma|} \int_\sigma v dS \text{ for } \sigma \in \mathcal{E}_{\text{int}} \text{ and } v_h = \sum_{\sigma \in \mathcal{E}_{\text{int}}} v_\sigma \varphi_\sigma, \quad (3.8)$$

so that $v_h \in V_h(\Omega)$. Finally, if $v \in W^{1,p}(\Omega)$, we set

$$v_\sigma = \frac{1}{|\sigma|} \int_\sigma v dS \text{ for } \sigma \in \mathcal{E}. \quad (3.9)$$

3.3 Discrete equations

Let us consider a partition $0 = t_0 < t_1 < \dots < t_N = T$ of the time interval $[0, T]$, which, for the sake of simplicity, we suppose uniform. Let k be the constant time step $k = t_n - t_{n-1}$ for $n = 1, \dots, N$. The density field $\varrho(t_n, x)$ and the velocity field $\mathbf{u}(t_n, x)$ will be approximated by the quantities

$$\varrho^n(x) = \sum_{K \in \mathcal{T}} \varrho_K^n 1_K(x), \quad \mathbf{u}^n(x) = \sum_{\sigma \in \mathcal{E}} \mathbf{u}_\sigma^n \varphi_\sigma(x), \quad (3.10)$$

where the approximate densities $(\varrho_K^n)_{K \in \mathcal{T}, n=1, \dots, N}$ and velocities $(\mathbf{u}_\sigma^n)_{\sigma \in \mathcal{E}_{\text{int}}, n=1, \dots, N}$ are the discrete unknowns (with $\varrho_K^n \in \mathbb{R}^+$ and $\mathbf{u}_\sigma^n \in \mathbb{R}^d$). The numerical scheme consists in writing the equations that are solved to determine these discrete unknowns. In order to ensure the positivity of the approximate densities, we shall use an upwinding technique for the density in the mass equation. For $q \in L_h(\Omega)$ and $\mathbf{u} \in \mathbf{W}_h(\Omega)$, the upwinding of q with respect to \mathbf{u} is defined, for $\sigma = K|L \in \mathcal{E}_{\text{int}}$ by

$$q_\sigma^{\text{up}} = \begin{cases} q_K & \text{if } \mathbf{u}_\sigma \cdot \mathbf{n}_{\sigma, K} > 0 \\ q_L & \text{if } \mathbf{u}_\sigma \cdot \mathbf{n}_{\sigma, K} \leq 0, \end{cases} \quad (3.11)$$

so that

$$\sum_{\sigma \in \mathcal{E}(K)} q_\sigma^{\text{up}} \mathbf{u}_\sigma \cdot \mathbf{n}_{\sigma, K} = \sum_{\substack{\sigma \in \mathcal{E}(K) \\ \sigma = K|L}} \left(q_K [\mathbf{u}_\sigma \cdot \mathbf{n}_{\sigma, K}]^+ - q_L [\mathbf{u}_\sigma \cdot \mathbf{n}_{\sigma, K}]^- \right),$$

where $a^+ = \max(a, 0)$, $a^- = -\min(a, 0)$.

Let us then consider the following numerical scheme [26]:

Given $(\varrho^0, \mathbf{u}^0) \in L_h^+(\Omega) \times \mathbf{W}_h(\Omega)$ find $(\varrho^n)_{1 \leq n \leq N} \subset (L_h(\Omega))^N$, $(\mathbf{u}^n)_{1 \leq n \leq N} \subset (\mathbf{W}_h(\Omega))^N$ such that for all $n = 1, \dots, N$

$$|K| \frac{\varrho_K^n - \varrho_K^{n-1}}{k} + \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \varrho_\sigma^{n, \text{up}} [\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma, K}] = 0, \quad \forall K \in \mathcal{T}, \quad (3.12a)$$

$$\begin{aligned} \sum_{K \in \mathcal{T}} \frac{|K|}{k} \left(\varrho_K^n \mathbf{u}_K^n - \varrho_K^{n-1} \mathbf{u}_K^{n-1} \right) \cdot \mathbf{v}_K + \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \varrho_\sigma^{n, \text{up}} \hat{\mathbf{u}}_\sigma^{n, \text{up}} [\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma, K}] \cdot \mathbf{v}_K \\ - \sum_{K \in \mathcal{T}} p(\varrho_K^n) \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \mathbf{v}_\sigma \cdot \mathbf{n}_{\sigma, K} + \mu \sum_{K \in \mathcal{T}} \int_K \nabla \mathbf{u}^n : \nabla \mathbf{v} \, dx \\ + (\mu + \lambda) \sum_{K \in \mathcal{T}} \int_K \operatorname{div} \mathbf{u}^n \operatorname{div} \mathbf{v} \, dx = 0, \quad \forall \mathbf{v} \in \mathbf{W}_h(\Omega). \end{aligned} \quad (3.12b)$$

Note that the boundary condition $\mathbf{u}_\sigma^n = 0$ if $\sigma \in \mathcal{E}_{\text{ext}}$ is ensured by the definition of the space $V_h(\Omega)$. Note also that if $\sigma \in \mathcal{E}_{\text{int}}$, $\sigma = K|L$, one has, following (3.7) and (3.11),

$$\hat{\mathbf{u}}_\sigma^{n, \text{up}} = \mathbf{u}_K^n = \frac{1}{|K|} \int_K \mathbf{u}^n(x) \, dx \text{ if } \mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma, K} > 0 \text{ and } \hat{\mathbf{u}}_\sigma^{n, \text{up}} = \mathbf{u}_L^n = \frac{1}{|L|} \int_L \mathbf{u}^n(x) \, dx \text{ if } \mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma, K} < 0.$$

It is well known that any solution $(\varrho^n)_{1 \leq n \leq N} \subset (L_h(\Omega))^N$ satisfies $\varrho^n > 0$ thanks to the upwind choice in (3.12a). Furthermore, summing (3.12a) over $K \in \mathcal{T}$ immediately yields the total conservation of mass, which reads:

$$\forall n = 1, \dots, N, \quad \int_\Omega \varrho^n \, dx = \int_\Omega \varrho^0 \, dx. \quad (3.13)$$

We finally state in this section the existence result, which can be proved by a topological degree argument [17, 26].

Proposition 3.1 (Existence). *Let $(\varrho^0, \mathbf{u}^0) \in L_h^{++}(\Omega) \times \mathbf{W}_h(\Omega)$. Under assumptions (1.4) and (1.5), Problem (3.12) admits at least one solution*

$$(\varrho^n)_{1 \leq n \leq N} \in [L_h^{++}(\Omega)]^N, (\mathbf{u}^n)_{1 \leq n \leq N} \in [\mathbf{W}_h(\Omega)]^N.$$

3.4 Main result: error estimate

Let $(r, \mathbf{U}) : [0, T] \times \bar{\Omega} \mapsto (0, \infty) \times \mathbb{R}^3$ be C^2 functions such that $\mathbf{U} = \mathbf{0}$ on $\partial\Omega$. Let (ϱ, \mathbf{u}) be a solution of the discrete problem (3.12). Inspired by (2.6), we introduce the discrete relative energy functional

$$\begin{aligned} \mathcal{E}(\varrho^n, \mathbf{u}^n | r^n, \mathbf{U}^n) &= \int_\Omega \left(\frac{1}{2} \varrho^n |\hat{\mathbf{u}}^n - \hat{\mathbf{U}}_h^n|^2 + E(\varrho^n | r^n) \right) dx \\ &= \sum_{K \in \mathcal{T}} |K| \left(\frac{1}{2} \varrho_K^n |\mathbf{u}_K^n - \mathbf{U}_{h, K}^n|^2 + E(\varrho_K^n | r_K^n) \right), \end{aligned} \quad (3.14)$$

where

$$r^n(x) = r(t_n, x), \quad \mathbf{U}^n(x) = \mathbf{U}(t_n, x), \quad n = 0, \dots, N, \quad (3.15)$$

$(\varrho^n, \mathbf{u}^n)$ is defined in (3.10), and E is defined by (2.5). Let us finally introduce the notations

$$M_0 = \sum_{K \in \mathcal{K}} |K| \varrho_K^0, \quad \text{and} \quad E_0 = \sum_{K \in \mathcal{K}} |K| \left(\frac{1}{2} \varrho_K^0 |\mathbf{u}_K^0|^2 + H(\varrho_K^0) \right).$$

Now, we are ready to state the main result of this paper. For the sake of clarity, we shall state the theorem and perform the proofs only in the most interesting three dimensional case. The modifications to be done for the two dimensional case, which is in fact more simple, are mostly due to the different Sobolev embeddings, and are left to the interested reader.

Theorem 3.1 (Error estimate). *Let $\theta_0 > 0$ and \mathcal{T} be a regular triangulation of a bounded polyhedral domain $\Omega \subset \mathbb{R}^3$ introduced in Section 3.1 such that $\theta \geq \theta_0$, where θ is defined in (3.1). Let p be a twice continuously differentiable function satisfying assumptions (1.5), (1.6) with $\gamma \geq 3/2$, and the additional assumption (1.7) in the case $\gamma < 2$. Let the viscosity coefficients satisfy assumptions (1.4). Suppose that $(\varrho^0, \mathbf{u}^0) \in L_h^+(\Omega) \times \mathbf{W}_h(\Omega)$ and that $(\varrho^n)_{1 \leq n \leq N} \subset [L_h^+(\Omega)]^N$, $(\mathbf{u}^n)_{1 \leq n \leq N} \subset [\mathbf{W}_h(\Omega)]^N$ is a solution of the discrete problem (3.12). Let (r, \mathbf{U}) in the class*

$$r \in C^2([0, T] \times \bar{\Omega}), \quad 0 < \underline{r} := \min_{(t,x) \in \bar{Q}_T} r(t, x) \leq r(t, x) \leq \bar{r} := \max_{(t,x) \in \bar{Q}_T} r(t, x), \quad (3.16a)$$

$$\mathbf{U} \in C^2([0, T] \times \bar{\Omega}; \mathbb{R}^3), \quad \mathbf{U}|_{\partial\Omega} = 0 \quad (3.16b)$$

be a (strong) solution of problem (1.1). Then there exists

$$c = c(T, |\Omega|, \text{diam}(\Omega), \theta_0, \gamma, M_0, E_0, \underline{r}, \bar{r}),$$

$$|p'|_{C^1([\underline{r}, \bar{r}])}, \|(\nabla r, \partial_t r, \partial_t \nabla r, \partial_t^2 r, \mathbf{U}, \nabla \mathbf{U}, \nabla^2 \mathbf{U}, \partial_t \mathbf{U}, \partial_t \nabla \mathbf{U})\|_{L^\infty(Q_T; \mathbb{R}^{65})} \in (0, +\infty)$$

(independent of h, k) such that for any $m = 1, \dots, N$,

$$\mathcal{E}(\varrho^n, \mathbf{u}^n | r^n, \mathbf{U}^n) \leq c \left(\mathcal{E}(\varrho^0, \mathbf{u}^0 | r^0, \mathbf{U}^0) + h^A + \sqrt{k} \right), \quad (3.17)$$

where

$$A = \begin{cases} \frac{2\gamma-3}{\gamma} & \text{if } \gamma \in (3/2, 2], \\ 1/2 & \text{if } \gamma > 2. \end{cases} \quad (3.18)$$

Remark 3.1.

1. Theorem 3.1 holds also for two dimensional bounded polyhedral domains under the assumption that $\gamma \geq 1$. Assumption(1.7) on the asymptotic behavior of pressure near 0 is no more necessary in this case. The value of A in the error estimate (3.17) is

$$A = \begin{cases} \frac{2\gamma-2}{\gamma} & \text{if } \gamma \in (1, 2], \\ 1 & \text{if } \gamma > 2. \end{cases}$$

2. Theorem 3.1 can be viewed as a discrete version of Proposition 2.1. It is to be noticed that the assumptions on the constitutive law for pressure guaranteeing the error estimates for the scheme (3.12) are somewhat stronger ($\gamma \geq 3/2$) than the assumptions needed for the stability in the continuous case ($\gamma \geq 1$). The threshold value $\gamma = 3/2$ is however in accordance with the existence theory of weak solutions. The assumptions on the regularity of the strong solution to be compared with the discrete solution in the scheme are slightly stronger than those needed to establish the stability estimates in the continuous case.
3. If $d = 3$, we notice that the assumptions on the pressure (as function of the density) in Theorem 3.1 are compatible with the isentropic case $p(\varrho) = \varrho^\gamma$ for all values $\gamma \geq 3/2$.
4. The scheme [26] contains in addition artificial stabilizing terms both in the continuity and momentum equations. These terms are necessary for the convergence proof in [26] even for the large values of γ . It is to be noticed that the error estimate in Theorem 3.1 is formulated for the numerical scheme without these stabilizing terms. Of course similar error estimate is a fortiori valid also for the scheme with the stabilizing terms, however, this issue is not discussed in the present paper.

The rest of the paper is devoted to the proof of Theorem 3.1. For the sake of simplicity, and in order to simplify notation, we present the proof for the uniformly regular mesh meaning that the relation (3.3) holds globally, namely there exist positive numbers $c_i = c_i(\theta_0)$ such that

$$c_1 h_K \leq h \leq c_2 h_\sigma \leq c_3 h_K, \quad c_1 |K| \leq |\sigma| h \leq c_2 |\sigma| h_K \leq c_3 |\sigma| h_\sigma \leq c_4 |K| \quad (3.19)$$

for any $K \in \mathcal{T}$ and any $\sigma \in \mathcal{E}$. The necessary (small) modifications needed to accommodate the regular mesh satisfying only (3.3) are left to the reader.

4 - Mesh independent estimates

We start by a remark on the notation. From now on, the letter c denotes positive numbers that may tacitly depend on T , $|\Omega|$, $\text{diam}(\Omega)$, γ , α , θ_0 , λ and μ , and on other parameters; the dependency on these other parameters (if any) is always explicitly indicated in the arguments of these numbers. These numbers can take different values even in the same formula. They are always independent of the size of the discretisation k and h .

4.1 Energy Identity

Our analysis starts with an energy inequality, which is crucial both in the convergence analysis and in the error analysis. We recall this energy estimate which is already given in [26], along with its proof for the sake of completeness.

Lemma 4.1. *Let $(\varrho^0, \mathbf{u}^0) \in L_h^+(\Omega) \times \mathbf{W}_h(\Omega)$ and suppose that $(\varrho^n)_{1 \leq n \leq N} \in [L_h^+(\Omega)]^N$, $(\mathbf{u}^n)_{1 \leq n \leq N} \in [\mathbf{W}_h(\Omega)]^N$ is a solution of the discrete problem (3.12) with the pressure p satisfying condition (1.5). Then there exist*

$$\begin{aligned} \bar{\varrho}_\sigma^n &\in [\min(\varrho_K^n, \varrho_L^n), \max(\varrho_K^n, \varrho_L^n)], \quad \sigma = K|L \in \mathcal{E}_{\text{int}}, \quad n = 1, \dots, N \\ \bar{\varrho}_K^{n-1, n} &\in [\min(\varrho_K^{n-1}, \varrho_K^n), \max(\varrho_K^{n-1}, \varrho_K^n)], \quad K \in \mathcal{T}, \quad n = 1, \dots, N \end{aligned}$$

such that

$$\begin{aligned} \sum_{K \in \mathcal{T}} |K| \left(\frac{1}{2} \varrho_K^m |\mathbf{u}_K^m|^2 + H(\varrho_K^m) \right) - \sum_{K \in \mathcal{T}} |K| \left(\frac{1}{2} \varrho_K^0 |\mathbf{u}_K^0|^2 + H(\varrho_K^0) \right) \\ + k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \left(\mu \int_K |\nabla_x \mathbf{u}^n|^2 dx + (\mu + \lambda) \int_K |\text{div} \mathbf{u}^n|^2 dx \right) \\ + [D_{\text{time}}^{m, |\Delta \mathbf{u}|}] + [D_{\text{time}}^{m, |\Delta \varrho|}] + [D_{\text{space}}^{m, |\Delta \mathbf{u}|}] + [D_{\text{space}}^{m, |\Delta \varrho|}] = 0, \quad (4.1) \end{aligned}$$

for all $m = 1, \dots, N$, where

$$[D_{\text{time}}^{m, |\Delta \mathbf{u}|}] = \sum_{n=1}^m \sum_{K \in \mathcal{T}} |K| \varrho_K^{n-1} \frac{|\mathbf{u}_K^n - \mathbf{u}_K^{n-1}|^2}{2} \quad (4.2a)$$

$$[D_{\text{time}}^{m, |\Delta \varrho|}] = \sum_{n=1}^m \sum_{K \in \mathcal{T}} |K| H''(\bar{\varrho}_K^{n-1, n}) \frac{|\varrho_K^n - \varrho_K^{n-1}|^2}{2}, \quad (4.2b)$$

$$[D_{\text{space}}^{m, |\Delta \mathbf{u}|}] = k \sum_{n=1}^m \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} |\sigma| \varrho_\sigma^{n, \text{up}} \frac{(\mathbf{u}_K^n - \mathbf{u}_L^n)^2}{2} |\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma, K}|, \quad (4.2c)$$

$$[D_{\text{space}}^{m, |\Delta \varrho|}] = k \sum_{n=1}^m \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} |\sigma| H''(\bar{\varrho}_\sigma^n) \frac{(\varrho_K^n - \varrho_L^n)^2}{2} |\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma, K}|. \quad (4.2d)$$

Proof. Mimicking the formal derivation of the total energy conservation in the continuous case we take as test function $\mathbf{v} = \mathbf{u}^n$ in the discrete momentum equation (3.12b)ⁿ and obtain

$$I_1 + I_2 + I_3 + I_4 = 0, \quad (4.3)$$

where

$$\begin{aligned} I_1 &= \sum_{K \in \mathcal{T}} \frac{|K|}{k} (\varrho_K^n \mathbf{u}_K^n - \varrho_K^{n-1} \mathbf{u}_K^{n-1}) \cdot \mathbf{u}_K^n, & I_2 &= \sum_{K \in \mathcal{T}} \sum_{\substack{\sigma \in \mathcal{E}(K) \\ \sigma = K|L}} |\sigma| \varrho_\sigma^{n,\text{up}} \hat{\mathbf{u}}_\sigma^{n,\text{up}} \cdot \mathbf{u}_K^n [\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}], \\ I_3 &= - \sum_{K \in \mathcal{T}} \sum_{\substack{\sigma \in \mathcal{E}(K) \\ \sigma = K|L}} |\sigma| p(\varrho_K^n) [\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}], & I_4 &= \sum_{K \in \mathcal{T}} \int_K (\mu \nabla \mathbf{u}^n : \nabla \mathbf{u}^n + (\mu + \lambda) \operatorname{div} \mathbf{u}^n \operatorname{div} \mathbf{u}^n) dx. \end{aligned}$$

Next, we multiply the continuity equation (3.12a)ⁿ_K by $\frac{1}{2} |\mathbf{u}_K^n|^2$ and sum over all $K \in \mathcal{T}$. We get

$$I_5 + I_6 = 0 \quad (4.4)$$

$$\text{with } I_5 = - \sum_{K \in \mathcal{T}} \frac{1}{2} \frac{|K|}{k} (\varrho_K^n - \varrho_K^{n-1}) |\mathbf{u}_K^n|^2 \text{ and } I_6 = - \sum_{K \in \mathcal{T}} \sum_{\substack{\sigma \in \mathcal{E}(K) \\ \sigma = K|L}} \frac{1}{2} |\sigma| \varrho_\sigma^{n,\text{up}} [\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}] |\mathbf{u}_K^n|^2.$$

Finally, we multiply the continuity equation (3.12a)ⁿ_K by $H'(\varrho_K^n)$ and sum over all $K \in \mathcal{T}$. We obtain

$$I_7 + I_8 = 0, \quad (4.5)$$

$$\text{with } I_7 = \sum_{K \in \mathcal{T}} \frac{|K|}{k} (\varrho_K^n - \varrho_K^{n-1}) H'(\varrho_K^n) \text{ and } I_8 = \sum_{K \in \mathcal{T}} \sum_{\substack{\sigma \in \mathcal{E}(K) \\ \sigma = K|L}} |\sigma| \varrho_\sigma^{n,\text{up}} [\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}] H'(\varrho_K^n).$$

We now sum formulas (4.3)–(4.5) in several steps.

Step 1: *Term* $I_1 + I_7$. We verify by a direct calculation that

$$I_1 = \sum_{K \in \mathcal{T}} \frac{|K|}{k} \left(\frac{1}{2} \varrho_K^n |\mathbf{u}_K^n|^2 - \frac{1}{2} \varrho_K^{n-1} |\mathbf{u}_K^{n-1}|^2 \right) + \sum_{K \in \mathcal{T}} \frac{|K|}{k} \varrho_K^{n-1} \frac{|\mathbf{u}_K^n - \mathbf{u}_K^{n-1}|^2}{2}.$$

In order to transform the term I_7 , we employ the Taylor formula

$$H'(\varrho_K^n) (\varrho_K^n - \varrho_K^{n-1}) = H(\varrho_K^n) - H(\varrho_K^{n-1}) + \frac{1}{2} H''(\bar{\varrho}_K^{n-1,n}) (\varrho_K^n - \varrho_K^{n-1})^2,$$

where $\bar{\varrho}_K^{n-1,n} \in [\min(\varrho_K^{n-1}, \varrho_K^n), \max(\varrho_K^{n-1}, \varrho_K^n)]$. Consequently,

$$\begin{aligned} I_1 + I_7 &= \sum_{K \in \mathcal{T}} \frac{|K|}{k} \left(\frac{1}{2} \varrho_K^n |\mathbf{u}_K^n|^2 - \frac{1}{2} \varrho_K^{n-1} |\mathbf{u}_K^{n-1}|^2 \right) + \sum_{K \in \mathcal{T}} \frac{|K|}{k} \left(H(\varrho_K^n) - H(\varrho_K^{n-1}) \right) \\ &\quad + \sum_{K \in \mathcal{T}} \frac{|K|}{k} \varrho_K^{n-1} \frac{|\mathbf{u}_K^n - \mathbf{u}_K^{n-1}|^2}{2} + \sum_{K \in \mathcal{T}} \frac{|K|}{k} H''(\bar{\varrho}_K^{n-1,n}) \frac{|\varrho_K^n - \varrho_K^{n-1}|^2}{2}. \end{aligned} \quad (4.6)$$

Step 2: *Term* $I_2 + I_6$. The contribution of the face $\sigma = K|L$ to the sum $I_2 + I_6$ reads, by virtue of (3.11),

$$\begin{aligned} |\sigma| [\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}]^+ \varrho_K \left(|\mathbf{u}_K^n|^2 - \mathbf{u}_K^n \cdot \mathbf{u}_L^n - \frac{1}{2} |\mathbf{u}_K^n|^2 + \frac{1}{2} |\mathbf{u}_L^n|^2 \right) \\ + |\sigma| [\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,L}]^+ \varrho_L \left(|\mathbf{u}_L^n|^2 - \mathbf{u}_K^n \cdot \mathbf{u}_L^n - \frac{1}{2} |\mathbf{u}_L^n|^2 + \frac{1}{2} |\mathbf{u}_K^n|^2 \right). \end{aligned}$$

Consequently,

$$I_2 + I_6 = \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} |\sigma| |\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}| \varrho_\sigma^{n,\text{up}} \frac{(\mathbf{u}_K^n - \mathbf{u}_L^n)^2}{2}. \quad (4.7)$$

Step 3: *Term $I_3 + I_8$.* We have

$$\begin{aligned} I_8 = \sum_{K \in \mathcal{T}} \sum_{\substack{\sigma \in \mathcal{E}(K) \\ \sigma=K|L}} |\sigma| [\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}] \left(H'(\varrho_K^n)(\varrho_\sigma^{n,\text{up}} - \varrho_K^n) + H(\varrho_K^n) \right) \\ + \sum_{K \in \mathcal{T}} \sum_{\substack{\sigma \in \mathcal{E}(K) \\ \sigma=K|L}} |\sigma| [\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}] \left(\varrho_K^n H'(\varrho_K^n) - H(\varrho_K^n) \right). \end{aligned}$$

Recalling (3.11), we may write the contribution of the face $\sigma = K|L$ to the first sum in I_8 ; it reads

$$\begin{aligned} |\sigma| [\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}]^+ \left(H(\varrho_K^n) - H'(\varrho_L^n)(\varrho_K^n - \varrho_L^n) - H(\varrho_L^n) \right) \\ + |\sigma| [\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,L}]^+ \left(H(\varrho_L^n) - H'(\varrho_K^n)(\varrho_L^n - \varrho_K^n) - H(\varrho_K^n) \right). \end{aligned}$$

Recalling that $rH'(r) - H(r) = p(r)$, we get, employing the Taylor formula

$$I_3 + I_8 = \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} |\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}| H''(\bar{\varrho}_\sigma^n) \frac{(\varrho_K^n - \varrho_L^n)^2}{2}$$

with some $\bar{\varrho}_\sigma^n \in [\min(\varrho_K^n, \varrho_L^n), \max(\varrho_K^n, \varrho_L^n)]$.

Step 4: *Conclusion*

Collecting the results of Steps 1-3 we arrive at

$$\begin{aligned} \sum_{K \in \mathcal{T}} \frac{1}{2} \frac{|K|}{k} \left(\varrho_K^n |\mathbf{u}_K^n|^2 - \varrho_K^{n-1} |\mathbf{u}_K^{n-1}|^2 \right) + \sum_{K \in \mathcal{T}} \frac{|K|}{k} \left(H(\varrho_K^n) - H(\varrho_K^{n-1}) \right) + \sum_{K \in \mathcal{T}} \left(\mu \int_K |\nabla_x \mathbf{u}^n|^2 dx \right. \\ \left. + (\mu + \lambda) \int_K |\text{div} \mathbf{u}^n|^2 dx \right) + \sum_{K \in \mathcal{T}} \frac{|K|}{k} \varrho_K^{n-1} \frac{|\mathbf{u}_K^n - \mathbf{u}_K^{n-1}|^2}{2} + \sum_{K \in \mathcal{T}} \frac{|K|}{k} H''(\bar{\varrho}_K^{n-1,n}) \frac{|\varrho_K^n - \varrho_K^{n-1}|^2}{2} \\ + \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma=K|L}} |\sigma| \varrho_\sigma^{n,\text{up}} \frac{(\mathbf{u}_K^n - \mathbf{u}_L^n)^2}{2} |\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}| + \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma=K|L}} |\sigma| H''(\bar{\varrho}_\sigma^n) \frac{(\varrho_K^n - \varrho_L^n)^2}{2} |\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}| = 0. \quad (4.8) \end{aligned}$$

At this stage, we get the statement of Lemma 4.1 by multiplying (4.8) ^{n} by k and summing from $n = 1$ to $n = m$. Lemma 4.1 is proved. \square

4.2 Estimates

In order to simplify the notations, we denote here and hereafter,

$$\varrho(t, x) = \sum_{n=1}^N \varrho_n(x) 1_{[n-1, n)}(t), \quad \mathbf{u}(t, x) = \sum_{n=1}^N \mathbf{u}^n(x) 1_{[n-1, n)}(t) \quad (4.9)$$

and recall that the usual Lebesgue norms of these functions read

$$\|\varrho\|_{L^\infty(0, T; L^p(\Omega))} \equiv \max_{n=1, \dots, N} \|\varrho^n\|_{L^p(\Omega)}, \quad \|\mathbf{u}\|_{L^p(0, T; L^q(\Omega; \mathbb{R}^3))} \equiv k \left(\sum_{n=1}^N \|\mathbf{u}^n\|_{L^q(\Omega; \mathbb{R}^3)}^p \right)^{1/p} \quad (4.10)$$

We have the following corollary of Lemma 4.1.

Corollary 4.1. (1) Under assumptions of Lemma 4.1, there exists $c = c(M_0, E_0) > 0$ (independent of h and k) such that

$$|\mathbf{u}|_{L^2(0,T;V_h^2(\Omega;\mathbb{R}^3))} \leq c \quad (4.11)$$

$$\|\mathbf{u}\|_{L^2(0,T;L^6(\Omega;\mathbb{R}^3))} \leq c \quad (4.12)$$

$$\|\varrho \hat{\mathbf{u}}^2\|_{L^\infty(0,T;L^1(\Omega))} \leq c. \quad (4.13)$$

(2) If in addition the pressure satisfies assumption (1.6) then

$$\|\varrho\|_{L^\infty(0,T;L^\gamma(\Omega))} \leq c \quad (4.14)$$

(3) If the pair (r, \mathbf{U}) belongs to the class (3.16) there exists $c = c(M_0, E_0, \underline{r}, \bar{r}, \|\mathbf{U}, \nabla \mathbf{U}\|_{L^\infty(Q_T;\mathbb{R}^{12})}) > 0$ such that for all $n = 1, \dots, N$,

$$\mathcal{E}(\varrho^n, \mathbf{u}^n | r^n, \mathbf{U}^n) \leq c, \quad (4.15)$$

where the discrete relative energy \mathcal{E} is defined in (3.14).

Proof. Recall that

$$|\mathbf{u}|_{L^2(0,T;V_h^2(\Omega;\mathbb{R}^3))}^2 = k \sum_{n=1}^N \sum_{K \in \mathcal{T}} \int_K |\nabla_x \mathbf{u}^n|^2 dx;$$

the estimate (4.11) follows from (4.1). The estimate (4.12) holds due to imbedding (9.29) in Lemma 9.3 and bound (4.11). The estimate (4.13) is just a short transcription of the bound for the kinetic energy in (4.1). The estimate (4.14) involving the density follows from the boundedness of $\sum_{K \in \mathcal{T}} |K| H(\varrho_K^m)$ and $\sum_{K \in \mathcal{T}} |K| \varrho_K^m$, by virtue of hypotheses (1.5) and (1.6). Finally, to get (4.15), we have employed (3.14), (2.5), (4.14) to estimate $\int_\Omega E(\varrho^n | \hat{r}^n) dx$ and (9.3), (9.21), (4.13) to evaluate $\sum_{K \in \mathcal{T}} \int_K \varrho_K^n |\mathbf{U}_{h,K}^n - \mathbf{u}_K^n|^2 dx$. \square

The following estimates are obtained thanks to the numerical diffusion due to the upwinding, as is classical in the framework of hyperbolic conservation laws, see e.g. [9].

Lemma 4.2 (Dissipation estimates on the density). *Let $(\varrho^0, \mathbf{u}^0) \in L_h^+(\Omega) \times \mathbf{W}_h(\Omega)$. Suppose that $(\varrho^n)_{1 \leq n \leq N} \subset [L_h^+(\Omega)]^N$, $(\mathbf{u}^n)_{1 \leq n \leq N} \subset [\mathbf{W}_h]^N(\Omega)$ is a solution of problem (3.12). Finally assume that the pressure satisfies hypotheses (1.5) and (1.6). Then we have:*

(1) If $\gamma \geq 2$ then there exists $c = c(\gamma, \theta_0, E_0) > 0$ such that

$$k \sum_{n=1}^N \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} |\sigma| \frac{(\varrho_K^n - \varrho_L^n)^2}{\max(\varrho_K^n, \varrho_L^n)} |\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}| \leq c. \quad (4.16)$$

(2) If $\gamma \in [1, 2)$ and the pressure satisfies additionally assumption (1.7) then there exists $c = c(M_0, E_0) > 0$ such that

$$\begin{aligned} k \sum_{n=1}^N \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} |\sigma| \frac{(\varrho_K^n - \varrho_L^n)^2}{[\max(\varrho_K^n, \varrho_L^n)]^{2-\gamma}} 1_{\{\bar{\varrho}_\sigma^n \geq 1\}} |\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}| \\ + k \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} |\sigma| (\varrho_K^n - \varrho_L^n)^2 1_{\{\bar{\varrho}_\sigma^n < 1\}} |\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}| \leq c, \end{aligned} \quad (4.17)$$

where the numbers $\bar{\varrho}_\sigma^n$ are defined in Lemma 4.1.

Proof. We start by proving the simpler statement (2). Taking into account the continuity of the pressure, we deduce from assumptions (1.6) and (1.7) that there exist numbers $\bar{p}_0 > 0$, $\bar{p}_\infty > 0$ such that

$$H''(s) \geq \begin{cases} \frac{\bar{p}_\infty}{s^{2-\gamma}}, & \text{if } s \geq 1, \\ \bar{p}_0 s^\alpha \geq \bar{p}_0, & \text{if } s < 1, . \end{cases}$$

whence, splitting the sum in the definition of the term $[D_{\text{space}}^{N,\Delta} \varrho]$ (see (4.2d)) into two sums, where (σ, n) satisfies $\bar{\varrho}_\sigma^n \geq 1$ for the first one and $\bar{\varrho}_\sigma^n < 1$ for the second, we obtain the desired result.

Let us now turn to the proof of statement (1). Multiplying the discrete continuity equation (3.12a) $_K^n$ by $\ln \varrho_K^n$ and summing over $K \in \mathcal{T}$, we get

$$\sum_{K \in \mathcal{T}} |K| \frac{\varrho_K^n - \varrho_K^{n-1}}{k} \ln \varrho_K^n + \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K), \sigma=K|L} (\ln \varrho_K^n) \varrho_\sigma^{n,\text{up}} \mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K} = 0.$$

By virtue of the convexity of the function $\varrho \mapsto \varrho \ln \varrho - \varrho$ on the positive real line, and due to the Taylor formula, we have

$$\varrho_K^n \ln \varrho_K^n - \varrho_K^{n-1} \ln \varrho_K^{n-1} - (\varrho_K^n - \varrho_K^{n-1}) \leq \ln \varrho_K^n (\varrho_K^n - \varrho_K^{n-1});$$

whence, thanks to the mass conservation (3.13) and the definition of $\varrho_\sigma^{\text{up}}$, we arrive at

$$\begin{aligned} \sum_{K \in \mathcal{T}} |K| \frac{\varrho_K^n \ln \varrho_K^n - \varrho_K^{n-1} \ln \varrho_K^{n-1}}{k} + \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma=K|L}} |\sigma| \varrho_K^n [\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}]^+ (\ln \varrho_K^n - \ln \varrho_L^n) \\ + \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma=K|L}} |\sigma| \varrho_L^n [\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,L}]^+ (\ln \varrho_L^n - \ln \varrho_K^n) \leq 0, \end{aligned}$$

or equivalently

$$\begin{aligned} k \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma=K|L}} |\sigma| [\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}]^+ (\varrho_K^n (\ln \varrho_K^n - \ln \varrho_L^n) - (\varrho_K^n - \varrho_L^n)) + k \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma=K|L}} |\sigma| [\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,L}]^+ (\varrho_L^n (\ln \varrho_L^n - \ln \varrho_K^n) - (\varrho_L^n - \varrho_K^n)) \leq \\ - \sum_{K \in \mathcal{T}} |K| (\varrho_K^n \ln \varrho_K^n - \varrho_K^{n-1} \ln \varrho_K^{n-1}) + k \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma=K|L}} |\sigma| ([\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}]^+ (\varrho_L^n - \varrho_K^n) + [\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,L}]^+ (\varrho_K^n - \varrho_L^n)). \end{aligned} \quad (4.18)$$

From [16, Lemma C.5], we know that if φ and ψ are functions in $C^1((0, \infty); \mathbb{R})$ such that $s\psi'(s) = \varphi'(s)$ for all $s \in (0, \infty)$, then for any $(a, b) \in (0, \infty)^2$ there exists $c \in [a, b]$ such that

$$(\psi(b) - \psi(a))b - (\varphi(b) - \varphi(a)) = \frac{1}{2}(b - a)^2 \psi'(c).$$

Applying this result with $\psi(s) = \ln s$, $\varphi(s) = s$ we obtain that the left hand side of (4.18) is greater or equal to

$$k \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma=K|L}} |\sigma| ([\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}]^+ + [\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,L}]^+) \frac{(\varrho_K^n - \varrho_L^n)^2}{\max(\varrho_K^n, \varrho_L^n)}.$$

On the other hand, the first term at the right hand side is bounded from above by $\|\varrho^n\|_{L^\gamma(\Omega)}^\gamma$. Finally the second term at the right hand side is equal to

$$-k \sum_{K \in \mathcal{T}} \int_K \varrho_K^n \operatorname{div} \mathbf{u}^n,$$

whence bounded from above by $k \|\mathbf{u}^n\|_{V_h^2(\Omega; \mathbb{R}^3)} \|\varrho^n\|_{L^2(\Omega)}$. The statement (1) of Lemma 4.2 now follows from the estimates of Corollary 4.1. \square

5 - Exact relative energy inequality for the discrete problem

The goal of this section is to prove the discrete version of the relative energy inequality.

Theorem 5.1. *Suppose that $\Omega \subset \mathbb{R}^3$ is a polyhedral domain and \mathcal{T} its regular triangulation introduced in Section 3.1. Let p satisfy hypotheses (1.5) and the viscosity coefficient μ, λ obey (1.4). Let $(\varrho^0, \mathbf{u}^0) \in L_h^+(\Omega) \times \mathbf{W}_h(\Omega)$ and suppose that $(\varrho^n)_{1 \leq n \leq N} \in [L_h^+(\Omega)]^N$, $(\mathbf{u}^n)_{1 \leq n \leq N} \in [\mathbf{W}_h(\Omega)]^N$ is a solution of the discrete problem (3.12). Then there holds for all $m = 1, \dots, N$,*

$$\begin{aligned}
& \sum_{K \in \mathcal{T}} \frac{1}{2} |K| \left(\varrho_K^n |\mathbf{u}_K^m - \mathbf{U}_{h,K}^m|^2 - \varrho_K^0 |\mathbf{u}_K^0 - \mathbf{U}_{h,K}^0|^2 \right) + \sum_{K \in \mathcal{T}} |K| \left(E(\varrho_K^m | r_K^m) - E(\varrho_K^0 | r_K^0) \right) \\
& + k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \left(\mu \int_K |\nabla_x (\mathbf{u}^n - \mathbf{U}_h^n)|^2 dx + (\mu + \lambda) \int_K |\operatorname{div}(\mathbf{u}^n - \mathbf{U}_h^n)|^2 dx \right) \\
& \leq k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \left(\mu \int_K \nabla_x \mathbf{U}_h^n : \nabla_x (\mathbf{U}_h^n - \mathbf{u}^n) dx + (\mu + \lambda) \int_K \operatorname{div} \mathbf{U}_h^n \operatorname{div} (\mathbf{U}_h^n - \mathbf{u}^n) dx \right) \\
& + k \sum_{n=1}^m \sum_{K \in \mathcal{T}} |K| \varrho_K^{n-1} \frac{\mathbf{U}_{h,K}^n - \mathbf{U}_{h,K}^{n-1}}{k} \cdot \left(\frac{\mathbf{U}_{h,K}^{n-1} + \mathbf{U}_{h,K}^n}{2} - \mathbf{u}_K^{n-1} \right) \\
& - k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \sum_{\substack{\sigma \in \mathcal{E}(K) \\ \sigma = K|L}} |\sigma| \varrho_\sigma^{n,\text{up}} \left(\frac{\mathbf{U}_{h,K}^n + \mathbf{U}_{h,L}^n}{2} - \hat{\mathbf{u}}_\sigma^{n,\text{up}} \right) \cdot \mathbf{U}_{h,K}^n [\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}] \\
& - k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \sum_{\substack{\sigma \in \mathcal{E}(K) \\ \sigma = K|L}} |\sigma| p(\varrho_K^n) [\mathbf{U}_{h,\sigma}^n \cdot \mathbf{n}_{\sigma,K}] \\
& + k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \frac{|K|}{k} (r_K^n - \varrho_K^n) \left(H'(r_K^n) - H'(r_K^{n-1}) \right) \\
& + k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \sum_{\substack{\sigma \in \mathcal{E}(K) \\ \sigma = K|L}} |\sigma| \varrho_\sigma^{n,\text{up}} H'(r_K^{n-1}) [\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}],
\end{aligned} \tag{5.1}$$

for any $0 < r \in C^1([0, T] \times \bar{\Omega})$, $\mathbf{U} \in C^1([0, T] \times \bar{\Omega})$, $\mathbf{U}|_{\partial\Omega} = 0$, where we have used notation (3.15) for r^n , \mathbf{U}^n .

We notice, comparing the terms in the “discrete” formula (5.1) with the terms in the “continuous” formula (2.7), that Theorem 5.1 represents a discrete counterpart of the “continuous” relative energy inequality (2.7). The rest of this section is devoted to its proof. To this end, we shall follow the proof of the “continuous” relative energy inequality (see [12] and [15]) and adapt it to the discrete case.

Proof. First, noting that the numerical diffusion in the energy identity (4.8) is positive, we infer

$$I_1 + I_2 + I_3 \leq 0, \tag{5.2}$$

with

$$\begin{aligned}
I_1 & := \sum_{K \in \mathcal{T}} \frac{1}{2} \frac{|K|}{k} \left(\varrho_K^n |\mathbf{u}_K^n|^2 - \varrho_K^{n-1} |\mathbf{u}_K^{n-1}|^2 \right), & I_2 & := \sum_{K \in \mathcal{T}} \frac{|K|}{k} \left(H(\varrho_K^n) - H(\varrho_K^{n-1}) \right), \\
I_3 & := \sum_{K \in \mathcal{T}} \left(\mu \int_K |\nabla_x \mathbf{u}^n|^2 dx + (\mu + \lambda) \int_K |\operatorname{div} \mathbf{u}^n|^2 dx \right).
\end{aligned}$$

Next, we multiply the discrete continuity equation (3.12a) $_K^n$ by $\frac{1}{2}|U_{h,K}^n|^2$ and sum over $K \in \mathcal{T}$ to obtain

$$I_4 := \sum_{K \in \mathcal{T}} \frac{1}{2} \frac{|K|}{k} (\varrho_K^n - \varrho_K^{n-1}) |U_{h,K}^n|^2 = - \sum_{K \in \mathcal{T}} \sum_{\substack{\sigma \in \mathcal{E}(K) \\ \sigma = K|L}} \frac{1}{2} |\sigma| \varrho_\sigma^{n,\text{up}} [\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}] |U_{h,K}^n|^2 := J_1 \quad (5.3)$$

In the next step, taking $-U^n$ as test function in the discrete momentum equation (3.12b); we get

$$I_5 = - \sum_{K \in \mathcal{T}} \frac{|K|}{k} (\varrho_K^n \mathbf{u}_K^n - \varrho_K^{n-1} \mathbf{u}_K^{n-1}) \cdot \mathbf{U}_{h,K}^n = J_2 + J_3 + J_4,$$

with

$$\begin{aligned} J_2 &= \sum_{K \in \mathcal{T}} \sum_{\substack{\sigma \in \mathcal{E}(K) \\ \sigma = K|L}} |\sigma| \varrho_\sigma^{n,\text{up}} \hat{\mathbf{u}}_\sigma^{n,\text{up}} \cdot \mathbf{U}_{h,K}^n [\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}], \\ J_3 &= \mu \sum_{K \in \mathcal{T}} \int_K \nabla \mathbf{u}^n : \nabla \mathbf{U}_h^n \, dx + (\mu + \lambda) \sum_{K \in \mathcal{T}} \int_K \operatorname{div} \mathbf{u}^n \operatorname{div} \mathbf{U}_h^n \, dx \\ &\text{and} \\ J_4 &= - \sum_{K \in \mathcal{T}} \sum_{\substack{\sigma \in \mathcal{E}(K) \\ \sigma = K|L}} |\sigma| p(\varrho_K^n) [\mathbf{U}_\sigma^n \cdot \mathbf{n}_{\sigma,K}]. \end{aligned}$$

We then multiply the continuity equation (3.12a) $_K^n$ by $H'(r_K^{n-1})$ and sum over all $K \in \mathcal{T}$ and obtain

$$- \sum_{K \in \mathcal{T}} \frac{|K|}{k} (\varrho_K^n - \varrho_K^{n-1}) H'(r_K^{n-1}) = \sum_{K \in \mathcal{T}} \sum_{\substack{\sigma \in \mathcal{E}(K) \\ \sigma = K|L}} |\sigma| \varrho_\sigma^{n,\text{up}} [\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}] H'(r_K^{n-1}).$$

Observing that $\varrho_K^n H'(r_K^n) - \varrho_K^{n-1} H'(r_K^{n-1}) = \varrho_K^n (H'(r_K^n) - H'(r_K^{n-1})) + (\varrho_K^n - \varrho_K^{n-1}) H'(r_K^{n-1})$, we rewrite the last identity in the form

$$\begin{aligned} I_6 &:= - \sum_{K \in \mathcal{T}} \frac{|K|}{k} (\varrho_K^n H'(r_K^n) - \varrho_K^{n-1} H'(r_K^{n-1})) = J_5 + J_6 \\ &\text{with } J_5 = - \sum_{K \in \mathcal{T}} \frac{|K|}{k} \varrho_K^n (H'(r_K^n) - H'(r_K^{n-1})) \text{ and } J_6 = \sum_{K \in \mathcal{T}} \sum_{\substack{\sigma \in \mathcal{E}(K) \\ \sigma = K|L}} |\sigma| \varrho_\sigma^{n,\text{up}} [\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}] H'(r_K^{n-1}). \end{aligned} \quad (5.4)$$

Finally, thanks to the the convexity of the function H , we have

$$\begin{aligned} I_7 &:= \sum_{K \in \mathcal{T}} \frac{|K|}{k} \left[(r_K^n H'(r_K^n) - H(r_K^n)) - (r_K^{n-1} H'(r_K^{n-1}) - H(r_K^{n-1})) \right] \\ &= \sum_{K \in \mathcal{T}} \frac{|K|}{k} r_K^n (H'(r_K^n) - H'(r_K^{n-1})) - \sum_{K \in \mathcal{T}} \frac{|K|}{k} (H(r_K^n) - (r_K^n - r_K^{n-1}) H'(r_K^{n-1}) - H(r_K^{n-1})) \\ &\leq \sum_{K \in \mathcal{T}} \frac{|K|}{k} r_K^n (H'(r_K^n) - H'(r_K^{n-1})) := J_7, \end{aligned} \quad (5.5)$$

Now, we gather the expressions (5.2)-(5.5); this is performed in several steps.

Step 1: *Term* $I_1 + I_4 + I_5$. We observe that

$$\begin{cases} \frac{|U_{h,K}^n|^2}{2} (\varrho_K^n - \varrho_K^{n-1}) = \frac{\varrho_K^n |U_{h,K}^n|^2 - \varrho_K^{n-1} |U_{h,K}^{n-1}|^2}{2} + \varrho_K^{n-1} \frac{U_{h,K}^{n-1} + U_{h,K}^n}{2} \cdot (U_{h,K}^{n-1} - U_{h,K}^n), \\ - (\varrho_K^n \mathbf{u}_K^n - \varrho_K^{n-1} \mathbf{u}_K^{n-1}) \cdot \mathbf{U}_{h,K}^n = - (\varrho_K^n \mathbf{u}_K^n \cdot \mathbf{U}_{h,K}^n - \varrho_K^{n-1} \mathbf{u}_K^{n-1} \cdot \mathbf{U}_{h,K}^{n-1}) - \varrho_K^{n-1} \mathbf{u}_K^{n-1} \cdot (U_{h,K}^{n-1} - U_{h,K}^n). \end{cases}$$

Consequently,

$$I_1 + I_4 + I_5 = \sum_{K \in \mathcal{T}} \frac{1}{2} \frac{|K|}{k} \left(\varrho_K^n |\mathbf{u}_K^n - \mathbf{U}_{h,K}^n|^2 - \varrho_K^{n-1} |\mathbf{u}_K^{n-1} - \mathbf{U}_{h,K}^{n-1}|^2 \right) - \sum_{K \in \mathcal{T}} |K| \varrho_K^{n-1} \frac{\mathbf{U}_{h,K}^n - \mathbf{U}_{h,K}^{n-1}}{k} \cdot \left(\frac{\mathbf{U}_{h,K}^{n-1} + \mathbf{U}_{h,K}^n}{2} - \mathbf{u}_K^{n-1} \right) \quad (5.6)$$

Step 2: *Term $J_1 + J_2$.* The contribution of the face $\sigma = K|L$ to J_1 reads

$$-|\sigma| \varrho_K^n \frac{\mathbf{U}_{h,K}^n + \mathbf{U}_{h,L}^n}{2} \cdot (\mathbf{U}_{h,K}^n - \mathbf{U}_{h,L}^n) [\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}]^+ - |\sigma| \varrho_L^n \frac{\mathbf{U}_{h,K}^n + \mathbf{U}_{h,L}^n}{2} \cdot (\mathbf{U}_{h,L}^n - \mathbf{U}_{h,K}^n) [\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,L}]^+.$$

Similarly, the contribution of the face $\sigma = K|L$ to J_2 is

$$|\sigma| \varrho_K^n \mathbf{u}_K^n \cdot (\mathbf{U}_{h,K}^n - \mathbf{U}_{h,L}^n) [\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}]^+ + |\sigma| \varrho_L^n \mathbf{u}_L^n \cdot (\mathbf{U}_{h,L}^n - \mathbf{U}_{h,K}^n) [\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,L}]^+.$$

Consequently,

$$J_1 + J_2 = - \sum_{K \in \mathcal{T}} \sum_{\sigma=K|L \in \mathcal{E}_K} |\sigma| \varrho_\sigma^{n,\text{up}} \left(\frac{\mathbf{U}_{h,K}^n + \mathbf{U}_{h,L}^n}{2} - \hat{\mathbf{u}}_\sigma^{n,\text{up}} \right) \cdot \mathbf{U}_{h,K}^n [\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}]. \quad (5.7)$$

Step 3: *Term $I_3 - J_3$.* This term can be written in the form

$$I_3 - J_3 = \sum_{K \in \mathcal{T}} \left(\mu \int_K |\nabla_x (\mathbf{u}^n - \mathbf{U}_h^n)|^2 dx + (\mu + \lambda) \int_K |\operatorname{div}(\mathbf{u}^n - \mathbf{U}_h^n)|^2 dx \right) - \sum_{K \in \mathcal{T}} \mu \int_K \left(\nabla \mathbf{U}_h^n : \nabla (\mathbf{U}_h^n - \mathbf{u}^n) + (\mu + \lambda) \operatorname{div} \mathbf{U}_h^n \operatorname{div} (\mathbf{U}_h^n - \mathbf{u}^n) \right). \quad (5.8)$$

Step 4: *Term $I_2 + I_6 + I_7$.* By virtue of (5.2), (5.4–5.5), we easily find that

$$I_2 + I_6 + I_7 = \sum_{K \in \mathcal{T}} \frac{|K|}{k} \left(E(\varrho_K^n | r_K^n) - E(\varrho_K^{n-1} | r_K^{n-1}) \right), \quad (5.9)$$

where the function E is defined in (2.5).

Step 5: *Term $J_5 + J_6 + J_7$.* Coming back to (5.4–5.5), we deduce that

$$J_5 + J_6 + J_7 = \sum_{K \in \mathcal{T}} \frac{|K|}{k} (r_K^n - \varrho_K^n) \left(H'(r_K^n) - H'(r_K^{n-1}) \right) + \sum_{K \in \mathcal{T}} \sum_{\substack{\sigma \in \mathcal{E}(K) \\ \sigma=K|L}} |\sigma| \varrho_\sigma^{n,\text{up}} [\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}] H'(r_K^{n-1}). \quad (5.10)$$

Step 6: *Conclusion*

According to (5.2)–(5.5), we have

$$\sum_{i=1}^7 I_i \leq \sum_{i=1}^7 J_i;$$

whence, writing this inequality by using expressions (5.6)–(5.10) calculated in steps 1-5, we get

$$\begin{aligned}
& \sum_{K \in \mathcal{T}} \frac{1}{2} \frac{|K|}{k} \left(\varrho_K^n |\mathbf{u}_K^n - \mathbf{U}_{h,K}^n|^2 - \varrho_K^{n-1} |\mathbf{u}_K^{n-1} - \mathbf{U}_{h,K}^{n-1}|^2 \right) + \sum_{K \in \mathcal{T}} \frac{|K|}{k} \left(E(\varrho_K^n |r_K^n) - E(\varrho_K^{n-1} |r_K^{n-1}) \right) \\
& \quad + \sum_{K \in \mathcal{T}} \left(\mu \int_K |\nabla_x (\mathbf{u}^n - \mathbf{U}_h^n)|^2 dx + (\mu + \lambda) \int_K |\operatorname{div}(\mathbf{u}^n - \mathbf{U}_h^n)|^2 dx \right) \\
& \leq \sum_{K \in \mathcal{T}} \left(\mu \int_K \nabla_x \mathbf{U}_h^n : \nabla_x (\mathbf{U}_h^n - \mathbf{u}^n) dx + (\mu + \lambda) \int_K \operatorname{div} \mathbf{U}_h^n \operatorname{div} (\mathbf{U}_h^n - \mathbf{u}^n) dx \right) \\
& \quad + \sum_{K \in \mathcal{T}} |K| \varrho_K^{n-1} \frac{\mathbf{U}_{h,K}^n - \mathbf{U}_{h,K}^{n-1}}{k} \cdot \left(\frac{\mathbf{U}_{h,K}^{n-1} + \mathbf{U}_{h,K}^n}{2} - \mathbf{u}_K^{n-1} \right) \\
& \quad - \sum_{K \in \mathcal{T}} \sum_{\sigma=K|L \in \mathcal{E}_K} |\sigma| \varrho_\sigma^{n,\text{up}} \left(\frac{\mathbf{U}_{h,K}^n + \mathbf{U}_{h,L}^n}{2} - \hat{\mathbf{u}}_\sigma^{n,\text{up}} \right) \cdot \mathbf{U}_{h,K}^n [\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}] \\
& \quad - \sum_{K \in \mathcal{T}} \sum_{\sigma=K|L \in \mathcal{E}_K} |\sigma| p(\varrho_K^n) [\mathbf{U}_{h,\sigma}^n \cdot \mathbf{n}_{\sigma,K}] + \sum_{K \in \mathcal{T}} \frac{|K|}{k} (r_K^n - \varrho_K^n) \left(H'(r_K^n) - H'(r_K^{n-1}) \right) \\
& \quad + \sum_{K \in \mathcal{T}} \sum_{\sigma=K|L \in \mathcal{E}_K} |\sigma| \varrho_\sigma^{n,\text{up}} H'(r_K^{n-1}) [\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}].
\end{aligned} \tag{5.11}$$

We obtain formula (5.1) by summing (5.11)ⁿ from $n = 1$ to $n = m$ and multiplying the resulting inequality by k . \square

6 - Approximate relative energy inequality for the discrete problem

The exact relative energy inequality as stated in Section 5 is a general inequality for the given numerical scheme, however it does not immediately provide a comparison of the discrete solution with the strong solution of the compressible Navier-Stokes equations. Its right hand side has to be conveniently transformed (modulo the possible appearance of residual terms vanishing as the space and time steps tend to 0) to provide such comparison tool via a Gronwall type argument.

The goal of this section is to derive a version of the discrete relative energy inequality, still with arbitrary (sufficiently regular) test functions (r, \mathbf{U}) , that will be convenient for the comparison of the discrete solution with the strong solution.

Lemma 6.1 (Approximate relative energy inequality). *Suppose that $\Omega \subset \mathbb{R}^3$ is a bounded polyhedral domain and \mathcal{T} its regular triangulation introduced in Section 3.1. Let the pressure p be a $C^2(0, \infty)$ function satisfying hypotheses (1.5), (1.6) with $\gamma \geq 6/5$ and satisfying the additional condition (1.7) if $\gamma < 2$.*

Let $(\varrho^0, \mathbf{u}^0) \in L_h^+(\Omega) \times \mathbf{W}_h(\Omega)$ and suppose that $(\varrho^n)_{1 \leq n \leq N} \in [L_h^+(\Omega)]^N$, $(\mathbf{u}^n)_{1 \leq n \leq N} \in [\mathbf{W}_h(\Omega)]^N$ is a solution of the discrete problem (3.12) with the viscosity coefficients μ, λ obeying (1.4).

Then there exists

$$c = c(M_0, E_0, \underline{r}, \bar{r}, |p'|_{C^1[\underline{r}, \bar{r}]}, \|(\partial_t r, \partial_t^2 r, \nabla r, \partial_t \nabla r, \mathbf{U}, \partial_t \mathbf{U}, \nabla \mathbf{U}, \partial_t \nabla \mathbf{U})\|_{L^\infty(Q_T; \mathbb{R}^{31})}) > 0$$

(where $\bar{r} = \max_{(t,x) \in \overline{Q_T}} r(t,x)$, $\underline{r} = \min_{(t,x) \in \overline{Q_T}} r(t,x)$), such that for all $m = 1, \dots, N$, we have:

$$\begin{aligned}
& \mathcal{E}(\varrho^m, \mathbf{u}^m | r^m, \mathbf{U}^m) - \mathcal{E}(\varrho^0, \mathbf{u}^0 | r(0), \mathbf{U}(0)) \\
& \quad + k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \left(\mu \int_K |\nabla_x(\mathbf{u}^n - \mathbf{U}_h^n)|^2 dx + (\mu + \lambda) \int_K |\operatorname{div}(\mathbf{u}^n - \mathbf{U}_h^n)|^2 dx \right) \\
& \leq k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \left(\mu \int_K \nabla_x \mathbf{U}_h^n : \nabla_x(\mathbf{U}_h^n - \mathbf{u}^n) dx + (\mu + \lambda) \int_K \operatorname{div} \mathbf{U}_h^n \operatorname{div}(\mathbf{U}_h^n - \mathbf{u}^n) dx \right) \\
& \quad + k \sum_{n=1}^m \sum_{K \in \mathcal{T}} |K| \varrho_K^{n-1} \frac{\mathbf{U}_{h,K}^n - \mathbf{U}_{h,K}^{n-1}}{k} \cdot (\mathbf{U}_{h,K}^n - \mathbf{u}_K^n) \\
& \quad + k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \varrho_\sigma^{n,\text{up}} (\hat{\mathbf{U}}_{h,\sigma}^{n,\text{up}} - \hat{\mathbf{u}}_\sigma^{n,\text{up}}) \cdot (\mathbf{U}_\sigma^n - \mathbf{U}_{h,K}^n) \hat{\mathbf{U}}_{h,\sigma}^{n,\text{up}} \cdot \mathbf{n}_{\sigma,K} \\
& \quad - k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \int_K p(\varrho_K^n) \operatorname{div} \mathbf{U}^n dx + k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \int_K (r_K^n - \varrho_K^n) \frac{p'(r_K^n)}{r_K^n} [\partial_t r]^n dx \\
& \quad - k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \int_K \frac{\varrho_K^n}{r_K^n} p'(r_K^n) \mathbf{u}^n \cdot \nabla r^n dx + R_{h,k}^m + G^m
\end{aligned} \tag{6.1}$$

for any pair (r, \mathbf{U}) belonging to the class (3.16), where

$$|G^m| \leq c k \sum_{n=1}^m \mathcal{E}(\varrho^n, \mathbf{u}^n | r^n, \mathbf{U}^n), \quad |R_{h,k}^m| \leq c(\sqrt{k} + h^A), \quad \text{and } A = \begin{cases} \frac{3\gamma-2}{2\gamma} & \text{if } \gamma \in [3/2, 2) \\ 1/2 & \text{if } \gamma \geq 3/2. \end{cases} \tag{6.2}$$

Proof. The right hand side of the relative energy inequality (5.1) is a sum $\sum_{i=1}^6 T_i$, where

$$\begin{aligned}
T_1 &= k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \left(\mu \int_K \nabla_x \mathbf{U}_h^n : \nabla_x(\mathbf{U}_h^n - \mathbf{u}^n) dx + (\mu + \lambda) \int_K \operatorname{div} \mathbf{U}_h^n \operatorname{div}(\mathbf{U}_h^n - \mathbf{u}^n) dx \right), \\
T_2 &= k \sum_{n=1}^m \sum_{K \in \mathcal{T}} |K| \varrho_K^{n-1} \frac{\mathbf{U}_{h,K}^n - \mathbf{U}_{h,K}^{n-1}}{k} \cdot \left(\frac{\mathbf{U}_{h,K}^{n-1} + \mathbf{U}_{h,K}^n}{2} - \mathbf{u}_K^{n-1} \right), \\
T_3 &= -k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \sum_{\sigma=K | L \in \mathcal{E}(K)} |\sigma| \varrho_\sigma^{n,\text{up}} \left(\frac{\mathbf{U}_{h,K}^n + \mathbf{U}_{h,L}^n}{2} - \hat{\mathbf{u}}_\sigma^{n,\text{up}} \right) \cdot \mathbf{U}_{h,K}^n [\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}], \\
T_4 &= -k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \sum_{\sigma=K | L \in \mathcal{E}(K)} |\sigma| p(\varrho_K^n) [\mathbf{U}_{h,\sigma}^n \cdot \mathbf{n}_{\sigma,K}], \\
T_5 &= k \sum_{n=1}^m \sum_{K \in \mathcal{T}} |K| (r_K^n - \varrho_K^n) \frac{H'(r_K^n) - H'(r_K^{n-1})}{k}, \\
T_6 &= k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \sum_{\sigma=K | L \in \mathcal{E}(K)} |\sigma| \varrho_\sigma^{n,\text{up}} H'(r_K^{n-1}) [\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}].
\end{aligned}$$

The term T_1 will be kept as it is; all the other terms T_i will be transformed to a more convenient form, as described in the following steps.

Step 1: *Term T_2 .* We have

$$T_2 = T_{2,1} + R_{2,1}, \quad \text{with } T_{2,1} = k \sum_{n=1}^m \sum_{K \in \mathcal{T}} |K| \varrho_K^{n-1} \frac{\mathbf{U}_{h,K}^n - \mathbf{U}_{h,K}^{n-1}}{k} \cdot (\mathbf{U}_{h,K}^{n-1} - \mathbf{u}_K^{n-1}), \quad \text{and } R_{2,1} = k \sum_{n=1}^m \sum_{K \in \mathcal{T}} R_{2,1}^{n,K},$$

where $R_{2,1}^{n,K} = \frac{|K|}{2} \varrho_K^{n-1} \frac{(\mathbf{U}_{h,K}^n - \mathbf{U}_{h,K}^{n-1})^2}{k}$; thanks to the mass conservation (3.13), the Taylor formula and the property (9.20) of the projection onto the space $V_h(\Omega)$, we get $|R_{2,1}^{n,K}| \leq \frac{M_0}{2} |K| k \|\partial_t \mathbf{U}\|_{L^\infty(0,T;W^{1,\infty}(\Omega;\mathbb{R}^3))}^2$. Therefore,

$$|R_{2,1}| \leq k c(M_0, \|(\partial_t \mathbf{U}, \partial_t \nabla \mathbf{U})\|_{L^\infty(Q_T;\mathbb{R}^{12})}) \quad (6.3)$$

Let us now decompose the term $T_{2,1}$ as

$$T_{2,1} = T_{2,2} + R_{2,2}, \text{ with } T_{2,2} = k \sum_{n=1}^m \sum_{K \in \mathcal{T}} |K| \varrho_K^{n-1} \frac{\mathbf{U}_{h,K}^n - \mathbf{U}_{h,K}^{n-1}}{k} \cdot (\mathbf{U}_{h,K}^n - \mathbf{u}_K^n), \text{ and } R_{2,2} = k \sum_{n=1}^m R_{2,2}^n, \quad (6.4)$$

$$\text{where } R_{2,2}^n = \sum_{K \in \mathcal{T}} |K| \varrho_K^{n-1} \frac{\mathbf{U}_{h,K}^n - \mathbf{U}_{h,K}^{n-1}}{k} \cdot (\mathbf{U}_{h,K}^{n-1} - \mathbf{U}_{h,K}^n) - \sum_{K \in \mathcal{T}} |K| \varrho_K^{n-1} \frac{\mathbf{U}_{h,K}^n - \mathbf{U}_{h,K}^{n-1}}{k} \cdot (\mathbf{u}_K^{n-1} - \mathbf{u}_K^n).$$

By the same token as above, we may estimate the residual term as follows

$$|R_{2,2}^n| \leq k c M_0 \|\partial_t \mathbf{U}\|_{L^\infty(0,T;W^{1,\infty}(\Omega;\mathbb{R}^3))}^2 + c M_0^{1/2} \left(\sum_{K \in \mathcal{T}} |K| \varrho_K^{n-1} |\mathbf{u}_K^{n-1} - \mathbf{u}_K^n|^2 \right)^{1/2} \|\partial_t \mathbf{U}\|_{L^\infty(0,T;W^{1,\infty}(\Omega;\mathbb{R}^3))};$$

whence, by virtue of estimate (4.2a),

$$|R_{2,2}| \leq \sqrt{k} c(M_0, E_0, \|(\partial_t \mathbf{U}, \partial_t \nabla \mathbf{U})\|_{L^\infty(Q_T;\mathbb{R}^{12})}). \quad (6.5)$$

Step 2: *Term T_3 .* Employing the definition (3.11) of upwind quantities, we easily establish that

$$T_3 = T_{3,1} + R_{3,1},$$

$$\text{with } T_{3,1} = k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \varrho_\sigma^{n,\text{up}} (\hat{\mathbf{u}}_\sigma^{n,\text{up}} - \hat{\mathbf{U}}_{h,\sigma}^{n,\text{up}}) \cdot \mathbf{U}_{h,K}^n \mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}, \quad R_{3,1} = k \sum_{n=1}^m \sum_{\sigma \in \mathcal{E}_{\text{int}}} R_{3,1}^{n,\sigma},$$

$$\text{and } R_{3,1}^{n,\sigma} = |\sigma| \varrho_K^n \frac{|\mathbf{U}_{h,K}^n - \mathbf{U}_{h,L}^n|^2}{2} [\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}]^+ + |\sigma| \varrho_L^n \frac{|\mathbf{U}_{h,L}^n - \mathbf{U}_{h,K}^n|^2}{2} [\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,L}]^+, \quad \forall \sigma = K|L \in \mathcal{E}_{\text{int}}.$$

Employing estimates (9.1) and (9.21)_{s=1} and the continuity of the mean value $\mathbf{U}_\sigma^n = \mathbf{U}_{h,\sigma}^n$ of \mathbf{U}_h^n over faces σ , we infer

$$|R_{3,1}^{n,\sigma}| \leq h^2 c \|\nabla \mathbf{U}\|_{L^\infty(Q_T;\mathbb{R}^9)}^2 |\sigma| (\varrho_K^n + \varrho_L^n) |\mathbf{u}_\sigma^n|, \quad \forall \sigma = K|L \in \mathcal{E}_{\text{int}},$$

whence

$$\begin{aligned} |R_{3,1}| &\leq h c \|\nabla \mathbf{U}\|_{L^\infty(Q_T;\mathbb{R}^9)}^2 \left(\sum_{K \in \mathcal{T}} \sum_{\sigma=K|L \in \mathcal{E}(K)} h |\sigma| (\varrho_K^n + \varrho_L^n)^{6/5} \right)^{5/6} \left[k \sum_{n=1}^m \left(\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} h |\sigma| |\mathbf{u}_\sigma^n|^6 \right)^{1/3} \right]^{1/2} \\ &\leq h c(M_0, E_0, \|\nabla \mathbf{U}\|_{L^\infty(Q_T;\mathbb{R}^9)}), \end{aligned} \quad (6.6)$$

provided $\gamma \geq 6/5$, thanks to the Hölder inequality, the equivalence relation (3.19), the equivalence of norms (9.34) and energy bounds listed in Corollary 4.1.

Evidently, for each face $\sigma = K|L \in \mathcal{E}_{\text{int}}$, $\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K} + \mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,L} = 0$; whence, finally

$$T_{3,1} = k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \varrho_\sigma^{n,\text{up}} (\hat{\mathbf{u}}_\sigma^{n,\text{up}} - \hat{\mathbf{U}}_{h,\sigma}^{n,\text{up}}) \cdot (\mathbf{U}_{h,K}^n - \mathbf{U}_\sigma^n) \mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K} \quad (6.7)$$

Let us now decompose the term $T_{3,1}$ as

$$\begin{aligned} T_{3,1} &= T_{3,2} + R_{3,2}, \text{ with } R_{3,2} = k \sum_{n=1}^m R_{3,2}^n, \\ T_{3,2} &= k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \varrho_\sigma^{n,\text{up}} \left(\hat{\mathbf{U}}_{h,\sigma}^{n,\text{up}} - \hat{\mathbf{u}}_\sigma^{n,\text{up}} \right) \cdot \left(\mathbf{U}_\sigma^n - \mathbf{U}_{h,K}^n \right) \hat{\mathbf{u}}_\sigma^{n,\text{up}} \cdot \mathbf{n}_{\sigma,K}, \text{ and} \\ R_{3,2}^n &= \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \varrho_\sigma^{n,\text{up}} \left(\hat{\mathbf{U}}_{h,\sigma}^{n,\text{up}} - \hat{\mathbf{u}}_\sigma^{n,\text{up}} \right) \cdot \left(\mathbf{U}_\sigma^n - \mathbf{U}_{h,K}^n \right) \left(\mathbf{u}_\sigma^n - \hat{\mathbf{u}}_\sigma^{n,\text{up}} \right) \cdot \mathbf{n}_{\sigma,K}. \end{aligned}$$

By virtue of Hölder's inequality and the Taylor formula, we get

$$\begin{aligned} |R_{3,2}^n| &\leq c \|\nabla \mathbf{U}\|_{L^\infty(Q_T; \mathbb{R}^9)} \left(\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} h |\sigma| \varrho_\sigma^{n,\text{up}} \left| \hat{\mathbf{u}}_\sigma^{n,\text{up}} - \hat{\mathbf{U}}_{h,\sigma}^{n,\text{up}} \right|^2 \right)^{1/2} \\ &\quad \times \left(\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} h |\sigma| |\varrho_\sigma^{n,\text{up}}|^{\gamma_0} \right)^{1/(2\gamma_0)} \left(\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} h |\sigma| \left| \mathbf{u}_\sigma^n - \hat{\mathbf{u}}_\sigma^{n,\text{up}} \right|^q \right)^{1/q}, \end{aligned}$$

where $\frac{1}{2} + \frac{1}{2\gamma_0} + \frac{1}{q} = 1$, $\gamma_0 = \min\{\gamma, 2\}$ and $\gamma \geq 3/2$. For the sum in the last term of the above product, we have

$$\begin{aligned} &\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} h |\sigma| \left| \mathbf{u}_\sigma^n - \hat{\mathbf{u}}_\sigma^{n,\text{up}} \right|^q \leq c \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} h |\sigma| \left| \mathbf{u}_\sigma^n - \mathbf{u}_K^n \right|^q \\ &\leq c \left(\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} \left(\|\mathbf{u}_\sigma^n - \mathbf{u}^n\|_{L^q(K; \mathbb{R}^3)}^q + \sum_{K \in \mathcal{T}} \|\mathbf{u}^n - \mathbf{u}_K^n\|_{L^q(K; \mathbb{R}^3)}^q \right) \right) \leq ch^{\frac{2\gamma_0-3}{2\gamma_0}q} |\mathbf{u}^n|_{V_h^2(\Omega; \mathbb{R}^3)}^q, \end{aligned}$$

where we have used the definition (3.11), the Minkowski inequality and the interpolation inequalities (9.18–9.19). Now we can go back to the estimate of $R_{3,2}^n$ taking into account the upper bounds (4.11), (4.14–4.15), in order to get

$$|R_{3,2}| \leq h^A c(M_0, E_0, \|\nabla \mathbf{U}\|_{L^\infty(Q_T; \mathbb{R}^9)}) \quad (6.8)$$

provided $\gamma \geq 3/2$, where A is given in (6.2).

Finally, we rewrite term $T_{3,2}$ as

$$\begin{aligned} T_{3,2} &= T_{3,3} + R_{3,3}, \text{ with } R_{3,3} = k \sum_{n=1}^m R_{3,3}^n, \\ T_{3,3} &= k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \varrho_\sigma^{n,\text{up}} \left(\hat{\mathbf{U}}_{h,\sigma}^{n,\text{up}} - \hat{\mathbf{u}}_\sigma^{n,\text{up}} \right) \cdot \left(\mathbf{U}_\sigma^n - \mathbf{U}_{h,K}^n \right) \hat{\mathbf{U}}_{h,\sigma}^{n,\text{up}} \cdot \mathbf{n}_{\sigma,K}, \text{ and} \\ R_{3,3}^n &= \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \varrho_\sigma^{n,\text{up}} \left(\hat{\mathbf{U}}_{h,\sigma}^{n,\text{up}} - \hat{\mathbf{u}}_\sigma^{n,\text{up}} \right) \cdot \left(\mathbf{U}_\sigma^n - \mathbf{U}_{h,K}^n \right) \left(\hat{\mathbf{u}}_\sigma^{n,\text{up}} - \hat{\mathbf{U}}_{h,\sigma}^{n,\text{up}} \right) \cdot \mathbf{n}_{\sigma,K}; \end{aligned} \quad (6.9)$$

whence

$$|R_{3,3}| \leq c(\|\nabla \mathbf{U}\|_{L^\infty(Q_T; \mathbb{R}^9)}) k \sum_{n=1}^m \mathcal{E}(\varrho^n, \mathbf{u}^n | r^n, \mathbf{U}^n). \quad (6.10)$$

Step 3: *Term T_4 .* Using the Stokes formula and the property (9.22) in Lemma 9.2, we easily see that

$$T_4 = -k \sum_{n=1}^m \sum_{K \in \mathcal{T}} p(\varrho_K^n) \operatorname{div} \mathbf{U}^n \, dx. \quad (6.11)$$

Step 4: *Term T_5 .* Using the Taylor formula, we get

$$H'(r_K^n) - H'(r_K^{n-1}) = H''(r_K^n)(r_K^n - r_K^{n-1}) - \frac{1}{2} H'''(\bar{r}_K^n)(r_K^n - r_K^{n-1})^2,$$

where $\bar{r}_K^n \in [\min(r_K^{n-1}, r_K^n), \max(r_K^{n-1}, r_K^n)]$; we infer

$$T_5 = T_{5,1} + R_{5,1}, \quad \text{with } T_{5,1} = k \sum_{n=1}^m \sum_{K \in \mathcal{T}} |K| (r_K^n - \varrho_K^n) \frac{p'(r_K^n)}{r_K^n} \frac{r_K^n - r_K^{n-1}}{k}, \quad R_{5,1} = k \sum_{n=1}^m \sum_{K \in \mathcal{T}} R_{5,1}^{n,K}, \quad \text{and}$$

$$R_{5,1}^{n,K} = \frac{1}{2} |K| H'''(\bar{r}_K^n) \frac{(r_K^n - r_K^{n-1})^2}{k} (\varrho_K^n - r_K^n).$$

Consequently, by the Taylor formula and thanks to the mass conservation (3.13)

$$|R_{5,1}| \leq k c(M_0, \underline{r}, \bar{r}, |p'|_{C^1([\underline{r}, \bar{r}])}, \|\partial_t r\|_{L^\infty(Q_T)}), \quad (6.12)$$

where \underline{r}, \bar{r} are defined in (3.16).

Let us now decompose $T_{5,1}$ as follows:

$$T_{5,1} = T_{5,2} + R_{5,2}, \quad \text{with } T_{5,2} = k \sum_{n=1}^m \sum_{K \in \mathcal{T}} |K| (r_K^n - \varrho_K^n) \frac{p'(r_K^n)}{r_K^n} [\partial_t r]^n, \quad R_{5,2} = k \sum_{n=1}^m \sum_{K \in \mathcal{T}} R_{5,2}^{n,K}, \quad \text{and}$$

$$R_{5,2}^{n,K} = |K| (r_K^n - \varrho_K^n) \frac{p'(r_K^n)}{r_K^n} \left(\frac{r_K^n - r_K^{n-1}}{k} - [\partial_t r]^n \right). \quad (6.13)$$

In accordance with (3.15), here and in the sequel, $[\partial_t r]^n(x) = \partial_t r(t_n, x)$.

By the same means as the preceding residual term, we may estimate

$$|R_{5,2}| \leq k c(M_0, \underline{r}, \bar{r}, |p'|_{C^1([\underline{r}, \bar{r}])}, \|(\partial_t^2 r, \partial_t \nabla r)\|_{L^\infty(Q_T; \mathbb{R}^4)}). \quad (6.14)$$

Step 5: *Term T_6 .* Using the same argumentation as in formula (6.7), we may write

$$T_6 = T_{6,1} + R_{6,1}, \quad R_{6,1} = k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} R_{6,1}^{n,\sigma,K}, \quad \text{with}$$

$$T_{6,1} = k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \sum_{\sigma=K|L \in \mathcal{E}(K)} |\sigma| \varrho_K^n \left(H'(r_K^{n-1}) - H'(r_\sigma^{n-1}) \right) \mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}, \quad \text{and} \quad (6.15)$$

$$R_{6,1}^{n,\sigma,K} = |\sigma| \left(\varrho_\sigma^{n,\text{up}} - \varrho_K^n \right) \left(H'(r_K^{n-1}) - H'(r_\sigma^{n-1}) \right) \mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}, \quad \text{for } \sigma = K|L.$$

We estimate this term separately for $\gamma \leq 2$ and $\gamma > 2$. If $\gamma \leq 2$, motivated by Lemma 4.2, we may write

$$|R_{6,1}^{n,\sigma,K}| \leq \sqrt{h} \|\nabla H'(r)\|_{L^\infty(Q_T; \mathbb{R}^3)} |\sigma|$$

$$\times \left(\frac{|\varrho_\sigma^{n,\text{up}} - \varrho_K^n|}{\max(\varrho_K, \varrho_L)^{(2-\gamma)/2}} \sqrt{|\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}|} 1_{\bar{\varrho}_\sigma^n \geq 1} \sqrt{h} (\varrho_K^n + \varrho_L^n)^{(2-\gamma)/2} \sqrt{|\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}|} \right.$$

$$\left. + |\varrho_\sigma^{n,\text{up}} - \varrho_K^n| \sqrt{|\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}|} 1_{\bar{\varrho}_\sigma^n < 1} \sqrt{h} \sqrt{|\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}|} \right), \quad (6.16)$$

where we again use the Taylor formula and where the numbers $\bar{\varrho}_\sigma^n$ are defined in Lemma 4.1. Conse-

quently, an application of the Hölder and Young inequalities yields

$$\begin{aligned}
|R_{6,1}| &\leq \sqrt{h} c \|\nabla H'(r)\|_{L^\infty(Q_T; \mathbb{R}^3)} k \sum_{n=1}^m \left[\left(\sum_{K \in \mathcal{T}} \sum_{\sigma=K} \sum_{L \in \mathcal{E}(K)} |\sigma| \frac{(\varrho_\sigma^{n,\text{up}} - \varrho_K^n)^2}{\max(\varrho_K, \varrho_L)^{(2-\gamma)}} |\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}| 1_{\bar{\varrho}_\sigma^n \geq 1} \right)^{1/2} \right. \\
&\quad \times \left. \left(\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| h \varrho_K^{2-\gamma} |\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}| \right)^{1/2} \right. \\
&\quad \left. + \left(\sum_{K \in \mathcal{T}} \sum_{\sigma=K} \sum_{L \in \mathcal{E}(K)} |\sigma| h (\varrho_\sigma^{n,\text{up}} - \varrho_K^n)^2 |\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}| 1_{\bar{\varrho}_\sigma^n < 1} \right)^{1/2} \left(\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| h |\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}| \right)^{1/2} \right] \\
&\leq \sqrt{h} c \|\nabla H'(r)\|_{L^\infty(Q_T; \mathbb{R}^3)} k \sum_{n=1}^m \left[\left(\sum_{K \in \mathcal{T}} \sum_{\sigma=K} \sum_{L \in \mathcal{E}(K)} |\sigma| \frac{(\varrho_\sigma^{n,\text{up}} - \varrho_K^n)^2}{\max(\varrho_K, \varrho_L)^{(2-\gamma)}} |\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}| 1_{\bar{\varrho}_\sigma^n \geq 1} \right. \right. \\
&\quad \left. \left. + \left(\sum_{K \in \mathcal{T}} |K| \varrho_K^{6(2-\gamma)/5} \right)^{5/6} \left(\sum_{\sigma \in \mathcal{E}} |\sigma| h |\mathbf{u}_\sigma^n|^6 \right)^{1/6} \right. \right. \\
&\quad \left. \left. + \sum_{K \in \mathcal{T}} \sum_{\sigma=K} \sum_{L \in \mathcal{E}(K)} |\sigma| h (\varrho_\sigma^{n,\text{up}} - \varrho_K^n)^2 |\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}| 1_{\bar{\varrho}_\sigma^n < 1} + |\Omega|^{5/6} \left(\sum_{\sigma \in \mathcal{E}} |\sigma| h |\mathbf{u}_\sigma^n|^6 \right)^{1/6} \right] \\
&\leq \sqrt{h} c(M_0, E_0, \underline{r}, \bar{r}, |p'|_{C([\underline{r}, \bar{r}])}, \|\nabla r\|_{L^\infty(Q_T; \mathbb{R}^3)})
\end{aligned} \tag{6.17}$$

provided $\gamma \geq 12/11$, where we use estimate (4.17), estimates (4.12), (4.14) of Corollary 4.1 and equivalence relation (9.34). In the case $\gamma > 2$, the same final bound may be obtained by a similar argument, replacing the estimate (4.17) by (4.16).

Let us now decompose the term $T_{6,1}$ as

$$\begin{aligned}
T_{6,1} &= T_{6,2} + R_{6,2}, \text{ with } T_{6,2} = k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \sum_{\sigma=K} \sum_{L \in \mathcal{E}(K)} |\sigma| \varrho_K^n H''(r_K^{n-1})(r_K^{n-1} - r_\sigma^{n-1}) [\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}], \\
R_{6,2} &= k \sum_{n=1}^m \sum_{K \in \mathcal{K}} \sum_{\sigma \in \mathcal{E}(K)} R_{6,2}^{n,\sigma,K}, \text{ and} \\
R_{6,2}^{n,\sigma,K} &= |\sigma| \varrho_K^n \left(H'(r_K^{n-1}) - H'(r_\sigma^{n-1}) - H''(r_K^{n-1})(r_K^{n-1} - r_\sigma^{n-1}) \right) [\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}]
\end{aligned}$$

Therefore, by virtue of the Taylor formula, Hölder's inequality, (9.29), (9.34), and (4.11), (4.14) in Corollary 4.1, we have, provided $\gamma \geq 6/5$,

$$\begin{aligned}
|R_{6,2}| &\leq hc \left(|H''|_{C([\underline{r}, \bar{r}])} + |H'''|_{C([\underline{r}, \bar{r}])} \right) \|\nabla r\|_{L^\infty(Q_T; \mathbb{R}^3)} \|\varrho\|_{L^\infty(0,T; L^\gamma(\Omega))} \|\mathbf{u}\|_{L^2(0,T; V_h^2(\Omega; \mathbb{R}^3))} \\
&\leq h c(M_0, E_0, \underline{r}, \bar{r}, |p'|_{C^1([\underline{r}, \bar{r}])}, \|\nabla r\|_{L^\infty(Q_T; \mathbb{R}^3)}),
\end{aligned} \tag{6.18}$$

where in the first line we have used notation (4.9).

Let us now deal with the term $T_{6,2}$. Noting that $\int_K \nabla r^{n-1} dx = \sum_{\sigma \in \mathcal{E}(K)} |\sigma| (r_\sigma^{n-1} - r_K^{n-1}) \mathbf{n}_{\sigma,K}$, we may write

$$\begin{aligned}
&\sum_{\sigma \in \mathcal{E}(K)} |\sigma| \varrho_K^n H''(r_K^{n-1})(r_K^{n-1} - r_\sigma^{n-1}) [\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}] \\
&= - \int_K \varrho_K^n H''(r_K^{n-1}) \mathbf{u}_K^n \cdot \nabla r^{n-1} dx + \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \varrho_K^n H''(r_K^{n-1})(r_K^{n-1} - r_\sigma^{n-1}) (\mathbf{u}_\sigma^n - \mathbf{u}_K^n) \cdot \mathbf{n}_{\sigma,K}.
\end{aligned}$$

Consequently, $T_{6,2} = T_{6,3} + R_{6,3}$, with

$$\begin{aligned} T_{6,3} &= -k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \int_K \varrho_K^n H''(r_K^{n-1}) \mathbf{u}^n \cdot \nabla r^{n-1} \, dx, \\ R_{6,3} &= k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \int_K \varrho_K^n H''(r_K^{n-1}) (\mathbf{u}^n - \mathbf{u}_K^n) \cdot \nabla r^{n-1} \, dx \\ &\quad + k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \varrho_K^n H''(r_K^{n-1}) (r_K^{n-1} - r_\sigma^{n-1}) (\mathbf{u}_\sigma^n - \mathbf{u}_K^n) \cdot \mathbf{n}_{\sigma,K}, \end{aligned}$$

where, by virtue of Hölder's inequality, (9.18), (9.19), and (4.11), (4.14) in Corollary 4.1,

$$|R_{6,3}| \leq h^A c(M_0, E_0, \underline{r}, \bar{r}, |p'|_{C^1([\underline{r}, \bar{r}] \| \nabla r \|_{L^\infty(Q_T; \mathbb{R}^3)}), \quad (6.19)$$

where A is defined in (6.2).

Finally we write $T_{6,3} = T_{6,4} + R_{6,4}$, with

$$\begin{aligned} T_{6,4} &= -k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \int_K \varrho_K^n \frac{p'(r_K^n)}{r_K^n} \mathbf{u}^n \cdot \nabla r^n \, dx, \\ R_{6,4} &= k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \int_K \varrho_K^n \left(H''(r_K^n) \nabla r^n - H''(r_K^{n-1}) \nabla r^{n-1} \right) \cdot \mathbf{u}^n \, dx, \end{aligned} \quad (6.20)$$

where

$$|R_{6,4}| \leq k c(M_0, E_0, \underline{r}, \bar{r}, |p'|_{C^1([\underline{r}, \bar{r}] \| \nabla r, \partial_t r, \partial_t \nabla r \|_{L^\infty(Q_T; \mathbb{R}^7)}). \quad (6.21)$$

We are now in position to conclude the proof of Lemma 6.1: we obtain the inequality (6.1) by gathering the principal terms (6.4), (6.9), (6.11), (6.13), (6.20) and the residual terms estimated in (6.3), (6.5), (6.6), (6.8), (6.10), (6.12), (6.14), (6.16), (6.17), (6.18), (6.19), (6.21) at the right hand side $\sum_{i=1}^6 T_i$ of the discrete relative energy inequality (5.1). \square

7 - A discrete identity satisfied by the strong solution

This section is devoted to the proof of a discrete identity satisfied by any strong solution. This identity is stated in Lemma 7.1 below. It will be used in combination with the approximate relative energy inequality stated in Lemma 6.1 to deduce the convenient form of the relative energy inequality verified by any function being a strong solution to the compressible Navier-Stokes system. This last step is performed in the next section.

Lemma 7.1 (A discrete identity for strong solutions). *Suppose that $\Omega \subset \mathbb{R}^3$ is a bounded polyhedral domain and \mathcal{T} a regular triangulation introduced in Section 3.1. Let the pressure p be a $C^2(0, \infty)$ function satisfying hypotheses (1.5) and (1.6) with $\gamma \geq 3/2$. Let (r, \mathbf{U}) belong to the class (3.16) satisfy equation (1.1) with the viscosity coefficients μ, λ obeying (1.4).*

Let $(\varrho^0, \mathbf{u}^0) \in L_h^+(\Omega) \times \mathbf{W}_h(\Omega)$ and suppose that $(\varrho^n)_{1 \leq n \leq N} \in [L_h^+(\Omega)]^N$, $(\mathbf{u}^n)_{1 \leq n \leq N} \in [\mathbf{W}_h(\Omega)]^N$ is a solution of the discrete problem (3.12). Then for any $\delta > 0$ there exists

$$c = c\left(M_0, E_0, \underline{r}, \bar{r}, |p'|_{C^1([\underline{r}, \bar{r}] \| (\nabla r, \partial_t r, \mathbf{U}, \nabla \mathbf{U}, \nabla^2 \mathbf{U}, \partial_t \mathbf{U}, \partial_t^2 \mathbf{U}, \partial_t \nabla \mathbf{U},) \|_{L^\infty(Q_T; \mathbb{R}^{58})})} \right) > 0,$$

such that for any $m = 1, \dots, N$, the following identity holds:

$$\begin{aligned}
& k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \int_K \left(\mu \nabla \mathbf{U}^n \cdot \nabla (\mathbf{u}^n - \mathbf{U}_h^n) + (\mu + \lambda) \operatorname{div} \mathbf{U}^n \operatorname{div} (\mathbf{u}^n - \mathbf{U}_h^n) \right) dx \\
& + k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \int_K r_K^{n-1} \frac{\mathbf{U}_{h,K}^n - \mathbf{U}_{h,K}^{n-1}}{k} \cdot (\mathbf{u}_K^n - \mathbf{U}_{h,K}^n) dx \\
& + k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \hat{r}_\sigma^{n,\text{up}} [\hat{\mathbf{U}}_{h,\sigma}^{n,\text{up}} \cdot \mathbf{n}_{\sigma,K}] (\mathbf{U}_\sigma^n - \mathbf{U}_{h,K}^n) \cdot (\hat{\mathbf{u}}_\sigma^{n,\text{up}} - \hat{\mathbf{U}}_{h,\sigma}^{n,\text{up}}) \\
& + k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \int_K p(r_K^n) \operatorname{div} \mathbf{U}^n dx + k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \int_K p'(r_K^n) \mathbf{u}^n \cdot \nabla r^n dx + \mathcal{R}_{h,k}^m = 0,
\end{aligned} \tag{7.1}$$

where

$$|\mathcal{R}_{h,k}^m| \leq c(h + k).$$

Before starting the proof we recall an auxiliary algebraic inequality whose straightforward proof is left to the reader, and introduce some notations.

Lemma 7.2. *Let p satisfies assumptions (1.5) and (1.6). Let $0 < a < b < \infty$. Then there exists $c = c(a, b) > 0$ such that for all $\varrho \in [0, \infty)$ and $r \in [a, b]$ there holds*

$$E(\varrho|r) \geq c(a, b) \left(1_{\mathcal{O}_{\text{res}}} + \varrho^\gamma 1_{\mathcal{O}_{\text{res}}} + (\varrho - r)^2 1_{\mathcal{O}_{\text{ess}}} \right),$$

where $E(\varrho|r)$ is defined in (2.5) and

$$\mathcal{O}_{\text{ess}} = [a/2, 2b], \quad \mathcal{O}_{\text{res}} = R_+ \setminus [a/2, 2b].$$

For \underline{r} and \bar{r} defined in (3.16), we define for a.e. $t \in (0, T)$ the residual and essential subsets of Ω as follows:

$$N_{\text{ess}}^n = \left\{ x \in \Omega \mid \frac{1}{2}\underline{r} \leq \varrho^n(x) \leq 2\bar{r} \right\}, \quad N_{\text{res}}^n = \Omega \setminus N_{\text{ess}}^n, \tag{7.2}$$

where $\varrho^n \in L_h(\Omega)$, and where, for a function g defined a.e. in Ω , we define:

$$[g]_{\text{ess}} = g 1_{N_{\text{ess}}^n}, \quad [g]_{\text{res}} = g 1_{N_{\text{res}}^n}.$$

Lemma 7.2 implies that for any pair (r, \mathbf{U}) belonging to the class (3.16) and any $\varrho^n \in L_h(\Omega)$, there exists $c(\underline{r}, \bar{r}) > 0$ such that

$$c(\underline{r}, \bar{r}) \sum_{K \in \mathcal{T}} \int_K \left([1]_{\text{res}} + [(\varrho^n)^\gamma]_{\text{res}} + [\varrho^n - \hat{r}^n]_{\text{ess}}^2 \right) dx \leq \mathcal{E}(\varrho^n, \mathbf{u}^n | r^n, \mathbf{U}^n). \tag{7.3}$$

We are now ready to proceed to the proof of Lemma 7.1.

Proof. We start by projecting the momentum equation to the discrete spaces. Since (r, \mathbf{U}) satisfies (1.1) and belongs to the class (3.16), Equation (1.1b) can be rewritten in the form

$$r \partial_t \mathbf{U} + r \mathbf{U} \cdot \nabla \mathbf{U} + \nabla p(r) = \mu \Delta \mathbf{U} + (\mu + \lambda) \nabla \operatorname{div} \mathbf{U}. \tag{7.4}$$

We write equation (7.4) at $t = t_n$, multiply scalarly by $\mathbf{u}^n - \mathbf{U}_h^n$, and integrate over Ω . We get, after summation from $n = 1$ to m ,

$$\begin{aligned}
\sum_{i=1}^5 \mathcal{T}_i &= 0, & \text{with} & \quad \mathcal{T}_1 = -k \sum_{n=1}^m \int_{\Omega} \left(\mu \Delta \mathbf{U}^n + (\mu + \lambda) \nabla \operatorname{div} \mathbf{U}^n \right) \cdot (\mathbf{u}^n - \mathbf{U}_h^n) dx, \\
\mathcal{T}_2 &= k \sum_{n=1}^m \int_{\Omega} r^n [\partial_t \mathbf{U}]^n \cdot (\mathbf{u}^n - \mathbf{U}_h^n) dx, & \mathcal{T}_3 &= k \sum_{n=1}^m \int_{\Omega} r^n \mathbf{U}^n \cdot \nabla \mathbf{U}^n \cdot (\mathbf{u}^n - \mathbf{U}_h^n) dx \\
\mathcal{T}_4 &= k \sum_{n=1}^m \int_{\Omega} \nabla p(r^n) \cdot \mathbf{u}^n dx, & \mathcal{T}_5 &= -k \sum_{n=1}^m \int_{\Omega} \nabla p(r^n) \cdot \mathbf{U}_h^n dx = 0.
\end{aligned} \tag{7.5}$$

In the steps below, we deal with each of the terms \mathcal{T}_i .

Step 1: *Term \mathcal{T}_1 .* Integrating by parts, we get:

$$\mathcal{T}_1 = \mathcal{T}_{1,1} + \mathcal{R}_{1,1},$$

$$\text{with } \mathcal{T}_{1,1} = k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \int_K \left(\mu \nabla \mathbf{U}^n : \nabla (\mathbf{u}^n - \mathbf{U}_h^n) + (\mu + \lambda) \operatorname{div} \mathbf{U}^n \operatorname{div} (\mathbf{u}^n - \mathbf{U}_h^n) \right) dx$$

and

$$\begin{aligned} \mathcal{R}_{1,1} &= -k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} \int_{\sigma} \left(\mu \mathbf{n}_{\sigma,K} \cdot \nabla \mathbf{U}^n \cdot (\mathbf{u}^n - \mathbf{U}_h^n) + (\lambda + \mu) \operatorname{div} \mathbf{U}^n (\mathbf{u}^n - \mathbf{U}_h^n) \cdot \mathbf{n}_{\sigma,K} \right) dS \\ &= -k \sum_{n=1}^m \sum_{\sigma \in \mathcal{E}} \int_{\sigma} \left(\mu \mathbf{n}_{\sigma} \cdot \nabla \mathbf{U}^n \cdot [\mathbf{u}^n - \mathbf{U}_h^n]_{\sigma, \mathbf{n}_{\sigma}} + (\lambda + \mu) \operatorname{div} \mathbf{U}^n [\mathbf{u}^n - \mathbf{U}_h^n]_{\sigma, \mathbf{n}_{\sigma}} \cdot \mathbf{n}_{\sigma} \right) dS, \end{aligned} \quad (7.6)$$

where in the last line \mathbf{n}_{σ} is a unit normal to σ and $[\cdot]_{\sigma, \mathbf{n}_{\sigma}}$ is the jump over sigma (with respect to \mathbf{n}_{σ}) defined in Lemma 9.5. Since the integral over any face σ of the jump of a function from $V_h(\Omega)$ is zero, we may write

$$\mathcal{R}_{1,1} = -k \sum_{n=1}^m \sum_{\sigma \in \mathcal{E}} \int_{\sigma} \left(\mu \mathbf{n}_{\sigma} \cdot (\nabla \mathbf{U}^n - [\nabla \mathbf{U}^n]_{\sigma}) \cdot [\mathbf{u}^n - \mathbf{U}_h^n]_{\sigma, \mathbf{n}_{\sigma}} + (\lambda + \mu) (\operatorname{div} \mathbf{U}^n - [\operatorname{div} \mathbf{U}^n]_{\sigma}) [\mathbf{u}^n - \mathbf{U}_h^n]_{\sigma, \mathbf{n}_{\sigma}} \cdot \mathbf{n}_{\sigma} \right) dS;$$

whence by using the Taylor formula and Hölder's inequality

$$\begin{aligned} |\mathcal{R}_{1,1}| &\leq k h c \|\nabla^2 \mathbf{U}\|_{L^\infty(Q_T; \mathbb{R}^{27})} \sum_{n=1}^m \sum_{\sigma \in \mathcal{E}} \sqrt{|\sigma|} \sqrt{h} \left(\frac{1}{\sqrt{h}} \left\| [\mathbf{u}^n - \mathbf{U}_h^n]_{\sigma, \mathbf{n}_{\sigma}} \right\|_{L^2(\sigma; \mathbb{R}^3)} \right) \\ &\leq k h c \|\nabla^2 \mathbf{U}\|_{L^\infty(Q_T; \mathbb{R}^{27})} \sum_{n=1}^m \sum_{\sigma \in \mathcal{E}} \left(|\sigma| h + \frac{1}{h} \left\| [\mathbf{u}^n - \mathbf{U}_h^n]_{\sigma, \mathbf{n}_{\sigma}} \right\|_{L^2(\sigma; \mathbb{R}^3)}^2 \right) \\ &\leq h c (M_0, E_0, \|\mathbf{U}, \nabla \mathbf{U}, \nabla^2 \mathbf{U}\|_{L^\infty(Q_T; \mathbb{R}^{39})}), \end{aligned} \quad (7.7)$$

where we have employed Lemma 9.5 and (4.11) in Corollary 4.1 in order to get the last line.

Step 2: *Term \mathcal{T}_2 .* Let us now decompose the term \mathcal{T}_2 as

$$\mathcal{T}_2 = \mathcal{T}_{2,1} + \mathcal{R}_{2,1},$$

$$\text{with } \mathcal{T}_{2,1} = k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \int_K r^{n-1} \frac{\mathbf{U}^n - \mathbf{U}^{n-1}}{k} \cdot (\mathbf{u}^n - \mathbf{U}_h^n) dx, \quad \mathcal{R}_{2,1} = k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \mathcal{R}_{2,1}^{n,K},$$

$$\text{and } \mathcal{R}_{2,1}^{n,K} = \int_K (r^n - r^{n-1}) [\partial_t \mathbf{U}]^n \cdot (\mathbf{u}^n - \mathbf{U}_h^n) dx + \int_K r^{n-1} \left([\partial_t \mathbf{U}]^n - \frac{\mathbf{U}^n - \mathbf{U}^{n-1}}{k} \right) \cdot (\mathbf{u}^n - \mathbf{U}_h^n) dx.$$

We have

$$|\mathcal{R}_{2,1}^{n,K}| \leq k \left[(\|r\|_{L^\infty(Q_T)} + \|\partial_t r\|_{L^\infty(Q_T)}) (\|\partial_t \mathbf{U}\|_{L^\infty(Q_T; \mathbb{R}^3)} + \|\partial_t^2 \mathbf{U}\|_{L^\infty(Q_T; \mathbb{R}^3)}) \sqrt{|K|} (\|\mathbf{u}^n\|_{L^2(K)} + \|\mathbf{U}_h^n\|_{L^2(K)}); \right]$$

whence

$$|\mathcal{R}_{2,1}| \leq k c (M_0, E_0, \bar{r}, \|(\partial_t r, \mathbf{U}, \partial_t \mathbf{U}, \nabla \mathbf{U}, \partial_t^2 \mathbf{U})\|_{L^\infty(Q_T; \mathbb{R}^{19})}), \quad (7.8)$$

thanks to the Hölder and Young inequalities, to the estimates (9.20), (9.23), (9.28), (9.29), and to the energy bound (4.11) from Corollary 4.1.

Step 2a: *Term $\mathcal{T}_{2,1}$.* We decompose the term $\mathcal{T}_{2,1}$ as

$$\mathcal{T}_{2,1} = \mathcal{T}_{2,2} + \mathcal{R}_{2,2},$$

$$\text{with } \mathcal{T}_{2,2} = k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \int_K r_K^{n-1} \frac{\mathbf{U}^n - \mathbf{U}^{n-1}}{k} \cdot (\mathbf{u}^n - \mathbf{U}_h^n) dx, \quad \mathcal{R}_{2,2} = k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \mathcal{R}_{2,2}^{n,K},$$

$$\text{and } \mathcal{R}_{2,2}^{n,K} = \int_K (r^{n-1} - r_K^{n-1}) \frac{\mathbf{U}^n - \mathbf{U}^{n-1}}{k} \cdot (\mathbf{u}^n - \mathbf{U}_h^n) dx;$$

therefore,

$$|\mathcal{R}_{2,2}^n| = \left| \sum_{K \in \mathcal{T}} \mathcal{R}_{2,2}^{n,K} \right| \leq h c \|\nabla r\|_{L^\infty(Q_T; \mathbb{R}^3)} \|\partial_t \mathbf{U}\|_{L^\infty(Q_T; \mathbb{R}^3)} \|\mathbf{u}^n - \mathbf{U}_h^n\|_{L^6(\Omega; \mathbb{R}^3)}.$$

Consequently, by virtue of formula (4.12) in Corollary 4.1 and estimates (9.29), (9.24),

$$|\mathcal{R}_{2,2}| \leq h c(M_0, E_0, \|(\nabla r, \mathbf{U}, \partial_t \mathbf{U}, \nabla \mathbf{U})\|_{L^\infty(Q_T; \mathbb{R}^{18})}). \quad (7.9)$$

Step 2b: *Term* $\mathcal{T}_{2,2}$. We decompose the term $\mathcal{T}_{2,2}$ as

$$\begin{aligned} \mathcal{T}_{2,2} &= \mathcal{T}_{2,3} + \mathcal{R}_{2,3}, \\ \text{with } \mathcal{T}_{2,3} &= k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \int_K r_K^{n-1} \frac{\mathbf{U}_{h,K}^n - \mathbf{U}_{h,K}^{n-1}}{k} \cdot (\mathbf{u}^n - \mathbf{U}_h^n) \, dx, \quad \mathcal{R}_{2,3} = k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \mathcal{R}_{2,3}^{n,K}, \\ \text{and } \mathcal{R}_{2,3}^{n,K} &= \int_K r_K^{n-1} \left(\frac{\mathbf{U}^n - \mathbf{U}^{n-1}}{k} - \left[\frac{\mathbf{U}^n - \mathbf{U}^{n-1}}{k} \right]_h \right) \cdot (\mathbf{u}^n - \mathbf{U}_h^n) \, dx \\ &+ \int_K r_K^{n-1} \left(\left[\frac{\mathbf{U}^n - \mathbf{U}^{n-1}}{k} \right]_h - \left[\frac{\mathbf{U}^n - \mathbf{U}^{n-1}}{k} \right]_{h,K} \right) \cdot (\mathbf{u}^n - \mathbf{U}_h^n) \, dx. \end{aligned}$$

Therefore,

$$|\mathcal{R}_{2,3}^n| = \left| \sum_{K \in \mathcal{T}} \mathcal{R}_{2,3}^{n,K} \right| \leq h c \|r\|_{L^\infty(Q_T)} \|\partial_t \mathbf{U}, \partial_t \nabla \mathbf{U}\|_{L^\infty(Q_T; \mathbb{R}^{13})} \|\mathbf{u}^n - \mathbf{U}_h^n\|_{L^6(\Omega; \mathbb{R}^3)},$$

where we have used the Taylor formula, Hölder's inequality, (9.20)_{s=1}, (9.1), (9.14), (9.28). Consequently, by virtue of formula (4.12) in Corollary 4.1 and estimates (9.29), (9.24),

$$|\mathcal{R}_{2,3}| \leq h c(M_0, E_0, \|(\nabla r, \mathbf{U}, \nabla \mathbf{U}, \partial_t \mathbf{U}, \partial_t \nabla \mathbf{U})\|_{L^\infty(Q_T; \mathbb{R}^{25})}). \quad (7.10)$$

Step 2c: *Term* $\mathcal{T}_{2,3}$. We rewrite this term in the form

$$\begin{aligned} \mathcal{T}_{2,3} &= \mathcal{T}_{2,4} + \mathcal{R}_{2,4}, \quad \mathcal{R}_{2,4} = k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \mathcal{R}_{2,4}^{n,K}, \\ \text{with } \mathcal{T}_{2,4} &= k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \int_K r_K^{n-1} \frac{\mathbf{U}_{h,K}^n - \mathbf{U}_{h,K}^{n-1}}{k} \cdot (\mathbf{u}_K^n - \mathbf{U}_{h,K}^n) \, dx, \\ \text{and } \mathcal{R}_{2,4}^{n,K} &= \int_K r_K^{n-1} \frac{\mathbf{U}_{h,K}^n - \mathbf{U}_{h,K}^{n-1}}{k} \cdot \left((\mathbf{u}^n - \mathbf{u}_K^n) - (\mathbf{U}_h^n - \mathbf{U}_{h,K}^n) \right) \, dx. \end{aligned} \quad (7.11)$$

Thanks to the Hölder inequality, to the estimates (9.1), (9.14), (9.21)_{s=1}, (9.28) and finally to the estimate (4.11) in Corollary 4.1, we then get:

$$|\mathcal{R}_{2,4}| \leq h c(M_0, E_0, \bar{r}, \|(\partial_t \mathbf{U}, \mathbf{U}, \nabla \mathbf{U}, \partial_t \nabla \mathbf{U})\|_{L^\infty(Q_T; \mathbb{R}^{24})}). \quad (7.12)$$

Step 3: *Term* \mathcal{T}_3 . Let us first decompose \mathcal{T}_3 as

$$\begin{aligned} \mathcal{T}_3 &= \mathcal{T}_{3,1} + \mathcal{R}_{3,1}, \\ \text{with } \mathcal{T}_{3,1} &= k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \int_K r_K^n \mathbf{U}_{h,K}^n \cdot \nabla \mathbf{U}^n \cdot (\mathbf{u}_K^n - \mathbf{U}_{h,K}^n) \, dx, \quad \mathcal{R}_{3,1} = k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \mathcal{R}_{3,1}^{n,K}, \\ \text{and } \mathcal{R}_{3,1}^{n,K} &= \int_K (r^n - r_K^n) \mathbf{U}^n \cdot \nabla \mathbf{U}^n \cdot (\mathbf{u}^n - \mathbf{U}_h^n) \, dx + \int_K r_K^n (\mathbf{U}^n - \mathbf{U}_h^n) \cdot \nabla \mathbf{U}^n \cdot (\mathbf{u}^n - \mathbf{U}_h^n) \, dx \\ &+ \int_K r_K^n (\mathbf{U}_h^n - \mathbf{U}_{h,K}^n) \cdot \nabla \mathbf{U}^n \cdot (\mathbf{u}^n - \mathbf{U}_h^n) \, dx + \int_K r_K^n \mathbf{U}_{h,K}^n \cdot \nabla \mathbf{U}^n \cdot \left(\mathbf{u}^n - \mathbf{u}_K^n - (\mathbf{U}_h^n - \mathbf{U}_{h,K}^n) \right) \, dx. \end{aligned}$$

We find that

$$|\mathcal{R}_{3,1}^{n,K}| \leq h \left[|K|^{1/2} (\|\mathbf{u}^n\|_{L^2(K;\mathbb{R}^3)} + \|\mathbf{U}_h^n\|_{L^2(K;\mathbb{R}^3)}) + |K|^{1/2} (\|\nabla \mathbf{u}^n\|_{L^2(K;\mathbb{R}^3)} + \|\nabla \mathbf{U}_h^n\|_{L^2(K;\mathbb{R}^3)}) \right] \\ \times \left(\|r\|_{L^\infty(Q_T)} + \|\nabla r\|_{L^\infty(Q_T;\mathbb{R}^3)} \right) \left(\|\mathbf{U}\|_{L^\infty(Q_T;\mathbb{R}^3)} + \|\nabla \mathbf{U}\|_{L^\infty(Q_T;\mathbb{R}^9)} \right)^2,$$

where we have used several times Hölder's inequality and the standard first order Taylor formula, along with the estimates (9.20) (to evaluate $\mathbf{U}^n - \mathbf{U}_h^n$), (9.1), (9.21)_{s=1} (to evaluate $\mathbf{U}_h^n - \mathbf{U}_{h,K}^n$), (9.1) (to evaluate $\mathbf{u}^n - \mathbf{u}_K^n$).

Consequently, using again (9.21)_{s=1} (to estimate $\|\nabla \mathbf{U}_h^n\|_{L^2(K;\mathbb{R}^3)}$), the definition of $|\cdot|_{V_h^2(\Omega)}$ norm, the Sobolev inequality (9.29) and the energy bound (4.11) from Corollary 4.1, we conclude that

$$|\mathcal{R}_{3,1}| \leq h c(M_0, E_0, \bar{r}, \|(\nabla r, \mathbf{U}, \nabla \mathbf{U})\|_{L^\infty(Q_T;\mathbb{R}^{15})}). \quad (7.13)$$

Now we shall deal with term $\mathcal{T}_{3,1}$. Integrating by parts, we get:

$$\int_K r_K^n \mathbf{U}_{h,K}^n \cdot \nabla \mathbf{U}^n \cdot (\mathbf{u}_K^n - \mathbf{U}_{h,K}^n) dx = \sum_{\sigma \in \mathcal{E}(K)} |\sigma| r_K^n [\mathbf{U}_{h,K}^n \cdot \mathbf{n}_{\sigma,K}] \mathbf{U}_\sigma^n \cdot (\mathbf{u}_K^n - \mathbf{U}_{h,K}^n) \\ = \sum_{\sigma \in \mathcal{E}(K)} |\sigma| r_K^n [\mathbf{U}_{h,K}^n \cdot \mathbf{n}_{\sigma,K}] (\mathbf{U}_\sigma^n - \mathbf{U}_{h,K}^n) \cdot (\mathbf{u}_K^n - \mathbf{U}_{h,K}^n),$$

thanks to the fact that $\sum_{\sigma \in \mathcal{E}(K)} \int_\sigma \mathbf{U}_{h,K}^n \cdot \mathbf{n}_{\sigma,K} dS = 0$.

Next we write

$$\mathcal{T}_{3,1} = \mathcal{T}_{3,2} + \mathcal{R}_{3,2}, \quad \mathcal{R}_{3,2} = k \sum_{n=1}^m \mathcal{R}_{3,2}^n,$$

$$\mathcal{T}_{3,2} = k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \hat{r}_\sigma^{n,\text{up}} [\hat{\mathbf{U}}_{h,\sigma}^{n,\text{up}} \cdot \mathbf{n}_{\sigma,K}] (\mathbf{U}_\sigma^n - \mathbf{U}_{h,K}^n) \cdot (\hat{\mathbf{u}}_\sigma^{n,\text{up}} - \hat{\mathbf{U}}_{h,\sigma}^{n,\text{up}}), \quad (7.14)$$

$$\text{and } \mathcal{R}_{3,2}^n = \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| (r_K^n - \hat{r}_\sigma^{n,\text{up}}) [\mathbf{U}_{h,K}^n \cdot \mathbf{n}_{\sigma,K}] (\mathbf{U}_\sigma^n - \mathbf{U}_{h,K}^n) \cdot (\mathbf{u}_K^n - \mathbf{U}_{h,K}^n) \\ + \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \hat{r}_\sigma^{n,\text{up}} \left[(\mathbf{U}_{h,K}^n - \hat{\mathbf{U}}_{h,\sigma}^{n,\text{up}}) \cdot \mathbf{n}_{\sigma,K} \right] (\mathbf{U}_\sigma^n - \mathbf{U}_{h,K}^n) \cdot (\mathbf{u}_K^n - \mathbf{U}_{h,K}^n) \\ + \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \hat{r}_\sigma^{n,\text{up}} [\hat{\mathbf{U}}_{h,\sigma}^{n,\text{up}} \cdot \mathbf{n}_{\sigma,K}] (\mathbf{U}_\sigma^n - \mathbf{U}_{h,K}^n) \cdot \left((\mathbf{u}_K^n - \hat{\mathbf{u}}_{h,\sigma}^{n,\text{up}}) - (\mathbf{U}_{h,K}^n - \hat{\mathbf{U}}_{h,\sigma}^{n,\text{up}}) \right).$$

We may use several times the Taylor formula to get the bound

$$|\mathcal{R}_{3,2}^n| \leq h c \|r\|_{W^{1,\infty}(\Omega)} \left(1 + \|\mathbf{U}\|_{W^{1,\infty}(\Omega;\mathbb{R}^3)} \right)^3 \sum_{K \in \mathcal{T}} h |\sigma| |\mathbf{u}_K^n| \\ + c \|r\|_{W^{1,\infty}(\Omega)} \left(1 + \|\mathbf{U}\|_{W^{1,\infty}(\Omega;\mathbb{R}^3)} \right)^2 \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} h |\sigma| |\mathbf{u}_K^n - \mathbf{u}_\sigma^n|,$$

where by virtue of Hölder's inequality, (9.16), (9.30), (9.18) (9.19),

$$\sum_{K \in \mathcal{T}} h |\sigma| |\mathbf{u}_K^n| \leq \left(\sum_{\sigma \in \mathcal{T}} h |\sigma| |\mathbf{u}_K^n|^6 \right)^{1/6} \leq c \left[\left(\sum_{K \in \mathcal{T}} \|\mathbf{u}^n - \mathbf{u}_K^n\|_{L^6(K;\mathbb{R}^3)}^6 \right)^{1/6} + \|\mathbf{u}^n\|_{L^6(K;\mathbb{R}^3)} \right] \leq c |\mathbf{u}^n|_{V_h^2(\Omega;\mathbb{R}^3)},$$

$$\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} h |\sigma| |\mathbf{u}_k^n - \mathbf{u}_\sigma^n| \leq c \left[\left(\sum_{K \in \mathcal{T}} \|\mathbf{u}^n - \mathbf{u}_K^n\|_{L^2(K; \mathbb{R}^3)}^2 \right)^{1/2} + \left(\sum_{K \in \mathcal{T}} \|\mathbf{u}^n - \hat{\mathbf{u}}_\sigma^n\|_{L^6(K; \mathbb{R}^3)}^2 \right)^{1/2} \right] \leq h c |\mathbf{u}_n|_{V_h^2(\Omega; \mathbb{R}^3)}.$$

Consequently, we may use (4.11) to conclude

$$|\mathcal{R}_{3,2}| \leq h c (M_0, E_0, \bar{r}, \|\nabla r, \mathbf{U}, \nabla \mathbf{U}\|_{L^\infty(Q_T; \mathbb{R}^{15})}). \quad (7.15)$$

Step 4: *Terms \mathcal{T}_4 and \mathcal{T}_5 .* We decompose \mathcal{T}_4 as

$$\begin{aligned} \mathcal{T}_4 &= \mathcal{T}_{4,1} + \mathcal{R}_{4,1}, \\ \text{with } \mathcal{T}_{4,1} &= k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \int_K p'(r_K^n) \mathbf{u}^n \cdot \nabla r^n \, dx, \quad \mathcal{R}_{4,1} = k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \int_K (p'(r^n) - p'(r_K^n)) \mathbf{u}^n \cdot \nabla r^n \, dx; \end{aligned} \quad (7.16)$$

whence

$$|\mathcal{R}_{4,1}| \leq h c (M_0, E_0, \underline{r}, \bar{r}, |p'|_{C^1([\underline{r}, \bar{r})]}, \|\nabla r\|_{L^\infty(Q_T; \mathbb{R}^3)}). \quad (7.17)$$

Employing integration by parts, we infer

$$\begin{aligned} \mathcal{T}_5 &= \mathcal{T}_{5,1} + \mathcal{R}_{5,1}, \quad \text{with } \mathcal{T}_{5,1} = k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \int_K p(r_K^n) \operatorname{div} \mathbf{U}^n \, dx, \\ \mathcal{R}_{5,1} &= k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \int_K \left[\nabla p(r^n) \cdot (\mathbf{U}^n - \mathbf{U}_h^n) + (p(r^n) - p(r_K^n)) \operatorname{div} \mathbf{U}^n \right] dx, \end{aligned} \quad (7.18)$$

and

$$|\mathcal{R}_{5,1}| \leq h c (\bar{r}, |p'|_{C([\underline{r}, \bar{r})]}, \|\nabla r, \nabla \mathbf{U}\|_{L^\infty(Q_T; \mathbb{R}^{12})}). \quad (7.19)$$

Gathering the formulae (7.6), (7.11), (7.14), (7.16), (7.18) and estimates for the residual terms (7.7), (7.8–7.12), (7.13–7.15), (7.17), (7.19) concludes the proof of Lemma 7.1. \square

8 - End of the proof of the error estimate (Theorem 3.1)

In this Section we put together the relative energy inequality (6.1) and the identity (7.1) derived in the previous section. The final inequality resulting from this manipulation is formulated in the following lemma.

Lemma 8.1. *Under assumptions of Theorem 3.1 there exists a positive number*

$$c = c(M_0, E_0, \underline{r}, \bar{r}, |p'|_{C^1([\underline{r}, \bar{r})]}, \|(\nabla r, \partial_t r, \partial_t \nabla r, \partial_t^2 r, \mathbf{U}, \nabla \mathbf{U}, \nabla^2 \mathbf{U}, \partial_t \mathbf{U}, \partial_t \nabla \mathbf{U})\|_{L^\infty(Q_T; \mathbb{R}^{65})})$$

(depending tacitly also on $T, \theta_0, \gamma, \operatorname{diam}(\Omega), |\Omega|$), such that for all $m = 1, \dots, N$, there holds:

$$\mathcal{E}(\varrho^m, \mathbf{u}^m | r^m, \mathbf{u}^m) \leq c \left[h^A + \sqrt{k} + \mathcal{E}(\varrho^0, \mathbf{u}^0 | r^0, \mathbf{u}^0) \right] + c k \sum_{n=1}^m \mathcal{E}(\varrho^n, \mathbf{u}^n | r^n, \mathbf{u}^n),$$

where A is defined in (6.2).

Proof. Gathering the formulae (6.1) and (7.1), one gets

$$\mathcal{E}(\varrho^m, \mathbf{u}^m | r^m, \mathbf{U}^m) - \mathcal{E}(\varrho^0, \mathbf{u}^0 | r(0), \mathbf{U}(0)) + \mu k \sum_{n=1}^m \left| \mathbf{u}^n - \mathbf{U}_h^n \right|_{V_h^2(\Omega; \mathbb{R}^3)}^2 \leq \mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3 + \mathcal{Q} \quad (8.1)$$

where

$$\begin{aligned}
\mathcal{P}_1 &= k \sum_{n=1}^m \sum_{K \in \mathcal{T}} |K| (\varrho_K^{n-1} - r_K^{n-1}) \frac{\mathbf{U}_{h,K}^n - \mathbf{U}_{h,K}^{n-1}}{k} \cdot (\mathbf{U}_{h,K}^n - \mathbf{u}_K^n), \\
\mathcal{P}_2 &= k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \sum_{\sigma=K|L \in \mathcal{E}_K} |\sigma| (\varrho_\sigma^{n,\text{up}} - \hat{r}_\sigma^{n,\text{up}}) (\hat{\mathbf{U}}_{h,\sigma}^{n,\text{up}} - \hat{\mathbf{u}}_\sigma^{n,\text{up}}) \cdot (\mathbf{U}_\sigma^n - \mathbf{U}_{h,K}^n) \mathbf{U}_{h,\sigma}^{n,\text{up}} \cdot \mathbf{n}_{\sigma,K}, \\
\mathcal{P}_3 &= k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \int_K (p(r_K^n) - p(\varrho_K^n)) \operatorname{div} \mathbf{U}^n \, dx \\
&\quad + k \sum_{n=1}^m \sum_{K \in \mathcal{T}} \left[\int_K \frac{r_K^n - \varrho_K^n}{r_K^n} p'(r_K^n) \mathbf{u}^n \cdot \nabla r^n \, dx + \int_K \frac{r_K^n - \varrho_K^n}{r_K^n} p'(r_K^n) [\partial_t r]^n \, dx \right], \\
\mathcal{Q} &= \mathcal{R}_{h,k}^m + R_{h,k}^m + G^m.
\end{aligned}$$

Now, we estimate conveniently the terms $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ in three steps.

Step 1: *Term \mathcal{P}_1 .* Due to Hölder's inequality, Taylor's formula we have with $\gamma_0 = \min\{\gamma, 2\}$,

$$\begin{aligned}
&\left| \sum_{K \in \mathcal{T}} |K| (\varrho_K^{n-1} - r_K^{n-1}) \frac{\mathbf{U}_{h,K}^n - \mathbf{U}_{h,K}^{n-1}}{k} \cdot (\mathbf{U}_{h,K}^n - \mathbf{u}_K^n) \right| \leq c \left(\|\partial_t \mathbf{U}\|_{L^\infty(Q_T; \mathbb{R}^3)} + \|\partial_t \nabla \mathbf{U}\|_{L^\infty(Q_T; \mathbb{R}^9)} \right) \times \\
&\left[\left(\sum_{K \in \mathcal{T}} |K| |\varrho_K^{n-1} - r_K^{n-1}|^2 \mathbf{1}_{\mathcal{O}_{\text{ess}}}(\varrho_K) \right)^{1/2} + \left(\sum_{K \in \mathcal{T}} |K| |\varrho_K^{n-1} - r_K^{n-1}|^{\gamma_0} \mathbf{1}_{\mathcal{O}_{\text{res}}}(\varrho_K) \right)^{1/\gamma_0} \right] \left(\sum_{K \in \mathcal{T}} |K| |\mathbf{U}_{h,K}^n - \mathbf{u}_K^n|^6 \right)^{1/6} \\
&\leq c \left(\|\partial_t \mathbf{U}, \partial_t \nabla \mathbf{U}\|_{L^\infty(Q_T; \mathbb{R}^{12})} \right) \left(\mathcal{E}^{1/2}(\varrho^{n-1}, \mathbf{u}^{n-1} | r^{n-1}, \mathbf{U}^{n-1}) \right. \\
&\quad \left. + \mathcal{E}^{1/\gamma_0}(\varrho^{n-1}, \mathbf{u}^{n-1} | r^{n-1}, \mathbf{U}^{n-1}) \right) \left(\sum_{K \in \mathcal{T}} \|\mathbf{U}_{h,K}^n - \mathbf{u}_K^n\|_{L^6(K; \mathbb{R}^3)}^6 \right)^{1/6},
\end{aligned}$$

where we have used Lemma 7.2 and estimate (4.15) to obtain the last line. Now, by the Minkowski inequality

$$\begin{aligned}
&\left(\sum_{K \in \mathcal{T}} \|\mathbf{U}_{h,K}^n - \mathbf{u}_K^n\|_{L^6(K; \mathbb{R}^3)}^6 \right)^{1/6} \leq \left(\sum_{K \in \mathcal{T}} \|(\mathbf{U}_{h,K}^n - \mathbf{u}_K^n) - (\mathbf{U}_h^n - \mathbf{u}^n)\|_{L^6(K; \mathbb{R}^3)}^6 \right)^{1/6} \\
&\quad + \|\mathbf{U}_h^n - \mathbf{u}^n\|_{L^6(\Omega; \mathbb{R}^3)} \leq c \left| \mathbf{u}^n - \mathbf{U}_h^n \right|_{V_h^2(\Omega; \mathbb{R}^3)},
\end{aligned}$$

where we have used estimate (9.14) and the Sobolev inequality (9.29). Finally, employing Young's inequality, and estimate (4.15), we arrive at

$$\begin{aligned}
|\mathcal{P}_1| &\leq c(\delta, M_0, E_0, \underline{r}, \bar{r}, \|(\mathbf{U}, \nabla \mathbf{U}, \partial_t \mathbf{U}, \partial_t \nabla \mathbf{U})\|_{L^\infty(Q_T, \mathbb{R}^{24})}) \\
&\quad \times \left(k \mathcal{E}(\varrho^0, \mathbf{u}^0 | r^0, \mathbf{U}^0) + k \sum_{n=1}^m \mathcal{E}(\varrho^n, \mathbf{u}^n | r^n, \mathbf{U}^n) \right) + \delta k \sum_{n=1}^m \left| \mathbf{u}^n - \mathbf{U}_h^n \right|_{V_h^2(\Omega; \mathbb{R}^3)}^2 \quad (8.2)
\end{aligned}$$

with any $\delta > 0$.

Step 2: *Term \mathcal{P}_2 .* We write $\mathcal{P}_2 = k \sum_{n=1}^m \mathcal{P}_2^n$ where Lemma 7.2 and the Hölder inequality yield, similarly as in the previous step,

$$\begin{aligned}
|\mathcal{P}_2^n| &\leq c(\underline{r}, \bar{r}, \|\nabla \mathbf{U}\|_{L^\infty(Q_T; \mathbb{R}^9)}) \times \\
&\quad \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| h \left(E^{1/2}(\varrho_\sigma^{n,\text{up}}, \hat{r}_\sigma^{n,\text{up}}) + E^{2/3}(\varrho_\sigma^{n,\text{up}}, \hat{r}_\sigma^{n,\text{up}}) \right) |\hat{\mathbf{U}}_{h,\sigma}^{n,\text{up}}| |\hat{\mathbf{U}}_{h,\sigma}^{n,\text{up}} - \hat{\mathbf{u}}_\sigma^{n,\text{up}}| \\
&\leq c(\underline{r}, \bar{r}, \|(\mathbf{U}, \nabla \mathbf{U})\|_{L^\infty(Q_T; \mathbb{R}^{12})}) \left[\left(\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| h \left(E(\varrho_\sigma^{n,\text{up}} | \hat{r}_\sigma^{n,\text{up}}) \right) \right)^{1/2} \right. \\
&\quad \left. + \left(\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| h E(\varrho_\sigma^{n,\text{up}} | \hat{r}_\sigma^{n,\text{up}}) \right)^{2/3} \right] \times \left(\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| h |\hat{\mathbf{U}}_{h,\sigma}^{n,\text{up}} - \hat{\mathbf{u}}_\sigma^{n,\text{up}}|^6 \right)^{1/6}
\end{aligned}$$

provided $\gamma \geq 3/2$. Next, we observe that the contribution of the face $\sigma = K|L$ to the sums $\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| h E(\varrho_\sigma^{n,\text{up}} | \hat{r}_\sigma^{n,\text{up}})$ and $\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| h |\hat{\mathbf{U}}_{h,\sigma}^{n,\text{up}} - \hat{\mathbf{u}}_\sigma^{n,\text{up}}|^6$ is less or equal than $2|\sigma| h (E(\varrho_K^n | \hat{r}_K^n) + E(\varrho_L^n | \hat{r}_L^n))$, and than $2|\sigma| h (|\mathbf{U}_{h,K}^n - \mathbf{u}_K^n|^6 + |\mathbf{U}_{h,L}^n - \mathbf{u}_L^n|^6)$, respectively. Consequently, we get by the same reasoning as in the previous step, under assumption $\gamma \geq 3/2$,

$$|\mathcal{P}_2| \leq c(\delta, M_0, E_0, \underline{\tau}, \bar{\tau}, \|(\mathbf{U}, \nabla \mathbf{U})\|_{L^\infty(Q_T; \mathbb{R}^{12})}) k \sum_{n=1}^m \mathcal{E}(\varrho^n, \mathbf{u}^n | r^n, \mathbf{U}^n) + \delta k \sum_{n=1}^m |\mathbf{u}^n - \mathbf{U}_h^n|_{V_h^2(\Omega; \mathbb{R}^3)}^2. \quad (8.3)$$

Step 3: *Term \mathcal{P}_3 .* Since the pair (r, \mathbf{U}) satisfies continuity equation (1.1a) in the classical sense, we have for all $n = 1, \dots, N$,

$$[\partial_t r]^n + \mathbf{U}^n \cdot \nabla r^n = -r^n \operatorname{div} \mathbf{U}^n,$$

where we recall that $[\partial_t r]^n(x) = \partial_t r(t_n, x)$ in accordance with (3.15). Using this identity we write

$$\begin{aligned} \mathcal{P}_3^n &= \mathcal{P}_{3,1} + \mathcal{P}_{3,2}, \quad \mathcal{P}_{3,i} = k \sum_{n=1}^m \mathcal{P}_{3,i}^n, \\ \text{with } \mathcal{P}_{3,1}^n &= - \sum_{K \in \mathcal{T}} \int_K \left(p(\varrho_K^n) - p'(r_K^n)(\varrho_K^n - r_K^n) - p(r_K^n) \right) \operatorname{div} \mathbf{U}^n \, dx \\ \text{and } \mathcal{P}_{3,2}^n &= \sum_{K \in \mathcal{T}} \left[\int_K \frac{r_K^n - \varrho_K^n}{r_K^n} p'(r_K^n) (\mathbf{u}^n - \mathbf{U}^n) \cdot \nabla r^n \, dx. \right. \end{aligned}$$

Now, we apply Lemma 7.2 in combination with assumption (1.6) to deduce

$$|\mathcal{P}_{3,1}| \leq c \|\operatorname{div} \mathbf{U}\|_{L^\infty(Q_T)} k \sum_{n=1}^m \mathcal{E}(\varrho^n, \mathbf{u}^n | r^n, \mathbf{U}^n). \quad (8.4)$$

Finally, the same reasoning as in Step 2 leads to the estimate

$$\begin{aligned} |\mathcal{P}_{3,2}| &\leq h c(M_0, E_0, \underline{\tau}, \bar{\tau}, |p'|_{C([\underline{\tau}, \bar{\tau}]])}) \|(\nabla r, \nabla \mathbf{U})\|_{L^\infty(\Omega; \mathbb{R}^9)} \\ &+ c(\delta, \|\underline{\tau}, \bar{\tau}, |p'|_{C([\underline{\tau}, \bar{\tau}]])}\| \|\nabla r\|_{L^\infty(\Omega; \mathbb{R}^3)}) k \sum_{n=1}^m \mathcal{E}(\varrho^n, \mathbf{u}^n | r^n, \mathbf{U}^n) + \delta k \sum_{n=1}^m |\mathbf{u}^n - \mathbf{U}_h^n|_{V_h^2(\Omega; \mathbb{R}^3)}^2. \end{aligned} \quad (8.5)$$

Gathering the formulae (8.1) and (8.2)-(8.5) with δ sufficiently small (with respect to μ), we conclude the proof of Lemma 8.1. \square

Finally, Lemma 8.1 in combination with the bound (4.15) yields

$$\mathcal{E}(\varrho^m, \mathbf{u}^m | r^m, \mathbf{u}^m) \leq c \left[h^A + \sqrt{k} + k + \mathcal{E}(\varrho^0, \mathbf{u}^0 | r^0, \mathbf{u}^0) \right] + c k \sum_{n=1}^{m-1} \mathcal{E}(\varrho^n, \mathbf{u}^n | r^n, \mathbf{u}^n);$$

whence Theorem 3.1 is a direct consequence of the standard discrete version of Gronwall's lemma. Theorem 3.1 is thus proved.

9 - Appendix: Fundamental auxiliary lemmas and estimates

In this section we report several results related to the properties of the Sobolev spaces on tetrahedra and of the Crouzeix-Raviart (C-R) space. We refer to the book Brezzi, Fortin [3] for the general introduction to the subject.

We start with the inequalities that can be obtained by rescaling from the standard inequalities on a reference tetrahedron of size equivalent to one.

Lemma 9.1 (Poincaré, Sobolev and interpolation inequalities on tetrahedra). *Let $1 \leq p \leq \infty$. Let $\theta_0 > 0$ and \mathcal{T} be a triangulation of Ω such that $\theta \geq \theta_0$ where θ is defined in (3.1). Then we have:*

(1) *Poincaré type inequalities on tetrahedra*

Let $1 \leq p \leq \infty$. There exists $c = c(\theta_0, p) > 0$ such that for all $K \in \mathcal{T}$ and for all $v \in W^{1,p}(K)$ we have

$$\|v - v_K\|_{L^p(K)} \leq ch \|\nabla v\|_{L^p(K)}, \quad (9.1)$$

$$\forall \sigma \in \mathcal{E}(K), \|v - v_\sigma\|_{L^p(K)} \leq ch \|\nabla v\|_{L^p(K)}. \quad (9.2)$$

(2) *Sobolev type inequalities on tetrahedra*

Let $1 \leq p < d$. There exists $c = c(\theta_0, p) > 0$ such that for all $K \in \mathcal{T}$ and for all $v \in W^{1,p}(K)$, $1 \leq p < d$ we have

$$\|v - v_K\|_{L^{p^*}(K)} \leq c \|\nabla v\|_{L^p(K)}, \quad (9.3)$$

$$\forall \sigma \in \mathcal{E}(K), \|v - v_\sigma\|_{L^{p^*}(K)} \leq c \|\nabla v\|_{L^p(K)}, \quad (9.4)$$

where $p^* = \frac{dp}{d-p}$.

(3) *Interpolation inequalities on the tetrahedra*

Let $1 \leq p < d$ and $p \leq q \leq p^*$. There exists $c = c(\theta_0, p) > 0$ such that for all $K \in \mathcal{T}$ and $v \in W^{1,p}(K)$, $1 \leq p < d$ we have

$$\|v - v_K\|_{L^q(K)} \leq ch^\beta \|\nabla v\|_{L^p(K; \mathbb{R}^d)}, \quad (9.5)$$

$$\|v - v_\sigma\|_{L^q(K)} \leq ch^\beta \|\nabla v\|_{L^p(K; \mathbb{R}^d)}, \quad (9.6)$$

where $\frac{1}{q} = \frac{\beta}{p} + \frac{1-\beta}{p^*}$.

Combining estimates (9.1–9.6) with the algebraic inequality

$$\left(\sum_{i=1}^L |a_i|^p \right)^{1/p} \leq \left(\sum_{i=1}^L |a_i|^q \right)^{1/q} \quad (9.7)$$

for all $(a = (a_1, \dots, a_L) \in \mathbb{R}^L, 1 \leq q \leq p < \infty$, we obtain the following corollaries.

Corollary 9.1 (Poincaré and Sobolev type inequalities on the Sobolev spaces). *Under the assumptions of Lemma 9.1, we have:*

(1) *Poincaré type inequalities on the domain Ω*

Let $1 \leq p \leq \infty$. There exists $c = c(\theta_0, p) > 0$ such that for all $v \in W^{1,p}(\Omega)$ we have

$$\|v - \hat{v}\|_{L^p(\Omega)} \equiv \left(\sum_{K \in \mathcal{T}} \|v - v_K\|_{L^p(K)}^p \right)^{1/p} \leq ch \|\nabla v\|_{L^p(\Omega; \mathbb{R}^d)}, \quad (9.8)$$

$$\left(\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} \|v - v_\sigma\|_{L^p(K)}^p \right)^{1/p} \leq ch \|\nabla v\|_{L^p(\Omega; \mathbb{R}^d)} \quad (9.9)$$

where \hat{v} and v_σ are defined by (3.7) and (3.9).

(2) *Sobolev type inequalities on the domain Ω*

Let $1 \leq p < d$. There exists $c = c(\theta_0, p) > 0$ such that for all $v \in W^{1,p}(\Omega)$, $1 \leq p < d$ we have

$$\|v - \hat{v}\|_{L^{p^*}(\Omega)} \leq c \|\nabla v\|_{L^p(\Omega)}, \quad (9.10)$$

$$\left(\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} \|v - v_\sigma\|_{L^{p^*}(K)}^{p^*} \right)^{1/p^*} \leq c \|\nabla v\|_{L^p(\Omega; \mathbb{R}^d)}. \quad (9.11)$$

(3) *Interpolation inequalities on the domain Ω*

Let $1 \leq p < d$ and $p \leq q \leq p^*$. There exists $c = c(\theta_0, p) > 0$ such that for all $v \in W^{1,p}(\Omega)$, $1 \leq p < d$ we have

$$\|v - \hat{v}\|_{L^q(\Omega)} \leq ch^\beta \|\nabla v\|_{L^p(\Omega)}, \quad (9.12)$$

$$\left(\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} \|v - v_\sigma\|_{L^q(K)}^q \right)^{\frac{1}{q}} \leq ch^\beta \|\nabla v\|_{L^p(\Omega; \mathbb{R}^d)}, \quad (9.13)$$

where $\frac{1}{q} = \frac{\beta}{p} + \frac{1-\beta}{p^*}$.

Corollary 9.2 (Poincaré and Sobolev type inequalities on V_h). *Under assumptions of Lemma 9.1, there holds:*

(1) *Poincaré type inequality in $V_h(\Omega)$: Let $1 \leq p < \infty$. There exists $c = c(\theta_0, p)$ such that for all $v \in V_h$,*

$$\|v - \hat{v}\|_{L^p(\Omega)} \leq ch|v|_{V_h^p(\Omega)}, \quad (9.14)$$

$$\left(\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} \|v - v_\sigma\|_{L^p(K)}^p \right)^{\frac{1}{p}} \leq ch|v|_{V_h^p(\Omega)}. \quad (9.15)$$

(2) *Sobolev type inequality in $V_h(\Omega)$: Let $1 \leq p < d$. There exists $c = c(\theta_0, p)$ such that for all $v \in V_h(\Omega)$,*

$$\|v - \hat{v}\|_{L^{p^*}(\Omega)} \leq c|v|_{V_h^p(\Omega)}, \quad (9.16)$$

$$\left(\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} \|v - v_\sigma\|_{L^{p^*}(K)}^{p^*} \right)^{\frac{1}{p^*}} \leq c|v|_{V_h^p(\Omega)}. \quad (9.17)$$

(3) *Interpolation type inequalities in $V_h(\Omega)$*

Let $1 \leq p < d$ and $p \leq q \leq p^*$. There exists $c = c(\theta_0, p) > 0$ such that for all $v \in V_h(\Omega)$, $1 \leq p < d$ we have

$$\|v - \hat{v}\|_{L^q(\Omega)} \leq ch^\beta |v|_{V_h^p(\Omega)}, \quad (9.18)$$

$$\left(\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} \|v - v_\sigma\|_{L^q(K)}^q \right)^{\frac{1}{q}} \leq ch^\beta |v|_{V_h^p(\Omega)}, \quad (9.19)$$

where $\frac{1}{q} = \frac{\beta}{p} + \frac{1-\beta}{p^*}$.

The next fundamental lemma deals with the properties of the projection v_h defined by (3.8).

Lemma 9.2 (Projection on V_h). *Let $\theta_0 > 0$ and \mathcal{T} be a triangulation of Ω such that $\theta \geq \theta_0$ where θ is defined in (3.1).*

(1) *Approximation estimates on the tetrahedra*

Let $1 \leq p \leq \infty$. There exists $c = c(\theta_0, p) > 0$ such that

$$\forall v \in W_0^{1,p} \cap W^{s,p}(\Omega), \forall K \in \mathcal{T}, \|v - v_h\|_{L^p(K)} \leq ch^s \|\nabla^s v\|_{L^p(K; \mathbb{R}^{d^s})}, \quad (9.20)$$

$$\|\nabla(v - v_h)\|_{L^p(K; \mathbb{R}^d)} \leq ch^{s-1} \|\nabla^s v\|_{L^p(K; \mathbb{R}^{d^s})}, s = 1, 2. \quad (9.21)$$

(2) *Preservation of divergence*

$$\forall \mathbf{v} \in W_0^{1,2}(\Omega, \mathbb{R}^d), \forall q \in L_h(\Omega), \sum_{K \in \mathcal{T}} \int_K q \operatorname{div} \mathbf{v}_h \, dx = \int_\Omega q \operatorname{div} \mathbf{v} \, dx \quad (9.22)$$

(3) *Approximation estimates of the Poincaré type on the whole domain*

Let $1 \leq p < \infty$. There exists $c = c(\theta_0, p) > 0$ such that for all $v \in W_0^{1,p}(\Omega)$,

$$\|v - v_h\|_{L^p(\Omega)} \leq ch \|\nabla v\|_{L^p(\Omega; \mathbb{R}^d)}. \quad (9.23)$$

(4) *Approximation estimates of the Sobolev type on the whole domain*

Let $1 \leq p < d$. There exists $c = c(\theta_0, p) > 0$ such that for all $v \in W_0^{1,p}(\Omega)$,

$$\|v - v_h\|_{L^{p^*}(\Omega)} \leq c \|\nabla v\|_{L^p(\Omega; \mathbb{R}^d)}. \quad (9.24)$$

Statement (2) of Lemma 9.2 is proved in [6], where one can find also the proof of item (1) for $p = 2$. We present here the proof of statements (1), (3), (4) for arbitrary p for the reader's convenience, since a straightforward reference is not available.

Proof. Step 1: We start with some generalities. First we complete the Crouzeix-Raviart basis (3.6) by functions ϕ_σ indexed also with $\sigma \in \mathcal{E}_{\text{ext}}$ saying

$$\frac{1}{|\sigma'|} \int_{\sigma'} \phi_\sigma dS = \delta_{\sigma, \sigma'}, \quad (\sigma, \sigma') \in \mathcal{E}^2$$

and observe that

$$\sum_{\sigma \in \mathcal{E}(K)} \phi_\sigma(x) = 1 \quad \text{for any } x \in K. \quad (9.25)$$

A scaling argument yields

$$\|\phi_\sigma\|_{L^\infty(\Omega)} \leq c(\theta_0), \quad h \|\nabla \phi_\sigma\|_{L^\infty(\Omega; \mathbb{R}^d)} \leq c(\theta_0). \quad (9.26)$$

Second, we define the projection $v \rightarrow v_h$ for any $v \in W^{1,p}(\Omega)$ by saying

$$v_h = \sum_{\sigma \in \mathcal{E}} v_\sigma \phi_\sigma.$$

We notice that if $v \in W_0^{1,p}(\Omega)$ then v_h coincides with (3.5). Moreover,

$$v_h = v \quad \text{for any affine function } v. \quad (9.27)$$

Third, due to the density argument, it is enough to show the remaining statements (1), (3), (4) for $v \in W_0^{1,p}(\Omega) \cap W^{s,\infty}(\Omega)$, $s = 1, 2$, according to the case.

Step 2: We denote by $x_K = \frac{1}{|K|} \int_K x dx$ the center of gravity of the tetrahedron K . We calculate by using (9.27) and the first order Taylor formula

$$\begin{aligned} v(x) - v_h(x) &= v(x) - v(x_K) - [v - v(x_K)]_h(x) \\ &= (x - x_K) \cdot \int_0^1 \nabla v(x_K + t(x - x_K)) dt - \sum_{\sigma \in \mathcal{E}(K)} \phi_\sigma(x) \frac{1}{|\sigma|} \int_\sigma [(x - x_K) \cdot \int_0^1 \nabla v(x_K + t(x - x_K)) dt] dS, \end{aligned}$$

where $x \in K$. This formula yields immediately the upper bound stated in (9.20)_{s=1} if $p = \infty$. If $1 \leq p < \infty$ we calculate the upper bound of the L^p -norm of each term at the right-hand side separately by using (9.26), Fubini's theorem, Hölder's inequality and the change of variables $y = x_K + t(x - x_K)$ together with the convexity of K .

The same reasoning can be applied to prove (9.20)_{s=2}. Indeed, we observe that

$$v(x) - v_h(x) = v(x) - (x - x_K) \cdot \nabla v(x_K) - v(x_K) - [v - (x - x_K) \cdot \nabla v(x_K) - v(x_K)]_h(x)$$

by virtue of (9.27). Now we apply to the right hand side of the last expression the second order Taylor formula in the integral form, and proceed exactly as described before.

Finally, one applies the same straightforward argumentation to get (9.21). This completes the proof of statement (1).

Step 3: Statement (3) follows easily from (9.20)_{s=1} and the algebraic inequality (9.7).

Step 4: We use (9.25) and (9.27) to write

$$v(x) - v_h(x) = \sum_{\sigma \in \mathcal{E}(K)} (v(x) - v_\sigma) \phi_\sigma(x), \quad x \in K;$$

whence

$$\|v - v_h\|_{L^{p^*}(K)} \leq c \|\nabla v\|_{L^p(K; \mathbb{R}^d)}$$

where we have used the Sobolev inequality (9.4) on the tetrahedron $K \in \mathcal{T}$ and the L^∞ -bound (9.26). We conclude the proof of statement (4) by using the relation (9.7). The proof of Lemma 9.2 is complete. \square

The following corollary is a direct consequence of (9.21).

Corollary 9.3 (Continuity of the projection onto V_h). *Under assumptions of Lemma 9.2, there exists $c = c(\theta_0, p) > 0$ such that*

$$\forall v \in W_0^{1,p}(\Omega), \quad |v_h|_{V_h^p(\Omega)} \leq c \|\nabla v\|_{L^p(\Omega; \mathbb{R}^d)}, \quad (9.28)$$

where $1 \leq p < \infty$.

Although the non conforming finite element space V_h is not a subspace of any Sobolev space, its elements enjoy the Sobolev type inequalities. This important fact is formulated in the next lemma.

Lemma 9.3 (Sobolev inequality on V_h). *Let Ω be a bounded domain of \mathbb{R}^d . Let \mathcal{T} be a triangulation of the domain Ω in simplices such that $\theta \geq \theta_0 > 0$ where θ is defined in (3.1). Then we have:*

(1) *Sobolev inequality in $V_h(\Omega)$ (case $1 \leq p < d$):*

There exists $c = c(\theta_0, p)$ such that for all $v \in V_h(\Omega)$,

$$\|v\|_{L^{p^*}(\Omega)} \leq c |v|_{V_h^p(\Omega)}. \quad (9.29)$$

(2) *Sobolev inequality in $V_h(\Omega)$, case $p \geq d$*

Let $1 \leq q < \infty$. There here exists $c = c(\theta_0, p, q) > 0$ such that for all $v \in V_h(\Omega)$,

$$\|v\|_{L^q(\Omega)} \leq c |v|_{V_h^p(\Omega)} \quad (9.30)$$

Proof. Step 1 Let $1 \leq r \leq \alpha < \infty$. Let $u \in V_h$. We call v the element of V_h such that $v_\sigma = |u_\sigma|^\alpha$. Then there exists C only depending on d, r, α such that

$$\|u\|_{L^r(\Omega)}^\alpha \leq \|u\|_{L^{\frac{r}{\alpha}}(\Omega)}. \quad (9.31)$$

To prove (9.31) we remark that, using a change of variable, it is enough to show to prove the existence of C for only the unit simplex \hat{K} . Let $u \in \mathbb{P}_1(\hat{K})$ and we call v the element of $\mathbb{P}_1(\hat{K})$ such that $v_\sigma = |u_\sigma|^\alpha$. Let $T(u) = \|u\|_{L^r(\hat{K})}$ and $S(u) = \|u\|_{L^{\frac{r}{\alpha}}(\hat{K})}$. These two functions are continuous, homogeneous of degree

1 and non zero if $u \neq 0$. Since $\mathbb{P}_1(\hat{K})$ is a finite dimensional space, we can choose a norm on $\mathbb{P}_1(\hat{K})$ and take $C = (\frac{M}{m})^\alpha$ where $M = \max\{T(u), \|u\|_{\mathbb{P}_1(\hat{K})} = 1\}$ and $m = \min\{T(u), \|u\|_{\mathbb{P}_1(\hat{K})} = 1\}$.

Step 2: Proof for $p = 1$.

We set $u = 0$ outside Ω . For $\sigma \in \mathcal{E}_{\text{int}}, \sigma = K|L$, we set $||[u(x)]|| = |u_K(x) - u_L(x)|$ for $x \in \sigma$. For $\sigma \in \mathcal{E}_{\text{ext}} \cap \mathcal{E}(K)$, we set $||[u(x)]|| = |u_K(x)|$ for $x \in \sigma$. We first remark that there exists $C_{1,1}$ and $C_{1,2}$ only depending on d such that

$$\|u\|_{L^{\frac{d}{d-1}}(\Omega)} \leq C_{1,1} \|u\|_{BV(\mathbb{R}^d)} \leq C_{1,2} \|\nabla_h u\|_{L^1(\Omega)} + C_{1,2} \sum_{\sigma \in \mathcal{E}} \int_{\sigma} |[u]| \, d\gamma.$$

We now prove that there exists $C_{1,3}$ only depending on d and θ_0 such that

$$\sum_{\sigma \in \mathcal{E}} \int_{\sigma} |[u]| \, d\gamma \leq C_{1,3} \|\nabla_h u\|_{L^1(\Omega)}.$$

Let $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}(K)$. Let x_σ be the center of mass of σ . We have, with $u_K = u$ in K ,

$$u_K(x) - u(x_\sigma) = \int_0^1 \nabla u_K \cdot (x - x_\sigma) \, dx.$$

Then if $\sigma = K|L$ we have

$$|u_K(x) - u_L(x)| \leq h_\sigma (|\nabla u_K| + |\nabla u_L|).$$

Integrating this inequality on σ gives

$$\int_{\sigma} |[u]| \, d\gamma \leq |\sigma| h_\sigma (|\nabla u_K| + |\nabla u_L|) \leq \frac{2}{\theta_0^d} (\|\nabla u\|_{L^1(K)} + \|\nabla u\|_{L^1(L)}).$$

Similarly for $\sigma \in \mathcal{E}_{\text{ext}} \cap \mathcal{E}(K)$ we have

$$\int_{\sigma} |[u]| \, d\gamma \leq \frac{2}{\theta_0^d} \|\nabla u\|_{L^1(K)}$$

Then there exists $C_{1,3} = C(d, \theta_0)$ such that

$$\sum_{i, \sigma \in \mathcal{E}} \int_{\sigma} |[u]| \, d\gamma \leq C_{1,3} \|\nabla_h u\|_{L^1(\Omega)}.$$

and then,

$$\|u\|_{L^{1^*}(\Omega)} \leq c(d, \theta_0) \|\nabla_h u\|_{L^1(\Omega)}.$$

Step 3: Proof for $1 < p < d$.

Let $1 < p < d$ and $p^* = \frac{pd}{d-p}$ and let $u \in V_h$. We set $u = 0$ outside Ω . Let $\alpha = \frac{p(d-1)}{d-p}$, so that $\alpha > 1$ and $\alpha 1^* = p^*$. We call v the element of V_h such that $v_\sigma = |u_\sigma|^\alpha$ for $\sigma \in \mathcal{E}$. One has $v \neq |u|^\alpha$ but there exists $C_{2,1}$ only depending on d and p (see lemma 9.31) such that

$$\|u\|_{L^{p^*}(\Omega)}^\alpha \leq C_{2,1} \|v\|_{L^{1^*}(\Omega)} \leq c(d, p, \theta_0) \|\nabla_h v\|_{L^1(\Omega)}.$$

Moreover using a scaling argument we obtain

$$\|\nabla_h v\|_{L^1(K)} \leq c(d, p, \theta_0) \sum_{\sigma \in \mathcal{E}(K)} |u_\sigma|^{\alpha-1} |\nabla u_K| |K|.$$

Then, using Hölder Inequality, we have, with $q = \frac{p}{p-1}$ (so that $q(\alpha-1) = p^*$),

$$\|\nabla_h v\|_{L^1(K)} \leq c(d, p, \theta_0) \|\nabla u\|_{L^p(K)} \|u\|_{L^{p^*}(K)}^{\frac{p^*}{q}}.$$

Summing on $K \in \mathcal{T}$ we obtain

$$\|u\|_{L^{p^*}(\Omega)} \leq C_2 \|\nabla_h u\|_{L^p(\Omega)}.$$

Step 4: Proof for $p \geq d$.

Let $1 \leq q < \infty$. There exists $r = r(d, q)$ such that $r < d$ and $r^* \geq q$. We have

$$\|u\|_{L^{r^*}(\Omega)} \leq c(r, d, q, \theta_0) \|\nabla_h u\|_{L^r(\Omega)}.$$

Moreover

$$\|u\|_{L^q(\Omega)} \leq |\Omega|^{\frac{1}{q} - \frac{1}{r^*}} \|u\|_{L^{r^*}(\Omega)} \leq c(d, q, \theta_0) |\Omega|^{\frac{1}{q} - \frac{1}{r^*}} \|\nabla_h u\|_{L^r(\Omega)}$$

and

$$\|\nabla_h u\|_{L^r(\Omega)} \leq |\Omega|^{\frac{1}{r} - \frac{1}{p}} \|\nabla_h u\|_{L^p(\Omega)}.$$

Finally

$$\|u\|_{L^q(\Omega)} \leq c(\Omega, d, p, q, \theta_0) \|\nabla_h u\|_{L^p(\Omega)}.$$

□

A Combination of Lemma 9.3 with estimates (9.14), (9.16) and the Hölder inequality yields the following corollary.

Corollary 9.4 (Estimates of the norms of mean values). *We have under the assumptions of Lemma 9.3:*

(1) *Poincaré type inequality involving mean values on tetrahedra*

There exists $c = c(\theta_0, p)$ such that for all $v \in V_h$,

$$\|\hat{v}\|_{L^p(\Omega)} \equiv \left(\sum_{K \in \mathcal{T}} |K| |v_K|^p \right)^{1/p} \leq c(\|v\|_{L^p(\Omega)} + h|v|_{V_h^p(\Omega)}). \quad (9.32)$$

(2) *Sobolev type inequality involving mean values on tetrahedrons*

Let $1 \leq p < d$, there exists $c = c(\theta_0, p)$ such that for all $v \in V_h$,

$$\|\hat{v}\|_{L^{p^*}(\Omega)} \equiv \left(\sum_{K \in \mathcal{T}} |K| |v_K|^{p^*} \right)^{1/p^*} \leq c(\|v\|_{L^{p^*}(\Omega)} + |v|_{V_h^p}). \quad (9.33)$$

Note that the Last but not least, we recall a result on equivalence of norms in the space $V_h(\Omega)$ which is a consequence of a discrete Poincaré inequality on the broken Sobolev space V_h [31, proposition 4.13].

Lemma 9.4 (Discrete and continuous norms in V_h). *Let $1 \leq p < \infty$. Let $\theta_0 > 0$ and \mathcal{T} be a triangulation of Ω such that $\theta \geq \theta_0$ where θ is defined in (3.1). Then the norms*

$$\left(\sum_{\sigma \in \mathcal{E}_{\text{int}}} |\sigma| h |v_\sigma|^p \right)^{1/p} \quad \text{and} \quad \|v\|_{L^p(\Omega)}^p \quad (9.34)$$

are equivalent on $V_h(\Omega)$ uniformly with respect to $h > 0$.

The last lemma in this overview deals with the estimates of jumps over faces. The reader can consult [8, Lemma 3.32] or [18, Lemma 2.2] for its proof.

Lemma 9.5 (Jumps over faces in the Crouzeix-Raviart space). *Let $\theta_0 > 0$ and \mathcal{T} be a triangulation of Ω such that $\theta \geq \theta_0$ where θ is defined in (3.1). Then there exists $c = c(\theta_0) > 0$ such that for all $v \in V_h(\Omega)$,*

$$\sum_{\sigma \in \mathcal{E}} \frac{1}{h} \int_\sigma [v]_{\sigma, \mathbf{n}_\sigma}^2 \, dS \leq c |v|_{V_h^2(\Omega)}^2, \quad (9.35)$$

where $[v]_{\sigma, \mathbf{n}_\sigma}$ is a jump of v with respect to a normal \mathbf{n}_σ to the face σ ,

$$\forall x \in \sigma = K|L \in \mathcal{E}_{\text{int}}, \quad [v]_{\sigma, \mathbf{n}_\sigma}(x) = \begin{cases} v|_K(x) - v|_L(x) & \text{if } \mathbf{n}_\sigma = \mathbf{n}_{\sigma, K} \\ v|_L(x) - v|_K(x) & \text{if } \mathbf{n}_\sigma = \mathbf{n}_{\sigma, L} \end{cases}$$

and

$$\forall x \in \sigma \in \mathcal{E}_{\text{ext}}, \quad [v]_{\sigma, \mathbf{n}_\sigma}(x) = v(x), \quad \text{with } \mathbf{n}_\sigma \text{ an exterior normal to } \partial\Omega.$$

9 - References

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