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REGULARITY RESULTS FOR A CLASS OF HYPERBOLIC EQUATIONS WITH VMO COEFFICIENTS

MAİTINE BERGOUNIOUX AND ERICA L. SCHWINDT

Abstract. In this note we show a regularity result for an hyperbolic system with discontinuous coefficients. More precisely, we deal with coefficients in the function space VMO and we prove the existence and uniqueness of a solution $u \in L^\infty(0, T; H^2(\Omega))$ with also suitable regularity for $\frac{\partial u}{\partial t}$, $\frac{\partial^2 u}{\partial t^2}$, and $\frac{\partial^3 u}{\partial t^3}$.

1. Introduction

Let $\Omega$ be a bounded open subset of $\mathbb{R}^d$ with $d \geq 3$. In the context of photoacoustic tomography process modelling [1], we are led to study the following wave equation

\begin{equation}
\begin{cases}
\frac{\partial^2 p}{\partial t^2}(t, x) - \text{div}(v_s^2 \nabla p)(t, x) = f(t, x) & \text{in } (0, T) \times \Omega \\
p(t, x) = 0 & \text{on } (0, T) \times \partial \Omega \\
p(0, x) = \frac{\partial p}{\partial t}(0, x) = 0 & \text{in } \Omega,
\end{cases}
\end{equation}

where $p = p(t, x)$ is an acoustic pressure wave, $v_s = v_s(x)$ is the speed of sound, $f$ is a distributed source that comes from a lightning process and $\Omega$ is the domain where the wave propagates. The coefficient $v_s$ is generally unknown and not smooth. We are interested in establishing new results of regularity of the solution $p$ in the case of discontinuous coefficient $v_s$.

Hereafter we will assume that $\partial \Omega$ is of class $C^2$ and we consider the following initial/boundary value problem:

\begin{equation}
\begin{cases}
\frac{\partial^2 u}{\partial t^2} + Lu = f & \text{in } (0, T) \times \Omega \\
u = 0 & \text{on } (0, T) \times \partial \Omega \\
u(0, x) = u_0, \quad \frac{\partial u}{\partial t}(0, x) = u_1 & \text{in } \Omega,
\end{cases}
\end{equation}

where $f : (0, T) \times \Omega \to \mathbb{R}$, $u_0, u_1 : \Omega \to \mathbb{R}$ are given and $L$ denotes a second order partial differential operator in the divergence form.
We introduce some definitions and notations and the variational formulation of System (1.1) has been extensively studied. Classical results of well-posedness and regularity can be found in [9, §7.2]. In this reference a regularity result similar to our Theorem 3.1 is obtained under coefficient smoothness assumptions, namely $a_{ij} \in C^1(\Omega)$ and $\nabla (a_{ij}) \in [C^1(\Omega)]^d$.

In this work, we consider discontinuous coefficients $a_{ij}$ such that $a_{ij} \in VMO \cap L^\infty(\Omega)$ and $\nabla a_{ij} \in [L^p(\Omega)]^d$ with $p > d$. Assuming the coefficients $a_{ij}$ belong to $L^\infty(\Omega)$, it can be proved that System (1.1) admits a unique solution $u \in C^0(0, T; H^1_0(\Omega))$ with $\frac{\partial u}{\partial t} \in C^0(0, T; L^2(\Omega))$ (see first part of the proof of Theorem 3.1).

Roughly speaking, the improved regularity, with respect to space, of the solution $u$ is associated with the elliptic regularity of the equation for almost every $t \in [0, T]$, that is, with the regularity of $Lu(t) = f(t) - \frac{\partial^2 u}{\partial t^2}(t)$. Several regularity results for elliptic operators $L$ have been obtained with more general elliptic operators of type $\tilde{L}u = - \sum_{i,j=1}^d a_{ij}(t, x)u_{x_i x_j} + \sum_{i=1}^d b_i(t, x)u_{x_i} + c(t, x)u$, and there exists a non-exhaustive list of papers devoted to results of regularity associated with the operator $\tilde{L}$ with different hypothesis on the coefficients $a_{ij}, b_i$ and $c$ (see for example [4, 5, 6, 8, 7, 9, 14, 15] and references therein). Other results for parabolic equations with VMO coefficients can be found in [2, 11].

In Section 2 we introduce some definitions and notations and the variational formulation of System (1.1). Section 3 is devoted to the proof of Theorem 3.1 which is based on the regularity results obtained in [14].

2. Preliminaries

In the sequel, $L^p(\Omega)$ is the space of measurable functions $u$ on $\Omega$ such that $\int_\Omega |u|^p < +\infty$ for $1 \leq p < \infty$. $L^\infty(\Omega)$ is the space of essentially bounded functions on $\Omega$. $C^k(\Omega)$ is the set of all functions $k$-times continuously differentiable and its derivates of order $|\alpha|$ are continuous for all multiindex $\alpha$ such that $|\alpha| \leq k$. $C^\infty_c(\Omega)$ denotes the subspace of all functions $u$ infinitely differentiable with compact support in $\Omega$. We will denote $H^k(\Omega)$ the usual Sobolev space of all functions $u$ such that $D^\alpha u$ exists in the distributional sense and belongs to $L^2(\Omega)$ for all multiindex $\alpha$ with $|\alpha| \leq k$. The subspace $H^1_0(\Omega)$ is the closure of $C^\infty_c(\Omega)$ in $H^1(\Omega)$ and the subspace $H^{-1}(\Omega)$ denotes the dual subspace to $H^1_0(\Omega)$. Let $X$ be a Banach space: we will denote by $L^p(0, T; X)$ the space of all measurable functions $u$ such that $u : [0, T] \rightarrow X$ defined by $u(t)(x) = u(t, x)$ (by abuse of notation) satisfies

$$
\|u\|_{L^p(0, T; X)} = \left( \int_0^T \|u(t)\|_X^p \, dt \right)^{1/p} < +\infty, \quad \text{if } p \in [1, +\infty)
$$

and

$$
\|u\|_{L^\infty(0, T; X)} = \operatorname{ess sup}_{0 \leq t \leq T} \|u(t)\|_X < +\infty, \quad \text{if } p = +\infty.
$$
The space $W^{1,p}(0,T;X)$ denotes all the functions $u \in L^p(0,T;X)$ such that \( \frac{\partial u}{\partial t} \in L^p(0,T;X) \). For simplicity, we will use often the notation $W^{1,p}(X)$ instead of $W^{1,p}(0,T;X)$.

Recall that the partial differential operator $L$ is elliptic if there exists a constant $\kappa > 0$ such that

\[
\sum_{i,j=1}^{d} a_{ij}(x)\xi_i \xi_j \geq \kappa |\xi|^2
\]

for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^d$. Moreover, we assume

\[
a_{ij} = a_{ji} \quad \text{and} \quad 0 < a_{\min} \leq a_{ij} \leq a_{\max}, \quad \text{for all} \ i, j \in \{1, 2, \ldots, d\}.
\]

so that the operator defined by (1.2) is elliptic.

### 2.1. Elliptic regularity results

Here, we recall the results obtained in [14]. We first introduce useful functional spaces.

**Definition 2.1.** A function $u$ is a bounded mean oscillation (BMO) function, if $u$ is a real-valued function whose mean oscillation is bounded (finite). This function space is also called John–Nirenberg space. More precisely, we say that a locally integrable function $u$ is a BMO function if

\[
\sup_B \left( \int_B |u(x) - u_B| \, dx \right) =: \|u\|_* < +\infty
\]

where $B$ ranges in the class of the balls of $\mathbb{R}^d$ and $u_B = \int_B u(x) \, dx = \frac{1}{|B|} \int_B u(x) \, dx$.

If $u$ a BMO function and $r > 0$ we set

\[
\eta(r) = \sup_{\rho \leq r} \frac{1}{|B_{\rho}|} \int_{B_{\rho}} |u(x) - u_{B_{\rho}}| \, dx
\]

where $B_{\rho}$ ranges in the class of the balls with radius $\rho$ less than or equal to $r$.

**Definition 2.2.** A function $u$ is a vanishing mean oscillation (VMO) function, if $u$ belongs to the subspace of the BMO functions whose BMO norm over a ball vanishes as the radius of the ball tends to zero:

\[
\lim_{r \to 0} \eta(r) = 0.
\]

The space VMO was introduced by D. Sarason in [12]. The characterization of the VMO functions via the norm of the function over balls implies a number of good features of VMO functions not shared by general BMO functions; for example a VMO function can be approximated by smooth functions. The space BMO can be characterized as the dual space to $H^1$. Furthermore, if $f$ is a BMO function then for any $q < +\infty$ $f$ is locally in $L^q$ and if $f$ belongs to the Sobolev space $W^{d/\theta,d/\theta}$ then $f$ is a VMO function, for any $\theta \in (0,1)$. For more details and properties of BMO and VMO functions we refer [10, 12, 13].

The following theorem have been proved for C. Vitanza in [14]. We consider the elliptic equation in non divergence form

\[
\tilde{L}u = - \sum_{i,j=1}^{d} \tilde{a}_{ij}(x)u_{x_i x_j} + \sum_{i=1}^{d} \tilde{b}_i(x)u_{x_i} + \tilde{c}(t, x)u = \tilde{f}
\]
and the associated Dirichlet problem

\begin{equation}
\begin{aligned}
\bar{L}u &= \bar{f} \\
\quad u \in W^{2,q}(\Omega) \cap W^{1,d}_0(\Omega), & \quad \bar{f} \in L^q(\Omega).
\end{aligned}
\end{equation}

**Theorem 2.1.** Let \( \Omega \) be \( C^{1,1} \). Assume \( \bar{a}_{ij} = \bar{a}_{ji} \), \( \bar{a}_{ij} \in VMO \cap L^\infty(\Omega) \) and that there exists \( \lambda > 0 \) such that

\[ \forall \xi \in \mathbb{R}^d \quad \lambda^{-1} |\xi|^2 \leq \sum_{i,j=1}^{d} \bar{a}_{ij}(x)\xi_i\xi_j \leq \lambda |\xi|^2 \text{ a.e. in } \Omega. \]

We also suppose \( \bar{b}_i \in L^s(\Omega) \), \( s > d \) for \( 1 < q \leq d \), \( s = q \) for \( q > d \), and \( \bar{c} \in L^r(\Omega) \) with \( r = \begin{cases} d & \text{if } 1 < q \leq d, \\ q & \text{if } q > d \end{cases} \) and \( \bar{c} \leq 0 \text{ a.e. in } \Omega. \) Then the Dirichlet problem (2.4) has a unique solution \( u \). Furthermore there exists a positive constant \( C \) such that

\[ ||u||_{W^{2,q}(\Omega) \cap W^{1,d}_0(\Omega)} \leq C \|\bar{f}\|_{L^q(\Omega)} \]

where the constant \( C \) depend on \( d, \partial \Omega, \lambda, \) on the VMO modulus of \( \bar{a}_{ij} \), on the \( L^s \) and \( L^d \) norms respectively of \( \bar{b}_i \) and \( \bar{c} \) and their AC modulus (see [14] for definition of AC modulus).

Here \( W^{k,q}(\Omega) \) denotes the space of all functions \( u \) such that \( D^\alpha u \in L^q(\Omega) \) for all multiindex \( \alpha \) with \( |\alpha| \leq k \) and \( 1 \leq q \leq +\infty \).

**Remark 2.1.** In this work, we will use Theorem 2.1 with no lower order term \( (\bar{c} = 0) \).

2.2. **Variational formulation of (1.1).** Let \( u \in C^2([0,T] \times \Omega) \) be a classical solution of (1.1), i.e., \( u \) satisfies equation (1.1) at any \( (t,x) \in (0,T) \times \Omega \). Multiplying the main equation of (1.1) by \( \phi \in C_0^\infty(\Omega) \) and integrate by parts, we obtain

\begin{equation}
\begin{aligned}
\int_\Omega \frac{\partial^2 u}{\partial t^2}(t,x)\phi(x) \, dx + \int_\Omega a_{ij}(x)\nabla u(t,x) \cdot \nabla \phi(x) \, dx = \int_\Omega f(t,x)\phi(x) \, dx
\end{aligned}
\end{equation}

a.e. \( t \in (0,T) \). Hence, from the density of \( C_0^\infty(\Omega) \) in \( H^1_0(\Omega) \), we have (2.5) for all \( \phi \in H^1_0(\Omega) \). Now, we recall the definition of a weak solution for (1.1) (see [9])

**Definition 2.3.** We say a function

\[ u \in L^2(0,T;H^1_0(\Omega)) \]

with \( \frac{\partial u}{\partial t} \in L^2(0,T;L^2(\Omega)) \) and \( \frac{\partial^2 u}{\partial t^2} \in L^2(0,T;H^{-1}(\Omega)) \)

is a weak solution of Problem (1.1) provided (2.5) holds true for all \( \phi \in H^1_0(\Omega) \) and \( 0 \leq t \leq T \) a.e., and \( u(0,x) = u_0(x) \) and \( \frac{\partial u}{\partial t}(0,x) = u_1(x) \).

We remark that the initial conditions \( u(0,x) = u_0(x) \) and \( \frac{\partial u}{\partial t}(0,x) = u_1(x) \) make sense because of regularity of a weak solution; indeed we have \( u \in C(0,T;L^2(\Omega)) \) and \( \frac{\partial u}{\partial t} \in C(0,T;H^{-1}(\Omega)) \).
3. The main result

Now, we may give the main result:

**Theorem 3.1.** Suppose $a_{ij} \in VMO \cap L^\infty(\Omega)$, $\nabla a_{ij} \in [L^p(\Omega)]^d$ with $p > d$ such that conditions (2.3) are ensured. We also suppose $f \in H^1(L^2(\Omega))$, $u_0 \in H^2(\Omega)$ and $u_1 \in H^1_0(\Omega)$. Then there exists a unique solution $u$ of (1.1) such that

$$u \in L^\infty(H^2(\Omega)), \quad \frac{\partial u}{\partial t} \in L^p(H^1_0(\Omega)), \quad \frac{\partial^2 u}{\partial t^2} \in L^\infty(L^2(\Omega)), \quad \frac{\partial^3 u}{\partial t^3} \in L^2(H^{-1}(\Omega))$$

with the estimate

$$\max_{0 \leq t \leq T} \left( \|u(t)\|_{H^2(\Omega)} + \left\| \frac{\partial u}{\partial t}(t) \right\|_{H^1_0(\Omega)} + \left\| \frac{\partial^2 u}{\partial t^2}(t) \right\|_{L^2(\Omega)} + \left\| \frac{\partial^3 u}{\partial t^3} \right\|_{L^2(H^{-1}(\Omega))} \right) \leq C \left( \|f\|_{H^1(L^2(\Omega))} + \|u_0\|_{H^2(\Omega)} + \|u_1\|_{H^1(\Omega)} \right)$$

with the constant $C$ depending on $\Omega, T$ and the coefficients $a_{ij}$.

**Proof.** We split the proof in several steps.

**Step 1:** Finite-dimensional approximate solutions.

For sake of simplicity, we denote $u' = \frac{\partial u}{\partial t}$, $u'' = \frac{\partial^2 u}{\partial t^2}$, $u''' = \frac{\partial^3 u}{\partial t^3}$ and $f' = \frac{\partial f}{\partial t}$ in the proof. We construct finite-dimensional approximate solutions of (2.5) by the method of Faedo–Galerkin.

As $H^1_0(\Omega)$ is a separable Hilbert space, there exist a family of functions $\{w_m\}_{m \geq 1}$ in $H^1_0(\Omega)$ such that

$$\{w_m\}_{m \geq 1}$$

is an orthogonal basis of $H^1_0(\Omega)$

and

$$\{w_m\}_{m \geq 1}$$

is an orthonormal basis of $L^2(\Omega)$.

Fix now a positive $m$, we look for approximate solutions of (2.5) $u_m : [0, T] \to H^1_0(\Omega)$, as

$$u_m(t) = \sum_{i=1}^{m} g_{im}(t)w_i$$

with $g_m := (g_{1m}, g_{2m}, \ldots, g_{mm})$ satisfying

$$\left\{ \begin{array}{l}
\left( u_m''(t), w_j \right) + \left( a_{ij} \nabla u_m(t), \nabla w_j \right) = \left( f(t), w_j \right) \\
g_{im}(0) = (u_0, w_i), \quad g_{im}'(0) = (u_1, w_i) \quad (i = 1, 2, \ldots, m)
\end{array} \right.$$

(3.7)

where $(\cdot, \cdot)$ denotes the scalar product in $L^2(\Omega)$. The initial conditions in system (3.7) mean that $u_m(0)$ and $u_m'(0)$ are the respective projections of $u_0$ and $u_1$ onto the subspace spanned by $\{w_1, w_2, \ldots, w_m\}$; thus we have $\lim_{m \to +\infty} u_m(0) = u_0$ and $\lim_{m \to +\infty} u_m'(0) = u_1$ (see, for example [3, Chapter 5]). From the classical theory of ordinary differential equations and assumptions of $w_i$, system (3.7) admits a unique local solution $g_m$ such that $g_{jm} \in C^2(0, T_m)$ for $j = 1, 2, \ldots, m$. Then, for each fixed $m$, $u_m$ defined by (3.6) is solution of (3.7).

**Step 2:** A priori estimates.

Multiplying (3.7) by $g_{jm}'$, summing for $j = 1, \ldots, m$ and taking relation (3.6) into account, we get

$$\left( u_m''(t), u_m'(t) \right) + \left( a_{ij} \nabla u_m(t), \nabla u_m'(t) \right) = \left( f(t), u_m'(t) \right), \quad \text{a.e.} t \in [0, T]$$

(3.8)
or equivalently

\[
(3.9) \quad \frac{\partial}{\partial t} \left\| u_m(t) \right\|_{L^2(\Omega)}^2 + \frac{\partial}{\partial t} \left\| a_{ij} \nabla u_m(t) \right\|_{L^2(\Omega)}^2 \leq \left( \left\| f(t) \right\|_{L^2(\Omega)}^2 + \left\| u_m'(t) \right\|_{L^2(\Omega)}^2 \right).
\]

Integrating on \((0, s)\), we deduce

\[\left\| u_m(s) \right\|_{L^2(\Omega)}^2 \leq \int_0^s \left\| u_m'(t) \right\|_{L^2(\Omega)}^2 \, dt + \left\| u_m(0) \right\|_{L^2(\Omega)}^2\]

and with Gronwall’s inequality

\[\left\| u_m'(t) \right\|_{L^2(\Omega)}^2 \leq \exp(T) \left( \left\| f \right\|_{L^2(\Omega)}^2 + \left\| u_m(0) \right\|_{L^2(\Omega)}^2 \right).
\]

Therefore,

\[
(3.10) \quad \max_{0 \leq t \leq T} \left\| u_m'(t) \right\|_{L^2(\Omega)}^2 \leq C \left( \left\| f \right\|_{L^2(\Omega)}^2 + \left\| u_1 \right\|_{L^2(\Omega)}^2 \right)
\]

with \(C\) depending on \(T\) and \(\Omega\). Here, we have used that \(u_m'(0)\) is the projection of \(u_1\) onto the subspace spanned by \(\{w_1, \ldots, w_m\}\).

Using (3.9) again, integrating on \((0, s)\) and using (3.10), we obtain

\[
\left\| a_{ij} \nabla u_m(t) \right\|_{L^2(\Omega)}^2 \leq 2 \left( \left\| f \right\|_{L^2(\Omega)}^2 + \left\| u_0 \right\|_{L^2(\Omega)}^2 \right) + TC \left( \left\| f \right\|_{L^2(\Omega)}^2 + \left\| u_1 \right\|_{L^2(\Omega)}^2 \right)
\]

with \(C\) depending on \(T\), \(\Omega\) and \(a_{ij}^{\text{max}}\). From hypothesis on \(a_{ij}\) and Poincaré inequality we get

\[
\left\| a_{ij} \nabla u_m(t) \right\|_{L^2(\Omega)}^2 \geq C_{\text{min}} \left\| u_m(t) \right\|_{H^1_0(\Omega)}^2.
\]

So,

\[
(3.11) \quad \max_{0 \leq t \leq T} \left\| u_m(t) \right\|_{H^1_0(\Omega)}^2 \leq C \left( \left\| f \right\|_{L^2(\Omega)}^2 + \left\| u_0 \right\|_{H^1_0(\Omega)}^2 + \left\| u_1 \right\|_{L^2(\Omega)}^2 \right)
\]

with \(C\) depending on \(\Omega\), \(T\), \(a_{\text{min}}\) and \(a_{ij}^{\text{max}}\).

Now, we estimate \(\left\| u_m'' \right\|_{L^2(\Omega)}\):

\[
\left\| u_m''(t) \right\|_{H^{-1}(\Omega)} = \sup_{\phi \in H_0^1(\Omega)} \left\langle u_m''(t), \phi \right\rangle_{H^{-1}, H_0^1} = \sup_{\phi \in H_0^1(\Omega)} \left[ \left\langle f(t), \phi \right\rangle - \left\langle a_{ij} \nabla u_m(t), \nabla \phi \right\rangle \right]
\]

\[
\leq \left\| f(t) \right\|_{L^2(\Omega)} + \left\| a_{ij} \nabla u_m(t) \right\|_{L^2(\Omega)}.
\]

Thus

\[
(3.12) \quad \left\| u_m'' \right\|_{L^2(\Omega)} \leq C \left( \left\| f \right\|_{L^2(\Omega)}^2 + \left\| u_0 \right\|_{H^1_0(\Omega)}^2 + \left\| u_1 \right\|_{L^2(\Omega)}^2 \right)
\]

where \(C\) depends on \(\Omega\), \(T\), \(a_{\text{min}}\) and \(a_{ij}^{\text{max}}\).

From these estimates we can conclude that \(T_m = T\), that is \(g_m = (g_{m1}, g_{m2}, \ldots, g_{mm})\) is a global solution of system (3.7) and consequently a global solution \(u_m\).
Step 3: passage to the limit.
The estimates of step 2 allow us to conclude there exists a subsequence \( u_{nk} \) still denoted \( u_m \) and a function \( u \) such that

\[
\begin{align*}
  u_m &\rightharpoonup u \quad L^2(0, T; H^1_0(\Omega)) \\
  u'_m &\rightharpoonup u' \quad L^2(0, T; L^2(\Omega)) \\
  u''_m &\rightharpoonup u'' \quad L^2(0, T; H^{-1}(\Omega))
\end{align*}
\]

where \( \rightharpoonup \) stands for the weak convergence. This yields

\[
\int_\Omega \frac{\partial^2 u_m}{\partial t^2}(t)w_j \, dx \rightharpoonup \int_\Omega \frac{\partial^2 u}{\partial t^2}(t)w_j \, dx \quad \text{as } m \to +\infty
\]

\[
\int_\Omega a_{ij}(x) \nabla u_m(t) \cdot \nabla w_j \, dx \rightharpoonup \int_\Omega a_{ij}(x) \nabla u(t) \cdot \nabla w_j \, dx \quad \text{as } m \to +\infty
\]

for every \( w_j \), by a density argument, for every \( H^1_0 \) function so equation (2.5) is satisfied. Furthermore, by standard arguments is possible to show that \( u(0) = u_0 \) and \( u'(0) = u_1 \). This proves that \( u \) is a weak solution of (1.1). Moreover from (3.10)-(3.12), we have \( u \in L^\infty(0, T; H^1_0(\Omega)) \), \( u' \in L^\infty(0, T; L^2(\Omega)) \) and \( u'' \in L^2(0, T; H^{-1}(\Omega)) \).

Step 4: The uniqueness solution of (1.1) follows similarly to the classical results for hyperbolic equations (for example [9, §7.2]) and from the conditions (2.3) for \( a_{ij} \).

Step 5: Regularity improvement.
Let us differentiate the main equation of (3.7) with respect to \( t \) and multiply by \( g_{jm} \)

\[
\left( u''_m(t), u''_m(t) \right) + \left( a_{ij} \nabla u'_m(t), \nabla u''_m(t) \right) = \left( f'(t), u''_m(t) \right),
\]

that is,

\[
(3.13) \quad \frac{\partial}{\partial t} \left\| u''_m(t) \right\|_{L^2(\Omega)}^2 + \frac{\partial}{\partial t} \left\| a_{ij} \nabla u'_m(t) \right\|_{L^2(\Omega)}^2 \leq \left( \left\| f'(t) \right\|_{L^2(\Omega)}^2 + \left\| u''_m(t) \right\|_{L^2(\Omega)}^2 \right).
\]

Integrating on \((0, s)\) gives

\[
\left\| u''_m(t) \right\|_{L^2(\Omega)}^2 \leq \left\| f \right\|_{L^2(\Omega)}^2 + \int_0^t \left\| u''_m(t) \right\|_{L^2(\Omega)}^2 \, dt + \left\| u''_m(0) \right\|_{L^2(\Omega)}^2
\]

and with (3.8) we deduce

\[
\left\| u''_m(0) \right\|_{L^2(\Omega)}^2 \leq C \left( \left\| f \right\|_{H^1(\Omega)}^2 + \left\| u_m(0) \right\|_{H^2(\Omega)}^2 \right) \leq C \left( \left\| f \right\|_{H^1(\Omega)}^2 + \left\| u_0 \right\|_{H^2(\Omega)}^2 \right).
\]

Then Gronwall’s inequality gives

\[
\max_{0 \leq t \leq T} \left\| u''_m(t) \right\|_{L^2(\Omega)}^2 \leq C \left( \left\| f \right\|_{H^1(\Omega)}^2 + \left\| u_0 \right\|_{H^2(\Omega)}^2 \right)
\]

where \( C \) depends on \( T, a_{\min} \), and \( \Omega \).
On the other hand, by integrating on \((0, s)\) in \((3.13)\) and using the last inequality, we obtain
\[
\left\| a_{ij} \nabla u_m'(t) \right\|_{L^2(\Omega)} \leq \| f \|_{L^2(L^2(\Omega))}^2 + \| u_m'' \|_{L^2(L^2(\Omega))}^2 + \left\| a_{ij} \nabla u_m'(0) \right\|_{L^2(\Omega)}^2
\]
\[
\leq C \left( \| f \|_{H^1(L^2(\Omega))}^2 + \| u_m'' \|_{H^2(\Omega)}^2 + \| u_m'(0) \|_{H^1(\Omega)}^2 \right)
\]
\[
\leq C \left( \| f \|_{H^1(L^2(\Omega))}^2 + \| u_0 \|_{H^2(\Omega)}^2 + \| u_1 \|_{H^1(\Omega)}^2 \right)
\]
with \(C\) depending on \(T, a_{\text{min}}, a_{\text{max}}\) and \(\Omega\). Therefore, \((3.14)\)
\[
\max_{0 \leq t \leq T} \left( \| u_m''(t) \|_{L^2(\Omega)}^2 + \| u_m'(t) \|_{H^1_0(\Omega)}^2 \right) \leq C \left( \| f \|_{H^1(L^2(\Omega))}^2 + \| u_0 \|_{H^2(\Omega)}^2 + \| u_1 \|_{H^1(\Omega)}^2 \right)
\]
where \(C\) depends on \(T, a_{\text{min}}, a_{\text{max}}\) and \(\Omega\).

In order to establish the higher regularity for \(u\), we remark that, from \((3.8)\), for a.e. \(t \in [0, T]\) we have
\[
(- \operatorname{div}(a_{ij} \nabla u_m(t)), \phi) = (f(t) - u_m''(t), \phi)
\]
for every \(\phi \in H^1_0(\Omega)\). We taking \(q = 2\), \(a_{ij} = a_{ij}, \tilde{b}_i = \frac{\partial a_{i1}}{\partial x_1} + \ldots + \frac{\partial a_{id}}{\partial x_d}\) and \(\bar{c} = 0\) in Theorem 2.1 and from hypothesis for \(a_{ij}\) and \(\nabla a_{ij}\), we get \(u_m(t) \in H^2(\Omega)\) and
\[
\| u_m(t) \|_{H^2(\Omega)} \leq C \| f(t) - u_m''(t) \|_{L^2(\Omega)}
\]
where \(C\) depends on \(\Omega\) and the coefficients \(a_{ij}\) (via \(\| \nabla a_{ij} \|_{L^p}\) and the VMO modulus of \(a_{ij}\)).

Hence, by using \((3.14)\) we deduce
\[
\max_{0 \leq t \leq T} \| u_m(t) \|_{H^2(\Omega)} \leq C \max_{0 \leq t \leq T} \| f(t) - u_m''(t) \|_{L^2(\Omega)}
\]
\[
\leq C \left( \| f \|_{H^1(L^2(\Omega))} + \| u_0 \|_{H^2(\Omega)} + \| u_1 \|_{L^2(\Omega)} \right).
\]
with \(C\) depending on \(\Omega, T\) and the coefficients \(a_{ij}\).

Last, we estimate \(\| u_m'' \|_{L^2(H^{-1}(\Omega))}\)
\[
\| u_m''(t) \|_{H^{-1}(\Omega)} = \sup_{\phi \in H^1_0(\Omega)} \left\langle u_m''(t), \phi \right\rangle_{H^{-1}, H^1_0} \frac{1}{\| \phi \|_{H^1_0(\Omega)}}
\]
\[
\leq \sup_{\phi \in H^1_0(\Omega)} \left[ \left\langle f'(t), \phi \right\rangle - \left\langle a_{ij} \nabla u_m'(t), \nabla \phi \right\rangle \right] \frac{1}{\| \phi \|_{H^1_0(\Omega)}}
\]
\[
\leq \| f'(t) \|_{L^2(\Omega)} + \| a_{ij} \nabla u_m'(t) \|_{L^2(\Omega)}.
\]
Thus, from \((3.14)\)
\[
\| u_m'' \|_{L^2(H^{-1}(\Omega))} \leq C \left( \| f \|_{H^1(L^2(\Omega))}^2 + \| u_0 \|_{H^2(\Omega)}^2 + \| u_1 \|_{H^1(\Omega)}^2 \right)
\]
where \(C\) depends on \(\Omega, T\) and the coefficients \(a_{ij}\). Passing to limit as \(m \to +\infty\), we obtain the same regularity and bounds for \(u\). This concludes the proof of theorem. \(\square\)

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