



On the zeros of a class of generalised Dirichlet series-IX

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ON THE ZEROS OF A CLASS OF GENERALISED
DIRICHLET SERIES-IX

BY

R. BALASUBRAMANIAN AND K. RAMACHANDRA

§ 1. **INTRODUCTION.** In paper VII^[4] of this series K. Ramachandra raised the question "Let $s = \sigma + it$, $T \geq T_0$ a large positive constant. For what values $\alpha = \alpha(T)$ the rectangle $(\sigma \geq \alpha(T), T \leq t \leq 2T)$ contains infinity of zeros of a generalised Dirichlet series of a certain type?" (In the earlier papers of this series he and R. Balasubramanian, sometimes individually and sometimes jointly considered the problem where $\alpha = \alpha(T)$ does not depend on T). Since the series considered in that paper were too general the answer $(\alpha(T) = \frac{1}{2} - \frac{D}{\log \log T})$ was perhaps too weak. In paper VIII^[2] of the series we considered somewhat restricted series and obtained the result that we can take $\alpha(T) = \frac{1}{2} - C_0(\log \log T)^{\frac{3}{2}}(\log T)^{-\frac{1}{2}}$. In the present paper we assume a serious restrictive condition namely an Euler product and show that we can take $\alpha(T) = \frac{1}{2} - C_0(\log \log T)(\log T)^{-1}$. (Actually we work with shorter rectangles and also obtain a quantitative result on the number of zeros). Accordingly we prove the following

THEOREM 1. *Let p run over primes and $\omega(p)$ complex numbers whose absolute value is 1 except for a finite set of primes where we define $\omega(p)$ to*

be zero. Let $s = \sigma + it$ as usual, and let

$$F(s) = \sum_{n=1}^{\infty} (a_n n^{-s}) = \prod_p \left(1 - \frac{\omega(p)}{p^s}\right)^{-1}, (\sigma > 1), \quad (1.1)$$

be continuable analytically in

$$\left\{ \sigma \geq \frac{1}{2} - \frac{C \log \log T}{\log H}, T \leq t \leq T + H \right\} ((\log T)^{C'} \leq H \leq T, C' \text{ to be specified}) \quad (1.2)$$

and there $\max |F(s)| \leq T^A$ where $A(\geq 1)$ is a constant. Let $C(\geq 1)$ be a large constant depending on A and let H exceed $(\log T)^{C'}$ where $C'(\geq 1)$ is a large constant. Then $F(s)$ has at least $(C'')^{-1} H (\log \log \log T)^{-1}$ zeros in the rectangle (1.2), where $C''(\geq 1)$ is a large constant.

REMARK 1. The condition on the Euler product (1.1) covers the Riemann zeta-function and the ordinary L -series. But it can be stated in such a way that it covers zeta and L -functions of algebraic number fields. The restriction (1.1) and the restriction $|F(s)| \leq T^A$ practically force us to give these (and perhaps only these, with trivial changes) as examples and in these cases the series have a functional equation and so in fact we can take $\alpha(T) = \frac{1}{2}$. Hence the main content of the present paper is the emphasis on results which can be proved without the assumption of a functional equation for $F(s)$. Thus we stress that *Theorem 1 does not depend on the assumption of a functional equation for $F(s)$.*

REMARK 2. In the case where $F(s)$ is the Riemann zeta-function or ordinary L -series we can prove (without using the functional equation) that

$$\frac{1}{T} \int_T^{2T} |F(\frac{1}{2} + it)|^2 dt \leq (\log T)^2 \quad (1.3)$$

and from this we can deduce that the number of zeros of $F(s)$ in $(\sigma \geq \beta, T \leq t \leq 2T)$ is $\leq T^\theta (\log T)^{A_1}$ where $\frac{1}{2} \leq \beta \leq 1, \theta = 4(1 - \beta)(3 - 2\beta)^{-1}$ and $A_1 \geq 1$ is a certain numerical constant. (For a proof of this fact see [5]). Thus in these cases and with the extra condition $H = T$, the zeros of the theorem "already belong" to the rectangle

$$\left(\sigma \leq \frac{1}{2} + C \frac{\log \log T}{\log T}, T \leq t \leq 2T \right) \quad (1.4)$$

§ 2. **PROOF OF THEOREM 1.** In the proof we use the Ramachandra kernel $\text{Exp}((\sin z)^2)$, a well-known theorem due to Montgomery and Vaughan and the well-known

THEOREM 2 (Borel-Caratheodory Theorem). *Suppose $G(z)$ is analytic in $|z - z_0| \leq R$ and on $|z - z_0| = R$ we have $\text{Re } G(z) \leq U$. Then in $|z - z_0| \leq r < R$, we have,*

$$|G(z)| \leq \frac{2rU}{R-r} + \frac{R+r}{R-r} |G(z_0)|. \quad (2.1)$$

PROOF. See [6], page 174.

We break up the proof into a few lemmas. The rough idea of the proof is as follows. We first prove that on the line $\sigma = \frac{1}{2} - 2\delta$, where $\delta = \frac{1}{4}C(\log \log T)(\log H)^{-1}$, we have (between T and $T+H$) $\gg H$ well-spaced points which are all heavy for $\log F(s)$ in a certain sense. Out of these we select $\gg H(\log \log \log T)^{-1}$ points with the property that any two (distinct) points are at a distance $\geq L = D \log \log \log T$, where $D(\geq 1)$ is a large constant. We then prove that if $\frac{1}{2} - 2\delta + it_j$ is any such point the rectangle $(\sigma \geq \frac{1}{2} - 4\delta, |t_j - t| \leq \frac{1}{3}L)$ contains a zero of $F(s)$. This will prove the theorem completely. We begin with

LEMMA 1. *For $u > 0$ let*

$$\Delta(u) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} u^w \text{Exp} \left(\left(\sin \frac{w}{1000} \right)^2 \right) \frac{dw}{w}. \quad (2.2)$$

Then

$$\Delta(u) = O(u^2) \text{ and also } \Delta(u) = 1 + O(u^{-2})$$

where the implied constants are absolute.

PROOF. By moving the line of integration to $\text{Re } w = 2$ (resp. $\text{Re } w = -2$) the lemma follows by easy estimations.

LEMMA 2. *Let $\sigma = \frac{1}{2} - 2\delta$, $X = H^{\frac{1}{2}}$ and*

$$F_1(s) = \sum_p \omega(p) p^{-s} \Delta \left(\frac{X}{p} \right). \quad (2.3)$$

Then

$$\frac{1}{H} \int_T^{T+H} |F_1(s)|^2 dt \gg H^{2\delta} (\delta \log H)^{-1} \quad (2.4)$$

and

$$\frac{1}{H} \int_T^{T+H} |F_1(s)|^4 dt \ll H^{4\delta} (\delta \log H)^{-2}. \quad (2.5)$$

PROOF. By a well-known theorem of H.L. Montgomery and R.C. Vaughan (see [3] for a simpler proof of a special case of their result which we need here) LHS of (2.4) is

$$\sum_p \frac{|\omega(p)|^2}{p^{2\sigma}} \left| \Delta \left(\frac{X}{p} \right) \right|^2 \left(1 + O \left(\frac{p}{H} \right) \right) \quad (2.6)$$

which is $\gg H^{2\delta} (\delta \log H)^{-1}$ by the use of Lemma 1. (Since the proof of this is routine we postpone its proof to § 3). This proves (2.4).

Also LHS of (2.5) is

$$\leq \sum_{p_1, p_2} (p_1 p_2)^{-2\sigma} \left| \Delta \left(\frac{X}{p_1} \right) \Delta \left(\frac{X}{p_2} \right) \right|^2 \left(1 + O \left(\frac{p_1 p_2}{H} \right) \right) \quad (2.7)$$

which is $\ll H^{4\delta} (\delta \log H)^{-2}$ by the use of Lemma 1. (Since the proof of this is routine we postpone its proof to § 3). This proves (2.5).

Thus Lemma 2 is completely proved.

LEMMA 3. Let T and H be positive integers (this can be assumed without loss of generality in Theorem 1). Then there exist $\gg H$ integers M satisfying $T \leq M \leq T + H - 1$ for which

$$\int_M^{M+1} |F_1(s)|^2 dt \gg H^{2\delta} (\delta \log H)^{-1}. \quad (2.8)$$

PROOF. By (2.4) we have

$$\sum_M \int_M^{M+1} |F(s)|^2 dt \geq C_1 H \psi$$

where C_1 is a positive constant and $\psi = H^{2\delta} (\delta \log H)^{-1}$. Here in the sum we drop those M for which the integral from M to $M + 1$ does not exceed $\frac{1}{2} C_1 \psi$ and obtain

$$\sum'_M \int_M^{M+1} |F_1(s)|^2 dt \geq \frac{1}{2} C_1 H \psi$$

where the slash indicates that we restrict only to those integrals which exceed $\frac{1}{2}C_1\psi$. By applying Hölder's inequality we obtain

$$\left(\sum'_M 1\right)^{\frac{1}{2}} \left(\sum'_M \left(\int_M^{M+1} |F_1(s)|^2 dt\right)^2\right)^{\frac{1}{2}} \geq \frac{1}{2}C_1H\psi.$$

The inequality

$$\int_M^{M+1} |F_1(s)|^2 dt \leq \left(\int_M^{M+1} |F_1(s)|^4 dt\right)^{\frac{1}{2}}$$

completes the proof of the lemma on using (2.5).

REMARK. Actually the deduction of Lemma 3 from Lemma 2 involves a general principle which was first observed and applied for discussing the zeros of Dirichlet series by R. Balasubramanian and K. Ramachandra in paper III^[1] of this series.

LEMMA 4. *In the interval $[T, T+H]$ there are $R \gg H(\log\log\log T)^{-1}$ points t_1, t_2, \dots, t_R with the properties $|t_j - t_{j'}| \geq L$ for all $j \neq j'$ where $L = D \log\log\log T$ and further*

$$|F_1\left(\frac{1}{2} - 2\delta + t_j\right)| \gg H^\delta (\delta \log H)^{-\frac{1}{2}}. \quad (2.9)$$

As has been said before D is a large positive constant.

PROOF. The proof follows from Lemma 3.

LEMMA 5. *Consider any of the points $t_j (= t_0$ say) given by Lemma 4 excluding the upper and the lower extreme points. Assume that the region $(\sigma \geq \frac{1}{2} - 4\delta, |t - t_0| \leq \frac{1}{3}L)$ is zero-free for $F(s)$. Then in $(\sigma \geq \frac{1}{2} - 3\delta, |t - t_0| \leq \frac{1}{4}L)$ we have,*

$$|\log F(s)| \leq (\log T)^3. \quad (2.10)$$

PROOF. Apply Borel-Caratheodory theorem as follows. Take $z_0 = 2 + it$ at which definitely $|\log F(z_0)| \leq 10$, whatever be t . Choose R such that the circle $|z - z_0| = R$ touches $\sigma = \frac{1}{2} - 4\delta$ (we mean the line $\operatorname{Re} z = \frac{1}{2} - 4\delta$)

and r to be $R - \delta$. This proves the lemma.

LEMMA 6. Put $s_0 = \frac{1}{2} - 2\delta + it_0$. Then

$$F_1(s_0) = \log F(s_0) + O(1). \quad (2.11)$$

PROOF. Since the powers of primes higher than the first contribute $O(1)$, LHS of (2.11) is

$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \log F(s_0 + w) X^w \exp \left(\left(\sin \frac{w}{1000} \right)^2 \right) \frac{dw}{w} + O(1).$$

We now break off the portion $| \operatorname{Im} w | \geq \frac{1}{4}L$ of the integral with an error $O(1)$ and in the rest of the integral move the line of integration to $\operatorname{Re} w = -\delta$. This involves an error $O(X^{-\delta}(\log T)^4) = O(1)$ if C is large. The pole at $w = 0$ contributes $\log F(s_0)$. This proves the lemma.

LEMMA 7. The equations (2.9), (2.10) and (2.11) are contradictory and hence the assumption of Lemma 5 is not true.

PROOF. The equations referred to in the lemma give

$$H^\delta (\delta \log H)^{-\frac{1}{2}} \ll (\log T)^3.$$

This is false if $\delta = \frac{1}{4}C(\log \log T)(\log H)^{-1}$ (as specified already) and C is a large positive constant.

This completes the proof of Theorem 1.

§ 3. EXPLANATIONS REGARDING (2.6) AND (2.7).

(A) In (2.6) the main term is

$$\begin{aligned} &= \sum_{p \leq X} p^{-2\sigma} \left(1 + O \left(\frac{p^2}{X^2} \right) \right) \left(1 + O \left(\frac{p}{X^2} \right) \right) + O(1) \\ &= \sum_{p \leq X} p^{-2\sigma} \left(1 + O \left(\frac{p^2}{X^2} \right) \right) + O(1) \\ &= \sum_{p \leq X} p^{-2\sigma} + O(X^{1-2\sigma}(\log H)^{-1}) \end{aligned}$$

$$= \sum_{p \leq X} p^{-2\sigma} + O(H^{2\delta}(\log H)^{-1}).$$

The error term is

$$\begin{aligned} &\ll \sum_{X \leq p \leq H} p^{-2\sigma} \left(\frac{X}{p}\right)^2 \left(1 + \frac{p}{H}\right) + \sum_{p > H} p^{-2\sigma} \left(\frac{X}{p}\right)^2 \left(\frac{p}{X^2}\right) \\ &\ll \sum_{X \leq p \leq H} p^{-2\sigma} \left(\frac{X}{p}\right)^2 + \sum_{p > H} p^{-1-2\sigma} \\ &\ll X^2(\log H)^{-1} X^{-1-2\sigma} + (\log H)^{-1} H^{-2\sigma} \\ &\ll (\log H)^{-1} (X H^{-\sigma} + H^{-2\sigma}) \ll H^{2\delta}(\log H)^{-1}. \end{aligned}$$

Thus (2.6) is $\sum_{p \leq X} p^{-2\sigma} + O(H^{2\delta}(\log H)^{-1})$. Now the first term here is \asymp

$\sum_{U \leq X} U^{4\delta} (\log U)^{-1}$ (U runs over powers of 2) which is

$$\begin{aligned} &\asymp \sum_{X^{\frac{1}{2}} \leq U \leq X} U^{4\delta} (\log H)^{-1} + O(H^{\delta} \log \log H) \\ &\asymp H^{2\delta} (\log H)^{-1} \left(\sum_{\nu=0}^Q 2^{-4\nu\delta} + O(2^{-4Q\delta} \delta^{-1}) \right) + O(H^{\delta} \log \log H) \text{ (where } Q \\ &\quad \left[\frac{1}{4} \frac{\log H}{\log 2} \right]), \\ &\asymp H^{2\delta} (\log H)^{-1} \left(\frac{1}{\delta} + O(H^{-\delta} \delta^{-1}) \right) + O(H^{\delta} \log \log H) \\ &= H^{2\delta} (\delta \log H)^{-1} (1 + O(H^{-\delta} + H^{-\delta} (\delta \log H) \log \log H)) \\ &= H^{2\delta} (\delta \log H)^{-1} (1 + O(H^{-\delta} + H^{-\delta/2} \log \log H)), \text{ (since } H^{\delta/2} \geq \frac{\delta}{2} \log H), \\ &\asymp H^{2\delta} (\delta \log H)^{-1} \text{ provided } H^{\delta} \geq (\log \log H)^3 \text{ (which is certainly} \\ &\text{satisfied since } \delta = \frac{1}{4} C (\log \log T) (\log H)^{-1} \text{ and } C \text{ is assumed to be} \\ &\text{large}). \end{aligned}$$

(B) The quantity in (2.7) is

$$\ll \left(\sum_p \left| \Delta \left(\frac{X}{p} \right) \right|^2 p^{-2\sigma} \right)^2 + \frac{1}{H} \left(\sum_p \left| \Delta \left(\frac{X}{p} \right) \right|^2 p^{1-2\sigma} \right)^2$$

$$\begin{aligned}
&\ll \left(\sum_{p \leq X} p^{-2\sigma} \right)^2 + \left(\sum_{p > X} \frac{X^4}{p^{4+2\sigma}} \right)^2 + \frac{1}{X^2} \left(\sum_{p \leq X} p^{1-2\sigma} \right)^2 + \frac{1}{X^2} \left(\sum_{p > X} \frac{X^4}{p^{3+2\sigma}} \right)^2 \\
&\ll H^{4\delta} (\delta \log H)^{-2} + \left(\frac{X^4}{X^{3+2\sigma} \log H} \right)^2 + \frac{1}{X^2} \left(\frac{X^{2-2\sigma}}{\log H} \right)^2 \\
&\quad + \frac{1}{X^2} \left(\frac{X^4}{X^{2+2\sigma} \log H} \right)^2 \text{ (if } H^\delta \geq (\log \log H)^3 \text{ which is satisfied by our assumption on } H) \\
&\ll H^{4\delta} (\delta \log H)^{-2} + H^{4\delta} (\log H)^{-2} \\
&\ll H^{4\delta} (\delta \log H)^{-2} \text{ if } \delta \text{ is small.}
\end{aligned}$$

This completes explanations regarding (2.6) and (2.7).

Thus Theorem 1 is completely proved.

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