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Hamiltonicity of large generalized de Bruijn Cycles

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Abstract

In this article, we determine when the large generalized de Bruijn cycles $BGC(p, d, n)$ are Hamiltonian. These digraphs have been introduced by Gómez, Padró and Pérennes as large interconnection networks with small diameter and they are a family of generalized p-cycles. They are the Kronecker product of the generalized de Bruijn digraph $GB(d, n)$ and the dicycle of length p, where $GB(d, n)$ is the digraph whose vertices are labeled with the integers modulo n such that there is an arc from vertex i to vertex j if, and only if, $j \equiv di + \alpha \pmod{n}$, for every α with $0 \leq \alpha \leq d - 1$.

Keywords: Hamiltonian digraphs, p-cycles, de Bruijn digraphs, interconnection networks

1. Introduction

A Hamiltonian dicycle in a digraph D is a dicycle C such that each vertex of D appears exactly once in C (see [2]). Hamiltonian properties have been studied for digraphs modeling interconnection networks. For example, the so called de Bruijn digraphs (see [3]) were introduced to show the existence of de Bruijn sequences, that is circular sequences of d^D elements such that any subsequence of length D appears exactly once. To prove the existence of such sequences, it was proved that de Bruijn digraphs are Hamiltonian (see [3]).

Since then, many generalizations of the de Bruijn digraphs have been proposed in the literature to build interconnection networks with small diameter. One of them is about *generalized p-cycles*: that is digraphs whose set of vertices is partitioned in p parts, that can be ordered in such a way that a vertex is adjacent only to vertices in the next part (see [11]). Those specific extensions of the de Bruijn digraphs are sometimes called *large generalized de Bruijn Cycles*, and they are denoted $BGC(p, d, n)$.

It was proved that the digraphs $BGC(p, d, n)$ are among the largest known p-cycles with given degree and diameter (see [11]). Their connectivity properties have been studied in [1, 10, 15]; but unlike many other variants of the original de Bruijn digraphs, the Hamiltonian properties of this class of digraphs have not been studied yet. The aim of this article consists in determining when $BGC(p, d, n)$ is Hamiltonian.

2. Definitions and earlier results

We refer to [2] for graph theory notions. Especially, we use the notion of *line digraph*:

Definition 1 ([2])**.** *Given a digraph $G = (V, E)$, the line digraph $L(G)$ of G has as vertices the arcs in G. There is an arc from $e = (u, v)$ to $e' = (u', v')$ in $L(G)$ if, and only if, $v = u'$.*

In this article, we use the following well-known property of line digraph operation:

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Proposition 1 ([5]). *The digraph G is Eulerian if, and only if, its line digraph $L(G)$ is Hamiltonian.*

We now define the so-called generalized de Bruijn digraphs, using arithmetical relations:

Definition 2 ([14, 16]). *The* generalized de Bruijn digraph *$GB(d, n)$ (also called Reddy-Pradhan-Kuhl digraph), is the digraph whose vertices are labeled with the integers modulo n; there is an arc from vertex i to vertex j if, and only if, $j \equiv di + \alpha \pmod{n}$, for every α with $0 \leq \alpha \leq d-1$.*

Those generalized de Bruijn digraphs can be extended in many ways. We briefly present two of them.

Definition 3 ([7]). *Let $1 \leq d, q \leq n-1$, and $0 \leq r \leq n-1$, the* consecutive-d digraph *$G(d, n, q, r)$ is the digraph whose vertices are labeled with the integers modulo n, such that there is an arc from vertex i to vertex j if, and only if, $j \equiv qi + r + \alpha \pmod{n}$, for every α with $0 \leq \alpha \leq d-1$.*

When $q = d$ and $r = 0$, $G(d, n, d, 0) = GB(d, n)$.

The consecutive-d digraphs also include another subfamily of digraphs that was introduced in [8]. Let λ be a positive integer, with $1 \leq \lambda \leq d$. Then, $GB_\lambda(d, n)$ is the subdigraph of $GB(d, n)$ such that there is an arc from i to j if, and only if, $j \equiv di + \alpha \pmod{n}$, for every α with $0 \leq \alpha \leq \lambda - 1$. Actually, the digraph $GB_\lambda(d, n)$ is nothing else than the consecutive-λ digraph $G(\lambda, n, d, 0)$, but the notation of $GB_\lambda(d, n)$ helps to understand that it is a subdigraph of $GB(d, n)$.

The characterization of the Hamiltonian consecutive-d digraphs is nearly complete:

Theorem 2 ([4, 6, 7, 13]). *Let $G = G(d, n, q, r)$ be a consecutive-d digraph.*

- *If $d = 1$, then G is Hamiltonian if, and only if, all of the four following conditions hold:*
  1. $\gcd(n, q) = 1$;
  2. *for every prime number p such that $p|n$, we have $p|q-1$;*
  3. *if $4|n$, then $4|q-1$ too;*
  4. $\gcd(n, q-1, r) = 1$.
- *If $d = 2$, then G is Hamiltonian if, and only if, one of the two following conditions holds:*
  1. $\gcd(n, q) = 2$;
  2. $\gcd(n, q) = 1$ *and either $G(1, n, q, r)$ or $G(1, n, q, r+1)$ is Hamiltonian.*
- *If $d = 3$, then:*
  1. *if $\gcd(n, q) \geq 2$, then G is Hamiltonian if, and only if, $\gcd(n, q) \leq 3$;*
  2. *if $1 \leq |q| \leq 3$ and n and q are relatively prime, then G is Hamiltonian.*
- *If $d \geq 4$, then G is Hamiltonian if, and only if, $\gcd(n, q) \leq d$.*

Corollary 1 ([6]). *Let $G = G(d, n, q, r)$ be a consecutive-d digraph. If $\lambda = \gcd(n, q) \geq 2$, then G is Hamiltonian if, and only if, $\lambda \leq d$.*

For the extension we consider here, we need to introduce the Kronecker product, also called conjunction or direct product (see [12]):

Definition 4. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two digraphs. Their Kronecker product $G_1 \otimes G_2$ is a digraph $G = (V, E)$ such that:*

1. $V = V_1 \times V_2$
2. $E = \{((u_1, u_2), (v_1, v_2)) : (u_1, v_1) \in E_1 \text{ and } (u_2, v_2) \in E_2\}$

Definition 5 ([11]). *The* large generalized de Bruijn Cycle *$BGC(p, d, n)$ is the Kronecker product of the generalized de Bruijn digraph $GB(d, n)$ with the dicycle C_p whose length is p. In other words, $BGC(p, d, n) = GB(d, n) \otimes C_p$.*

If $p = 1$, then $BGC(1, d, n)$ is isomorphic to the generalized de Bruijn digraph $GB(d, n)$.

3. Existence of Hamiltonian dicycles

We now completely characterize the Hamiltonicity of the digraphs $BGC(p,d,n)$:

Theorem 3. *For $p \geq 1$, the digraph $BGC(p,d,n)$ is Hamiltonian if, and only if, one of the four following conditions holds:*

1. $d \geq 3$;
2. $d = 2$ *and* n *is even;*
3. $d = 2$ *and, for every prime number* q *such that* $q|n$, $q|2^p - 1$;
4. $d = n = 1$.

When $p = 1$, these conditions are exactly the necessary and sufficient conditions of Hamiltonicity of the generalized de Bruijn digraphs, and they are proved in [6, 8]. Furthermore, if $d = 1$, then we trivially verify that $BGC(p,1,n)$ is Hamiltonian if, and only if, $GB(1,n)$ is Hamiltonian. So, the only solution is the degenerate case when $n = 1$.

For the rest of this paper, we assume that $p \geq 2$ and $d > 1$. The proof of Theorem 3 follows from the four following lemmas. Observe that we omit the notation (mod n) in some parts of the proofs, when the context is clear. The vertices of $BGC(p,d,n)$ will be labeled by $\mathbb{Z}_n \times \mathbb{Z}_p$ in the proofs.

Lemma 1. *If $\lambda = \gcd(n,d) \geq 2$, then $BGC(p,d,n)$ is Hamiltonian.*

Proof. Let $n' = \dfrac{n}{\lambda}$. We denote by $BGC_\lambda(p,d,n)$ the digraph $GB_\lambda(d,n) \otimes C_p$. In [8] it is proven that $L(GB_\lambda(d,n')) = GB_\lambda(d,n)$. Note that $L(G \otimes C_p) = L(G) \otimes C_p$; hence $L(BGC_\lambda(p,d,n')) = BGC_\lambda(p,d,n)$. Moreover, $GB_\lambda(d,n')$ is Eulerian, and so strongly connected [8]. Since there is a loop in $GB_\lambda(d,n')$ in vertex 0, there is always a dipath in $BGC_\lambda(p,d,n')$ from $(0,0)$ to any vertex (i,j) and also from (i,j) to $(0,0)$. Consequently, $BGC_\lambda(p,d,n')$ is also strongly connected. Furthermore, it is a λ-regular digraph too. Therefore, $BGC_\lambda(p,d,n')$ is Eulerian, and so, its line digraph $BGC_\lambda(p,d,n)$ is Hamiltonian by Proposition 1. As a consequence, $BGC(p,d,n)$ is Hamiltonian too. □

If $d \geq 4$, and n and d are relatively prime, then we can use the Hamiltonian properties of the consecutive-d digraphs. We consider a Hamiltonian dicycle in any consecutive-d digraph as a circular permutation σ of $\mathbb{Z}_n$. If j is the vertex that follows vertex i in the Hamiltonian dicycle, then $\sigma(i) = j$; similarly, if k is the vertex that follows vertex j in the Hamiltonian dicycle, then $\sigma^2(i) = \sigma(j) = k$, and so on.

Lemma 2. *If $d \geq 4$ and n and d are relatively prime, then $BGC(p,d,n)$ is Hamiltonian.*

Proof. Consider the consecutive-d digraph $G = G(d,n,d^p,0)$. Since $\gcd(n,d^p) = 1$ and $d \geq 4$, we know by Theorem 2 that G is a Hamiltonian digraph. Let $0, \sigma(0), \sigma^2(0), \ldots, \sigma^{n-1}(0), 0$ be a Hamiltonian dicycle in G.

Then, we consider the following dicycle $C = (0,0),(0,1),\ldots,(0,p-1),(\sigma(0),0),(d\sigma(0),1),\ldots,(d^{p-1}\sigma(0),p-1),\ldots,(\sigma^i(0),0),(d\sigma^i(0),1),\ldots,(d^{p-1}\sigma^i(0),p-1),\ldots,(\sigma^{n-1}(0),0),(d\sigma^{n-1}(0),1),\ldots,(d^{p-1}\sigma^{n-1}(0),p-1),(0,0)$. It is effectively a dicycle as, by definition, there is an arc from vertex (i,k) to vertex $(di,k+1)$, when $0 \leq k \leq p-2$. Furthermore, there is also an arc from vertex $(d^{p-1}\sigma^i(0),p-1)$ to vertex $(\sigma^{i+1}(0),0)$, as, by the adjacency relations of G, there exists α, $0 \leq \alpha \leq d-1$, such that $\sigma^{i+1}(0) \equiv d^p\sigma^i(0) + \alpha \equiv d(d^{p-1}\sigma^i(0)) + \alpha \pmod n$.

Let us note that, since $\gcd(n,d) = 1$, the mapping (i,k) to its successor $(di,k+1)$ is one-to-one. So, it suffices to verify that all the vertices with a given k, for example $k = 0$, are different, which follows from the fact the $\sigma^i(0)$, $0 \leq i \leq n-1$, are all different in the Hamiltonian dicycle of G. □

The only remaining cases are $d = 2$ and $d = 3$. If $d = 2$ and n is even, then there is a Hamiltonian dicycle in $BGC(p,2,n)$ by Lemma 1. Else:

Lemma 3. *If n is odd, then $BGC(p,2,n)$ is Hamiltonian if, and only if, for all prime number q such that $q|n$, $q|2^p - 1$.*

Proof. Let us assume that there is a Hamiltonian dicycle C in $BGC(p, 2, n)$. Let k_0 be a fixed integer modulo p. We claim that if there exists a vertex (i, k_0) such that (i, k_0) precedes in C the vertex $(2i, k_0 + 1)$, then this holds for all the vertices. Indeed, as, for $j \equiv i - 2^{-1} \pmod{n}$, the successors of the vertex (j, k_0) are $(2j, k_0 + 1)$ and $(2j + 1, k_0 + 1) = (2i, k_0 + 1)$, the vertex (j, k_0) must precede the vertex $(2j, k_0 + 1)$ in C. Furthermore, since 2^{-1} is a generator element in $\mathbb{Z}_n$, each vertex (i, k_0) has to precede the vertex $(2i, k_0 + 1)$ in C.

Otherwise, if there does not exist a vertex (i, k_0) such that (i, k_0) precedes in C the vertex $(2i, k_0 + 1)$, then each vertex (i, k_0) has to precede the vertex $(2i + 1, k_0 + 1)$ in C. In summary, there are only two possibilities for a given k_0. At the end, there are only 2^p possible Hamiltonian dicycles, namely: $C_0, C_1, \ldots, C_{2^p-1}$, such that after a vertex $(i, 0)$, the next vertex in C_r whose label is also in $\mathbb{Z}_n \times \{0\}$ following $(i, 0)$ is $(2^p i + r, 0)$. Then, observe that C_r is a Hamiltonian dicycle if, and only if, $i \to 2^p i + r$ is a circular permutation of $\mathbb{Z}_n$, that is $G(1, n, 2^p, r)$ is Hamiltonian. By Theorem 2, $G(1, n, 2^p, r)$ is Hamiltonian if, and only if, every of the following conditions hold:

- n and 2^p are relatively prime;
- for all prime number q such that $q|n$, $q|2^p - 1$;
- $\gcd(n, 2^p - 1, r) = 1$.

The first condition is verified as n is odd, and the third one can always be held by taking $r = 1$. Consequently, a necessary and sufficient condition for $BGC(p, 2, n)$ to be Hamiltonian when n is odd is that for all prime number q such that $q|n$, $q|2^p - 1$. □

Remark 1. *Especially when $M_p = 2^p - 1$ is a Mersenne prime number, $BGC(p, 2, n)$ is Hamiltonian if, and only if, n is a power of M_p, and there are always $\varphi(M_p) = M_p - 1$ possible Hamiltonian dicycles, where φ denotes the Euler's function.*

Finally, if $d = 3$, we prove that $BGC(p, 3, n)$ is always Hamiltonian by using a method of link-interchange.

Definition 6. *Let C_1, C_2 be two dicycles that are subdigraphs of the same digraph D. A pair $\{x_1, x_2\}$ with $x_1 \in C_1$ and $x_2 \in C_2$ is called an* interchange pair *if the predecessor y_1 of x_1 in C_1 is incident to x_2 in D, and the predecessor y_2 of x_2 in C_2 is incident to x_1 in D too.*

If $\{x_1, x_2\}$ is an interchange pair, then we can build a dicycle containing all the vertices of $C_1 \cup C_2$ by deleting (y_1, x_1) and (y_2, x_2) and adding the arcs (y_1, x_2) and (y_2, x_1).

Lemma 4. *$BGC(p, 3, n)$ is Hamiltonian.*

Proof. When $3|n$, that is a direct consequence of Lemma 2. Consequently, we assume that 3 does not divide n. To every vertex (i, k) we associate the vertex $(3i + 1, k + 1)$. As n and 3 are relatively prime, the digraph $BGC(p, 3, n)$ is partitioned into pairwise vertex-disjoint dicycles $C_1, C_2, \ldots, C_m$. If there is only one dicycle, that is $m = 1$, we are done as it is Hamiltonian. Otherwise, we use interchange pairs to merge successively the dicycles till we have only one. But we have to be careful to do independent interchanges.

We first claim that, for every $i \in \mathbb{Z}_n$ and for every $k \in \mathbb{Z}_p$, if the vertices (i, k) and $(i+1, k)$ are not in the same dicycle, then the pair of vertices $\{(i, k), (i + 1, k)\}$ is an interchange pair. Indeed let $(j, k-1)$ be the the predecessor of (i, k). So, $3j + 1 \equiv i \pmod{n}$ and as $3j + 2 \equiv i + 1 \pmod{n}$, $(j, k-1)$ is incident to $(i+1, k)$. Similarly, let $(j', k-1)$ be the the predecessor of $(i+1, k)$. Then, $3j' + 1 \equiv i + 1 \pmod{n}$ and as $3j \equiv i \pmod{n}$, so $(j', k - 1)$ is incident to (i, k). Therefore, the claim is proved. However, we have to be careful not to use twice the same vertex in an interchange pair, as the predecessor has changed when doing the first merging.

Case 1: n even. Here we will only use some interchange pairs of the form: $g(i) = \{(2i, 0), (2i + 1, 0)\}$ and $f(j) = \{(2j + 1, 1), (2j + 2, 1)\}$, with $i, j \in \mathbb{Z}_n$. These pairs are pairwise independent, because n is even. While there exists an $i \in \mathbb{Z}_n$ such that $(2i, 0)$ and $(2i + 1, 0)$ are in different

dicycles, we merge these two dicycles using the interchange pair $g(i)$. After at most $n/2$ merge operations, we get a set of disjoint dicycles such that, for all $i \in \mathbb{Z}_n$, the vertices $(2i, 0)$ and $(2i + 1, 0)$ belong to the same dicycle. Then, for every $i \in \mathbb{Z}_n$, we now consider vertices $(2i, 0)$ and $(2(i + 3^{-1}), 0)$. Suppose that they are in two different dicycles, namely: C'_1 and C'_2. By construction, the vertex $(2i + 1, 0)$, which is also in C'_1, precedes the vertex $(6i + 4, 1)$ of C'_1, and the vertex $(2(i+3^{-1}), 0)$ precedes $(6i+3, 1)$ of C'_2. Moreover, we have that $\{(6i+3, 1), (6i+4, 1)\} = f(3i + 1)$ is an admissible interchange pair that we can use to merge the two dicycles, because $6i+3$ is odd whereas n is even, and so, $6i+3 \pmod n$ is odd. Finally, since 3 and n are relatively prime, 3^{-1} is a generator element in $\mathbb{Z}_n$, and so, we can successively consider the possible $i \in \mathbb{Z}_n$ such that $(2i, 0)$ and $(2(i + 3^{-1}), 0)$ belong to different dicycles, and merge their dicycles. At the end of that final step, we get that all the vertices $(i, 0)$, with $i \in \mathbb{Z}_n$, belong to the same dicycle. Therefore, all the remaining dicycles have been merged into one.

It is interesting to notice that no extra - interchange pairs $\{(i, k), (i + 1, k)\}$ with $k \notin \{0, 1\}$ are needed. We propose another set of interchange pairs when n is odd.

Case 2: n odd. The proof is quite the same as for the preceding case, except that we choose the set of interchange pairs $P = \{\{(2i + 1, k), (2i + 2, k)\} : 1 \leq 2i + 1 < n\}$. Note that all the pairs in P are pairwise independent. In [6], they proved that in the case $p = 1$, these interchange pairs $\{2i + 1, 2i + 2\}$, with $1 \leq 2i + 1 < n$, were enough to merge the dicycles (obtained by joining i to $3i + 1$) into one Hamiltonian dicycle. To do that, they proved the graph G_1, consisting of the undirected cycles of the decomposition plus the edges $\{2i+1, 2i+2\}$, $1 \leq 2i+1 < n$, was connected. Here, we similarly consider the graph G consisting of the undirected cycles $C_1, C_2, \ldots, C_m$ plus the edges $\{(2i + 1, k), (2i + 2, k)\}$, with $\{(2i + 1, k), (2i + 2, k)\} \in P$. To prove that we can merge these dicycles into one Hamiltonian dicycle, it suffices to prove that G is connected.

We have that $3 * (-2^{-1}) + 1 \equiv (1 + 2) * (-2^{-1}) + 1 \equiv -2^{-1} - 1 + 1 \equiv -2^{-1} \pmod n$ or, equivalently, one of the dicycles C_i consists of all the vertices $(-2^{-1}, k)$, $0 \leq k \leq p-1$. Let (i, k_1), (j, k_2) be any pair of vertices of G. As G_1 is connected, there is a path from i to -2^{-1} in G_1, and also a path from -2^{-1} to j in G_1. So, there is in G a path from (i, k_1) to some $(-2^{-1}, k'_1)$, and also a path from some $(-2^{-1}, k'_2)$ to (j, k_2). Moreover, as $(-2^{-1}, k'_1)$ and $(-2^{-1}, k'_2)$ are in the same dicycle C_i, there is a path in G from $(-2^{-1}, k'_1)$ to $(-2^{-1}, k'_2)$; hence, we also have a path in G from (i, k_1) to (j, k_2). In other words, G is connected. □

4. Conclusion

We completely characterized the Hamiltonian Properties of the digraphs $BGC(p, d, n)$. A closely related family is the large generalized Kautz Cycles $KGC(p, d, n)$, for which a partial characterization of their Hamiltonicity can be found in [9]. The Hamiltonian Properties of both $BGC(2, d, n)$ and $KGC(2, d, n)$ have immediate applications for directed hypergraphs, that are also studied in [9].

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