Adding constraints to BSDEs with jumps: an alternative to multidimensional reflections

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Abstract

This paper is dedicated to the analysis of backward stochastic differential equations (BSDEs) with jumps, subject to an additional global constraint involving all the components of the solution. We study the existence and uniqueness of a minimal solution for these so-called constrained BSDEs with jumps via a penalization procedure. This new type of BSDE offers a nice and practical unifying framework to the notions of constrained BSDEs presented in [22] and BSDEs with constrained jumps introduced in [17]. More remarkably, the solution of a multidimensional Brownian reflected BSDE studied in [16] and [14] can also be represented via a well chosen one-dimensional constrained BSDE with jumps. This last result is very promising from a numerical point of view for the resolution of high dimensional optimal switching problems and more generally for systems of coupled variational inequalities.

Keywords: Stochastic control, Switching problems, BSDE with jumps, Reflected BSDE.

MSC Classification (2000): 93E20, 60H30, 60J75.

1 Introduction

Since their introduction by Pardoux and Peng in [18], Backward Stochastic Differential Equations (BSDEs in short) have been widely studied. In particular, they appear as a very powerful tool to solve partial differential equations (PDEs) and corresponding stochastic optimization problems. Several generalizations of this notion are based on the addition of new constraints on the solution. First, El Karoui et al. [11] study the case where the component Y is forced to stay above a given process, leading to the notion of reflected BSDEs

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related to optimal stopping and obstacle problems. Motivated by super replication issues under portfolio constraints, Buckdahn and Hu [6, 7] followed by Cvitanic et al. [9] consider the case where the other component Z of the solution is constrained to stay in a fixed convex set. More recently, Kharroubi et al. [17] introduce a constraint on the jump component U of the BSDE, providing a representation of solutions for a class of PDE, called quasi-variational inequalities, arising from optimal impulse control problems. The generalization of the results of El Karoui et al. [11] to oblique reflections in a multi-dimensional framework was first given in a very special case (e.g. the generator does not depend on z) by Ramasubramanian [23], who studied a BSDE reflected in an orthant. Then, Hu and Tang [16] followed by Hamadène and Zhang [14] consider general BSDEs with oblique reflections and connect them with systems of variational inequalities and optimal switching problems. Our paper introduces the notion of constrained BSDEs with jumps, which offers in particular a nice and natural probabilistic representation for these types of switching problems. This new notion essentially unifies and extends the notions of constrained BSDE without jumps, BSDE with constrained jumps as well as multidimensional BSDE with oblique reflections.

Let us illustrate our presentation with the example of the following switching problem

$$\sup_{\alpha} \mathbf{E}\Big[g_{\alpha_T}(X_T) + \int_0^T \psi_{\alpha_s}(s, X_s)ds + \sum_{0<\tau_k\le T} c_{\alpha_{\tau_k^-},\alpha_{\tau_k}}\Big]\,, \tag{1.1}$$

where X is an underlying Itô diffusion process, α is a switching control process valued in $\mathcal{I} := \{1, \dots, m\}$, $m > 0$, and $(\tau_k)_k$ denotes the jump times of the control α. This type of stochastic control problem is typically encountered by an agent maximizing the production rentability of a given good by switching between m possible modes of production based on different commodities. A switch is penalized by a given cost function c and the production rentability functions ψ and g depend on the chosen mode of production. As observed in [10], the solution of problem (1.1) starting in mode $i_0 \in \mathcal{I}$ at time t rewrites $Y_t^{i_0}$ where $(Y^i, Z^i, K^i)_{i\in\mathcal{I}}$ solves the following multidimensional reflected BSDE

$$\left\{\begin{array}{l} Y_t^i = g_i(X_T) + \int_t^T \psi_i(s, X_s)ds - \int_t^T \langle Z_s^i, dW_s\rangle + K_T^i - K_t^i\,, \quad 0\le t\le T\,, \quad i\in\mathcal{I}\,,\\ Y_t^i \ge Y_t^j + c_{i,j}\,, \quad 0\le t\le T\,, \quad i,j\in\mathcal{I}\,,\\ \int_0^T [Y_t^i - \max_{j\in\mathcal{I}}\{Y_j + c_{i,j}\}]dK_t^i = 0\,, \quad i\in\mathcal{I}\,. \end{array}\right. \tag{1.2}$$

The main difficulty in the derivation of a one-dimensional BSDE representation for this type of problem relies on the dependence of the solution in mode $i \in \mathcal{I}$ with respect to the global solution in all possible modes. Nevertheless, Tang and Yong [26] interpret the value function associated to this problem as the unique viscosity solution of a given coupled system of variational inequalities. A clever observation of Bouchard [3] concludes that this unique viscosity solution represents also the value function of a well suited stochastic target problem associated to a diffusion with jumps. Using entirely probabilistic arguments, the BSDE representation provided in this paper heavily relies on this type of correspondence. In our approach, we let artificially the strategy jump randomly between the different modes of production. Similarly to the approach of Pardoux et al. [19], this allows to retrieve in the jump component of a one-dimensional backward process, some information regarding the solution in the other modes of production. Indeed, let us introduce a pure jump process $(I_t)_{0\le t\le T}$ based on an independent random measure μ and consider the following constrained BSDE associated to the two dimensional forward process (I, X) (called transmutation-diffusion process in [19]) and defined on $[0, T]$ by:

$$\begin{cases} \tilde{Y}_t &= g_{I_T}(X_T) + \int_t^T \psi_{I_s}(s, X_s)ds + \tilde{K}_T - \tilde{K}_t - \int_t^T \langle \tilde{Z}_s, dW_s\rangle - \int_t^T \int_{\mathcal{I}} \tilde{U}_s(i)\mu(ds, di)\,, \\ \tilde{U}_t(i) &\geq c_{i,I_{t^-}}\,, \quad d\mathbf{P}\otimes dt \otimes \lambda(di) \;\; a.e. \end{cases} \tag{1.3}$$

This BSDE enters into the class of constrained BSDEs studied in the paper and its unique minimal solution relates directly to the solution of (1.2) via the relation $(\tilde{Y}_t, \tilde{Z}_t, \tilde{U}_t) = (Y_t^{I_t}, Z_t^{I_t}, \{Y_t^i - Y_{t^-}^{I_{t^-}}\}_{i\in\mathcal{I}})$ for $t \in [0, T]$. In particular, the solution of the switching problem (1.1) starting in mode I_0 at time 0 rewrites $\tilde{Y}_0^{I_0}$.

In order to unify our results with the one based on multidimensional reflected BSDE considered in [16] or [14], we extend this approach and introduce the notion of constrained BSDE with jumps whose solution (Y, Z, U, K) satisfies the general dynamics

$$Y_t \;=\; \xi + \int_t^T f(s, Y_s, Z_s, U_s)ds + K_T - K_t - \int_t^T \langle Z_s, dW_s\rangle - \int_t^T \int_{\mathcal{I}} U_s(i)\mu(ds, di), \tag{1.4}$$

a.s., for $0 \le t \le T$, as well as the constraint

$$h_i(t, Y_{t^-}, Z_t, U_t(i)) \;\geq\; 0, \qquad d\mathbf{P}\otimes dt\otimes\lambda(di) \;\; a.e.\,, \tag{1.5}$$

where f and h are given random Lipschitz functions, and h is non-increasing in its last variable. Through a penalization argument, we provide in Section 2 the existence of a unique minimal solution to the constrained BSDE with jumps (1.4)-(1.5). This new type of BSDE mainly extends and unifies the existing literature on BSDEs in three interconnected directions:

- We generalize the notion of BSDE with constrained jumps considered in [17], letting the driver function f depend on U and considering a general constraint function h depending on all the components of the solution.
- We add some jumps in the dynamics of constrained BSDE studied in [22] and let the coefficients depend on the jump component U.
- Via the addition of artificial jumps, a well chosen one-dimensional constrained BSDE with jumps allows to represent the solution of a multidimensional reflected BSDE, in the framework of [14] or [16].

We believe that the representation of a multidimensional BSDE with oblique reflections by a one-dimensional constrained BSDE with jumps is also numerically very promising. As developed in [2], it offers the possibility to solve high-dimensional optimal switching problems via a natural extension of the entirely probabilistic numerical scheme studied in [4]. Such type of algorithm could also solve high dimensional systems of variational inequalities, which relate directly to multidimensional BSDEs with oblique reflections, see [16] for more details. The algorithm as well as the Feynman Kac representation of general constrained BSDEs with jumps are presented in [12].

The paper is organized as follows. The next section provides the existence of a unique minimal solution for the new class of constrained BSDEs with jumps (1.4)-(1.5). The connection with multidimensional reflected BSDEs is detailed in Section 3. We regroup in the last section of the paper some technical results on BSDEs, mainly extensions of existing results, which are not the main focus of the paper but present some interest in themselves: we provide a comparison theorem for super-solutions of BSDEs with jumps, as well as viability and comparison properties for multidimensional constrained BSDEs. We isolate these results in order to present them in a general framework and to simplify their possible future invocation. All the proofs of the paper only rely on probabilistic arguments and can be applied in a non-Markovian setting.

Notations. Throughout this paper we are given a finite terminal time T and a probability space $(\Omega, \mathcal{G}, \mathbf{P})$ endowed with a d-dimensional standard Brownian motion $W = (W_t)_{t\geq 0}$, and a Poisson random measure μ on $\mathbb{R}_+ \times \mathcal{I}$, where $\mathcal{I} = \{1, \ldots, m\}$, with intensity measure $\lambda(di)dt$ for some finite measure λ on $\mathcal{I}$ with $\lambda(i) > 0$ for all $i \in \mathcal{I}$. We set $\tilde{\mu}(dt, di) = \mu(dt, di) - \lambda(di)dt$ the compensated measure associated to μ. $\sigma(\mathcal{I})$ denotes the σ-algebra of subsets of $\mathcal{I}$. For $x = (x_1, \ldots, x_\ell) \in \mathbb{R}^\ell$ with $\ell \in \mathbb{N}$, we set $|x| = \sqrt{|x_1|^2 + \cdots + |x_\ell|^2}$ the Euclidean norm. We denote by $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ (resp. $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$) the augmentation of the natural filtration generated by W and μ (resp. by W), and by $\mathcal{P}_{\mathbb{G}}$ (resp. $\mathcal{P}_{\mathbb{F}}$, $\mathfrak{P}_{\mathbb{G}}$, $\mathfrak{P}_{\mathbb{F}}$) the σ-algebra of $\mathbb{G}$-predictable (resp. $\mathbb{F}$-predictable $\mathbb{G}$-progressive, $\mathbb{F}$-progressive) subsets of $\Omega \times [0, T]$. We denote by $\mathcal{S}^{\mathbf{2}}_{\mathbb{G}}$ (resp. $\mathcal{S}^{\mathbf{2}}_{\mathbb{F}}$) the set of real-valued càd-làg $\mathbb{G}$-adapted (resp. continuous $\mathbb{F}$-adapted) processes $Y = (Y_t)_{0\leq t\leq T}$ such that

$$\|Y\|_{\mathcal{S}^{\mathbf{2}}} := \left(\mathbf{E}\Big[\sup_{0\leq t\leq T} |Y_t|^2\Big]\right)^{\frac{1}{2}} < \infty.$$

$\mathbf{L^p(0, T)}$, $p \geq 1$, is the set of real-valued processes $\phi = (\phi_t)_{0\leq t\leq T}$ such that

$$\|\phi\|_{\mathbf{L^p(0,T)}} := \left(\mathbf{E}\Big[\int_0^T |\phi_t|^p dt\Big]\right)^{\frac{1}{p}} < \infty,$$

and $\mathbf{L^p_{\mathbb{F}}(0, T)}$ (resp. $\mathbf{L^p_{\mathbb{G}}(0, T)}$) is the subset of $\mathbf{L^p(0, T)}$ consisting of $\mathfrak{P}_{\mathbb{F}}$-measurable (resp. $\mathfrak{P}_{\mathbb{G}}$-measurable) processes. $\mathbf{L^p_{\mathbb{F}}(W)}$ (resp. $\mathbf{L^p_{\mathbb{G}}(W)}$), $p \geq 1$, is the set of $\mathbb{R}^d$-valued $\mathcal{P}_{\mathbb{F}}$-measurable (resp. $\mathcal{P}_{\mathbb{G}}$-measurable) processes $Z = (Z_t)_{0\leq t\leq T} \in \mathbf{L^p_{\mathbb{F}}(0, T)}$ (resp. $\mathbf{L^p_{\mathbb{G}}(0, T)}$) . $\mathbf{L^p}(\tilde{\mu})$, $p \geq 1$, is the set of $\mathcal{P} \otimes \sigma(\mathcal{I})$-measurable maps $U : \Omega \times [0, T] \times \mathcal{I} \to \mathbb{R}$ such that

$$\|U\|_{\mathbf{L^p}(\tilde{\mu})} := \left(\mathbf{E}\Big[\int_0^T \int_{\mathcal{I}} |U_t(i)|^p \lambda(di)dt\Big]\right)^{\frac{1}{p}} < \infty.$$

$\mathbf{A}^{\mathbf{2}}_{\mathbb{F}}$ (resp. $\mathbf{A}^{\mathbf{2}}_{\mathbb{G}}$) is the closed subset of $\mathcal{S}^{\mathbf{2}}_{\mathbb{F}}$ (resp. $\mathcal{S}^{\mathbf{2}}_{\mathbb{G}}$) consisting of nondecreasing processes $K = (K_t)_{0\leq t\leq T}$ with $K_0 = 0$. Finally, for $t \in [0, T]$, $\mathcal{T}_t$ denotes the set of $\mathbb{F}$-stopping times τ such that $\tau \in [t, T]$, $\mathbf{P}$-a.s.. For ease of notation, we omit in all the paper the dependence in $\omega \in \Omega$, whenever it is not relevant.

2 Constrained Backward SDEs with jumps

This section is devoted to the presentation of constrained Backward SDEs with jumps, generalizing the framework considered in [17] or [22]. Namely:

- We allow the driver function to depend on the jump component of the backward process,
- We extend the class of possible constraint functions by letting them depend on all the components of the solution to the BSDE.

We adapt the arguments developed in [17] in order to derive existence and uniqueness of a minimal solution for this new type of BSDE. No major difficulty appears for the obtention of these results and, from our point of view, the nice feature of such constrained BSDE relies on their relation with multidimensional reflected BSDE, developed in the next section.

2.1 Formulation

A constrained BSDE with jumps is characterized by three objects:

- a terminal condition, i.e. a $\mathcal{G}_T$-measurable random variable ξ,
- a driver function, i.e. a map $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$, which is $\mathfrak{P}_{\mathbb{G}} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^m)$-measurable,
- a constraint function, i.e. a $\sigma(\mathcal{I}) \otimes \mathfrak{P}_{\mathbb{G}} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R})$-measurable map $h : \mathcal{I} \times \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ such that $h_i(\omega,t,y,z,.)$ is non-increasing for all $(i,\omega,t,y,z) \in \mathcal{I} \times \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d$.

Definition 2.1. *(i) A solution to the corresponding constrained BSDE with jumps is a quadruple* $(Y,Z,U,K) \in \mathcal{S}^{\mathbf{2}}_{\mathbb{G}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L}^{\mathbf{2}}(\tilde{\mu}) \times \mathbf{A}^{\mathbf{2}}_{\mathbb{G}}$ *satisfying*

$$Y_t \;=\; \xi + \int_t^T f(s,Y_s,Z_s,U_s)ds + K_T - K_t - \int_t^T \langle Z_s, dW_s\rangle - \int_t^T\int_{\mathcal{I}} U_s(i)\mu(ds,di), \tag{2.1}$$

for $0 \le t \le T$ *a.s., as well as the constraint*

$$h_i(t,Y_{t^-},Z_t,U_t(i)) \;\geq\; 0, \qquad d\mathbf{P}\otimes dt\otimes\lambda(di) \;\; a.e.\,. \tag{2.2}$$

(ii) (Y,Z,U,K) *is a minimal solution to* (2.1)-(2.2) *whenever it is solution to* (2.1)-(2.2) *and for any other solution* $(\check{Y},\check{Z},\check{U},\check{K})$ *of* (2.1)-(2.2), *we have* $Y \le \check{Y}$ *a.s.*

We notice that for a minimal solution (Y,Z,U,K) to (2.1)-(2.2), the component Y naturally interprets in the terminology of Peng [20] as the smallest supersolution to (2.1)-(2.2).

Remark 2.1. As in the Brownian setting considered in [22] , since the constraint (2.2) involves all the components of the solution, the minimality of the solution can not be expressed in general through a Skorohod condition type. Nevertheless, in a Markovian setting and when the constraint does not involve the Z-component of the solution, the authors provide in Corollary 2.1 of [12] a Skorokhod condition under an extra regularity requirement.

Remark 2.2. In the case where the driver function f does not depend on U and the constraint function h is of the form $h_i(u + c(t, y, z))$, observe that this BSDE exactly fits in the framework considered in [17]. Similarly, in the Brownian case (i.e. no jump component), this type of BSDEs was studied in [22]. Therefore, our framework generalizes and unifies those considered in [17] and [22].

In order to work on this class of BSDE, we require the classical Lipschitz and linear growth conditions on the coefficients, as well as a control on the way the driver function depends on the jump component U of the BSDE. We regroup these conditions in the following assumption.

(H0)

(i) There exists a constant $k > 0$ such that the functions f and h satisfy $\mathbf{P}$-a.s. the uniform Lipschitz property:

$$\begin{aligned} |f(t,y,z,u) - f(t,y',z',u')| &\leq k|(y,z,u) - (y',z',u')| \,, \\ |h_i(t,y,z,u_i) - h_i(t,y',z',u'_i)| &\leq k|(y,z,u_i) - (y',z',u'_i)| \,, \end{aligned}$$

for all $\{i, t, (y,z,u), (y',z',u')\} \in \mathcal{I} \times [0,T] \times [\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^m]^2$.

(ii) The coefficients ξ, f and h satisfy the following integrability condition

$$\mathbf{E}|\xi|^2 + \int_0^T \mathbf{E}|f(t,0,0,0)|^2 dt + \sum_{i \in \mathcal{I}} \int_0^T \mathbf{E}|h_i(t,0,0,0)|^2 dt \quad < \quad \infty \,. \tag{2.3}$$

(iii) There exist two constants $C_1 \geq C_2 > -1$ such that we can find a $\mathcal{P}_{\mathbb{G}} \otimes \sigma(\mathcal{I}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^m) \otimes \mathcal{B}(\mathbb{R}^m)$-measurable map $\gamma \; : \Omega \times [0,T] \times \mathcal{I} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^m \to [C_2, C_1]$ satisfying

$$f(t,y,z,u) - f(t,y,z,u') \quad \leq \quad \int_{\mathcal{I}} (u_i - u'_i) \gamma_t^{y,z,u,u'}(i) \lambda(di),$$

for all $(i, t, y, z, u, u') \in \mathcal{I} \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \times [\mathbb{R}^m]^2$, $\mathbf{P}$-a.s..

Remark 2.3. Under Assumption **(H0)** (i) and (ii), existence and uniqueness of a solution (Y, Z, U, K) to the BSDE (2.1) with $K = 0$ follows from classical results on BSDEs with jumps, see Lemma 2.4 in [25]. In order to add the h-constraint (2.2), one needs as usual to relax the dynamics of Y by injecting the non-decreasing process K in (2.1). In mathematical finance, the purpose of this new process K is to increase the super replication price Y of a contingent claim, under additional portfolio constraints. In order to find a minimal solution to the constrained BSDE (2.1)-(2.2), the nondecreasing property of h is crucial for stating comparison principles needed in the penalization approach.

Remark 2.4. Part (iii) of Assumption **(H0)** constrains the dependence of the driver f with respect to the jump component of the BSDE. It is inspired by [24] and will ensure comparison results for BSDEs driven by this type of driver, as detailed in Section 4.1.

2.2 Approximation by penalization

This paragraph focuses on the existence of a unique minimal solution for the constrained BSDE with jumps (2.1)-(2.2). Our approach requires the addition of an increasing component to the comparison results for BSDEs with jumps, derived by Royer [24]. This comparison theorem for super-solutions of BSDEs with jumps is reported in Section 4.1.

The proof relies on a classical penalization argument and we introduce the following sequence of BSDEs with jumps

$$\begin{aligned} Y_t^n &= \xi + \int_t^T f(s, Y_s^n, Z_s^n, U_s^n)ds + n\int_t^T \int_{\mathcal{I}} h_i^-(s, Y_s^n, Z_s^n, U_s^n(i))\lambda(di)ds \\ &\qquad - \int_t^T \langle Z_s^n, dW_s\rangle - \int_t^T \int_{\mathcal{I}} U_s^n(i)\mu(ds, di), \quad 0 \le t \le T, \quad n \in \mathbb{N}, \end{aligned} \tag{2.4}$$

where $h_i^-(.) := \max(-h_i(.), 0)$ is the negative part of the function h_i, $i \in \mathcal{I}$. Under Assumption **(H0)**, the Lipschitz property of the coefficients f and h ensures existence and uniqueness of a solution $(Y^n, Z^n, U^n) \in \mathcal{S}^{\mathbf{2}}_{\mathbb{G}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L}^{\mathbf{2}}(\tilde{\mu})$ to (2.4), see Theorem 2.1 in [1].

In order to obtain the convergence of the sequence $(Y^n)_{n\in\mathbb{N}}$, we require:

(H1) There exists $(\check{Y}, \check{Z}, \check{U}, \check{K}) \in \mathcal{S}^{\mathbf{2}}_{\mathbb{G}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L}^{\mathbf{2}}(\tilde{\mu}) \times \mathbf{A}^{\mathbf{2}}_{\mathbb{G}}$ solution of (2.1)-(2.2).

This assumption, which may appear restrictive, is rather classical in the framework of BSDE with constraints on other component of the solution than Y (see e.g. [22] and [17]). We present in Section 3 a large class of cases where **(H1)** is satisfied, see Example 3.1 for instance.

A first result is that under **(H0)**-**(H1)**, the sequence $(Y^n)_{n\in\mathbb{N}}$ converges. More precisely, we have the following Lemma.

Lemma 2.1. *If **(H0)** holds, the sequence $(Y^n)_{n\in\mathbb{N}}$ is nondecreasing:*

$$Y_t^n \le Y_t^{n+1}, \quad t \in [0, T], \ \mathbf{P} - a.s.$$

*for all $n \in \mathbb{N}$. Moreover, under **(H1)**, the sequence $(Y^n)_{n\in\mathbb{N}}$ converges increasingly to a process $Y \in \mathcal{S}^2_{\mathbb{G}}$:*

$$Y_t^n \longrightarrow Y_t, \ as\ n \to \infty, \quad t \in [0, T], \ \mathbf{P} - a.s.$$

Proof. For $n \in \mathbb{N}$, we introduce the Lipschitz map $f^n := f + n\int_{\mathcal{I}} h^- d\lambda$. Since f satisfies **(H0)**(iii) and h is lipschitz and non-increasing, we deduce:

$$\begin{aligned} f^n(t, y, z, u) - f^n(t, y, z, u') &\le \int_{\mathcal{I}} \{(u_i - u_i')\gamma_t^{y,z,u,u'}(i) + n(h_i^-(t, y, z, u_i) - h_i^-(t, y, z, u_i'))\}\lambda(di), \\ &\le \int_{\mathcal{I}} (u_i - u_i')(\gamma_t^{y,z,u,u'}(i) + kn\mathbf{1}_{u_i \ge u_i'})\lambda(di), \qquad \mathbf{P}\text{- a.s}, \quad n \in \mathbb{N}, \end{aligned}$$

for any $(t,y,z,u,u')\in[0,T]\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^m\times\mathbb{R}^m$. Thus, for any $n\in\mathbb{N}$, the coefficients f^n and f^{n+1} satisfy **(H0)** as well as $f^n\leq f^{n+1}$. We deduce from a simplified version of Proposition 4.1 without the additional increasing process K, that the sequence $(Y^n)_{n\in\mathbb{N}}$ is non-decreasing.
Furthermore, for any quadruple $(\check{Y},\check{Z},\check{U},\check{K})\in\mathcal{S}^{\mathbf{2}}_{\mathbb{G}}\times\mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{W})\times\mathbf{L}^{\mathbf{2}}(\tilde{\mu})\times\mathbf{A}^{\mathbf{2}}_{\mathbb{G}}$ satisfying (2.1)-(2.2), we obtain $Y^n\leq\check{Y}$ a.s., $n\in\mathbb{N}$, applying once again Proposition 4.1 but with coefficients $f_1=f_2=f^n$ and $K^2=\check{K}$. Therefore, under **(H1)**, the sequence $(Y^n)_{n\in\mathbb{N}}$ is nondecreasing and upper bounded, ensuring its monotonic convergence to a process Y with $\|Y\|_{\mathcal{S}^2}<\infty$. □

We are now ready to study the convergence of the quadruple $(Y^n,Z^n,U^n,K^n)_{n\in\mathbb{N}}$, where the nondecreasing process $K^n\in\mathbf{A}^{\mathbf{2}}_{\mathbb{G}}$ is defined by

$$K_t^n \;:=\; n\int_0^t\int_{\mathcal{I}} h_i^-(s,Y_s^n,Z_s^n,U_s^n(i))\lambda(di)ds,\quad 0\leq t\leq T\;,\quad n\in\mathbb{N}.$$

We first provide a uniform bound for the sequence $(Y^n,Z^n,U^n,K^n)_{n\in\mathbb{N}}$.

Lemma 2.2. *Under* **(H0)** *and* **(H1)***, there exists a constant C such that*

$$\|Y^n\|_{\mathcal{S}^{\mathbf{2}}}+\|Z^n\|_{\mathbf{L}^{\mathbf{2}}(\mathbf{0},\mathbf{T})}+\|U^n\|_{\mathbf{L}^{\mathbf{2}}(\tilde{\mu})}+\|K^n\|_{\mathcal{S}^{\mathbf{2}}}\;\leq\;C\;,$$

for all $n\in\mathbb{N}$.

Proof. From Lemma 2.1, there exists a constant C such that

$$\sup_{n\in\mathbb{N}}\|Y^n\|_{\mathcal{S}^2}\;\leq\;\|Y^0\|_{\mathcal{S}^2}+\|Y\|_{\mathcal{S}^2}\;\leq\;C\;. \tag{2.5}$$

Applying Itô's formula to $|Y^n|^2$ and using **(H0)** (i), we have

$$\begin{aligned}
\mathbf{E}|Y_t^n|^2 \;&=\; \mathbf{E}|Y_T^n|^2+2\mathbf{E}\int_t^T Y_s^n f(s,Y_s^n,Z_s^n,U_s^n)ds-\mathbf{E}\int_t^T|Z_s^n|^2ds\\
&\quad -\mathbf{E}\int_t^T\!\!\int_{\mathcal{I}}\big(|Y_{s^-}^n+U_s^n(i)|^2-|Y_{s^-}^n|^2\big)\mu(di,ds)+2\mathbf{E}\int_t^T Y_s^n dK_s^n\\
&\leq\; \mathbf{E}|Y_T^n|^2+2\mathbf{E}\int_t^T|Y_s^n|\big(f(s,0,0,0)+k|Y_s^n|+k|Z_s^n|+k|U_s^n|\big)ds-\mathbf{E}\int_t^T|Z_s^n|ds\\
&\quad -\mathbf{E}\int_t^T\!\!\int_{\mathcal{I}}\big(2Y_s^nU_s^n(i)-|U_s^n(i)|^2\big)\lambda(di)ds+2\mathbf{E}\sup_{s\in[0,T]}|Y_s^n|\int_t^T dK_s^n\;.
\end{aligned}$$

Using the inequality $2ab\leq\eta|a|^2+\frac{|b|^2}{\eta}$ for $a,b\in\mathbb{R}$ and $\eta>0$ together with **(H0)** (ii), we get the existence of a constant C such that

$$\mathbf{E}\int_0^T|Z_s^n|^2ds+\mathbf{E}\int_0^T\int_{\mathcal{I}}|U_s^n(i)|^2\lambda(di)ds\leq C\Big(\mathbf{E}\sup_{t\in[0,T]}|Y_t^n|^2+1\Big)+2\mathbf{E}K_T^n\sup_{t\in[0,T]}|Y_t^n|\;. \tag{2.6}$$

Then, since

$$K_T^n \;=\; Y_0^n-Y_T^n-\int_0^T f(s,Y_s^n,Z_s^n,U_s^n)ds+\int_0^T\langle Z_s^n,dW_s\rangle+\int_0^T\int_{\mathcal{I}}U_s^n(i)\mu(di,ds)\;,$$

we have from **(H0)** (i) the existence of a positive constant C' such that

$$\mathbf{E}|K_T^n|^2 \ \le\ C'\Big(1+\mathbf{E}\sup_{t\in[0,T]}|Y_t^n|^2+\mathbf{E}\int_0^T|Z_t^n|^2dt+\mathbf{E}\int_0^T\int_{\mathcal{I}}|U_s^n(i)|^2\lambda(di)ds\Big)\ . \qquad (2.7)$$

Applying the inequality $2ab\le 2C'|a|^2+\frac{|b|^2}{2C'}$ for $a,b\in\mathbb{R}$, we obtain

$$2\mathbf{E}K_T^n\sup_{t\in[0,T]}|Y_t^n|\ \le\ \frac{1}{2}\mathbf{E}\int_0^T|Z_s^n|^2ds+\frac{1}{2}\mathbf{E}\int_0^T\int_{\mathcal{I}}|U_s^n(i)|^2\lambda(di)ds+C''\Big(1+\mathbf{E}\sup_{t\in[0,T]}|Y_t^n|^2\Big)\ .$$

Combining this last estimate with (2.5) and (2.6), we obtain a constant C such that

$$\|Y^n\|_{\mathcal{S}^2}+\|Z^n\|_{\mathbf{L^2(0,T)}}+\|U^n\|_{\mathbf{L^2}(\tilde\mu)}\ \ \le\ \ C\ , \qquad n\in\mathbb{N}\,.$$

Then, combining the previous inequality with (2.7), we get

$$\|Y^n\|_{\mathcal{S}^2}+\|Z^n\|_{\mathbf{L^2(0,T)}}+\|U^n\|_{\mathbf{L^2}(\tilde\mu)}+\|K^n\|_{\mathcal{S}^2}\ \ \le\ \ C\ , \qquad n\in\mathbb{N}\,. \qquad (2.8)$$

□

The next theorem states that the sequence $(Y^n,Z^n,U^n,K^n)_{n\in\mathbb{N}}$ converges indeed to the minimal solution of the constrained BSDE (2.1)-(2.2).

Theorem 2.1. *Under* **(H0)** *and* **(H1)***, there exists* $(Z,U,K)\in\mathcal{S}^\mathbf{2}_\mathbb{G}\times\mathbf{L}^\mathbf{2}_\mathbb{G}(\mathbf{W})\times\mathbf{L^2}(\tilde\mu)\times\mathbf{A}^\mathbf{2}_\mathbb{G}$ *such that*

- *(i)* (Y,Z,U,K) *is the unique minimal solution in* $\mathcal{S}^\mathbf{2}_\mathbb{G}\times\mathbf{L}^\mathbf{2}_\mathbb{G}(\mathbf{W})\times\mathbf{L^2}(\tilde\mu)\times\mathbf{A}^\mathbf{2}_\mathbb{G}$ *to* (2.1)-(2.2)*, with* K *predictable,*
- *(ii) the following convergence holds*

$$\big\|Y^n-Y\big\|_{\mathbf{L^2(0,T)}}+\big\|Z^n-Z\big\|_{\mathbf{L^p(0,T)}}+\big\|U^n-U\big\|_{\mathbf{L^p}(\tilde\mu)}\ \underset{n\to\infty}{\longrightarrow}\ 0, \qquad 1\le p<2\ .$$

Moreover, (Z,U,K) *is the weak limit of* $(Z^n,U^n,K^n)_{n\in\mathbb{N}}$ *in* $\mathbf{L}^\mathbf{2}_\mathbb{G}(\mathbf{W})\times\mathbf{L^2}(\tilde\mu)\times\mathbf{L}^\mathbf{2}_\mathbb{G}(\mathbf{0},\mathbf{T})$ *and* K_t *is the weak limit of* $(K_t^n)_{n\in\mathbb{N}}$ *in* $\mathbf{L^2}(\boldsymbol{\Omega},\mathcal{G}_\mathbf{t},\mathbf{P})$*, for all* $t\in[0,T]$*.*

Proof. We prove the statements of the theorem in a reverse order. First, we show the convergence of the sequence $(Y^n,Z^n,U^n,K^n)_{n\in\mathbb{N}}$. Second, we verify that the limit is a minimal solution to (2.1)-(2.2). Third, we tackle the uniqueness property.

Step 1: *Convergence of* $(Y^n,Z^n,U^n,K^n)_{n\in\mathbb{N}}$*.*
From Lemmata 2.1 and 2.2, we are in position to apply Theorem 3.1 in [13] and we deduce that the sequence $(Y^n,Z^n,U^n,K^n)_{n\in\mathbb{N}}$ converges in the sense specified above. Furthermore, the limit $(Y,Z,U,K)\in\mathcal{S}^\mathbf{2}_\mathbb{G}\times\mathbf{L}^\mathbf{2}_\mathbb{G}(\mathbf{W})\times\mathbf{L^2}(\tilde\mu)\times\mathbf{A}^\mathbf{2}_\mathbb{G}$ satisfies (2.1) and K is predictable.

Step 2: (Y,Z,U,K) *is a minimal solution to* (2.1)-(2.2)*.*
Since (Y,Z,U,K) solves (2.1), we now focus on the constraint property (2.2). From the

previous convergence result, we derive in particular that $(Y^n, Z^n, U^n)_{n\in\mathbb{N}}$ converges in $\mathbf{L^1_{\mathbb{G}}(0,T)} \times \mathbf{L^1_{\mathbb{G}}(0,T)} \times \mathbf{L^1}(\tilde{\mu})$ to (Y, Z, U). Since h is Lipschitz, we get

$$\frac{\mathbf{E}[K_T^n]}{n} = \mathbf{E}\left[\int_0^T\int_{\mathcal{I}} h_i^-(s, Y_s^n, Z_s^n, U_s^n(i))\lambda(di)ds\right] \to \mathbf{E}\left[\int_0^T\int_{\mathcal{I}} h_i^-(s, Y_s, Z_s, U_s(i))\lambda(di)ds\right] ,$$

as n goes to infinity. Since $(K_T^n)_{n\in\mathbb{N}}$ is uniformly bounded in $\mathbf{L^1}(\mathbf{\Omega}, \mathcal{G}_\mathbf{T}, \mathbf{P})$ according to Lemma 2.2, we deduce that the right hand side of the previous expression equals zero. Hence the constraint (2.2) is satisfied.

As observed in the previous step, for any quadruple $(\check{Y}, \check{Z}, \check{U}, \check{K}) \in \mathcal{S}^\mathbf{2}_{\mathbb{G}} \times \mathbf{L^2_{\mathbb{G}}(W)} \times \mathbf{L^2}(\tilde{\mu}) \times \mathbf{A^2_{\mathbb{G}}}$ satisfying (2.1)-(2.2), the sequence $(Y^n)_{n\in\mathbb{N}}$ is upper bounded by $\check{Y}$. Passing to the limit, we deduce that (Y, Z, U, K) is a minimal solution to (2.1)-(2.2).

Step 3: *Uniqueness of the minimal solution.*
From the minimality condition, the uniqueness for the component Y of the solution is obvious. Suppose now that we have two solutions (Y, Z, U, K) and (Y, Z', U', K') in $\mathcal{S}^\mathbf{2}_{\mathbb{G}} \times \mathbf{L^2_{\mathbb{G}}(W)} \times \mathbf{L^2}(\tilde{\mu}) \times \mathbf{A^2_{\mathbb{G}}}$ with K and K' predictable. Then we have

$$\begin{aligned}\int_0^t [f(s, Y_s, Z_s, U_s) - f(s, Y_s, Z'_s, U'_s)]ds + \int_0^t [Z'_s - Z_s]dW_s \\ + \int_0^t \int_{\mathcal{I}} [U'_s(i) - U_s(i)]\mu(di, ds) + K'_t - K_t \quad = \quad 0 \,, \quad 0 \le t \le T. \qquad (2.9)\end{aligned}$$

Since μ is a Poisson measure, it has unaccessible jumps. Recalling that K and K' are predictable and removing the predictable projection of the previous expression (2.9), we compute

$$\int_{\mathcal{I}} [U'_t(i) - U_t(i)]\mu(\{t\}, di) \quad = \quad 0 \,, \qquad 0 \le t \le T \,, \qquad \mathbf{P} - p.s.$$

Recalling that $\mu = \sum_k \delta_{\tau_k, \xi_k}$ and taking $t = \tau_k$ in the previous expression, we deduce $U' = U$. Plugging this identity in (2.9), we deduce

$$\int_0^t [f(s, Y_s, Z_s, U_s) - f(s, Y_s, Z'_s, U_s)]ds + \int_0^t [Z'_s - Z_s]dW_s + K'_t - K_t \quad = \quad 0 \,, \qquad (2.10)$$

for $0 \le t \le T$. Identifying the finite variation and the Brownian parts in (2.10) we get

$$\int_0^T [Z'_s - Z_s]dW_s \quad = \quad 0 \,,$$

which leads to $Z = Z'$. The uniqueness of K finally follows from (2.10). □

Remark 2.5. A key argument in the proof of Theorem 2.1 is the monotonic limit theorem for BSDEs with jumps proved in [13]. Let us mention that this theorem was initially proved by Peng [20] in a Brownian framework and that the idea of the proof of the extension to the mixed Brownian-Poisson framework was already given in Royer [24].

Remark 2.6. Observe that the purpose of Assumption **(H1)** is simply to ensure an upper bound in $\mathcal{S}^{\mathbf{2}}_{\mathbb{G}}$ on the sequence of solutions $(Y^n)_{n\in\mathbb{N}}$ to the penalized BSDEs. If such an upper bound already exists, there exists a minimal solution to (2.1)-(2.2) and **(H1)** is automatically satisfied. Hence, Theorem 2.1 also holds under **(H0)** and the uniform estimate

$$\sup_{n\in\mathbb{N}} \|Y^n\|_{\mathcal{S}^2} \quad < \quad +\infty\,. \tag{2.11}$$

Note that, contrary to the case where the constraint only involves the component Y of the solution, we cannot derive in general the uniform estimate (2.11), see, e.g., the counterexample provided in Remark 3.1 of [17].

Particular cases where Assumption **(H1)** is satisfied are for instance presented in Theorem 3.1, see Example 3.1 below. In a Markovian setting, sufficient conditions for this assumption are also provided in Remark 3.2 of [12].

3 Connection with multidimensional reflected BSDEs

In this section, we prove that one-dimensional constrained BSDEs with jumps offer a nice alternative for the representation of solutions to multidimensional reflected BSDEs studied in [16] and [14]. This representation has practical implications, since, for example, it opens the door to the numerical resolution of multi-dimensional reflected BSDEs via the approximation of a single one-dimensional constrained BSDE with additional artificial jumps. The arguments presented here are purely probabilistic and therefore apply in the non Markovian framework considered in [16]. Furthermore, the proofs require precise comparison results for reflected BSDEs based on viability properties that are reported in Section 4.2 for the convenience of the reader.

3.1 Multidimensional reflected BSDEs

Recall that solving a general multidimensional reflected BSDE consists in finding m triplets $(Y^i, Z^i, K^i)_{i\in\mathcal{I}} \in (\mathcal{S}^{\mathbf{2}}_{\mathbb{F}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{F}}(\mathbf{W}) \times \mathbf{A}^{\mathbf{2}}_{\mathbb{F}})^m$ satisfying, for all $i \in \mathcal{I}$,

$$\begin{cases} Y^i_t = \xi^i + \int_t^T \psi_i(s, Y^1_s, \ldots, Y^m_s, Z^i_s)ds - \int_t^T \langle Z^i_s, dW_s\rangle + K^i_T - K^i_t\,, \quad 0 \le t \le T\,, \\ Y^i_t \ge \max_{j\in A_i} h_{i,j}(t, Y^j_t)\,, \quad 0 \le t \le T\,, \\ \int_0^T [Y^i_t - \max_{j\in A_i}\{h_{i,j}(t, Y^j_t)\}]dK^i_t = 0\,, \end{cases} \tag{3.1}$$

where $\psi_i : \Omega \times [0,T] \times \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}$ is an $\mathbb{F}$-progressively measurable map, $\xi^i \in \mathbf{L}^2(\Omega, \mathcal{F}_T, \mathbf{P})$, A_i is a nonempty subset of $\mathcal{I}$ and, for any $j \in A_i \cup \{i\}$, $h_{i,j} : \Omega\times[0,T]\times\mathbb{R} \to \mathbb{R}$ is a given $\mathcal{P}_{\mathbb{F}} \otimes \mathcal{B}(\mathbb{R})$-measurable function satisfying $h_{i,i}(t, y) = y$ for all $(t, y) \in [0,T] \times \mathbb{R}$.

As detailed in Theorem 3.1 and Theorem 4.2 of [14], existence and uniqueness of a solution to (3.1) is ensured by the following assumption:

(H2)

(i) For any $i \in \mathcal{I}$ and $j \in A_i$, we have $\xi^i \geq h_{i,j}(T, \xi^j)$.

(ii) For any $i \in \mathcal{I}$, $\mathbf{E}|\xi^i|^2 + \mathbf{E}\int_0^T \sup_{y\in\mathbb{R}^m} |\psi_i(t,y,0)|^2 \mathbf{1}_{\{y_i=0\}} dt < +\infty$, and ψ_i is Lipschitz continuous: there exists a constant $k_\psi \geq 0$ such that

$$|\psi_i(t,y,z) - \psi_i(t,y',z')| \leq k_\psi(|y-y'| + |z-z'|) \ , \ \ (t,y,z,y',z') \in [0,T] \times [\mathbb{R}^m \times \mathbb{R}^d]^2 \, .$$

(iii) For any $i \in \mathcal{I}$, and $j \neq i$, ψ_i is nondecreasing in its $(j+1)-$th variable i.e. for any $(t,y,y',z) \in \mathcal{I} \times [\mathbb{R}^m]^2 \times \mathbb{R}^d$ such that $y_k = y'_k$ for $k \neq j$ and $y_j \leq y'_j$, we have

$$\psi_i(t,y,z) \quad \leq \quad \psi_i(t,y',z) \quad \mathbf{P}-a.s.$$

(iv) For any $(i,t,y) \in \mathcal{I} \times [0,T] \times \mathbb{R}$ and $j \in A_i$, $h_{i,j}$ is continuous, $h_{i,j}(t,.)$ is a 1-Lipschitz increasing function satisfying $h_{i,j}(t,y) \leq y$, $\mathbf{P}$-a.s. and we have $h_{i,j}(.,0) \in \mathbf{L^2(0,T)}$.

(v) For any $i \in \mathcal{I}$, $j \in A_i$ and $l \in A_j$, we have $l \in A_i \cup \{i\}$ and

$$h_{i,j}(t, h_{j,l}(t,y)) \quad < \quad h_{i,l}(t,y) \ , \qquad (t,y) \in [0,T] \times \mathbb{R} \, .$$

Remark 3.1. Part (ii) and (iii) of Assumption **(H2)** are classical Lipschitz and monotonicity properties of the driver. Parts (iv)-(v) ensure a tractable form for the domain of $\mathbb{R}^m$ where $(Y^i)_{i\in\mathcal{I}}$ lies, and (i) implies that the terminal condition is indeed in the domain. Recent results in [5] allow to relax the monotonicity condition (iii) for the case of constraint function h associated to switching problems.

Remark 3.2. Under Assumption **(H2)**, the increasing property of the functions $(h_{i,j})_{i,j}$ together with a straightforward recursive argument allows to generalize Assumption (H2)-(v) to the consideration of tuple of any size N: for any $N \in \mathbb{N}$, and $(i_1, \ldots, i_N) \in \mathcal{I}^N$ with $i_k \in A_{i_{k-1}}$ for any $k \leq N$, we have

$$h_{i_1,i_2}(t,.) \circ h_{i_2,i_3}(t,.) \circ \ldots \circ h_{i_{N-1},i_N}(t,y) \quad < \quad h_{i_1,i_N}(t,y) \ , \qquad (t,y) \in [0,T] \times \mathbb{R} \, .$$

3.2 Corresponding constrained BSDE with jumps

We consider now the following one-dimensional constrained BSDE with jumps : find a minimal quadruple $(\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{K}) \in \mathcal{S}^{\mathbf{2}}_{\mathbb{G}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{0},\mathbf{T}) \times \mathbf{L^2}(\tilde{\mu}) \times \mathbf{A}^{\mathbf{2}}_{\mathbb{G}}$ satisfying

$$\begin{aligned} \tilde{Y}_t \quad = \quad & \xi^{I_T} + \int_t^T \psi_{I_{s^-}}(s, \tilde{Y}_s + \tilde{U}_s(1)\mathbf{1}_{I_{s^-}\neq 1}, \ldots, \tilde{Y}_s + \tilde{U}_s(m)\mathbf{1}_{I_{s^-}\neq m}, \tilde{Z}_s)ds + \tilde{K}_T - \tilde{K}_t \\ & - \int_t^T \langle \tilde{Z}_s, dW_s\rangle - \int_t^T \int_{\mathcal{I}} \tilde{U}_s(i)\mu(ds,di), \quad 0 \leq t \leq T, \ a.s. \end{aligned} \tag{3.2}$$

together with the constraint

$$\mathbf{1}_{A_{I_{t^-}}}(i)\Big[\tilde{Y}_{t^-} - h_{I_{t^-},i}(t, \tilde{Y}_{t^-} + \tilde{U}_t(i))\Big] \quad \geq \quad 0, \qquad d\mathbf{P} \otimes dt \otimes \lambda(di) \ \ a.e. \ , \tag{3.3}$$

where the process I is a pure jump process defined by

$$I_t \quad = \quad I_0 + \int_0^t \int_{\mathcal{I}} (i - I_{s^-})\mu(ds,di) \, .$$

Remark 3.3. If the Poisson measure rewrites $\sum_{n\geq 0}\delta_{(\kappa_n,L_n)}$, where $(\kappa_n)_n$ are the jump times and $(L_n)_n$ the jump sizes, the pure jump process I simply coincides with L_n on each $[\kappa_n,\kappa_{n+1})$.

Considering I as an extra source of randomness, the BSDE (3.2)-(3.3) enters into the class of constrained BSDEs with jumps of the form (2.1)-(2.2) studied above, with the following correspondence

$$\begin{aligned} \xi &= \xi^{I_T}\,; \\ f(t,y,z,u) &= \psi_{I_{t^-}}(t,(y+u_i\mathbf{1}_{I_{t^-}\neq i})_{i\in\mathcal{I}},z)\,, \qquad (t,y,z,u)\in[0,T]\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^m\,; \\ h_i(t,y,z,v) &= \{y-h_{I_{t^-},i}(t,y+v)\}\mathbf{1}_{i\in A_{I_{t^-}}} \qquad (i,t,y,z,v)\in\mathcal{I}\times[0,T]\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}\,. \end{aligned}$$

As detailed below, Assumption **(H2)** is sufficient to ensure the existence of a one-dimensional minimal solution to the BSDE (3.2)-(3.3). Remarkably, we prove hereafter that this one-dimensional solution directly relates with the multidimensional solution of the reflected BSDE (3.1). Since the new constrained BSDE is one-dimensional, this alternative BSDE representation is promising for the numerical resolution of optimal switching problems. An entirely probabilistic numerical scheme for these equations is given in [12].

We are now ready to state the main result of the paper.

Theorem 3.1. *Suppose that Assumption* **(H2)** *is in force and denote by* $(Y^i,Z^i,K^i)_{i\in\mathcal{I}}$ *the unique solution of* (3.1). *Then, the constrained BSDE* (3.2)-(3.3) *satisfies* **(H0)**-**(H1)** *and its unique corresponding minimal solution* $(\tilde{Y},\tilde{Z},\tilde{U},\tilde{K})\in\mathcal{S}^{\mathbf{2}}_{\mathbb{G}}\times\mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{W})\times\mathbf{L}^{\mathbf{2}}(\tilde{\mu})\times\mathbf{A}^{\mathbf{2}}_{\mathbb{G}}$ *verifies*

$$\tilde{Y}_t=Y_t^{I_t}\,,\quad \tilde{Z}_t=Z_t^{I_{t^-}}\,,\quad \tilde{U}_t=(Y_t^i-Y_{t^-}^{I_{t^-}})_{i\in\mathcal{I}}\,,\qquad 0\leq t\leq T\,. \tag{3.4}$$

Proof. The proof divides in 3 steps. First we prove the existence of a unique minimal solution to (3.2)-(3.3). Then, we introduce a sequence of penalized BSDEs converging to the solution of the multidimensional reflected BSDE (3.1). Finally, we prove that a corresponding sequence of penalized BSDEs with jumps, built via a relation of the form of (3.4), converges indeed to the solution of (3.2)-(3.3).

Step 1: *Existence and uniqueness of a minimal solution to* (3.2)-(3.3).
In order to use Theorem 2.1, we need to verify that Assumptions **(H0)** and **(H1)** are satisfied in this context.

First, parts (i) and (ii) of Assumption **(H0)** are direct consequences of **(H2)**(ii) and (iv). Fix any $(t,y,z,u,u')\in[0,T]\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^m\times\mathbb{R}^m$, and define $v^{(k)}\in\mathbb{R}^m$ by

$$v^{(k)} \;=\; (u'_1,\ldots,u'_{k-1},u_k,\ldots,u_m),\qquad 1\leq k\leq m+1\,.$$

From the monotonicity assumption **(H2)**(iii) on the Lipschitz function ψ we get

$$\begin{aligned} f(t,y,z,u)-f(t,y,z,u') =& \sum_{k=1}^{m} \psi_{I_{t^-}}(t,(y+v_i^{(k)}\mathbf{1}_{I_{t^-}\neq i})_{i\in\mathcal{I}},z)-\psi_{I_{t^-}}(t,(y+v_i^{(k+1)}\mathbf{1}_{I_{t^-}\neq i})_{i\in\mathcal{I}},z) \\ &\leq k_\psi \sum_{k=1}^{m-1}(u_k-u'_k)\mathbf{1}_{u_k\geq u'_k}\mathbf{1}_{k\neq I_{t^-}} . \end{aligned}$$

Taking $\gamma_t^{y,z,u,u'}(i) = \frac{k_\psi}{\lambda(i)}\mathbf{1}_{u_k\geq u'_k}\mathbf{1}_{i\neq I_{t^-}}$ (which is well defined, since $\lambda(i)>0$ for any $i\in\mathcal{I}$), we get **(H0)**-(iii).

In order to prove that **(H1)** holds, one needs to verify the existence of a solution to (3.2)-(3.3). We indeed check hereafter that the candidate $(\tilde{Y},\tilde{Z},\tilde{U})$ defined in (3.4) satisfies (3.2) as well as (3.3). Let define $N_t := \mu(\mathcal{I}\times[0,t])$ for $t\in[0,T]$, the (random) number of stopping times κ_n, associated to the random measure μ, which satisfy $\kappa_n\in[0,t]$. Then, since Y is a solution of the reflected BSDE (3.1), we have

$$\begin{aligned} Y_{\kappa_{N_T}}^{L_{N_T}} &= \xi^{L_{N_T}} + \int_{\kappa_{N_T}}^{T} \psi_{L_{N_T}}(s,(Y_s^{L_{N_T}}+U_s(i)\mathbf{1}_{i\neq L_{N_T}})_{i\in\mathcal{I}},Z_s^{L_{N_T}})ds \\ &\quad - \int_{\kappa_{N_T}}^{T} Z_s^{L_{N_T}}dW_s + K_T^{L_{N_T}} - K_{\kappa_{N_T}}^{L_{N_T}} . \end{aligned}$$

Then, still using the equation (3.1) and identifying the jumps at time κ_{N_T}, we compute:

$$\begin{aligned} Y_{\kappa_{N_T-1}}^{L_{N_T-1}} &= Y_{\kappa_{N_T}}^{L_{N_T}} + \int_{\kappa_{N_T-1}}^{\kappa_{N_T}} \psi_{L_{N_T-1}}(s,(Y_s^{L_{N_T-1}}+U_s(i)\mathbf{1}_{i\neq L_{N_T-1}})_{i\in\mathcal{I}},Z_s^{L_{N_T-1}})ds \\ &\quad - \int_{\kappa_{N_t-1}}^{\kappa_{N_t}} Z_s^{L_{N_T-1}}dW_s + K_{\kappa_{N_T}}^{L_{N_T-1}} - K_{\kappa_{N_T-1}}^{L_{N_T-1}} + (Y_{\kappa_{N_T}}^{L_{N_T-1}} - Y_{\kappa_{N_T}}^{L_{N_T}}) \\ &= \xi^{I_T} + \int_{\kappa_{N_T-1}}^{T} \psi_{I_{s^-}}(s,(Y_s^{I_s}+U_s(i)\mathbf{1}_{i\neq I_{s^-}})_{i\in\mathcal{I}},Z_s^{I_{s^-}})ds - \int_{\kappa_{N_T-1}}^{T} Z_s^{I_{s^-}}dW_s \\ &\quad - \int_{\kappa_{N_T-1}}^{T}\int_{\mathcal{I}} U_s(i)\mu(di,ds) + K_T^{L_{N_T}} - K_{\kappa_{N_T}}^{L_{N_T}} + K_{\kappa_{N_T}}^{L_{N_T-1}} - K_{\kappa_{N_T-1}}^{L_{N_T-1}} . \end{aligned}$$

Repeating this procedure until time κ_{N_t+1} for $t\in[0,T]$, we get

$$\begin{aligned} Y_{\kappa_{N_t+1}}^{L_{N_t+1}} &= \xi^{I_T} + \int_{\kappa_{N_t+1}}^{T} \psi_{I_{s^-}}(s,(Y_s^{I_s}+U_s(i)\mathbf{1}_{i\neq I_{s^-}})_{i\in\mathcal{I}},Z_s^{I_{s^-}})ds - \int_{\kappa_{N_t+1}}^{T} Z_s^{I_{s^-}}dW_s \\ &\quad - \int_{\kappa_{N_t+1}}^{T}\int_{\mathcal{I}} U_s(i)\mu(di,ds) + K_T^{L_{N_T}} - K_{\kappa_{N_T}}^{L_{N_T}} + K_{\kappa_{N_T}}^{L_{N_T-1}} - K_{\kappa_{N_T-1}}^{L_{N_T-1}} \\ &\quad + \ldots + K_{\kappa_{N_t+2}}^{L_{N_t+1}} - K_{\kappa_{N_t+1}}^{L_{N_t+1}} . \end{aligned}$$

Combining this last expression with the equation satisfied by $Y^{L_{N_t}}$ between t and κ_{N_t+1}, we deduce the existence of a square integrable increasing process $\tilde{K}$ such that $(\tilde{Y},\tilde{Z},\tilde{U},\tilde{K})$ satisfies equation (3.2). The reflection constraint in (3.1) together with the identification

(3.4) imply directly that $(\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{K})$ satisfies the constraint (3.3).

Therefore **(H0)** and **(H1)** hold for (3.2)-(3.3) and the existence of a unique minimal solution follows from Theorem 2.1.

Step 2: *Penalization of the multidimensional BSDE* (3.1).
We now introduce the following sequence of multidimensional penalized BSDEs: for $n \in \mathbb{N}$, find m couples $(Y^{i,n}, Z^{i,n})_{i\in\mathcal{I}} \in (\mathcal{S}^{\mathbf{2}}_{\mathbb{F}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{F}}(\mathbf{W}))^m$ satisfying

$$Y_t^{i,n} \;=\; \xi^i + \int_t^T \psi_i^n(s, Y_s^{1,n}, \ldots, Y_s^{m,n}, Z_s^n)ds - \int_t^T \langle Z_s^{i,n}, dW_s\rangle \;, \quad 0 \le t \le T, \quad i \in \mathcal{I} \;, \tag{3.5}$$

where the random map ψ^n is defined on $[0,T] \times \mathbb{R}^m \times [\mathbb{R}^d]^m$ by

$$\psi_i^n(t,y,z) \quad = \quad \psi_i(t,y,z_i) + n\sum_{j\in A_i}[y_i - h_{i,j}(t,y_j)]^-\lambda(j) \;, \quad (i,t,y) \in \mathcal{I}\times[0,T]\times\mathbb{R}^d \;.$$

For any $n \in \mathbb{N}$, the existence of a unique solution to (3.5) is given in the seminal paper [18] and we prove now that the sequence of solutions to these BSDEs converges to the solution of the multidimensional reflected BSDE (3.1).

In order to prove that the sequence $(Y^{i,n})_{n\in\mathbb{N}}$ is nondecreasing and convergent for any $i \in \mathcal{I}$, we shall appeal to the multidimensional comparison theorem for reflected BSDEs presented in Section 4.2 of the paper. First, since $\psi_i^n \le \psi_i^{n+1}$ for any $i \in \mathcal{I}$ and $n \in \mathbb{N}$, Theorem 2.1 in [15] implies that the sequence $(Y^{\cdot,n})_{n\in\mathbb{N}}$ is nondecreasing componentwise. Second, we compute from the Lipschitz property of ψ that

$$-2\langle y, \psi^n(t,y',z) - \psi^n(t,y',z')\rangle = -2\langle y, \psi(t,y',z) - \psi(t,y',z')\rangle \le k_\psi^2\,|y|^2 + \sum_{i=1}^m |z_i - z_i'|^2 \;,$$

$\mathbf{P}$-a.s., for any $\{t,y,y',(z,z')\} \in [0,T]\times[\mathbb{R}^+]^m\times\mathbb{R}^m\times[\mathbb{R}^{d\times m}]^2$ and $n \in \mathbb{N}$. Therefore, since $\psi^n(t,Y_t,Z_t) = \psi(t,Y_t,Z_t)$ for $t \in [0,T]$, we deduce from Proposition 4.3 below that

$$Y_t^{i,n} \le Y_t^i \;, \quad \text{for all} \quad (i,t,n) \in \mathcal{I}\times[0,T]\times\mathbb{N}\,. \tag{3.6}$$

Introducing the sequence of processes $K^{i,n} := n\int_0^\cdot \int_{A_i}[Y_s^{i,n} - h_{i,j}(s,Y_s^{j,n})]^-\lambda(dj)ds$, for $i \in \mathcal{I}$ and $n \in \mathbb{N}$, we deduce from Peng's monotonic limit theorem [20] the existence of:

- $\hat{Y}^1, \ldots, \hat{Y}^m$ $\mathbb{F}$-adapted càdlàg processes with $\|\hat{Y}^i\|_{\mathcal{S}^2} < \infty$ for all $i \in \mathcal{I}$,
- $\hat{Z}^1, \ldots, \hat{Z}^m \in \mathbf{L}^{\mathbf{2}}_{\mathbb{F}}(\mathbf{W})$,
- $\hat{K}^1, \ldots, \hat{K}^m$ $\mathbb{F}$-adapted nondecreasing càdlàg processes with $\hat{K}_0^i = 0$ and $\|\hat{K}^i\|_{\mathcal{S}^2} < \infty$, for all $i \in \mathcal{I}$,

such that $Y^{i,n} \uparrow \hat{Y}^i$ a.e., $Y^{i,n} \to \hat{Y}^i$ in $\mathbf{L}^{\mathbf{2}}_{\mathbb{F}}(\mathbf{0},\mathbf{T})$, $Z^{i,n} \to \hat{Z}^i$ in $\mathbf{L}^{\mathbf{2}}_{\mathbb{F}}(\mathbf{W})$ weakly, $K_T^{i,n} \to \hat{K}_T^i$ in $\mathbf{L^2}(\boldsymbol{\Omega}, \boldsymbol{\mathcal{F}_T}, \mathbf{P})$ weakly and

$$\left\{\begin{array}{l} \hat{Y}_t^i = \xi^i + \int_t^T \psi_i(s,\hat{Y}_s^1,\ldots,\hat{Y}_s^m,\hat{Z}_s^i)ds - \int_t^T\langle\hat{Z}_s^i, dW_s\rangle + \hat{K}_T^i - \hat{K}_t^i\,, \quad i \in \mathcal{I}\,, \\ \hat{Y}_t^i \ge \max_{j\in A_i} h_{i,j}(t,\hat{Y}_t^j)\;, \quad 0 \le t \le T\;, \quad i \in \mathcal{I}\,. \end{array}\right. \tag{3.7}$$

Observe that the last inequality in (3.7) is not a direct consequence of Peng's monotonic limit theorem but follows instead from a similar argument as the one used in Step 2 of the proof of Theorem 2.1 above: for $i \in \mathcal{I}$, since the sequence $(K^{i,n})_n$ is uniformly bounded in $\mathbf{L^1(\Omega, \mathcal{F}_T, P)}$ we have

$$\begin{aligned} 0 \;=\; \lim_{n\to\infty} \frac{\mathbf{E}[|K_T^{i,n}|]}{n} &\;=\; \lim_{n\to\infty} \mathbf{E}\left[\int_0^T \int_{A_i} [Y_s^{i,n} - h_{i,j}(s, Y_s^{j,n})]^- \lambda(dj) ds\right] \\ &\;=\; \mathbf{E}\left[\int_0^T \int_{A_i} [\hat{Y}_s^i - h_{i,j}(s, \hat{Y}_s^j)]^- \lambda(dj) ds\right] , \qquad i \in \mathcal{I} , \end{aligned}$$

which easily rewrites as the constraint inequality in (3.7). It still remains to prove that $(\hat{Y}, \hat{Z}, \hat{K})$ also satisfies the minimality property of (3.1).

For this purpose, we consider the following RBSDE whose unique solution $(\tilde{Y}, \tilde{Z}, \tilde{K})$ in $(\mathcal{S}^\mathbf{2}_\mathbb{F} \times \mathbf{L}^\mathbf{2}_\mathbb{F}(\mathbf{W}) \times \mathbf{A}^\mathbf{2}_\mathbb{F})^m$ exists according to Theorem 2.1 in [21]:

$$\left\{ \begin{array}{l} \tilde{Y}_t^i = \xi^i + \int_t^T \psi_i(s, \hat{Y}_s^1, \ldots, \hat{Y}_s^{i-1}, \tilde{Y}_s^i, \hat{Y}_s^{i+1} \ldots, \hat{Y}_s^m, \tilde{Z}_s^i) ds \\ \qquad\qquad\qquad\qquad\qquad\qquad - \int_t^T \langle \tilde{Z}_s^i, dW_s \rangle + \tilde{K}_T^i - \tilde{K}_t^i , \\ \tilde{Y}_t^i \geq \max_{j\in A_i} h_{i,j}(t, \hat{Y}_t^j) , \quad 0 \leq t \leq T , \quad i \in \mathcal{I} , \\ \int_0^T [\tilde{Y}_{t^-}^i - \max_{j\in A_i} h_{i,j}(t, \hat{Y}_{t^-}^j)] d\tilde{K}_t^j = 0 , \quad i \in \mathcal{I} . \end{array} \right. \tag{3.8}$$

We note that (3.7) and (3.8) have the same lower barrier. For any $i \in \mathcal{I}$, since $\tilde{Y}^i$ is the smallest ψ_i-supermartingale with lower barrier $\max_{j\in A_i} h(., \hat{Y}_.^j)$, we know from Theorem 2.1 in [21] that $\tilde{Y}^i \leq \hat{Y}^i$.

On the other hand, we deduce from **(H2)** (iii) that

$$\psi_i^n(s, \hat{Y}_s^1, \ldots, \hat{Y}_s^{i-1}, y, \hat{Y}_s^{i+1}, \ldots, \hat{Y}_s^m) \quad \geq \quad \psi_i^n(s, Y_s^{1,n}, \ldots, Y_s^{i-1,n}, y, Y_s^{i+1,n}, \ldots, Y_s^{m,n}) ,$$

for all $(i, s, y, n) \in \mathcal{I} \times [0, T] \times \mathbb{R} \times \mathbb{N}$, $\mathbf{P} - a.s.$. For $i \in \mathcal{I}$, since $\tilde{Y}^i \geq \max_{j\in A_i} h_{i,j}(., Y_.^j)$, combining **(H2)** (iv) and a comparison theorem for one dimensional reflected BSDEs, we get $Y^{i,n} \leq \tilde{Y}^i$ for any $n \in \mathbb{N}$, and, sending n to infinity, deduce $\hat{Y}^i \leq \tilde{Y}^i$.

Therefore $\hat{Y} = \tilde{Y}$ and $(\hat{Y}, \hat{Z}, \hat{K})$ satisfies

$$\left\{ \begin{array}{l} \hat{Y}_t^i = \xi^i + \int_t^T \psi_i(s, \hat{Y}_s, \hat{Z}_s^i) ds - \int_t^T \langle \hat{Z}_s^i, dW_s \rangle + \hat{K}_T^i - \hat{K}_t^i , \quad 0 \leq t \leq T , \quad i \in \mathcal{I} , \\ \hat{Y}_t^i \geq \max_{j\in A_i} h_{i,j}(t, \hat{Y}_t^j) , \quad 0 \leq t \leq T , \quad i \in \mathcal{I} , \\ \int_0^T [\hat{Y}_{t^-}^i - \max_{j\in A_i} h_{i,j}(t, \hat{Y}_{t^-}^j)] d\hat{K}_t^j = 0 , \quad i \in \mathcal{I} . \end{array} \right. \tag{3.9}$$

Notice that the minimality condition in (3.9) differs from the expected one in (3.1). Nevertheless, those two coincide whenever $\hat{Y}$ is continuous, property that we verify now.

Suppose on the contrary that $\hat{Y}_t^{i_1} \neq \hat{Y}_{t^-}^{i_1}$ for some fixed $(i_1, t) \in \mathcal{I} \times [0, T]$. Then, we deduce from (3.9) that $\hat{Y}_t^{i_1} - \hat{Y}_{t^-}^{i_1} = \hat{K}_{t^-}^{i_1} - \hat{K}_t^{i_1} < 0$, which further implies

$$\hat{Y}_{t^-}^{i_1} = \max_{j\in A_i} h_{i_1,j}(t, \hat{Y}_{t^-}^j) = h_{i_1,i_2}(t, \hat{Y}_{t^-}^{i_2}) ,$$

for some $i_2 \neq i_1$. Using the constraint satisfied by $\hat{Y}$, we get

$$h_{i_1,i_2}(t, \hat{Y}_{t^-}^{i_2}) = \hat{Y}_{t^-}^{i_1} > \hat{Y}_t^{i_1} \geq \max_{i\in A_{i_1}} h_{i_1,i}(t, \hat{Y}_t^i) \geq h_{i_1,i_2}(t, \hat{Y}_t^{i_2}).$$

Thus $\hat{Y}^{i_2}_t < \hat{Y}^{i_2}_{t^-}$. Repeating this argument we get a finite cyclic sequence $(i_k)_{1\leq k\leq N}$ such that $i_N = i_1$ and

$$\hat{Y}^{i_{k-1}}_{t^-} \;=\; h_{i_{k-1},i_k}(t,\hat{Y}^{i_k}_{t^-})\,, \quad 2\leq k\leq N\,.$$

Since $h_{i_1,i_N}(t,\hat{Y}^{i_1}_{t^-}) \leq \hat{Y}^{i_1}_{t^-}$ according to **(H2)** (iv), we deduce that

$$h_{i_1,i_N}(t,\hat{Y}^{i_1}_{t^-}) \;\leq\; h_{i_1,i_2}(t,.)\circ h_{i_2,i_3}(t,.)\circ\ldots\circ h_{i_{N-1},i_N}(t,\hat{Y}^{i_1}_{t^-})\,, \qquad t\in[0,T]\,,$$

which contradicts **(H2)** (v), see Remark 3.2.

Step 3: *Link between solutions of BSDE* (3.1) *and BSDE* (3.2)-(3.3)*.*
For $n\in\mathbb{N}$, define the process $(Y^{I,n},Z^{I,n},U^{I,n})\in\mathcal{S}^{\mathbf{2}}_{\mathbb{G}}\times\mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{W})\times\mathbf{L}^{\mathbf{2}}(\tilde{\mu})$ by

$$Y^{I,n}_t \;:=\; Y^{I_t,n}_t\,,\;\; Z^{I,n}_t \;:=\; Z^{I_{t^-},n}_t \quad\text{and}\quad U^{I,n}_t \;:=\; (Y^{i,n}_t - Y^{I,n}_{t^-})_{i\in\mathcal{I}}\,, \quad 0\leq t\leq T. \qquad (3.10)$$

In order to obtain the correspondence (3.4), it only remains to prove that $(Y^{I,n},Z^{I,n},U^{I,n})_n$ converges to $(\tilde{Y},\tilde{Z},\tilde{U})$.

As in Step 1, writing the dynamics of (3.5) between each successive stopping times associated to the random measure μ, we easily check that $(Y^{I,n},Z^{I,n},U^{I,n})$ is the unique solution of the following penalized BSDE

$$\begin{aligned} Y^{I,n}_t \;=&\; \xi^{I_T} + \int_t^T \psi_{I_{s^-}}(s, Y^{I,n}_s + U^{I,n}_s(1)\mathbf{1}_{I_{s^-}\neq 1},\ldots,Y^{I,n}_s + U^{I,n}_s(m)\mathbf{1}_{I_{s^-}\neq m}, Z^{I,n}_s)ds \\ &- \int_t^T \langle Z^{I,n}_s, dW_s\rangle + n\int_t^T\!\!\int_{\mathcal{I}} h_i^-(s,Y^{I,n}_{s^-},Z^{I,n}_s,U^{I,n}_s(i))\lambda(di)ds + \int_t^T\!\!\int_{\mathcal{I}} U^{I,n}_s(i)\mu(ds,di), \end{aligned}$$

for $0\leq t\leq T$. Therefore, Step 1 ensures that we can apply Theorem 2.1 and we get

$$\|Y^{I,n}-\tilde{Y}\|_{\mathbf{L^2(0,T)}} + \|Z^{I,n}-\tilde{Z}\|_{\mathbf{L^p(0,T)}} + \|U^{I,n}-\tilde{U}\|_{\mathbf{L^p}(\tilde{\mu})} \;\longrightarrow\; 0\,, \quad p<2, \quad (3.11)$$

where we recall that $(\tilde{Y},\tilde{Z},\tilde{U})$ is the minimal solution to (3.2)-(3.3). Combining this result with (3.10) and Step 2 concludes the proof. □

Example 3.1. *Let consider the case of optimal switching problem developed in [16] where the constraint function h is given by $h_{i,j}: y^j\mapsto y^j - c^{ij}$ for $i,j\in\mathcal{I}$ and the cost matrix c satisfies*

$$c^{ij}+c^{jk} \;\geq\; c^{ik}\,, \qquad i,j,k\in\mathcal{I}\,.$$

*Then, if the driver vector function ψ satisfies the requirements of Assumption **(H2)**, Theorem 2.1 in [16] ensures the existence of a solution to the BSDE* (3.1)*. Thus, the corresponding constrained BSDE with jumps* (3.2)-(3.3) *has a solution. This framework provides hereby a practical example of interest where Assumption **(H1)** is satisfied.*

4 Subsidiary technical points

This section regroups technical properties which are mainly extensions of existing results but that we could not find as such in the literature. They are not the main focus of the paper but still present some interest in themselves. This dissociation allows to present them in a more abstract setting and simplifies their possible future quotation. We provide a comparison theorem for super-solutions to BSDEs with jumps, as well as viability and comparison properties for multidimensional reflected BSDEs.

4.1 A comparison theorem for super-solutions to BSDEs with jumps

We present here a general comparison theorem for super-solutions to BSDEs with jumps. This extends the results of Theorem 2.5 in [24] obtained in the non-reflected case.

Proposition 4.1. *Let $f_1, f_2 : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$ two generators satisfying Assumption **(H0)** and $\xi_1, \xi_2 \in \mathbf{L}^2(\Omega, \mathcal{G}_T, \mathbf{P})$. Let $(Y^1, Z^1, U^1) \in \mathcal{S}^2_{\mathbb{G}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu})$ satisfying on $[0,T]$*

$$Y^1_t \quad = \quad \xi^1 + \int_t^T f_1(s, Y^1_s, Z^1_s, U^1_s)ds - \int_t^T \langle Z^1_s, dW_s\rangle - \int_t^T \int_{\mathcal{I}} U^1_s(i)\mu(ds, di) \ , \quad (4.1)$$

and $(Y^2, Z^2, U^2, K^2) \in \mathcal{S}^2_{\mathbb{G}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu}) \times \mathbf{A}^{\mathbf{2}}_{\mathbb{G}}$ satisfying on $[0,T]$

$$Y^2_t = \quad \xi^2 + \int_t^T f_2(s, Y^2_s, Z^2_s, U^2_s)ds - \int_t^T \langle Z^2_s, dW_s\rangle - \int_t^T \int_{\mathcal{I}} U^2_s(i)\mu(ds, di) + K^2_T - K^2_t \ . \quad (4.2)$$

If $\xi^1 \le \xi^2$ and $f_1(t, Y^1_t, Z^1_t, U^1_t) \le f_2(t, Y^1_t, Z^1_t, U^1_t)$ for all $t \in [0,T]$, then we have

$$Y^1_t \quad \le \quad Y^2_t \ , \qquad 0 \le t \le T \ .$$

Proof. We notice that the scheme of the proof of Theorem 2.5 in [24] applies directly: the linearization of the BSDE satisfied by $Y^2 - Y^1$ provides a representation of the process $\Gamma(Y^2 - Y^1)$ as a supermartingale for a well chosen Doléans-Dade exponential process Γ. Since the additional process K^2 is increasing, this representation still holds and gives a positive sign for $Y^2 - Y^1$. □

4.2 Viability and comparison property for multi-dimensional BSDEs

We generalize in this paragraph some viability and comparison properties for multidimensional BSDEs in a closed convex cone $\mathcal{C}$ of $\mathbb{R}^{2m}$, whenever we add some reflections on the Y-component of the BSDE. The two following propositions are respectively extensions of Theorem 2.5 in [8] and a simplifying version of Theorem 2.1 in [15]. Their derivations do not present major difficulty and we choose to detail them for sake of completeness.

Let $(Y, Z) \in (\mathcal{S}^2_{\mathbb{F}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{F}}(\mathbf{W}))^{2m}$ satisfying

$$Y_t \quad = \quad Y_T + \int_t^T F(s, Y_s, Z_s)ds - \int_t^T \langle Z_s, dW_s\rangle + K_T - K_t \ , \quad 0 \le t \le T \ , \qquad (4.3)$$

where $F : \Omega \times [0,T] \times \mathbb{R}^{2m} \times \mathbb{R}^{2m\times d} \to \mathbb{R}^{2m}$ is a progressively measurable function satisfying **(H2)** (ii) and K is an $\mathbb{R}^{2m}$-valued finite variation process such that

$$K_t \quad = \quad \int_0^t k_s d|K|_s \ ,$$

with $k_t \in \mathcal{C}$ and $|K|_s$ the variation of K on $[0,s]$. We denote by $d_{\mathcal{C}}$ the distance to $\mathcal{C}$, i.e. $d_{\mathcal{C}} : x \mapsto \min_{y\in\mathcal{C}} |x-y|$, and introduce $\Pi_{\mathcal{C}}$ the projection operator onto $\mathcal{C}$.

Proposition 4.2. *Suppose $Y_T \in \mathcal{C}$ and there exists a constant C^0 such that F satisfies*

$$4\langle y - \Pi_{\mathcal{C}}(y), F(t,y,z)\rangle \quad \leq \quad \langle D^2|d_{\mathcal{C}}|^2(y)z, z\rangle + 2C^0|d_{\mathcal{C}}|^2(y) \qquad \mathbf{P}-a.s.\,, \tag{4.4}$$

for any $(t,y,z) \in [0,T] \times \mathbb{R}^{2m} \times \mathbb{R}^{2m\times d}$ such that $|d_{\mathcal{C}}|^2$ is twice differentiable at the point y. Then, we have

$$Y_t \quad \in \quad \mathcal{C}\,, \qquad 0 \leq t \leq T\ , \qquad \mathbf{P}-a.s.$$

Proof. The proof presented here is an adaptation of the one of Theorem 2.5 in [8], allowing to tackle the additional difficulty due to the dK term in the dynamics of Y.
Let $\eta \in C^\infty(\mathbb{R}^{2m})$ be a non-negative function, with support in the unit ball, such that $\int_{\mathbb{R}^{2m}} \eta(x)dx = 1$. For $\delta > 0$ and $x \in \mathbb{R}^{2m}$, we define

$$\eta_\delta(x) := \frac{1}{\delta^{2m}}\eta\Big(\frac{x}{\delta}\Big) \qquad \text{and} \qquad \phi_\delta(x) := \int_{\mathbb{R}^{2m}} |d_{\mathcal{C}}(x-y)|^2\eta_\delta(y)dy\ .$$

Via direct computation, one can verify that $\phi_\delta \in C^\infty(\mathbb{R}^{2m})$ and

$$\left\{\begin{array}{lll} 0 \ \leq \ \phi_\delta(x) \ \leq \ (d_{\mathcal{C}}(x)+\delta)^2\ , & & \\ D\phi_\delta(x) \ = \ \int_{\mathbb{R}^{2m}} D|d_{\mathcal{C}}(y)|^2\eta_\delta(x-y)dy \ \text{ and } |D\phi_\delta(x)| \ \leq \ 2(d_{\mathcal{C}}(x)+\delta)\ , & & \\ D^2\phi_\delta(x) \ = \ \int_{\mathbb{R}^{2m}} D^2|d_{\mathcal{C}}(y)|^2\eta_\delta(x-y)dy \text{ and } \ 0 \ \leq \ |D^2\phi_\delta(x)| \ \leq \ 2I_{2m}\ , & & \end{array}\right. \tag{4.5}$$

for any $x \in \mathbb{R}^{2m}$. An application of Itô's formula to $\phi_\delta(Y)$, combined with these estimates and $d_{\mathcal{C}}(Y_T) = 0$, leads to

$$\begin{aligned} \mathbf{E}\phi_\delta(Y_t) \quad = \quad & \mathbf{E}\phi_\delta(Y_T) + \mathbf{E}\int_t^T \langle D\phi_\delta(Y_s), F(s,Y_s,Z_s)\rangle ds - \frac{1}{2}\mathbf{E}\int_t^T \langle D^2\phi_\delta(Y_s)Z_s, Z_s\rangle ds \\ & +\mathbf{E}\int_t^T \langle D\phi_\delta(Y_s), k_s\rangle d|K|_s \\ \leq \quad & \delta^2 + \mathbf{E}\int_t^T\int_{\mathbb{R}^{2m}} \Big[\langle D|d_{\mathcal{C}}(y)|^2, F(s,y,Z_s)\rangle - \frac{1}{2}\langle D^2|d_{\mathcal{C}}(y)|^2Z_s, Z_s\rangle\Big]\eta_\delta(Y_s-y)dyds \\ & -\mathbf{E}\int_t^T\int_{\mathbb{R}^{2m}} \langle D|d_{\mathcal{C}}(y)|^2, F(s,y,Z_s) - F(s,Y_s,Z_s)\rangle\eta_\delta(Y_s-y)dyds \\ & +\mathbf{E}\int_t^T\int_{\mathbb{R}^{2m}} \langle D|d_{\mathcal{C}}(y)|^2, k_s\rangle\eta_\delta(Y_s-y)dyd|K|_s\,, \qquad 0 \leq t \leq T\ . \end{aligned} \tag{4.6}$$

Since k is valued in the closed convex cone $\mathcal{C}$, we observe that

$$\langle D|d_{\mathcal{C}}(y)|^2, k_s\rangle \quad \leq \quad 0\,, \qquad 0 \leq s \leq T\ , \quad y \in \mathbb{R}^{2m}\ .$$

Then, plugging this expression, (4.4) and inequality $2d_c(.) \leq 1 + d_c(.)^2$ in (4.6), we get

$$\begin{aligned}
\mathbf{E}\phi_\delta(Y_t) &\leq \delta^2 + C^0 \mathbf{E} \int_t^T \int_{\mathbb{R}^{2m}} |d_{\mathcal{C}}(y)|^2 \eta_\delta(y - Y_s) dy ds \\
&\quad + 2\mathbf{E} \int_t^T \int_{\mathbb{R}^{2m}} d_{\mathcal{C}}(y) \eta_\delta(Y_s - y) \max_{y' : \, |y' - Y_s| \leq \delta} |F(s, y', Z_s) - F(s, Y_s, Z_s)| dy ds \\
&\leq \delta^2 + C^0 \int_t^T \mathbf{E}\phi_\delta(Y_s) ds + \mathbf{E} \int_t^T (1 + \phi_\delta(Y_s)) \max_{y' : \, |y' - Y_s| \leq \delta} |F(s, y', Z_s) - F(s, Y_s, Z_s)| ds \,,
\end{aligned}$$

for any $t \in [0, T]$. Using the uniform Lipschitz property of F, we deduce

$$\mathbf{E}\phi_\delta(Y_t) \;\; \leq \;\; C \left\{ \delta^2 + \delta + \int_t^T \mathbf{E}\phi_\delta(Y_s) ds \right\} \,, \quad 0 \leq t \leq T \,, \;\; \delta > 0 \,,$$

and Gronwall's lemma leads to

$$\mathbf{E}\phi_\delta(Y_t) \;\; \leq \;\; C(\delta^2 + \delta), \quad 0 \leq t \leq T \,, \;\; \delta > 0 \,.$$

Finally, from Fatou's Lemma, we have

$$\mathbf{E}|d_{\mathcal{C}}(Y_t)|^2 \;\; \leq \;\; \liminf_{\delta \to 0} \mathbf{E}\phi_\delta(Y_t) \; = \; 0 \,, \quad 0 \leq t \leq T \,,$$

which concludes the proof. □

We now turn to the obtention of a multidimensional comparison result for BSDEs, whenever the dominating BSDE suffers additional reflections. This proposition also simplifies the results of Theorem 2.1 in [15] in the case where the i^{th} component of each driver only depends on the i^{th} component of Z, for any $i \leq d$.

Consider $(Y^1, Z^1, K^1) \in (\mathcal{S}^2_{\mathbb{F}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{F}}(\mathbf{W}) \times \mathbf{A}^{\mathbf{2}}_{\mathbb{F}})^m$ satisfying

$$Y_t^1 \;\; = \;\; Y_T^1 + \int_t^T F_1(s, Y_s^1, Z_s^1) ds - \int_t^T \langle Z_s^1, dW_s \rangle + K_T^1 - K_t^1 \,, \quad 0 \leq t \leq T \,,$$

and $(Y^2, Z^2) \in (\mathcal{S}^2_{\mathbb{F}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{F}}(\mathbf{W}))^m$ satisfying

$$Y_t^2 \;\; = \;\; Y_T^2 + \int_t^T F_2(s, Y_s^2, Z_s^2) ds - \int_t^T \langle Z_s^2, dW_s \rangle \,, \quad 0 \leq t \leq T \,,$$

where F_1 and F_2 are two driver functions satisfying **(H2)** (ii) and such that the i^{th} component of each driver only depends on the i^{th} component of the corresponding Z, for any $i \leq d$.

Proposition 4.3. *Suppose $Y_T^1 \geq Y_T^2$ and the existence of a constant C^1 such that*

$$-\,2\langle y, F_1(t, y', z) - F_2(t, y', z') \rangle \;\; \leq \;\; C^1 |y|^2 + \sum_{i=1}^m |z_i - z_i'|^2 \; \mathbf{P} - a.s. \,, \tag{4.7}$$

for any $(t, y, y', z, z') \in [0, T] \times (R^+)^m \times \mathbb{R}^m \times [\mathbb{R}^{m \times d}]^2$. Then $Y_t^1 \geq Y_t^2$, for all $t \in [0, T]$.

Proof. The process $(Y^1 - Y^2, Y^2)$ is valued in $\mathbb{R}^{2m}$ and solution of a BSDE of the form (4.3) associated to the driver

$$F \ : \ (t,(y,y'),(z,z')) \quad \mapsto \quad (F_1(t,y+y',z+z') - F_2(t,y',z'), F_2(t,y',z')) \ ,$$

for any $\{t,(y,y'),(z,z')\} \in [0,T] \times \mathbb{R}^{2m} \times \mathbb{R}^{2m\times d}$. Introducing the closed convex cone $\mathcal{C} := (\mathbb{R}^+)^m \times \mathbb{R}^m$ of $\mathbb{R}^{2m}$, we see that $d_{\mathcal{C}}(y,y') = |y^-|$ for $(y,y') \in \mathbb{R}^{2m}$. Therefore, we deduce from the Lipschitz property of F_1 and (4.7) that

$$\begin{aligned}
& 4\langle (y,y') - \Pi_{\mathcal{C}}(y,y'), F(t,(y,y'),(z,z'))\rangle \\
= \ & 4\langle -y^-, F_1(t,y+y',z+z') - F_1(t,y',z+z')\rangle + 4\langle -y^-, F_1(t,y',z+z') - F_2(t,y',z')\rangle \\
\leq \ & 4k|y^-|^2 + 2\sum_{i=1}^m \mathbf{1}_{y_i<0}|z_i|^2 + 2C^1|y^-|^2 \\
= \ & \langle D^2|d_{\mathcal{C}}|^2(y,y')(z,z'),(z,z')\rangle + (2C^1+4k)|d_{\mathcal{C}}|^2(y,y') \qquad \mathbf{P}-a.s.\,,
\end{aligned}$$

for any $\{t,(y,y'),(z,z')\} \in [0,T] \times \mathbb{R}^{2m} \times \mathbb{R}^{2m\times d}$. Applying Proposition 4.2 with $C^0 = C^1 + 2k$, we deduce that the process $(Y^1 - Y^2, Y^2)$ is valued in $\mathcal{C}$ and complete the proof. □

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