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# SOME REMARKS ON THE MEAN VALUE OF THE RIEMANN ZETA-FUNCTION AND OTHER DIRICHLET SERIES—I

By K. RAMACHANDRA

## § 1. Introduction

In the last section of my paper [2], I raised some questions on the mean value of generalised Dirichlet series. It looks too ambitious unless we limit ourselves to Dirichlet series  $F(s) = \sum_{n=1}^{\infty} a_n n^{-s}$  ( $s = \sigma + it$ ) and that too with some restrictions. We do not claim to solve all the problems raised. Let  $F(s)$  be convergent absolutely somewhere in the complex plane and let  $F(s)$  admit an analytic continuation in  $\sigma > \frac{1}{2}$ ,  $t > \tau_0$  and there  $F(s) = O(t^A)$ . If  $\sigma_1$  is large enough,  $F(s)$  is zero free in  $\sigma > \sigma_1$  and we define  $(F(s))^{2k}$  (for any positive real number  $2k$ ) as the analytic continuation of  $(F(s))^{2k}$  ( $\sigma > \sigma_1$ ) along lines parallel to the real axis. (If such lines contain a zero of  $F(s)$  we do not define  $(F(s))^{2k}$  on such lines). Let  $\delta$  be a positive constant not exceeding  $\frac{1}{10}$  and  $m$  a non-negative integer. Let  $T > T_0(\delta)$  be a real variable and  $H$  a real variable subject to  $(\log T)^\delta < H < T$ . Imposing the conditions  $\delta < 2k < \delta^{-1}$  and  $0 < m < \delta^{-1}$  we define for  $\sigma > \frac{1}{2}$ ,

$$Q(\sigma) = \frac{1}{H} \int_T^{T+H} \left| \frac{d^m}{ds^m} (F(s))^{2k} \right| dt.$$

Our main problem is to study lower bounds for  $Q(\sigma)$ . The only known progress so far, in this direction is a theorem of Ingham which gives for fixed  $\sigma > \frac{1}{2}$  and fixed  $2k$  ( $0 < 2k < 4$ ,  $m = 0$ ) an asymptotic formula for  $Q(\sigma)$  in the special case  $F(s) = \zeta(s)$ ,  $H = T$ . Ingham's proof was complicated and

Davenport gave a simpler proof of Ingham's Theorem (for references see [5]). Our first object in this note is to give in § 3, a satisfactory lower bound for  $\max Q(\sigma)$  as  $\sigma$  runs over all real numbers  $\geq \frac{1}{2} + \frac{\log \log H}{\log H}$ . However our proof of lower bound depends very much on the existence of an "Euler product" for  $F(s)$ . (The Euler product condition is general enough to include the case when  $F(s)$  is the zeta-function and Hecke L-series of algebraic number fields). From such a theorem we can also get satisfactory lower bounds for  $Q(\frac{1}{2})$  as will be seen. Two simple samples of our general results (subject to the condition  $2k > 1$ ) are

$$(*) \frac{1}{H} \int_T^{T+H} |\zeta(\frac{1}{2} + it)|^{2k} dt > (\log H)^{k^2} (\log \log H)^{-C}$$

$$(**) \frac{1}{H} \int_T^{T+H} |\zeta'(\frac{1}{2} + it)| dt > (\log H)^{\frac{5}{2}} (\log \log H)^{-C}$$

where  $C$  is a constant depending only on  $\delta$ . Our next object is to deal (in § 4 and § 5) with the case when  $F(s)$  has no Euler product. Here we are forced, for lack of better ideas, to limit ourselves to the case  $2k = 1$ . To compensate for the generality the lower bounds we obtain for  $Q(\sigma)$  are not so satisfactory, but they are still, I hope, of some interest.

The problem of upper bounds for  $Q(\sigma)$  seems hopelessly difficult. Even with the assumption of Riemann hypothesis it is not clear how to improve the trivial inequality

$$(***) \frac{1}{T} \int_T^{2T} |\zeta(\frac{1}{2} + it)| dt = O((\log T)^{\frac{1}{2}}).$$

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*Foot-Note:* The publication of this paper which was ready by the middle of 1977 was delayed due to various reasons. In the meanwhile I have replaced  $(\log \log H)^{-C}$  in (\*) and (\*\*) by  $\frac{1}{C}$ . Next I have replaced  $O((\log T)^{\frac{1}{2}})$  by  $O((\log T)^{\frac{1}{2}})$  in (\*\*\*) unconditionally. These (and other) results will appear in papers II and III with the same title as the present paper.

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**§ 2. Notation**

The letter  $C$  with or without subscripts denotes a positive constant depending only on  $\delta$ . The letter  $K$  with or without subscripts denotes constants to be chosen later appropriately in a proof. Also the constant  $C$  may not be the same at each occurrence.

**§ 3. The case of Euler product**

Let in  $\sigma > 1$ ,  $F(s)$  be defined by

$$F(s) = \prod_p \left( 1 - \frac{x(p)}{p^s} \right)^{-a_p} \quad (p \text{ runs over all primes})$$

where  $\{x(p)\}$  and  $\{a_p\}$  are bounded sequences of complex numbers, and  $|x(p)| < p^{\frac{1}{2}}$  for every prime  $p$ . We assume either of the following conditions on  $F(s)$ .

(i) There exists a constant  $p_0$  such that whenever  $a_p \neq 0$  and  $p > p_0$  we have  $|a_p| \geq a$ , where  $a$  is a positive constant.

Further the function  $\prod_p \left( 1 - \frac{|x(p)|^2}{p^s} \right)^{-|a_p|}$  which is analytic in  $\sigma > 1$  can be continued analytically in a neighbourhood of  $s = 1$  and has a simple pole at  $s = 1$ .

(ii) Suppose that as  $\sigma \rightarrow \frac{1}{2} + 0$ ,  $\left( \prod_p \left( 1 + \frac{|x(p) a_p|^2}{p^{2\sigma}} \right) \right)$

lies between  $\left( \log \frac{1}{\sigma - \frac{1}{2}} \right)^{b_1}$  and  $\left( \log \frac{1}{\sigma - \frac{1}{2}} \right)^{b_2}$

where  $a > 0$  and  $b_1, b_2$  are three real constants.

We can now state

**Theorem 1**

With the notation explained already,

$$\max_{\sigma \geq \frac{1}{2} + \frac{\log \log H}{\log H}} Q(\sigma) > (\log H)^{ak^a + m_1} (\log \log H)^{-c}$$

where  $C$  is a positive constant, and  $m_1 = m$  or  $\min(m, 1)$  according as we have (ii) or (i).

As a corollary we can deduce the results for  $\zeta(\frac{1}{2} + it)$  quoted in the introduction. More generally we have

**Theorem 2**

Let  $L(s)$  denote either the Dedekind zeta-function or the Hecke  $L$ -series of an algebraic number field of degree  $n$  and according as it is Galois or not we put  $a = n$  or  $1$  and also  $m_1 = m$  or  $\min(1, m)$ . Then we have with  $2k \geq 1$ ,

$$\frac{1}{H} \int_T^{T+H} |L(\frac{1}{2} + it)^{2k} dt| > (\log H)^{ak^a} (\log \log H)^{-c}$$

$$\frac{1}{H} \int_T^{T+H} |L^{(m)}(\frac{1}{2} + it)| dt > (\log H)^{\frac{a}{2} + m_1} (\log \log H)^{-c}$$

where  $C$  is a positive constant. Further the condition  $2k \geq 1$  is unnecessary if we assume the hypothesis that  $L(s) \neq 0$  for  $\sigma > \frac{1}{2}$ .

*Remark:* Theorem 2 gives an improvement on theorem 3 of my paper [3]. It will be seen later that  $m_1$  can also be replaced by  $m_a = \max(m_1, mn^{-1})$ .

**Deduction of Theorem 2 from Theorem 1**

Let  $F(s) = L(s)$ . We put  $s = \sigma + it$ ,  $w = u + iv$ , where  $\sigma$  is the number at which  $\max Q(\sigma)$  is attained. We limit ourselves to the case  $2k \geq 1$ ,  $m = 0$  or  $2k = 1$ ,  $m = 1$ . The proof in both the cases are similar and we consider the first case  $2k \geq 1$ ,  $m = 0$ . We impose  $T + \frac{H}{4} \leq t \leq T + \frac{3H}{4}$ .

If now  $b$  is an odd positive integer which is fixed to be a large integer depending on  $\delta$ , we have by Cauchy's Theorem

$$F(s) = \frac{1}{2\pi i} \int_R F(W) e^{(W-s)^{2b}} \frac{dW}{W-s}$$

where  $R$  is the rectangle with corners  $\frac{1}{2} + i(T + H)$ ,  $\frac{1}{2} + iT$ ,  $2 + iT$ ,  $2 + i(T + H)$ . Because  $b$  is large we can check that the horizontal portions contribute a bounded quantity to the integral. The same is trivially true of the line  $u = 2$ . Thus

$$\begin{aligned} |F(s)|^{2k} &= O\left(\left(\int_{u=\frac{1}{2}} \left|F(W) e^{(W-s)^{2b}} \frac{dW}{W-s}\right|\right)^{2k} + 1\right) \\ &= O\left(1 + \int_{u=\frac{1}{2}} \left|F(W)^{2k} e^{(W-s)^{2b}} \frac{dW}{W-s}\right|\right. \\ &\quad \left.\left(\int_{u=\frac{1}{2}} \left|e^{(W-s)^{2b}} \frac{dW}{W-s}\right|\right)^{2k-1}\right) \end{aligned}$$

Theorem 2 now follows on integrating with respect to  $s$  and also using the fact that  $\sigma - \frac{1}{2} > \frac{\log \log H}{\log H}$ . We have to remember that if the field is Galois, condition (ii) is satisfied; otherwise the condition (i) is satisfied.

*To prove Theorem 1, we assume that it is false with  $C = 1$ .*

We now proceed to prove by a series of lemmas the truth of the theorem for some  $C > 1$ . We can certainly choose the latter constant and this would prove the Theorem 1. Accordingly we begin with

**Lemma 1:** *We have*

$$\sigma > \frac{1}{2} + \frac{\max_{\tau} \log \log H}{\log H} \frac{1}{H} \int_{\tau}^{T+H} |F(\sigma + it)|^{2k} dt < (\log H)^{ok^2+m} \log \log H.$$

*Proof:* Trivial since, for

$$0 \leq j \leq \delta^{-1}, \left| \frac{d^j}{dt^j} (F(2 + it))^{2k} \right| \text{ is bounded.}$$

**Lemma 2:** *The maximum of  $|F(\sigma + it)|$  taken over all*

$$\sigma > \frac{1}{2} + \frac{2 \log \log H}{\log H}, T+1 \leq t \leq T+H-1, \text{ does not exceed } H^2.$$

*Proof:* Follows from the fact  $|F(s)|^{2k}$  is subharmonic. However we supply a proof. If  $F(s) \neq 0$  in  $|s - s_0| \leq r$

$$\text{we have } \log |F(s_0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |F(s_0 + re^{i\theta})| d\theta.$$

The first quantity is less than the second if  $F(s_0) = 0$ . Assuming now  $F(s_0) \neq 0$  and defining

$$\phi(s) = F(s) \pi \left( \frac{r^2 - (\bar{\rho} - \bar{s}_0)(s - s_0)}{(s - \rho)r} \right)$$

where  $\rho$  runs over all the zeros of  $F(s)$  in  $|s - s_0| \leq r$ , we see that  $\log |F(s_0)| \leq \log |\phi(s_0)|$  and that on  $|s - s_0| = r$  we have  $|F(s)| = |\phi(s)|$ . This proves

$$\log |F(s_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} \log |F(s_0 + re^{i\theta})| d\theta.$$

This gives easily

$$\log |F(s_0)| \leq \frac{1}{\pi r^2} \int_{|s-s_0| \leq r} \log |F(s)| dv$$

where  $dv$  is the element of area of the disc. We multiply this by  $2k$  and apply the arithmetico-geometric inequality (in the limiting form to suit integrals) and we obtain

$$|F(s_0)|^{2k} \leq \frac{1}{\pi r^2} \int_{|s-s_0| \leq r} |F(s)|^{2k} dv.$$

By taking a suitable radius say  $\frac{\log \log H}{\log H}$  we get the lemma.

**Lemma 3:** Let  $N(\alpha, T_1, T_2)$  denote the number of zeros of  $F(s)$  in  $\sigma > \alpha$   $\left( \alpha > \frac{1}{2} + \frac{3 \log \log H}{\log H} \right)$  and  $T_1 \leq t < T_2$ .

Then if  $T + (\log H)^8 \leq t < T + H - (\log H)^8$ ,

$$N(\alpha, t, t+1) \leq (\log H)^4.$$

*Proof:* Follows from Jensen's inequality

(see page 126 of [ 6 ])

$$\int_0^r \frac{n(x)}{x} dx = \frac{1}{2\pi} \int_0^{2\pi} \log | F(2 + it + re^{i\theta}) | d\theta - \log | F(2 + it) |,$$

where  $n(x)$  denotes the number of zeros of  $F(s)$  in a disc of radius  $x$  with centre  $2 + it$ . We have to select a suitable  $r$  and use Lemma 2. Note that if  $0 < x_1 < r$  we have

$$\int_0^r \frac{n(x)}{x} dx \geq n(x_1) \log \frac{r}{x_1}.$$

**Lemma 4:** For

$$\alpha \geq \frac{1}{2} + \frac{3 \log \log H}{\log H}, T_1 = T + (\log H)^8, \\ T_2 = T + H - (\log H)^8,$$

we have  $N(\alpha, T_1, T_2) < H^1 - C_1(\alpha - \frac{1}{2})(\log H)^{C_2}$ .

*Proof:* We select a "well-spaced" system of zeros connected in  $N(\alpha, T_1, T_2)$  and proceed to estimate their number by the zero detecting function  $F(s) M_H(s) - 1$  where  $M_H(s)$  is the sum of the first  $H$  terms of the Dirichlet series for  $(F(s))^{-1}$ . Note that if  $\phi_1(s)$  is the zero detecting function in question and  $Re s = 1$  then

$$\frac{1}{2\pi i} \int_{Re W=2} \phi_1(s+W) \Gamma(W) X^W dW$$

is a very good approximation to  $\phi_1(s)$  if we set  $X = H^{C_3}$  where  $C_3$  is a large constant. Fairly routine considerations lead to the lemma. (An excellent reference to our ideas of deducing "density estimates" which was also found by Gallagher in a more perfect form is Gallagher's paper [ 1 ]). In the proof we have of course to use Lemma 1.

**Lemma 5:** Let now  $\alpha \geq \alpha_0 = \frac{1}{2} + \frac{K_1 \log \log H}{\log H}$  where  $K_1$  is a large constant. Then  $N(\alpha, T_1, T_2) < H(\log H)^{-K_2}$  where  $K_2 \rightarrow \infty$  as  $K_1 \rightarrow \infty$ . (Hereafter  $T_1, T_2$  will be as in Lemma 4 and  $\alpha_0$  as in this Lemma).

*Proof:* This lemma is a Corollary to Lemma 4.

We next divide the  $t$  interval  $T_1 \leq t \leq T_2$  into equal intervals of length  $(\log H)^{K_3}$  ( $K_3 \geq 10$ ) ignoring a small bit at one end. Let  $I_1$  run through those intervals which do not contain a zero in  $\sigma \geq \frac{1}{2} + \frac{K_1 \log \log H}{\log H}$  and  $I_2$  the rest of the intervals. Let  $I_3$  run through the intervals  $I_1$  with  $t$  intervals of length  $(\log H)^2$  removed both above and below. Plainly the intervals  $I_3$  cover the interval  $T_1 \leq t \leq T_2$  except certain bits of total length not exceeding  $H(\log H)^{-K_2} + H(\log H)^{2-K_3}$ . We put  $s_0 = \alpha + it$  ( $\alpha$  fixed and  $\geq \alpha_0 + \frac{\log \log H}{\log H}$ ) and set out to obtain an asymptotic formula for

$$Q_1(\alpha) = \sum_{I_3} \int_{I_3} |F(s_0)|^{2k} dt.$$

We prove

**Lemma 6:** Let  $K_2 = K_3$  and  $\alpha \geq \alpha_0 + \frac{\log \log H}{\log H}$  where  $\alpha_0 = \frac{1}{2} + \frac{K_1 \log \log H}{\log H}$ . Then we have,

$$Q_1(\alpha) = H \sum_{n=1}^{\infty} |d_k(n)|^2 n^{-2\alpha} + O(H(\log H)^{C_4 - \frac{1}{2}K_2}) \\ + O(H(\log H)^{C_4} H^{-\frac{1}{8}(\alpha - \alpha_0)}),$$

where  $d_k(n)$  are defined by the expansion  $(F(s))^k = \sum_{n=1}^{\infty} d_k(n) n^{-s}$  valid in  $\sigma > 1$ . Also  $K_2 \rightarrow \infty$  as  $K_1 \rightarrow \infty$ .

*Proof:* The proof of this lemma is nearly standard. We will merely sketch the proof. For  $t$  in  $I_3$  we have with  $X = H^{\frac{1}{2}}$  and the notation introduced already,

$$F(s_0) = P(t) + E, P(t) = \sum_{n=1}^{\infty} d_k(n) n^{-s_0} e^{-\frac{n}{x}}$$

where  $E = \frac{1}{2\pi i} \int (F(s_0+W))^k \Gamma(W+1) X^W \frac{dW}{W} + O(H^{-10})$ ,  
 $|ImW| \leq (\log H)^2, Re(W+\alpha) = \alpha_0$

From Lemma 1,  $\int_{I_3} |E|^2 dt = O(H(\log H)^{C_5} X^{-\frac{1}{2}(\alpha-\alpha_0)})$

Next we note that  $|(F(s_0))^k|^2 = |P(t)|^2 + O(|(F(s_0))^k - P(t)|^2) + O(|(F(s_0) - P(t))P(t)|)$ .

Hence

$$\sum_{I_3} \int_{I_3} |F(s_0)|^{2k} dt = \sum_{I_3} \int_{I_3} |P(t)|^2 dt + E' \text{ where}$$

$$E' = O\left(\sum_{I_3} \int_{I_3} |E|^2 dt + \left(\int_T^{T+H} |E|^2 dt\right)^{\frac{1}{2}} \left(\sum_{I_3} \int_{I_3} |P(t)|^2 dt\right)^{\frac{1}{2}}\right)$$

It is easy to see that  $\int_T^{T+H} |P(t)|^2 dt = O(H(\log H)^{C_5})$ , and

$$\int_T^{T+H} |P(t)|^4 dt = O(H(\log H)^{C_5}) \text{ and so the } O\text{-term is}$$

$$O(H(\log H)^{C_4} X^{-\frac{1}{2}(\alpha-\alpha_0)}). \text{ Also}$$

$$\sum_{I_3} \int_{I_3} |P(t)|^2 dt = \int_T^{T+H} |P(t)|^2 dt + O(H(\log H)^{C_4 - \frac{1}{2}K_2}),$$

and

$$\int_T^{T+H} |P(t)|^2 dt = \int_T^{T+H} \left| \sum_{n \leq X(\log X)^2} d_k(n) n^{-s_0} \right|^2 dt + O(H^{-10})$$

and this by standard arguments (see for instance Lemma 9 below) is

$$(H + O(X(\log X)^3)) \sum_{n \leq X(\log X)^2} |d_k(n)|^2 n^{-2\alpha} e^{-\frac{2n}{X}} + O(H^{10})$$

$$\text{Let } S = \sum_{n=1}^{\infty} |d_k(n)|^2 n^{-2\alpha} \text{ and } S_1 = \sum_{n \leq X(\log X)^2} |d_k(n)|^2 n^{-2\alpha} e^{-\frac{2n}{X}}$$

Then using  $1 - e^{-\frac{n}{X}} = O\left(\frac{n}{X}\right)$  for  $n \leq X$  and  $O(1)$  for  $n > X$  we have

$$\begin{aligned} S - S_1 &= O\left(\frac{1}{X} \sum_{n < X} |d_k(n)|^2 n^{1-2\alpha} + \sum_{n \geq X} |d_k(n)|^2 n^{-2\alpha}\right) \\ &= O(X^{1-2\alpha} (\log H)^{C_4}). \end{aligned}$$

This proves Lemma 6.

**Lemma 7:** In case (i) we have  $\left| \frac{d^m}{d\alpha^m} S \right| \gg \left(\frac{1}{\alpha - \frac{1}{2}}\right)^{ak^2 + m}$ ,

and in case (ii) we have  $S (\alpha - \frac{1}{2})^{ak^2}$  is  $\gg \left(\log \frac{1}{\alpha - \frac{1}{2}}\right)^{b_2 k^2}$  and

$$\ll \left(\log \frac{1}{\alpha - \frac{1}{2}}\right)^{b_1 k^2}.$$

*Proof:* We leave this as an exercise to the reader.

**Lemma 8:** Let  $f(x)$  be  $m$  times continuously differentiable in the interval of the integration below. Then for any positive number  $d$  we have,

$$\begin{aligned} f(x) - \binom{m}{1} f(x+d) + \binom{m}{2} f(x+2d) + \dots \\ + (-1)^m \binom{m}{m} f(x+md) \\ = (-1)^m \int_0^d \int_0^d \int_0^d \dots \int_0^d f^{(m)}(x+u_1+u_2+\dots+u_m) du_1 \dots du_m \end{aligned}$$

where  $\binom{m}{1}, \binom{m}{2}, \dots$  are binomial coefficients.

*Proof:* This can be proved by induction on  $m$ . Details are left to the reader.

From Lemmas 6, 7 and 8 we can deduce Theorem 1 as follows. We first assume condition (iii). Define

$$\alpha_1 = \frac{1}{2} + \frac{K_4 \log \log H}{\log H}, \text{ and } \alpha_2, \dots, \alpha_{m+1} \text{ by } \alpha_j =$$

$$\alpha_1 + \frac{(j-1) (\log \log H)^{C_6}}{\log H} \quad (j = 2, \dots, m+1) \text{ where } C_6 \text{ is a}$$

large constant. Taking  $K_4$  a large constant we see that

$$\frac{1}{H} Q_1(\alpha_1) > (\log H)^{ak^2+m} (\log \log H)^{-C_7} \text{ and this leads to}$$

Theorem 1 with  $C = \max(1, C_7)$ , in case  $m = 0$ . If  $m \geq 1$ , we see that  $Q_1(\alpha_1)$  dominates  $Q_1(\alpha_j)$ , ( $j \geq 2$ ).

$$\text{So taking } f(\alpha) = (F(\alpha + it))^{2k}, \quad x = \alpha_1, \quad d = \frac{(\log \log H)^{C_6}}{\log H}$$

we see that

$$\begin{aligned} & \sum_{l_3} \int_{l_3} \left| f(\alpha_1) - \binom{m}{1} f(\alpha_2) + \binom{m}{2} f(\alpha_3) + \dots \right. \\ & \qquad \qquad \qquad \left. + (-1)^m \binom{m}{m} f(\alpha_{m+1}) \right| dt \\ & < \int_0^d \dots \int_0^d \left( \sum_{l_3} \int_{l_3} |f(\alpha_1 + u_1 + \dots + u_m)| dt \right) du_1 \dots du_m \end{aligned}$$

This shows that the max  $Q_1(\sigma)H^{-1}$  and so of  $Q(\sigma)$  in  $\sigma \geq \frac{1}{2} + \frac{\log \log H}{\log H}$  exceeds  $(\log H)^{ak^2+m} (\log \log H)^{-C_8}$

and this proves Theorem 1 with  $C = \max(1, C_8)$ .

Next we prove Theorem 1 subject to the condition (i). The case  $m = 0$  can be disposed off as before. Let  $m > 1$  and the theorem be false with a large constant  $C$ . Then in addition to Lemma 1 we also have

$$\max_{\sigma > \frac{1}{2} + \frac{\log \log H}{\log H}} \frac{1}{H} \int_{\tau}^{\tau+H} \left| \frac{d}{ds} (F(s))^{2k} \right| dt =$$

$$O((\log H)^{ak^2+m_1} (\log \log H)^{-c}).$$

We choose  $\mathfrak{A}_1$  and  $\mathfrak{A}_2$  as before and we have

$$\sum_{I_3} \int_{I_3} (|f(\mathfrak{A}_1)| - |f(\mathfrak{A}_2)|) dt \leq \sum_{I_3} \int_{I_3} |f(\mathfrak{A}_1) - f(\mathfrak{A}_2)| dt$$

Here the left side exceeds  $H(\log H)^{ak^2+m_1-1} (\log \log H)^{-C_9}$

while the right side is  $O(H(\log H)^{ak^2+m_1-1} (\log \log H)^{C_6-C})$

which is a contradiction if we choose  $C = C_6 + C_9 + 1$ . This proves Theorem 1 with  $C = C_6 + C_9 + 1$ .

Before leaving this section we remark that if instead of condition (ii) we are given that the product

$$\pi \left( 1 + \frac{|\chi(p)a_p|^2}{p^2\sigma} \right) \text{ lies between } (\sigma - \frac{1}{2})^{-a'} \left( \log \frac{1}{\sigma - \frac{1}{2}} \right)^{b_1}$$

and  $(\sigma - \frac{1}{2})^{-a} \left( \log \frac{1}{\sigma - \frac{1}{2}} \right)^{b_2}$ , where

$a' > a > 0$ ,  $b_1, b_2$  are four real constants then we can conclude

$$\max Q(\sigma) > (\log H)^{ak^2+ma(a')^{-1}} (\log \log H)^{-C}$$

$$\sigma \geq \frac{1}{2} + \frac{\log \log H}{\log H}$$

This explains the second part of our remark below Theorem 2 since for non-Galois number fields we can take  $a' = n$ .

**§ 4. The case of no Euler product :** In this and the next section we assume instead of the Euler product a condition of the type,  $\sum_{n \leq x} |a_n|^8 = O(x(\log x)^{k'})$  where  $k'$  is a positive

constant and  $x \geq 2$ . (The condition can be relaxed, but we do not want to go into such questions). The main result of this section is

**Theorem 3** Suppose that as  $\sigma \rightarrow \frac{1}{2} + 0$ ,  $|F(2\sigma)|$  exceeds  $\left( \frac{1}{\sigma - \frac{1}{2}} \right)^a \left( \log \frac{1}{\sigma - \frac{1}{2}} \right)^{b_3}$  where  $a > 1$  and  $b_3$  are two real constants. Then with  $2k = 1$ ,  $m = 0$ , we have,

$$\frac{1}{H} \int_T^{T+H} |F(\frac{1}{2} + it)| dt = Q(\frac{1}{2}) > (\log H)^{a-1} (\log \log H)^{-C}$$

*Proof:* It suffices to prove that  $\max Q(\sigma)$  for  $\sigma > \frac{1}{2} + \frac{\log \log H}{\log H}$ , exceeds  $(\log H)^{a-1} (\log \log H)^{-C_1}$  and

Theorem 3 follows from this just as Theorem 2 follows from Theorem 1. (The constants  $C, K$  with or without subscripts should not be confused with the earlier constants. Also to save space the proof will be only sketchy). We assume that this is false. Lemma 2 has its analogue without modification.

We put  $\alpha_1 = \frac{1}{2} + \frac{2 \log \log H}{\log H}$ . Let  $\alpha \geq \alpha_1 + \frac{\log \log H}{\log H}$ ,

and put  $s_0 = \alpha + it$ . We divide the interval

$T_1 = T + (\log H)^8 \leq t \leq T_2 = T + H - (\log H)^8$  into equal intervals  $J$  of length  $(\log H)^{K_1}$  ignoring a bit at one end. We put

$$\zeta_Y(s) = \sum_{n < Y} n^{-s}, \quad \zeta_Y^{-1}(s) = \sum_{n < Y} \mu(n) n^{-s}$$

where  $Y = H^{\frac{1}{2}}$ . It is not hard to prove that if  $\phi(s) = \zeta_Y(s) \zeta_Y^{-1}(s) - 1$  and  $M(J)$  denotes the maximum of  $|\phi(s_0)|$  for  $t$  in  $J$ ,

$$\sum_J (M(J))^2 = O(H (\log H)^{10} Y^{1-2\alpha})$$

We omit those intervals  $J$  for which  $M(J) \geq \frac{1}{2}$  and denote the rest of the intervals by  $I$ . The number of intervals  $J$  which are excluded is  $O(HY^{1-2\alpha} (\log H)^{10})$ . If  $K$  is a sufficiently

large constant and  $\alpha = \frac{1}{2} + \frac{K \log \log H}{\log H}$  we proceed to prove

that  $\sum_I \int_I |F(s_0)| dt > H (\log H)^{a-1} (\log \log H)^{-C_2}$ .

We write  $F(s_0) = F_Y(s_0) + E_Y(s_0)$  where  $F_Y(s_0) = \sum_{n=1}^{\infty} a_n n^{-s_0} e^{-\frac{n}{Y}}$ ,

and we have  $\int_T^{T+H} |E_Y| dt = O(H (\log H)^{C_3} Y^{\alpha - \alpha_1})$ .

We can replace the integrand

$$|F(s_0)| \text{ by } |F_Y(s_0) \zeta_Y^{-1}(s_0) \zeta_Y(s_0)| \text{ without}$$

disturbing the left side very much. Then we replace

$\sum_I \int_I |\dots| dt$  by  $\int_T^{T+H} |\dots| dt$  without much error. Next we

use the fact that the last integral is bounded below by

$$\left| \int_T^{T+H} F_Y(s_0) \zeta_Y^{-1}(s_0) \overline{\zeta_Y(s_0)} dt \right|.$$

To see that this is  $\gg H |F(2\alpha)| (\alpha - \frac{1}{2})$ , we use the following lemma and some simple computations and this would complete the proof of Theorem 3.

**Lemma 9:** *If  $\{x_n\}$  and  $\{y_n\}$  are two sequences of complex numbers, then,*

$$\begin{aligned} & \int_0^T \left( \sum_{n=1}^{\infty} x_n n^{-it} \right) \left( \sum_{n=1}^{\infty} \overline{y_n} n^{it} \right) dt \\ &= T \sum_{n=1}^{\infty} x_n \overline{y_n} + O \left( \left( \sum_{n=1}^{\infty} n |x_n|^2 \right)^{\frac{1}{2}} \left( \sum_{n=1}^{\infty} n |y_n|^2 \right)^{\frac{1}{2}} \right) \end{aligned}$$

*Remark.* For a simple proof of this lemma see [4].

### § 5. Balasubramanian's remark

**Theorem 4.** *In the case  $2k = 1$ ,  $m = 0$  as before, we have,*

$$\frac{1}{H} \int_T^{T+H} |F(\frac{1}{2} + it)| dt = O(\frac{1}{2}) \gg \max(1, S_2^{\frac{3}{2}} S_3^{-1} (\log \log H)^{-1})$$

$$\text{where } S_2 = \sum_{n=1}^{\infty} |a_n|^2 n^{-2\alpha}, \quad S_3 = \sum_{n=1}^{\infty} |a_n|^2 d(n) n^{-2\alpha},$$

$\alpha = \frac{1}{2} + \frac{C \log \log H}{\log H}$  (where  $C$  is a large positive constant) and  $d(n)$  is the usual divisor function.

*Proof:* As before we assume  $\max Q(\sigma)$  for  $\sigma \geq \frac{1}{2} + \frac{\log \log H}{\log H}$  does not exceed  $S_2^{\frac{3}{2}} S_3^{-1}$  (Note that this is

$O((\log H, C^4))$ . Next we write  $s_0 = \alpha + it$ ,  $F(s_0) = F_Y(s_0) + E_Y$ ,

where  $Y = H^{\frac{1}{2}}$ ,  $F_Y(s_0) = \sum_{n=1}^{\infty} a_n n^{-s_0} e^{-\frac{n}{Y}}$ . We have an

asymptotic formula for  $\int_T^{T+H} |F_Y(s_0)|^2 dt$ , and also a

good upper bound for  $\int_T^{T+H} |F_Y(s_0)|^4 dt$ . Using

$$\int_T^{T+H} |F_Y(s_0)|^2 dt \leq \left( \int_T^{T+H} |F_Y(s_0)| dt \right)^{\frac{3}{2}} \left( \int_T^{T+H} |F_Y(s_0)|^4 dt \right)^{\frac{1}{2}}$$

we are led to the theorem. The details are left to the reader.

### References

1. **P. X. Gallagher**, : "Local mean value and density estimates for Dirichlet's L-functions," *Indag. Math.*, 37 (1975), 259-264.
2. **K. Ramachandra**, : "On the zeros of a class of generalised Dirichlet series - V," *J. Reine U. Angew. Math.* 303 / 304 (1978), 295-313.
3. **K. Ramachandra**, : "On the frequency of Titchmarsh's phenomenon for  $\zeta(s)$ " *J. Lond. Math. Soc.* (2) 8 (1974), 683-690.
4. **K. Ramachandra**, : "Some remarks on a Theorem of Montgomery and Vaughan," *J. Number Theory*, Vol 11 (1979), 465-471,
5. **E. C. Titchmarsh**, : *The Theory of the Riemann zeta-function*, Oxford (1951).
6. **E. C. Titchmarsh**, : *The Theory of functions*, Oxford (1952).

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