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Abstract

In this work, we study the problem of mean-variance hedging with a random horizon $T \wedge \tau$, where T is a deterministic constant and τ is a jump time of the underlying asset price process. We first formulate this problem as a stochastic control problem and relate it to a system of BSDEs with a jump. We then provide a verification theorem which gives the optimal strategy for the mean-variance hedging using the solution of the previous system of BSDEs. Finally, we prove that this system of BSDEs admits a solution via a decomposition approach coming from filtration enlargement theory.

Keywords: Mean-variance hedging, Backward SDE, random horizon, jump processes, progressive enlargement of filtration, decomposition in the reference filtration.

AMS subject classifications: 91B30, 60G57, 60H10, 93E20.

1 Introduction

In most financial markets, the assumption that the market is complete fails to be true. In particular, investors cannot always hedge the financial products that they are interested in. One possible approach to deal with this problem is mean-variance hedging. That is, for a given financial product with terminal value H at a fixed horizon time T and an initial

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capital x, we need to find a strategy π^* such that the value V^{x,π^*} of the portfolio with initial amount x and strategy π^* minimizes the mean square error

$$\mathbb{E}\Big[\big|V_T^{x,\pi} - H\big|^2\Big]$$

over all possible investment strategies π.

In this paper, we are concerned with the mean-variance hedging problem over a random horizon. More precisely, we consider a random time τ and a contingent claim with a gain at time $T \wedge \tau$ of the form

$$H \quad = \quad H^b \mathbb{1}_{T<\tau} + H^a_\tau \mathbb{1}_{T\geq\tau}, \tag{1.1}$$

where $T < \infty$ is a fixed deterministic terminal time. We then study the mean-variance hedging problem over the horizon $[0, T \wedge \tau]$ defined by

$$\inf_\pi \mathbb{E}\Big[\big|V_{T\wedge\tau}^{x,\pi} - H\big|^2\Big] . \tag{1.2}$$

Financial products with gains of the form (1.1) naturally appear on financial markets, see e.g. Examples 2.1, 2.2 and 2.3 presented in Subsection 2.3.

The mean-variance hedging problem with deterministic horizon T is one of the classical problems from mathematical finance and has been considered by several authors via two main approaches. One of them is based on martingale theory and projection arguments and the other considers the problem as a quadratic stochastic control problem and describes the solution using BSDE theory.

The bulk of the literature primarily focuses on the continuous case where both approaches are used (see e.g. Delbaen and Schachermayer [6], Gouriéroux *et al.* [10], Laurent and Pham [23] and Schweizer [25] for the first approach, and Lim and Zhou [21] and Lim [20] for the second one).

In the discontinuous case, the mean-variance hedging problem is considered by Arai [2], Lim [22] and Jeanblanc *et al* [14]. In [2], the author uses the projection approach for general semimartingale price processes model whereas in [22] the problem is considered from the point of view of stochastic control for the case of diffusion price processes driven by Brownian motion and Poisson process. The author provides under a so-called "martingale condition" the existence of solutions to the associated BSDEs. In the recent paper [14], the authors combine tools from both approaches, which allows them to work in a general semimartingale model and to give a description of the optimal solution to the mean-variance hedging via the BSDE theory. More precisely the authors prove that the value process of the mean-variance hedging problem has a quadratic structure and that the coefficients appearing in this quadratic expression are related to some BSDEs. Then, they provide an equivalence between the existence of an optimal strategy and the existence of a solution to a BSDE associated to the control problem. They have also shown in some specific examples, via the control problem, the existence of solutions for BSDEs of interest. However the problem is still open in the general case.

In this paper, we study the mean-variance hedging with horizon $T \wedge \tau$ given by (1.2). We use a stochastic control approach and describe the optimal solution by a solution to a system of BSDEs.

We shall consider a model of diffusion price process driven by a Brownian motion and a random jump time τ. We follow the progressive enlargement approach initiated by Jacod, Jeulin and Yor (see [15] and [16]), which leads to considering an enlargement of the initial information given by the Brownian motion to make τ a stopping time. We note that this approach allows to work under wide class of assumptions, in particular, on contrary to the Poisson case, no a priori law is fixed for the random time τ.

Following the quadratic form obtained in [14], we use a martingale optimality principle to obtain an associated system of nonstandard BSDEs. We then establish a verification result (Theorem 3.2) which provides an explicit optimal investment strategy via the solution to the associated system of BSDEs. Our contribution is twofold.

• We link the mean-variance hedging problem on a random horizon with a system of BSDE, in a general filtration progressive enlargement setup which allows us to work without a priori knowledge of the law of jump part. We show that, under wide assumptions, the mean-variance hedging problem admits an optimal strategy described by the solution of the associated system of BSDEs.

• We prove that the associated system of BSDEs, which is nonstandard, admits a solution. The main difficulty here is that the obtained system of BSDEs is nonstandard since it is driven by a Brownian motion and a jump martingale and has generators with quadratic growth in the variable z and are undefined for some values of the variable y. To solve these BSDEs we follow a decomposition approach inspired by the result of Jeulin (see Proposition 2.1) which allows to consider BSDEs in the smallest filtration (see Theorem 4.3). Then using BMO properties, we provide solutions to the decomposed BSDEs which lead to the existence of a solution to the BSDEs in the enlarged filtration.

We notice that, for the problem at hand *i.e.* mean-variance hedging with horizon $T \wedge \tau$, the interest of our approach is that it provides a solution to the associated BSDEs, without supposing any additional specific assumptions to the studied BSDEs unlike in [22] where to prove existence of a solution to the BSDE the author introduces the "martingale condition" or in [14] where the existence of a solution to the BSDE is given in specific cases.

The paper is organized as follows. In Section 2, we present the details of the probabilistic model for the financial market, and setup the mean-variance hedging on random horizon. In Section 3, we show how to construct the associated BSDEs via the martingale optimality principle and we state the two main theorems of this paper. The first one concerns the existence of a solution to the associated system of BSDEs and the second one is a verification theorem which gives an optimal strategy via the solution of the BSDEs. Then, Section 4 is dedicated to the proof of the existence of solution to the associated system of BSDEs. Finally, some technical results are relegated to the appendix.

2 Preliminaries and market model

2.1 The probability space

Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a complete probability space. We assume that this space is equipped with a one-dimensional standard Brownian motion W and we denote by $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ the right continuous complete filtration generated by W. We also consider on this space a random time τ, which represents for example a default time in credit risk or in counterparty risk, or a death time in actuarial issues. The random time τ is not assumed to be an $\mathbb{F}$-stopping time. We therefore use in the sequel the standard approach of filtration enlargement by considering $\mathbb{G}$ the smallest right continuous extension of $\mathbb{F}$ that turns τ into a $\mathbb{G}$-stopping time (see e.g. [15, 16]). More precisely $\mathbb{G} := (\mathcal{G}_t)_{t\geq 0}$ is defined by

$$\mathcal{G}_t \quad := \quad \bigcap_{\varepsilon>0} \tilde{\mathcal{G}}_{t+\varepsilon} \ ,$$

for all $t \geq 0$, where $\tilde{\mathcal{G}}_s := \mathcal{F}_s \vee \sigma(\mathbb{1}_{\tau\leq u} \ , u \in [0, s])$, for all $s \geq 0$.

We denote by $\mathcal{P}(\mathbb{F})$ (resp. $\mathcal{P}(\mathbb{G})$) the σ-algebra of $\mathbb{F}$ (resp. $\mathbb{G}$)-predictable subsets of $\Omega \times \mathbb{R}_+$, i.e. the σ-algebra generated by the left-continuous $\mathbb{F}$ (resp. $\mathbb{G}$)-adapted processes.

We now introduce a decomposition result for $\mathcal{P}(\mathbb{G})$-measurable processes.

Proposition 2.1. *Any $\mathcal{P}(\mathbb{G})$-measurable process $X = (X_t)_{t\geq 0}$ is represented as*

$$X_t \quad = \quad X^b_t \mathbb{1}_{t\leq\tau} + X^a_t(\tau)\mathbb{1}_{t>\tau} \ ,$$

for all $t \geq 0$, where X^b is $\mathcal{P}(\mathbb{F})$-measurable and X^a is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable.

This result is proved in Lemma 4.4 of [15] for bounded processes and is easily extended to the case of unbounded processes. For the sake of completeness, we detail its proof in the appendix.

Remark 2.1. In the case where the studied process X depends on another parameter x evolving in a Borel subset $\mathcal{X}$ of $\mathbb{R}^p$, and if X is $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathcal{X})$, then, decomposition given by Proposition 2.1 is still true but where X^b is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathcal{X})$-mesurable and X^a is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathcal{X})$-measurable. Indeed, it is obvious for the processes generating $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathcal{X})$ of the form $X_t(\omega, x) = L_t(\omega)R(x)$, $(t, \omega, x) \in \mathbb{R}_+ \times \Omega \times \mathcal{X}$, where L is $\mathcal{P}(\mathbb{G})$-measurable and R is $\mathcal{B}(\mathcal{X})$-measurable. Then, the result is extended to any $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathcal{X})$ -measurable process by the monotone class theorem.

We then impose the following assumption, which is classical in the filtration enlargement theory.

(H) The process W remains a $\mathbb{G}$-Brownian motion.

We notice that under **(H)**, the stochastic integral $\int_0^t X_s dW_s$ is well defined for all $\mathcal{P}(\mathbb{G})$-measurable processes X such that $\int_0^t |X_s|^2 ds < \infty$.

In the sequel we denote by N the process $\mathbb{1}_{\tau\leq .}$ and we suppose

(Hτ) The process N admits an $\mathbb{F}$-compensator of the form $\int_0^{\cdot\wedge\tau} \lambda_t dt$, i.e. $N - \int_0^{\cdot\wedge\tau} \lambda_t dt$ is a $\mathbb{G}$-martingale, where λ is a bounded $\mathcal{P}(\mathbb{F})$-measurable process.

We then denote by M the $\mathbb{G}$-martingale defined by

$$M_t \quad := \quad N_t - \int_0^{t\wedge\tau} \lambda_s ds \ ,$$

for all $t \geq 0$. We also introduce the process $\lambda^{\mathbb{G}}$ which is defined by $\lambda_t^{\mathbb{G}} := (1 - N_t)\lambda_t$.

2.2 Financial model

We consider a financial market model on the time interval $[0, T]$ where $0 < T < \infty$ is a finite time horizon. We suppose that the financial market is composed by a riskless bond with zero interest rate and a risky asset S. The price process $(S_t)_{t\geq 0}$ of the risky asset is modeled by the linear stochastic differential equation

$$S_t \quad = \quad S_0 + \int_0^t S_{s^-}(\mu_s ds + \sigma_s dW_s + \beta_s dM_s) \ , \quad \forall t \in [0, T] \ , \tag{2.1}$$

where μ, σ and β are $\mathcal{P}(\mathbb{G})$-measurable processes and S_0 is a positive constant. For example S could be a Credit Default Swap on the firm whose default time is τ. We impose the following assumptions on the coefficients μ, σ and β.

(HS)

(i) The processes μ and σ are bounded: there exists a constant $C > 0$ such that

$$|\mu_t| + |\sigma_t| \quad \leq \quad C \ , \quad \forall t \in [0, T] \ , \quad \mathbb{P} - a.s.$$

(ii) The process σ is uniformly elliptic: there exists a constant $C > 0$ such that

$$|\sigma_t| \quad \geq \quad C \ , \quad \forall t \in [0, T] \ , \quad \mathbb{P} - a.s.$$

(iii) There exists a constant C such that

$$-1 \ \leq \ \beta_t \ \leq \ C \ , \quad \forall t \in [0, T] \ , \quad \mathbb{P} - a.s.$$

Under **(HS)**, we know from e.g. Theorem 1 in [9] that the process S defined by (2.1) is well defined.

2.3 Mean-variance hedging

We consider investment strategies which are $\mathcal{P}(\mathbb{G})$-measurable processes π such that

$$\int_0^{T\wedge\tau} |\pi_t|^2 dt \quad < \quad +\infty \ , \qquad \mathbb{P} - a.s.$$

This condition and $(\mathbf{H}S)$ ensure that the stochastic integral $\int_0^t \frac{\pi_r}{S_{r^-}} dS_r$ is well defined for such a strategy π and $t \in [0, T \wedge \tau]$. The wealth process $V^{x,\pi}$ corresponding to a pair (x, π), where $x \in \mathbb{R}$ is the initial amount, is defined by the stochastic integration

$$V_t^{x,\pi} \quad := \quad x + \int_0^t \frac{\pi_r}{S_{r^-}} dS_r \ , \quad \forall\, t \in [0, T \wedge \tau] \ .$$

We denote by $\mathcal{A}$ the set of admissible strategies π such that

$$\mathbb{E}\Big[\int_0^{T\wedge\tau} |\pi_t|^2 dt\Big] \quad < \quad \infty \ .$$

For $x \in \mathbb{R}$, the problem of mean-variance hedging consists in computing the quantity

$$\inf_{\pi\in\mathcal{A}} \mathbb{E}\Big[\big|V_{T\wedge\tau}^{x,\pi} - H\big|^2\Big] \ , \tag{2.2}$$

where H is a bounded $\mathcal{G}_{T\wedge\tau}$-measurable random variable of the form

$$H \quad = \quad H^b \mathbb{1}_{T<\tau} + H_\tau^a \mathbb{1}_{T\geq\tau} \ , \tag{2.3}$$

where H^b is an $\mathcal{F}_T$-measurable random variable valued in $\mathbb{R}$ and H^a is a càd-làg $\mathcal{P}(\mathbb{F})$-measurable process also valued in $\mathbb{R}$ and such that

$$\big\|H^b\big\|_\infty \ < \ \infty, \quad \text{and} \quad \Big\|\sup_{t\in[0,T]} \big|H_t^a\big|\Big\|_\infty \ < \ \infty \ , \tag{2.4}$$

where we recall that $\|.\|_\infty$ is defined by

$$\|X\|_\infty \quad := \quad \inf\Big\{C \geq 0 \ : \ \mathbb{P}\big(|X| \leq C\big) = 1\Big\} \ ,$$

for any random variable X.

Since the problem we are interested in uses the values of the coefficients μ, σ and β only on the interval $[0, T \wedge \tau]$, we can assume by Proposition 2.1 that μ, σ and β are $\mathcal{P}(\mathbb{F})$-measurable and we shall do that in the sequel.

Remark 2.2. For simplicity, we have supposed that the riskless interest rate is equal to zero. However, all the results can be extended to the case of a bounded $\mathcal{P}(\mathbb{G})$-measurable interest rate process r. Indeed, for such an interest rate process the mean-variance hedging problem becomes

$$\inf_{\pi\in\mathcal{A}} \mathbb{E}\Big[\big|\tilde{V}_{T\wedge\tau}^{x,\pi} - \tilde{H}\big|^2\Big] \ ,$$

where $\tilde{V}^{x,\pi}$ and $\tilde{H}$ are the discounted values of $V^{x,\pi}$ and H given by

$$\tilde{H} \quad := \quad H \exp\Big(-\int_0^{T\wedge\tau} r_s ds\Big)$$

and

$$\tilde{V}_t^{x,\pi} \quad := \quad V_t^{x,\pi} \exp\Big(-\int_0^t r_s ds\Big) \ , \quad t \in [0, T] \ .$$

From the dynamic of $V^{x,\pi}$ we see that $\tilde{V}^{x,\pi}$ satisfies

$$\tilde{V}_t^{x,\pi} \quad = \quad x + \int_0^t \pi_s \big(\tilde{\mu}_s ds + \tilde{\sigma}_s dW_s + \tilde{\beta}_s dM_s\big)$$

where

$$\tilde{\mu}_t \; := \; e^{-\int_0^t r_s ds}(\mu_t - r_t) \; , \; \tilde{\sigma}_t \; := \; e^{-\int_0^t r_s ds}\sigma_t \quad \text{and} \quad \tilde{\beta}_t \; := \; e^{-\int_0^t r_s ds}\beta_t$$

for $t \in [0, T]$. In particular, we get the same model but with coefficients $\tilde{\mu}$, $\tilde{\sigma}$ and $\tilde{\beta}$ instead of μ, σ and β. Since $\tilde{\mu}$, $\tilde{\sigma}$ and $\tilde{\beta}$ also satisfy (HS), we can extend the results to this model with new coefficients.

We end this section by two examples of financial products taking the form (2.3).

Example 2.1 (Insurance contract)**.** Consider a seller of an insurance policy which protects the buyer over the time horizon $[0, T]$ from some fixed loss L. Then if we denote by τ the time at which the loss appears, the losses of the seller are of the form

$$H \quad = \quad -p\mathbb{1}_{T<\tau} + (L-p)\mathbb{1}_{T\geq\tau} \; ,$$

where p denotes the premium that the insurance policy holder pays at time 0.

Example 2.2 (Credit Default Swap with counterparty risk)**.** Consider a protection seller who sells a CDS against a credit event to a protection buyer for a nominal N against a premium payments p with a maturity T. If the reference entity defaults, the protection seller pays the buyer the nominal N and the CDS contract is terminated. Moreover, both the buyer and seller of credit protection take on counterparty risk:

– the buyer takes the risk that the seller of credit protection may default, if the seller defaults the buyer loses its protection against default by the reference entity,

– the seller takes the risk that the buyer may default on the contract, depriving the seller of the expected revenue stream.

Denote by τ the first default time, and by ξ the random variable such that $\xi = 1$ if the first default is the reference entity one and $\xi = 0$ otherwise. The losses of the seller are of the form

$$H \quad = \quad -pNT\mathbb{1}_{T<\tau} + N\mathbb{1}_{\tau\leq T,\xi=1} - pN\Big(\sum_{k=0}^{T} k\mathbb{1}_{k\leq\tau<k+1}\Big)\mathbb{1}_{\tau\leq T} \; .$$

Example 2.3 (Credit contract)**.** Consider a bank which lends an amount A to a company over the period $[0, T]$. Suppose that the time horizon $[0, T]$ is divided on n subintervals $[k\frac{T}{n}, (k+1)\frac{T}{n}]$, $k = 0, \ldots, n-1$, and that the interest rate of the loan over a time subinterval is r. The company has then to pay $\frac{(1+r)^n}{n}A$ to the bank at each time $k\frac{T}{n}$, $k = 1, \ldots, n$. If we denote by τ the company default time, then the losses of the bank are given by

$$H \quad = \quad -((1+r)^n - 1)A\mathbb{1}_{T<\tau} + H_\tau^a \mathbb{1}_{T\geq\tau} \; ,$$

where the function H^a is given by

$$H_t^a \quad = \quad -\sum_{k=1}^{n-1}\Big(k\frac{(1+r)^n}{n} - 1\Big)A\mathbb{1}_{k\frac{T}{n}<t\leq(k+1)\frac{T}{n}} \; , \quad t \in [0, T] \, .$$

3 Solution of the mean-variance problem by BSDEs

3.1 Martingale optimality principle

To find the optimal value of the problem (2.2), we follow the approach initiated by Hu *et al.* [12] to solve the exponential utility maximization problem in the pure Brownian case. More precisely, we look for a family of processes

$$\Big\{\big(J_t^\pi\big)_{t\in[0,T]}\ :\ \pi\in\mathcal{A}\Big\}$$

satisfying the following conditions

(i) $J_{T\wedge\tau}^\pi = \big|V_{T\wedge\tau}^{x,\pi}-H\big|^2$, for all $\pi\in\mathcal{A}$.

(ii) $J_0^{\pi_1}=J_0^{\pi_2}$, for all $\pi_1,\pi_2\in\mathcal{A}$.

(iii) $\big(J_t^\pi\big)_{t\in[0,T]}$ is a $\mathbb{G}$-submartingale for all $\pi\in\mathcal{A}$.

(iv) There exists some $\pi^*\in\mathcal{A}$ such that $\big(J_t^{\pi^*}\big)_{t\in[0,T]}$ is a $\mathbb{G}$-martingale.

Under these conditions, we have

$$J_0^{\pi^*} \;=\; \inf_{\pi\in\mathcal{A}}\mathbb{E}\Big[\big|V_{T\wedge\tau}^{x,\pi}-H\big|^2\Big]\,.$$

Indeed, using (i), (iii) and Doob's optional stopping theorem, we have

$$J_0^\pi \;\le\; \mathbb{E}\big[J_{T\wedge\tau}^\pi\big] \;=\; \mathbb{E}\Big[\big|V_{T\wedge\tau}^{x,\pi}-H\big|^2\Big]\,, \tag{3.5}$$

for all $\pi\in\mathcal{A}$. Then, using (i), (iv) and Doob's optional stopping theorem, we have

$$J_0^{\pi^*} \;=\; \mathbb{E}\Big[\big|V_{T\wedge\tau}^{x,\pi^*}-H\big|^2\Big]\,. \tag{3.6}$$

Therefore, from (ii), (3.5) and (3.6), we get for any $\pi\in\mathcal{A}$

$$\mathbb{E}\Big[\big|V_{T\wedge\tau}^{x,\pi^*}-H\big|^2\Big] \;=\; J_0^{\pi^*} \;=\; J_0^\pi \;\le\; \mathbb{E}\Big[\big|V_{T\wedge\tau}^{x,\pi}-H\big|^2\Big]\,.$$

We can see that

$$J_0^{\pi^*} \;=\; \inf_{\pi\in\mathcal{A}}\mathbb{E}\Big[\big|V_{T\wedge\tau}^{x,\pi}-H\big|^2\Big]\,.$$

3.2 Related BSDEs

We now construct a family $\{(J_t^\pi)_{t\in[0,T]},\ \pi\in\mathcal{A}\}$ satisfying the previous conditions by using BSDEs as in [12]. To this end, we define the following spaces.

– $\mathcal{S}_{\mathbb{G}}^\infty$ is the subset of $\mathbb{R}$-valued càd-làg $\mathbb{G}$-adapted processes $(Y_t)_{t\in[0,T]}$ essentially bounded

$$\|Y\|_{\mathcal{S}^\infty} \;:=\; \Big\|\sup_{t\in[0,T]}|Y_t|\Big\|_\infty \;<\;\infty\,.$$

– $\mathcal{S}^{\infty,+}_{\mathbb{G}}$ is the subset of $\mathcal{S}^\infty_{\mathbb{G}}$ of processes $(Y_t)_{t\in[0,T]}$ valued in $(0,\infty)$, such that

$$\Big\|\frac{1}{Y}\Big\|_{\mathcal{S}^\infty} \;<\; \infty\,.$$

– $L^2_{\mathbb{G}}$ is the subset of $\mathbb{R}$-valued $\mathcal{P}(\mathbb{G})$-measurable processes $(Z_t)_{t\in[0,T]}$ such that

$$\|Z\|_{L^2} \;:=\; \Big(\mathbb{E}\Big[\int_0^T |Z_t|^2 dt\Big]\Big)^{\frac{1}{2}} \;<\; \infty\,.$$

– $L^2(\lambda)$ is the subset of $\mathbb{R}$-valued $\mathcal{P}(\mathbb{G})$-measurable processes $(U_t)_{t\in[0,T]}$ such that

$$\|U\|_{L^2(\lambda)} \;:=\; \Big(\mathbb{E}\Big[\int_0^{T\wedge\tau} \lambda_s|U_s|^2 ds\Big]\Big)^{\frac{1}{2}} \;<\; \infty\,.$$

To construct a family $\{(J^\pi_t)_{t\in[0,T]},\ \pi\in\mathcal{A}\}$ satisfying the previous conditions, we set

$$J^\pi_t \;=\; Y_t\big|V^{x,\pi}_{t\wedge\tau}-\mathcal{Y}_t\big|^2+\Upsilon_t\,,\quad t\in[0,T]\,,$$

where[1] (Y,Z,U) is solution in $\mathcal{S}^{\infty,+}_{\mathbb{G}}\times L^2_{\mathbb{G}}\times L^2(\lambda)$ to

$$Y_t \;=\; 1+\int_{t\wedge\tau}^{T\wedge\tau}\mathfrak{f}(s,Y_s,Z_s,U_s)ds-\int_{t\wedge\tau}^{T\wedge\tau}Z_s dW_s-\int_{t\wedge\tau}^{T\wedge\tau}U_s dM_s\,,\quad t\in[0,T]\,, \tag{3.7}$$

$(\mathcal{Y},\mathcal{Z},\mathcal{U})$ is solution in $\mathcal{S}^\infty_{\mathbb{G}}\times L^2_{\mathbb{G}}\times L^2(\lambda)$ to

$$\mathcal{Y}_t \;=\; H+\int_{t\wedge\tau}^{T\wedge\tau}\mathfrak{g}(s,\mathcal{Y}_s,\mathcal{Z}_s,\mathcal{U}_s)ds-\int_{t\wedge\tau}^{T\wedge\tau}\mathcal{Z}_s dW_s-\int_{t\wedge\tau}^{T\wedge\tau}\mathcal{U}_s dM_s\,,\quad t\in[0,T]\,, \tag{3.8}$$

and (Υ,Ξ,Θ) is solution in $\mathcal{S}^\infty_{\mathbb{G}}\times L^2_{\mathbb{G}}\times L^2(\lambda)$ to

$$\Upsilon_t \;=\; \int_{t\wedge\tau}^{T\wedge\tau}\mathfrak{h}(s,\Upsilon_s,\Xi_s,\Theta_s)ds-\int_{t\wedge\tau}^{T\wedge\tau}\Xi_s dW_s-\int_{t\wedge\tau}^{T\wedge\tau}\Theta_s dM_s\,,\quad t\in[0,T]\,. \tag{3.9}$$

Remark 3.3. We notice that the jump components U, $\mathcal{U}$ and Θ are also bounded since Y, $\mathcal{Y}$ and Υ are in $\mathcal{S}^\infty_{\mathbb{G}}$. Indeed, let C be a constant such that

$$\|Y\|_{\mathcal{S}^\infty} \;\leq\; C\,. \tag{3.10}$$

Then since $Y_{\cdot-}+U$ is $\mathbb{G}$-predictable, we have

$$\begin{aligned}\mathbb{E}\Big[\int_0^T \mathbb{1}_{|Y_{t-}+U_t|>C}\lambda^{\mathbb{G}}_t dt\Big] &= \mathbb{E}\Big[\int_0^T \mathbb{1}_{|Y_{t-}+U_t|>C}dN_t\Big]\\ &= \mathbb{E}\Big[\mathbb{1}_{|Y_{\tau-}+U_\tau|>C,\tau\leq T}\Big]\\ &= \mathbb{E}\Big[\mathbb{1}_{|Y_\tau|>C,\tau\leq T}\Big]\\ &= 0\,.\end{aligned}$$

Therefore, we have $|Y_{\cdot-}+U|\leq C$ in $L^2(\lambda)$. From (3.10) we get $|U|\leq 2C$ in $L^2(\lambda)$. The same argument can be applied for $\mathcal{U}$ and Θ.

[1] As commonly done for the integration w.r.t. jump processes, the integral $\int_a^b$ stands for $\int_{(a,b]}$.

In these terms, we are bound to choose three functions $\mathfrak{f}$, $\mathfrak{g}$ and $\mathfrak{h}$ for which J^π is a submartingale for all $\pi \in \mathcal{A}$, and there exists a $\pi^* \in \mathcal{A}$ such that J^{π^*} is a martingale. In order to calculate $\mathfrak{f}$, $\mathfrak{g}$ and $\mathfrak{h}$, we write J^π as the sum of a (local) martingale M^π and an (not strictly) increasing process K^π that is constant for some $\pi^* \in \mathcal{A}$.
To alleviate the notation we write $\mathfrak{f}(t)$ (resp. $\mathfrak{g}(t)$, $\mathfrak{h}(t)$) for $\mathfrak{f}(t, Y_t, Z_t, U_t)$ (resp. $\mathfrak{g}(t, \mathcal{Y}_t, \mathcal{Z}_t, \mathcal{U}_t)$, $\mathfrak{h}(t, \Upsilon_t, \Xi_t, \Theta_t)$) for $t \in [0, T]$.
Define for each $\pi \in \mathcal{A}$ the process X^π by

$$X_t^\pi \quad := \quad V_{t\wedge\tau}^{x,\pi} - \mathcal{Y}_t \;, \quad t \in [0, T] \;.$$

From Itô's formula, we get

$$dJ_t^\pi \quad = \quad dM_t^\pi + dK_t^\pi \;, \tag{3.11}$$

where M^π and K^π are defined by

$$\begin{aligned} dM_t^\pi \quad := \quad & \Big\{ 2X_{t^-}^\pi (\pi_t\beta_t - \mathcal{U}_t)(Y_{t^-} + U_t) + |\pi_t\beta_t - \mathcal{U}_t|^2 (Y_{t^-} + U_t) + |X_{t^-}^\pi|^2 U_t + \Theta_t \Big\} dM_t \\ & + \Big\{ 2Y_t X_t^\pi (\pi_t\sigma_t - \mathcal{Z}_t) + Z_t |X_t^\pi|^2 + \Xi_t \Big\} dW_t \;, \end{aligned}$$

$$\begin{aligned} dK_t^\pi \quad := \quad & \Big\{ Y_t \big[2X_t^\pi (\pi_t\mu_t + \mathfrak{g}(t)) + |\pi_t\sigma_t - \mathcal{Z}_t|^2 \big] - |X_t^\pi|^2 \mathfrak{f}(t) + 2X_t^\pi Z_t (\pi_t\sigma_t - \mathcal{Z}_t) \\ & + 2\lambda_t^{\mathbb{G}} X_t^\pi U_t (\pi_t\beta_t - \mathcal{U}_t) + \lambda_t^{\mathbb{G}} |\pi_t\beta_t - \mathcal{U}_t|^2 (U_t + Y_t) - \mathfrak{h}(t) \Big\} dt \;. \end{aligned}$$

We then write dK^π in the following form

$$dK_t^\pi \quad = \quad K_t(\pi_t) dt \;,$$

where K is defined by

$$K_t(\pi) \quad := \quad A_t |\pi|^2 + B_t \pi + C_t \;, \quad \pi \in \mathbb{R} \;, \quad t \in [0, T] \;,$$

with

$$\begin{aligned} A_t \quad & := \quad |\sigma_t|^2 Y_t + \lambda_t^{\mathbb{G}} |\beta_t|^2 (U_t + Y_t) \;, \\ B_t \quad & := \quad 2X_t^\pi (\mu_t Y_t + \sigma_t Z_t + \lambda_t^{\mathbb{G}} \beta_t U_t) - 2\sigma_t Y_t \mathcal{Z}_t - 2\lambda_t^{\mathbb{G}} \beta_t \mathcal{U}_t (Y_t + U_t) \;, \\ C_t \quad & := \quad -\mathfrak{f}(t) |X_t^\pi|^2 + 2X_t^\pi (Y_t \mathfrak{g}(t) - Z_t \mathcal{Z}_t - \lambda_t^{\mathbb{G}} U_t \mathcal{U}_t) + Y_t |\mathcal{Z}_t|^2 + \lambda_t^{\mathbb{G}} |\mathcal{U}_t|^2 (U_t + Y_t) - \mathfrak{h}(t) \;, \end{aligned}$$

for all $t \in [0, T]$. To ensure that K^π is nondecreasing for any $\pi \in \mathcal{A}$ and that K^{π^*} is constant for some $\pi^* \in \mathcal{A}$, we take K_t such that $\min_{\pi \in \mathbb{R}} K_t(\pi) = 0$. Using $Y \in \mathcal{S}_{\mathbb{G}}^{\infty,+}$ and **(H**S**)** (ii), we then notice that $A_t > 0$ for all $t \in [0, T]$. Indeed, we have

$$0 \quad = \quad \mathbb{E}[[Y_\tau]^- \mathbb{1}_{\tau \leq T}] \;=\; \mathbb{E}[[Y_{\tau^-} + U_\tau]^- \mathbb{1}_{\tau \leq T}] \;=\; \mathbb{E}\Big[\int_0^T [Y_{s^-} + U_s]^- dN_s \Big] \;,$$

therefore we get that

$$\mathbb{E}\Big[\int_0^T [Y_{s^-} + U_s]^- dM_s + \int_0^T [Y_s + U_s]^- \lambda_s^{\mathbb{G}} ds \Big] \quad = \quad 0 \;.$$

From Remark 3.3, the predictable process $[Y_{.^-}+U]^-$ is bounded. Thus we get that the first integral on the left is a true martingale thus we have

$$\mathbb{E}\Big[\int_0^T [Y_s+U_s]^-\lambda_s^{\mathbb{G}}ds\Big] \quad = \quad 0 \ , \tag{3.12}$$

which gives $(Y_s+U_s)\lambda_s^{\mathbb{G}}\geq 0$ for $s\in[0,T]$. Therefore, the minimum of K_t over $\pi\in\mathbb{R}$ is given by

$$\underline{K}_t \ := \ \min_{\pi\in\mathbb{R}} K_t(\pi) \quad = \quad C_t-\frac{|B_t|^2}{4A_t} \ .$$

We then obtain from the expressions of A, B and C that

$$\underline{K}_t \quad = \quad \mathfrak{A}_t|X_t^\pi|^2+\mathfrak{B}_tX_t^\pi+\mathfrak{C}_t \ ,$$

with

$$\begin{aligned}
\mathfrak{A}_t &:= -\mathfrak{f}(t)-\frac{|\mu_tY_t+\sigma_tZ_t+\lambda_t^{\mathbb{G}}\beta_tU_t|^2}{|\sigma_t|^2Y_t+\lambda_t^{\mathbb{G}}|\beta_t|^2(U_t+Y_t)} \ ,\\
\mathfrak{B}_t &:= 2\Big\{\frac{(\mu_tY_t+\sigma_tZ_t+\lambda_t^{\mathbb{G}}\beta_tU_t)(\lambda_t^{\mathbb{G}}\beta_t\mathcal{U}_t(Y_t+U_t)+\sigma_tY_t\mathcal{Z}_t)}{|\sigma_t|^2Y_t+\lambda_t^{\mathbb{G}}|\beta_t|^2(U_t+Y_t)}+\mathfrak{g}(t)Y_t-Z_t\mathcal{Z}_t-\lambda_t^{\mathbb{G}}U_t\mathcal{U}_t\Big\} \ ,\\
\mathfrak{C}_t &:= -\mathfrak{h}(t)+|\mathcal{Z}_t|^2Y_t+\lambda_t^{\mathbb{G}}(U_t+Y_t)|\mathcal{U}_t|^2-\frac{|\sigma_tY_t\mathcal{Z}_t+\lambda_t^{\mathbb{G}}\beta_t\mathcal{U}_t(U_t+Y_t)|^2}{|\sigma_t|^2Y_t+\lambda_t^{\mathbb{G}}|\beta_t|^2(U_t+Y_t)} \ .
\end{aligned}$$

For that the family $(J^\pi)_{\pi\in\mathcal{A}}$ satisfies the conditions (iii) and (iv) we choose $\mathfrak{f}$, $\mathfrak{g}$ and $\mathfrak{h}$ such that

$$\mathfrak{A}_t \ = \ 0 \ , \ \mathfrak{B}_t \ = \ 0 \quad \text{and} \quad \mathfrak{C}_t \ = \ 0 \ ,$$

for all $t\in[0,T]$. This leads to the following choice for the drivers $\mathfrak{f}$, $\mathfrak{g}$ and $\mathfrak{h}$

$$\left\{\begin{aligned}
\mathfrak{f}(t,y,z,u) &:= -\frac{|\mu_ty+\sigma_tz+\lambda_t^{\mathbb{G}}\beta_tu|^2}{|\sigma_t|^2y+\lambda_t^{\mathbb{G}}|\beta_t|^2(u+y)} \ ,\\
\mathfrak{g}(t,y,z,u) &:= \tfrac{1}{Y_t}\Big[Z_tz+\lambda_t^{\mathbb{G}}U_tu-\frac{(\mu_tY_t+\sigma_tZ_t+\lambda_t^{\mathbb{G}}\beta_tU_t)(\sigma_tY_tz+\lambda_t^{\mathbb{G}}\beta_t(U_t+Y_t)u)}{|\sigma_t|^2Y_t+\lambda_t^{\mathbb{G}}|\beta_t|^2(U_t+Y_t)}\Big] \ ,\\
\mathfrak{h}(t,y,z,u) &:= |\mathcal{Z}_t|^2Y_t+\lambda_t^{\mathbb{G}}(U_t+Y_t)|\mathcal{U}_t|^2-\frac{|\sigma_tY_t\mathcal{Z}_t+\lambda_t^{\mathbb{G}}\beta_t\mathcal{U}_t(U_t+Y_t)|^2}{|\sigma_t|^2Y_t+\lambda_t^{\mathbb{G}}|\beta_t|^2(U_t+Y_t)} \ .
\end{aligned}\right.$$

We then notice that the obtained system of BSDEs is not fully coupled, which allows to study each BSDE alone as soon as we start from the BSDE $(\mathfrak{f},1)$[2] and end with the BSDE $(\mathfrak{h},0)$. However the obtained generators are nonstandard since they involve the jump component and they are not Lipschitz continuous. Moreover, these generators are not defined on the whole space $\mathbb{R}\times\mathbb{R}\times\mathbb{R}$. Using a decomposition approach based on Proposition 2.1, we obtain the following result whose proof is detailed in Section 4.

Theorem 3.1. *The BSDEs* (3.7), (3.8) *and* (3.9) *admit solutions* (Y,Z,U), $(\mathcal{Y},\mathcal{Z},\mathcal{U})$ *and* (Υ,Ξ,Θ) *in* $\mathcal{S}_{\mathbb{G}}^\infty\times L_{\mathbb{G}}^2\times L^2(\lambda)$. *Moreover* $Y\in\mathcal{S}_{\mathbb{G}}^{\infty,+}$.

[2] The notation BSDE (f,H) holds for the BSDE with generator f and terminal condition H.

3.3 A verification Theorem

We now turn to the sufficient condition of optimality. As explained in Subsection 3.1, a candidate to be an optimal strategy is a process $\pi^* \in \mathcal{A}$ such that J^{π^*} is a martingale, which implies that $dK^{\pi^*} = 0$. This leads to

$$\pi^*_t \quad = \quad \arg\min_{\pi\in\mathbb{R}} K_t(\pi) \ ,$$

which gives the implicit equation in π^*

$$\pi^*_t \quad = \quad \big(\mathcal{Y}_{t^-} - V^{x,\pi^*}_{t^-}\big)\frac{\mu_t Y_{t^-} + \sigma_t Z_t + \lambda^{\mathbb{G}}_t\beta_t U_t}{|\sigma_t|^2 Y_{t^-} + \lambda^{\mathbb{G}}_t|\beta_t|^2(U_t + Y_{t^-})} + \frac{\sigma_t Y_{t^-}\mathcal{Z}_t + \lambda^{\mathbb{G}}_t\beta_t\mathcal{U}_t(Y_{t^-} + U_t)}{|\sigma_t|^2 Y_{t^-} + \lambda^{\mathbb{G}}_t|\beta_t|^2(U_t + Y_{t^-})} \ .$$

Integrating each side of this equality w.r.t. $\frac{dS_t}{S_{t^-}}$ leads to the following SDE

$$\begin{aligned} V^*_t \quad &= \quad x + \int_0^t \big(\mathcal{Y}_{r^-} - V^*_{r^-}\big)\frac{\mu_r Y_{r^-} + \sigma_r Z_r + \lambda^{\mathbb{G}}_r\beta_r U_r}{|\sigma_r|^2 Y_{r^-} + \lambda^{\mathbb{G}}_r|\beta_r|^2(U_r + Y_{r^-})}\frac{dS_r}{S_{r^-}} \\ &\quad + \int_0^t \frac{\sigma_r Y_{r^-}\mathcal{Z}_r + \lambda^{\mathbb{G}}_r\beta_r\mathcal{U}_r(Y_{r^-} + U_r)}{|\sigma_r|^2 Y_{r^-} + \lambda^{\mathbb{G}}_r|\beta_r|^2(U_r + Y_{r^-})}\frac{dS_r}{S_{r^-}} \ , \quad t \in [0, T\wedge\tau] \ . \end{aligned} \tag{3.13}$$

We first study the existence of a solution to SDE (3.13).

Proposition 3.2. *The SDE* (3.13) *admits a solution V^* which satisfies*

$$\mathbb{E}\Big[\sup_{t\in[0,T\wedge\tau]} |V^*_t|^2\Big] \quad < \quad \infty \ . \tag{3.14}$$

Proof. To alleviate the notation we rewrite (3.13) under the form

$$\begin{cases} V^*_0 &= \quad x \ , \\ dV^*_t &= \quad (E_t V^*_{t^-} - F_t)(\mu_t dt + \sigma_t dW_t + \beta_t dM_t) \ , \end{cases} \tag{3.15}$$

where E and F are defined by

$$\begin{aligned} E_t \quad &:= \quad -\frac{\mu_t Y_{t^-} + \sigma_t Z_t + \lambda^{\mathbb{G}}_t\beta_t U_t}{|\sigma_t|^2 Y_{t^-} + \lambda^{\mathbb{G}}_t|\beta_t|^2(U_t + Y_{t^-})} \ , \\ F_t \quad &:= \quad -\frac{\lambda^{\mathbb{G}}_t\beta_t\mathcal{U}_t(Y_{t^-} + U_t) + \mu_t Y_{t^-}\mathcal{Y}_{t^-} + \lambda^{\mathbb{G}}_t\beta_t U_t\mathcal{Y}_{t^-} + \sigma_t Z_t\mathcal{Y}_{t^-} + \sigma_t\mathcal{Z}_t Y_{t^-}}{|\sigma_t|^2 Y_{t^-} + \lambda^{\mathbb{G}}_t|\beta_t|^2(U_t + Y_{t^-})} \ , \end{aligned}$$

for all $t \in [0, T]$. We first notice that from **(H**S**)** (ii), and since $Y \in \mathcal{S}^{\infty,+}_{\mathbb{G}}$ and $\lambda^{\mathbb{G}}(Y + U)$ is nonnegative, there exists a constant $C > 0$ such that

$$|\sigma_t|^2 Y_t + \lambda^{\mathbb{G}}_t|\beta_t|^2(U_t + Y_{t^-}) \quad \geq \quad C \ , \quad \mathbb{P}\otimes dt - a.e.$$

Therefore, using (Y, Z, U), $(\mathcal{Y}, \mathcal{Z}, \mathcal{U})$, $(\Upsilon, \Xi, \Theta) \in \mathcal{S}^\infty_{\mathbb{G}} \times L^2_{\mathbb{G}} \times L^2(\lambda)$, Remark 3.3 and **(H**$S$**)**, we get that E and F are square integrable

$$\mathbb{E}\Big[\int_0^T \Big(|E_t|^2 + |F_t|^2\Big)dt\Big] \quad < \quad \infty \ .$$

Using Itô's formula, we obtain that the process V^* defined by

$$\begin{aligned} V_t^* &:= (x+\Psi_t)\Phi_t\ , \quad t\in[0,T\wedge\tau)\ , \\ \text{and}\ \ V_{T\wedge\tau}^* &= \mathbb{1}_{\tau\leq T}\Big[(1+E_\tau\beta_\tau)V_{\tau^-}^* - F_\tau\beta_\tau\Big] + \mathbb{1}_{\tau>T}(x+\Psi_T)\Phi_T\ , \end{aligned} \tag{3.16}$$

where

$$\Phi_t \ := \ \exp\Big(\int_0^t\Big(E_s(\mu_s-\lambda_s^{\mathbb{G}}\beta_s)-\frac{1}{2}|\sigma_sE_s|^2\Big)ds+\int_0^t\sigma_sE_sdW_s\Big)\ ,$$

and

$$\Psi_t \ := \ -\int_0^t\frac{F_s}{\Phi_s}\Big[\mu_s-\lambda_s^{\mathbb{G}}\beta_s-|E_s\sigma_s|^2\Big]ds-\int_0^t\frac{F_s}{\Phi_s}\sigma_sdW_s\ ,$$

for all $t\in[0,T]$, is solution to (3.13).
We now prove that V^* defined by (3.16) satisfies (3.14). We proceed in two steps.

Step 1: We prove that

$$\mathbb{E}\Big[|V_{T\wedge\tau}^*|^2\Big] \ < \ \infty\ . \tag{3.17}$$

Since V^* satisfies (3.15), we have $V^*=V^{x,\pi^*}$ where π^* is given by

$$\pi_t^* \ = \ E_tV_{t^-}^*-F_t\ , \quad t\in[0,T]\ .$$

We therefore have $Y|V_{.\wedge\tau}^*-\mathcal{Y}|^2=J^{\pi^*}-\Upsilon$ and from (3.11) and the dynamics of Υ given by (3.9), we have

$$d\big(Y_t|V_{t\wedge\tau}^*-\mathcal{Y}_t|^2\big) \ = \ dM_t^*+dK_t^{\pi^*}-\mathfrak{h}(t)dt$$

where M^* is a locally square integrable martingale with $M_0^*=0$. From the definition of K^{π^*} and using the fact that

$$\pi_t^* \ = \ \frac{X_{t^-}^{\pi^*}(\mu_tY_{t^-}+\sigma_tZ_t+\lambda_t^{\mathbb{G}}\beta_tU_t)+\sigma_tY_{t^-}\mathcal{Z}_t+\lambda_t^{\mathbb{G}}\beta_t\mathcal{U}_t(Y_{t^-}+U_t)}{|\sigma_t|^2Y_{t^-}+\lambda_t^{\mathbb{G}}|\beta_t|^2(U_t+Y_{t^-})}\ ,$$

we get $K_t^{\pi^*}=0$ for all $t\in[0,T\wedge\tau]$. Therefore, from the definition of $\mathfrak{h}$, we get

$$\begin{aligned} Y_{T\wedge\tau}|V_{T\wedge\tau}^*-\mathcal{Y}_{T\wedge\tau}|^2 \ &= \ Y_0|x-\mathcal{Y}_0|^2+M_{T\wedge\tau}^*+\int_0^{T\wedge\tau}\Big[|\mathcal{Z}_t|^2Y_t+\lambda_t^{\mathbb{G}}(U_t+Y_t)|\mathcal{U}_t|^2 \\ &\quad -\frac{|\sigma_tY_t\mathcal{Z}_t+\lambda_t^{\mathbb{G}}\beta_t\mathcal{U}_t(U_t+Y_t)|^2}{|\sigma_t|^2Y_t+\lambda_t^{\mathbb{G}}|\beta_t|^2(U_t+Y_t)}\Big]dt\ . \end{aligned}$$

Since M^* is a local martingale, there exists an increasing sequence of $\mathbb{G}$-stopping times $(\nu_i)_{i\in\mathbb{N}}$ such that $\nu_i\to+\infty$ as $i\to\infty$ and

$$\begin{aligned} \mathbb{E}\big[Y_{T\wedge\tau\wedge\nu_i}|V_{T\wedge\tau\wedge\nu_i}^*-\mathcal{Y}_{T\wedge\tau\wedge\nu_i}|^2\big] \ &= \ Y_0|x-\mathcal{Y}_0|^2+\mathbb{E}\int_0^{T\wedge\tau\wedge\nu_i}\Big[|\mathcal{Z}_t|^2Y_t+\lambda_t^{\mathbb{G}}(U_t+Y_t)|\mathcal{U}_t|^2 \\ &\quad -\frac{|\sigma_tY_t\mathcal{Z}_t+\lambda_t^{\mathbb{G}}\beta_t\mathcal{U}_t(U_t+Y_t)|^2}{|\sigma_t|^2Y_t+\lambda_t^{\mathbb{G}}|\beta_t|^2(U_t+Y_t)}\Big]dt\ . \end{aligned} \tag{3.18}$$

Since $Y \in \mathcal{S}^{\infty,+}_{\mathbb{G}}$, there exists a positive constant C such that

$$\mathbb{E}\big[|V^*_{T\wedge\tau\wedge\nu_i} - \mathcal{Y}_{T\wedge\tau\wedge\nu_i}|^2\big] \;\leq\; C\mathbb{E}\big[Y_{T\wedge\tau\wedge\nu_i}|V^*_{T\wedge\tau\wedge\nu_i} - \mathcal{Y}_{T\wedge\tau\wedge\nu_i}|^2\big]\;.$$

Therefore, using (3.18), we get that

$$\mathbb{E}\big[|V^*_{T\wedge\tau\wedge\nu_i} - \mathcal{Y}_{T\wedge\tau\wedge\nu_i}|^2\big] \;\leq\; C\Big(Y_0|x-\mathcal{Y}_0|^2 + \mathbb{E}\int_0^T \Big[|\mathcal{Z}_t|^2 Y_t + \lambda^{\mathbb{G}}_t(U_t+Y_t)|\mathcal{U}_t|^2\Big]dt\Big)\;.$$

Since Y, U and $\mathcal{U}$ are uniformly bounded and $\mathcal{Z} \in L^2_{\mathbb{G}}$, there exists a constant C such that

$$\mathbb{E}\big[\big|V^*_{T\wedge\tau\wedge\nu_i} - \mathcal{Y}_{T\wedge\tau\wedge\nu_i}\big|^2\big] \;\leq\; C\;.$$

From Fatou's lemma, we get that

$$\mathbb{E}\big[\big|V^*_{T\wedge\tau} - \mathcal{Y}_{T\wedge\tau}\big|^2\big] \;\leq\; \liminf_{i\to\infty}\mathbb{E}\big[\big|V^*_{T\wedge\tau\wedge\nu_i} - \mathcal{Y}_{T\wedge\tau\wedge\nu_i}\big|^2\big] \;\leq\; C\;.$$

Which implies that

$$\mathbb{E}\big[\big|V^*_{T\wedge\tau}\big|^2\big] \;\leq\; C + 2\mathbb{E}\big[V^*_{T\wedge\tau}\mathcal{Y}_{T\wedge\tau}\big]\;.$$

Finally, using the Young inequality and noting that $\mathcal{Y}$ is uniformly bounded, it follows that there exists a constant C such that

$$\mathbb{E}\big[\big|V^*_{T\wedge\tau}\big|^2\big] \;\leq\; C\;.$$

Step 2: We prove that

$$\mathbb{E}\Big[\sup_{t\in[0,T\wedge\tau]}|V^*_t|^2\Big] \;<\; \infty\;.$$

For that we remark that $V^*_{.\wedge\tau}$ is solution to the following linear BSDE

$$V^*_{t\wedge\tau} \;=\; V^*_{T\wedge\tau} - \int_{t\wedge\tau}^{T\wedge\tau}\frac{\mu_s}{\sigma_s}z_s ds - \int_{t\wedge\tau}^{T\wedge\tau} z_s dW_s - \int_{t\wedge\tau}^{T\wedge\tau} u_s dM_s\;, \quad t\in[0,T]\;, \tag{3.19}$$

with

$$z_t \;:=\; \sigma_t\frac{(\mathcal{Y}_{t^-} - V^*_{t^-})(\mu_t Y_{t^-} + \sigma_t Z_t + \lambda^{\mathbb{G}}_t\beta_t U_t) + \sigma_t Y_{t^-}\mathcal{Z}_t + \lambda^{\mathbb{G}}_t\beta_t\mathcal{U}_t(Y_{t^-}+U_t)}{|\sigma_t|^2 Y_{t^-} + \lambda^{\mathbb{G}}_t|\beta_t|^2(U_t+Y_{t^-})}\;,$$

$$u_t \;:=\; \beta_t\frac{(\mathcal{Y}_{t^-} - V^*_{t^-})(\mu_t Y_{t^-} + \sigma_t Z_t + \lambda^{\mathbb{G}}_t\beta_t U_t) + \sigma_t Y_{t^-}\mathcal{Z}_t + \lambda^{\mathbb{G}}_t\beta_t\mathcal{U}_t(Y_{t^-}+U_t)}{|\sigma_t|^2 Y_{t^-} + \lambda^{\mathbb{G}}_t|\beta_t|^2(U_t+Y_{t^-})}\;,$$

for all $t\in[0,T]$. Applying Itô's formula to $|V^*|^2$ we have

$$\mathbb{E}|V^*_{t\wedge\tau}|^2 \;=\; \mathbb{E}|V^*_{T\wedge\tau}|^2 - 2\mathbb{E}\int_{t\wedge\tau}^{T\wedge\tau} V^*_{s\wedge\tau}\frac{\mu_s}{\sigma_s}z_s ds - \mathbb{E}\int_{t\wedge\tau}^{T\wedge\tau}|z_s|^2 ds - \mathbb{E}\int_{t\wedge\tau}^{T\wedge\tau}|u_s|^2\lambda_s ds\;,$$

for all $t\in[0,T]$. Using (3.17), **(H**S**)** and the Young inequality we obtain the existence of a constant C such that

$$\mathbb{E}|V^*_{t\wedge\tau}|^2 + \mathbb{E}\int_{t\wedge\tau}^{T\wedge\tau}|z_s|^2 ds + \mathbb{E}\int_{t\wedge\tau}^{T\wedge\tau}|u_s|^2\lambda_s ds \;\leq\; C\Big(1+\mathbb{E}\int_t^T|V^*_{s\wedge\tau}|^2\Big)\;.$$

We then deduce from the Gronwall inequality that

$$\sup_{t\in[0,T]} \mathbb{E}|V^*_{t\wedge\tau}|^2 + \mathbb{E}\int_0^{T\wedge\tau} |z_s|^2 ds + \mathbb{E}\int_0^{T\wedge\tau} |u_s|^2\lambda_s ds \quad < \quad +\infty\ . \tag{3.20}$$

Now from (3.19), we have

$$\begin{aligned}\mathbb{E}\Big[\sup_{t\in[0,T]} |V^*_{t\wedge\tau}|^2\Big] \quad &\leq \quad 3\Big(|V^*_0|^2 + \mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\int_0^{t\wedge\tau}\frac{\mu_s}{\sigma_s}z_s ds\Big|^2\Big]\\ &\quad +\mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\int_0^{t\wedge\tau} z_s dW_s + \int_0^{t\wedge\tau} u_s dM_s\Big|^2\Big]\Big)\ .\end{aligned}$$

From **(H**S**)** and the BDG inequality, there exists a constant C such that

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |V^*_{t\wedge\tau}|^2\Big] \quad \leq \quad C\Big(1 + \mathbb{E}\int_0^{T\wedge\tau} |z_s|^2 ds + \mathbb{E}\int_0^{T\wedge\tau} |u_s|^2\lambda_s ds\Big)\ .$$

This last inequality with (3.20) gives (3.14). □

As explained previously, we now consider the strategy π^* defined by

$$\pi^*_t \quad = \quad \frac{(\mathcal{Y}_{t^-} - V^*_{t^-})(\mu_t Y_{t^-} + \sigma_t Z_t + \lambda^{\mathbb{G}}_t\beta_t U_t) + \sigma_t Y_{t^-}\mathcal{Z}_t + \lambda^{\mathbb{G}}_t\beta_t\mathcal{U}_t(Y_{t^-} + U_t)}{|\sigma_t|^2 Y_{t^-} + \lambda^{\mathbb{G}}_t|\beta_t|^2(U_t + Y_{t^-})}\ , \tag{3.21}$$

for all $t \in [0,T]$. We first notice from the expressions of π^* and V^* that

$$V^{x,\pi^*}_t \quad = \quad V^*_t\ , \tag{3.22}$$

for all $t \in [0,T]$. Using (3.14) and (3.22), we have

$$\mathbb{E}\Big[\sup_{t\in[0,T\wedge\tau]} |V^{x,\pi^*}_t|^2\Big] \quad < \quad \infty\ . \tag{3.23}$$

We can now state our verification theorem which is the main result of this section.

Theorem 3.2. *The strategy* π^* *given by* (3.21) *belongs to the set* $\mathcal{A}$ *and is optimal for the mean-variance problem* (2.2)*. Thus we have*

$$\mathbb{E}\Big[\big|V^{x,\pi^*}_{T\wedge\tau} - H\big|^2\Big] \quad = \quad \min_{\pi\in\mathcal{A}}\mathbb{E}\Big[\big|V^{x,\pi}_{T\wedge\tau} - H\big|^2\Big] \quad = \quad Y_0|x - \mathcal{Y}_0|^2 + \Upsilon_0\ ,$$

where $Y, \mathcal{Y}$ *and* Υ *are solutions to* (3.7)-(3.8)-(3.9).

To prove this verification theorem, we first need of the following lemma.

Lemma 3.1. *For any* $\pi \in \mathcal{A}$*, the process* $M^\pi_{.\wedge\tau}$ *defined by* (3.11) *is a* $\mathbb{G}$*-local martingale.*

Proof. Fix $\pi \in \mathcal{A}$. Then from the definition of $V^{x,\pi}$, **(H**S**)** and the BDG inequality, we have

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \big|V^{x,\pi}_{t\wedge\tau}\big|^2\Big] \quad < \quad \infty\ . \tag{3.24}$$

Define the sequence of $\mathbb{G}$-stopping times $(\nu_n)_{n\geq 1}$ by

$$\nu_n \quad := \quad \inf\Big\{s \geq 0 \ : \ \big|V^{x,\pi}_{s\wedge\tau}\big| \geq n\Big\} ,$$

for all $n \geq 1$. First, notice that $(\nu_n)_{n\geq 1}$ is nondecreasing and goes to infinity as n goes to infinity from (3.24). Moreover, from the definition of ν_n, we have

$$|V^{x,\pi}_s \mathbb{1}_{s\in[0,\nu_n\wedge\tau)}| \quad \leq \quad n$$

for all $s \in [0,T]$. Then, since $\pi \in \mathcal{A}$, $Y, \mathcal{Y} \in \mathcal{S}^\infty_{\mathbb{G}}$ and $Z, \mathcal{Z}, \Xi \in L^2_{\mathbb{G}}$, we get

$$\mathbb{E}\Big[\int_0^{\tau\wedge\nu_n\wedge T} \Big|2Y_t X^\pi_t(\pi_t\sigma_t - \mathcal{Z}_t) + Z_t|X^\pi_t|^2 + \Xi_t\Big|^2 dt\Big] \quad < \quad \infty ,$$

for all $n \geq 1$. Moreover, since $U, \mathcal{U}, \Theta \in L^2(\lambda)$, we get from Remark 3.3

$$\mathbb{E}\Big[\int_0^{\tau\wedge\nu_n\wedge T} \Big|(2X^\pi_{t^-} + \pi_t\beta_t - \mathcal{U}_t)(\pi_t\beta_t - \mathcal{U}_t)(Y_{t^-} + U_t) + |X^\pi_{t^-}|^2 U_t + \Theta_t\Big|\lambda^{\mathbb{G}}_t dt\Big] \quad < \quad \infty ,$$

for all $n \geq 1$. Therefore, we get that the stopped process $M^\pi_{.\wedge\tau\wedge\nu_n}$ is a $\mathbb{G}$-martingale. □

Proof of Theorem 3.2. As explained in Subsection 3.1, we check each of the points (i), (ii), (iii) and (iv).
(i) From the definition of Y, $\mathcal{Y}$ and Υ, we have

$$J^\pi_{T\wedge\tau} \quad = \quad Y_{T\wedge\tau}\big|V^{x,\pi}_{T\wedge\tau} - H\big|^2 + \Upsilon_{T\wedge\tau} \quad = \quad \big|V^{x,\pi}_{T\wedge\tau} - H\big|^2 ,$$

for all $\pi \in \mathcal{A}$.

(ii) From the definition of the family $(J^\pi)_{\pi\in\mathcal{A}}$, we have

$$J^\pi_0 \quad = \quad Y_0|V^{x,\pi}_0 - \mathcal{Y}_0|^2 + \Upsilon_0 \quad = \quad Y_0|x - \mathcal{Y}_0|^2 + \Upsilon_0 ,$$

for all $\pi \in \mathcal{A}$.

(iii) Fix $\pi \in \mathcal{A}$. Since $Y, \mathcal{Y}, \Upsilon \in \mathcal{S}^\infty_{\mathbb{G}}$, we have from the definition of J^π and the BDG inequality

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |J^\pi_t|\Big] \quad < \quad +\infty . \tag{3.25}$$

Now, fix $s, t \in [0,T]$ such that $s \leq t$. Using the decomposition (3.11) and Lemma 3.1, there exists an increasing sequence of $\mathbb{G}$-stopping times $(\nu_i)_{i\geq 1}$ such that $\nu_i \to +\infty$ as $i \to +\infty$ and

$$\mathbb{E}\Big[J^\pi_{t\wedge\nu_i}\big|\mathcal{G}_s\Big] \quad \geq \quad J^\pi_{s\wedge\nu_i} , \tag{3.26}$$

for all $i \geq 1$. Then, from (3.25), we can apply the conditional dominated convergence theorem and we get by sending i to ∞ in (3.26)

$$\mathbb{E}\Big[J^\pi_t\big|\mathcal{G}_s\Big] \quad \geq \quad J^\pi_s ,$$

for all $s,t \in [0,T]$ with $s \leq t$.

(iv) We now check that $\pi^* \in \mathcal{A}$ i.e. $\mathbb{E}\int_0^{T\wedge\tau} |\pi_s^*|^2 ds < \infty$. Using the definition of π^* and (3.22) we have that V^{x,π^*} is solution to the linear BSDE

$$V_t^{x,\pi*} \;=\; V_{T\wedge\tau}^{x,\pi^*} - \int_{t\wedge\tau}^{T\wedge\tau} \frac{\mu_s}{\sigma_s} z_s ds - \int_{t\wedge\tau}^{T\wedge\tau} z_s dW_s - \int_{t\wedge\tau}^{T\wedge\tau} u_s dM_s \;, \quad t \in [0,T] \;,$$

with

$$z_t \;=\; \sigma_t \pi_t^* \quad \text{and} \quad u_t \;=\; \beta_t \pi_t^* \;,$$

for all $t \in [0,T]$. Therefore, using (3.23), $(\mathbf{H}S)$, applying Itô's formula to $|V^{x,\pi*}|^2$, using the Young inequality, the BDG inequality and the Gronwall inequality (see e.g. the proof of Proposition 2.2 in [3]), we get

$$\mathbb{E}\Big[\int_0^{T\wedge\tau} |\pi_s^*|^2 ds\Big] \;<\; \infty \;.$$

We now check that J^{π^*} is a $\mathbb{G}$-martingale. Since K^{π^*} is constant, we obtain from Lemma 3.1 that J^{π^*} is a $\mathbb{G}$-local martingale. Then, from the expression of J^{π^*} and since $Y, \mathcal{Y}, \Upsilon \in \mathcal{S}_{\mathbb{G}}^\infty$, there exists a constant C such that

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |J_t^{\pi^*}|\Big] \;\leq\; C\Big(1 + \mathbb{E}\Big[\sup_{t\in[0,T\wedge\tau]} |V_t^{x,\pi^*}|^2\Big]\Big) \;.$$

Using (3.23), we get that

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |J_t^{\pi^*}|\Big] \;<\; +\infty \;.$$

Therefore, J^{π^*} is a true $\mathbb{G}$-martingale and π^* is optimal. □

4 A decomposition approach for solving BSDEs in the filtration $\mathbb{G}$

We now prove Theorem 3.1 via a decomposition procedure. We first provide a general result which gives existence of a solution to a BSDE in the enlarged filtration $\mathbb{G}$ as soon as an associated BSDE in the filtration $\mathbb{F}$ admits a solution. Actually the associated BSDE is defined by the terms appearing in the decomposition of the coefficients of the BSDE in $\mathbb{G}$ given by Proposition 2.1. We therefore introduce the spaces of processes where solutions in $\mathbb{F}$ classically lie.

– $\mathcal{S}_{\mathbb{F}}^\infty$ is the subset of $\mathbb{R}$-valued continuous $\mathbb{F}$-adapted processes $(Y_t)_{t\in[0,T]}$ essentially bounded

$$\|Y\|_{\mathcal{S}^\infty} \;:=\; \Big\|\sup_{t\in[0,T]} |Y_t|\Big\|_\infty \;<\; \infty \;.$$

– $\mathcal{S}^{\infty,+}_{\mathbb{F}}$ is the subset of $\mathcal{S}^{\infty}_{\mathbb{F}}$ of processes $(Y_t)_{t\in[0,T]}$ valued in $(0,\infty)$, such that

$$\Big\|\frac{1}{Y}\Big\|_{\mathcal{S}^\infty} \;<\; \infty\,.$$

– $L^2_{\mathbb{F}}$ is the subset of $\mathbb{R}$-valued $\mathcal{P}(\mathbb{F})$-measurable processes $(Z_t)_{t\in[0,T]}$ such that

$$\|Z\|_{L^2} \;:=\; \Big(\mathbb{E}\Big[\int_0^T |Z_t|^2dt\Big]\Big)^{\frac{1}{2}} \;<\; \infty\,.$$

Finally since the BSDEs associated to our mean-variance problem have generators with superlinear growth, we consider the additional space of BMO-martingales: BMO($\mathbb{P}$) is the subset of $(\mathbb{P},\mathbb{F})$-martingales m such that

$$\|m\|_{\mathrm{BMO}(\mathbb{P})} \;:=\; \sup_{\nu\in\mathcal{T}_{\mathbb{F}}[0,T]}\Big\|\mathbb{E}\big[\langle m\rangle_T-\langle m\rangle_\nu|\mathcal{F}_\nu\big]^{\frac{1}{2}}\Big\|_\infty \;<\; \infty\,,$$

where $\mathcal{T}_{\mathbb{F}}[0,T]$ is the set of $\mathbb{F}$-stopping times on $[0,T]$. This means local martingales of the form $m_t=\int_0^t Z_sdW_s$, $t\in[0,T]$, are BMO($\mathbb{P}$)-martingale if and only if

$$\Big\|\int_0^{\cdot} Z_sdW_s\Big\|_{\mathrm{BMO}(\mathbb{P})} \;:=\; \sup_{\nu\in\mathcal{T}_{\mathbb{F}}[0,T]}\Big\|\Big(\mathbb{E}\Big[\int_\nu^T |Z_t|^2dt\Big|\mathcal{F}_\nu\Big]\Big)^{\frac{1}{2}}\Big\|_\infty \;<\; \infty\,.$$

4.1 A general existence theorem for BSDEs with random horizon

We provide here a general result on existence of a solution to a BSDE driven by W and N with horizon $T\wedge\tau$. We consider a generator function $F:\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, which is $\mathcal{P}(\mathbb{G})\otimes\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R})$-measurable, and a terminal condition ξ which is a $\mathcal{G}_{T\wedge\tau}$-measurable random variable of the form

$$\xi \;=\; \xi^b\mathbb{1}_{T<\tau}+\xi^a_\tau\mathbb{1}_{T\geq\tau}\,, \tag{4.27}$$

where ξ^b is an $\mathcal{F}_T$-measurable bounded random variable and $\xi^a\in\mathcal{S}^\infty_{\mathbb{F}}$. From Proposition 2.1 and Remark 2.1, we can write

$$F(t,.)\mathbb{1}_{t\leq\tau} \;=\; F^b(t,.)\mathbb{1}_{t\leq\tau}\,,\quad t\geq 0\,, \tag{4.28}$$

where F^b is a $\mathcal{P}(\mathbb{F})\otimes\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R})$-measurable map. We then introduce the following BSDE

$$Y^b_t \;=\; \xi^b+\int_t^T F^b(s,Y^b_s,Z^b_s,\xi^a_s-Y^b_s)ds-\int_t^T Z^b_sdW_s\,,\quad t\in[0,T]\,. \tag{4.29}$$

Theorem 4.3. *Assume that the BSDE* (4.29) *admits a solution* $(Y^b,Z^b)\in\mathcal{S}^\infty_{\mathbb{F}}\times L^2_{\mathbb{F}}$. *Then BSDE*

$$Y_t \;=\; \xi+\int_{t\wedge\tau}^{T\wedge\tau} F(s,Y_s,Z_s,U_s)ds-\int_{t\wedge\tau}^{T\wedge\tau} Z_sdW_s-\int_{t\wedge\tau}^{T\wedge\tau} U_sdN_s\,,\quad t\in[0,T]\,, \tag{4.30}$$

admits a solution $(Y,Z,U)\in\mathcal{S}^\infty_{\mathbb{G}}\times L^2_{\mathbb{G}}\times L^2(\lambda)$ *given by*

$$\begin{aligned} Y_t &= Y^b_t\mathbb{1}_{t<\tau}+\xi^a_\tau\mathbb{1}_{t\geq\tau}\ ,\\ Z_t &= Z^b_t\mathbb{1}_{t\leq\tau}\ ,\\ U_t &= \big(\xi^a_t-Y^b_t\big)\mathbb{1}_{t\leq\tau}\ , \end{aligned}\tag{4.31}$$

for all $t\in[0,T]$.

Proof. We proceed in three steps.
Step 1: We prove that for $t\in[0,T]$, (Y,Z,U) defined by (4.31) satisfies the equation (4.30). We distinguish three cases.
Case 1: $\tau>T$.
From (4.31), we get $Y_t=Y^b_t$, $Z_t=Z^b_t$ and $U_t=\xi^a_t-Y^b_t$ for all $t\in[0,T]$. Then, using that (Y^b,Z^b) is a solution to (4.29), we have

$$Y_t \;=\; \xi^b+\int_t^T F^b(s,Y_s,Z_s,U_s)ds-\int_t^T Z^b_sdW_s\ .$$

Since the predictable processes Z and Z^b are indistinguishable on $\{\tau>T\}$, we have from Theorem 12.23 of [11], $\int_t^T Z_sdW_s=\int_t^T Z^b_sdW_s$ on $\{\tau>T\}$. Moreover since $\xi=\xi^b$ and $\int_{t\wedge\tau}^{T\wedge\tau}U_sdN_s=0$ on $\{\tau>T\}$ we get by using (4.28)

$$Y_t \;=\; \xi+\int_{t\wedge\tau}^{T\wedge\tau}F(s,Y_s,Z_s,U_s)ds-\int_{t\wedge\tau}^{T\wedge\tau}Z_sdW_s-\int_{t\wedge\tau}^{T\wedge\tau}U_sdN_s\ .$$

Case 2: $\tau\in(t,T]$.
From (4.31), we have $Y_t=Y^b_t$. Since (Y^b,Z^b) is solution to (4.29), we have

$$Y_t \;=\; Y^b_\tau+\int_t^\tau F^b(s,Y^b_s,Z^b_s,\xi^a_s-Y^b_s)ds-\int_t^\tau Z^b_sdW_s\ .$$

Still using (4.28) and (4.31), we get

$$Y_t \;=\; \xi^a_\tau+\int_t^\tau F(s,Y_s,Z_s,U_s)ds-\int_t^\tau Z^b_sdW_s-(\xi^a_\tau-Y^b_\tau)\ .$$

Since the predictable processes $Z\mathbb{1}_{.<\tau}$ and $Z^b\mathbb{1}_{.<\tau}$ are indistinguishable on $\{\tau>t\}\cap\{\tau\leq T\}$, we have from Theorem 12.23 of [11], $\int_t^{T\wedge\tau}Z_sdW_s=\int_t^{T\wedge\tau}Z^b_sdW_s$ on $\{\tau>t\}\cap\{\tau\leq T\}$. Therefore, we get

$$Y_t \;=\; \xi^a_\tau+\int_t^\tau F(s,Y_s,Z_s,U_s)ds-\int_t^\tau Z_sdW_s-(\xi^a_\tau-Y^b_\tau)\ .$$

Finally, we easily check from the definition of U that $\int_t^{T\wedge\tau}U_sdN_s=\xi^a_\tau-Y^b_\tau$. Therefore, we get using (4.27)

$$Y_t \;=\; \xi+\int_{t\wedge\tau}^{T\wedge\tau}F(s,Y_s,Z_s,U_s)ds-\int_{t\wedge\tau}^{T\wedge\tau}Z_sdW_s-\int_{t\wedge\tau}^{T\wedge\tau}U_sdN_s\ .$$

Case 3: $\tau\leq t$.

Then, from (4.31), we have $Y_t = \xi^a_\tau$. We therefore get on $\{\tau \leq t\}$ by using (4.27)

$$Y_t \;=\; \xi + \int_{t\wedge\tau}^{T\wedge\tau} F(s, Y_s, Z_s, U_s)ds - \int_{t\wedge\tau}^{T\wedge\tau} Z_s dW_s - \int_{t\wedge\tau}^{T\wedge\tau} U_s dN_s \;.$$

Step 2: We notice that Y is a càd-làg $\mathbb{G}$-adapted process and U is $\mathcal{P}(\mathbb{G})$-measurable since Y^b and ξ^a are continuous and $\mathbb{F}$-adapted. We also notice from its definition that the process Z is $\mathcal{P}(\mathbb{G})$-measurable, since Z^b is $\mathcal{P}(\mathbb{F})$-measurable.

Step 3: We now prove that the solution satisfies the integrability conditions.

– From the definition of Y, we have

$$|Y_t| \;\leq\; |Y^b_t| + |\xi^a_t| \;, \quad t \in [0, T] \;. \tag{4.32}$$

Since $Y^b \in \mathcal{S}^\infty_{\mathbb{F}}$ and $\xi^a \in \mathcal{S}^\infty_{\mathbb{F}}$, we get that $\|Y\|_{\mathcal{S}^\infty} < +\infty$.

– From the definition of the process Z, we have $Z \in L^2_{\mathbb{G}}$.

– From the definition of U, we have

$$|U_t| \;\leq\; |Y^b_t| + |\xi^a_t| \;, \quad t \in [0, T] \;.$$

Since $Y^b \in \mathcal{S}^\infty_{\mathbb{F}}$, $\xi^a \in \mathcal{S}^\infty_{\mathbb{F}}$ and λ is bounded, we get $U \in L^2(\lambda)$.

□

Using this abstract result we prove the existence of solutions to each of the BSDEs (3.7), (3.8) and (3.9) in the following subsections.

4.2 Solution to the BSDE $(\mathfrak{f}, 1)$

Following Theorem 4.3, we consider for coefficients $(\mathfrak{f}, 1)$ the BSDE in $\mathbb{F}$: find $(Y^b, Z^b) \in \mathcal{S}^\infty_{\mathbb{F}} \times L^2_{\mathbb{F}}$ such that

$$\left\{\begin{array}{rcl} dY^b_t &=& \Big\{\dfrac{|(\mu_t - \lambda_t\beta_t)Y^b_t + \sigma_t Z^b_t + \lambda_t\beta_t|^2}{|\sigma_t|^2 Y^b_t + \lambda_t|\beta_t|^2} - \lambda_t + \lambda_t Y^b_t\Big\}dt + Z^b_t dW_t \;, \quad t \in [0, T] \;, \\ Y^b_T &=& 1 \;. \end{array}\right. \tag{4.33}$$

To solve this BSDE, we have to deal with two main issues. The first is that the generator $\mathfrak{f}$ has a superlinear growth. The second difficulty is that the generator value is not defined for all the values that the process Y can take. In particular the generator may explode if the process Y goes to zero. Taking in consideration these issues we get the following result.

Proposition 4.3. *The BSDE* (4.33) *has a solution* (Y^b, Z^b) *in* $\mathcal{S}^{\infty,+}_{\mathbb{F}} \times L^2_{\mathbb{F}}$ *with* $\int_0^\cdot Z^b dW \in$ BMO($\mathbb{P}$).

Proof. We first notice that the BSDE (4.33) can be written under the form

$$\left\{\begin{array}{rcl} dY_t^b & = & \Big\{\frac{|\mu_t-\lambda_t\beta_t|^2}{|\sigma_t|^2}Y_t^b-\frac{\lambda_t|\beta_t|^2}{|\sigma_t|^4}|\mu_t-\lambda_t\beta_t|^2-\lambda_t+\lambda_tY_t^b+\frac{2(\mu_t-\lambda_t\beta_t)}{|\sigma_t|^2}(\sigma_tZ_t^b+\lambda_t\beta_t)\\ & & +\frac{\big|\sigma_tZ_t^b+\lambda_t\beta_t+(\lambda_t\beta_t-\mu_t)\frac{\lambda_t|\beta_t|^2}{|\sigma_t|^2}\big|^2}{|\sigma_t|^2Y_t^b+\lambda_t|\beta_t|^2}\Big\}dt+Z_t^bdW_t\;,\quad t\in[0,T]\;,\\ Y_T^b & = & 1\;. \end{array}\right.$$

Since the variable Y^b appears in the denominator we can not directly solve this BSDE. We then proceed in four steps. We first introduce a modified BSDE with a lower bounded denominator to ensure that the generator is well defined. We then prove via a change of probability and a comparison theorem that the solution of the modified BSDE satisfies the initial BSDE.

Step 1: *Introduction of the modified BSDE.*

Let $(Y^\varepsilon,Z^\varepsilon)$ be the solution in $\mathcal{S}^\infty_{\mathbb{F}}\times L^2_{\mathbb{F}}$ to the BSDE

$$\left\{\begin{array}{rcl} dY_t^\varepsilon & = & \Big\{\frac{|\mu_t-\lambda_t\beta_t|^2}{|\sigma_t|^2}Y_t^\varepsilon-\frac{\lambda_t|\beta_t|^2}{|\sigma_t|^4}|\mu_t-\lambda_t\beta_t|^2-\lambda_t+\lambda_tY_t^\varepsilon+\frac{2(\mu_t-\lambda_t\beta_t)}{|\sigma_t|^2}(\sigma_tZ_t^\varepsilon+\lambda_t\beta_t)\\ & & +\frac{\big|\sigma_tZ_t^\varepsilon+\lambda_t\beta_t+(\lambda_t\beta_t-\mu_t)\frac{\lambda_t|\beta_t|^2}{|\sigma_t|^2}\big|^2}{|\sigma_t|^2(Y_t^\varepsilon\vee\varepsilon)+\lambda_t|\beta_t|^2}\Big\}dt+Z_t^\varepsilon dW_t\;,\quad t\in[0,T]\;,\\ Y_T^\varepsilon & = & 1\;, \end{array}\right. \tag{4.34}$$

where ε is a positive constant such that

$$\exp\Big(-\int_0^T\Big(\lambda_t+\frac{|\mu_t-\lambda_t\beta_t|^2}{|\sigma_t|^2}\Big)dt\Big)\quad\geq\quad\varepsilon\;,\quad\mathbb{P}-a.s. \tag{4.35}$$

Such a constant exists from $(\mathbf{H}S)$. Since the BSDE (4.34) is a quadratic BSDE, there exists a solution $(Y^\varepsilon,Z^\varepsilon)$ in $\mathcal{S}^\infty_{\mathbb{F}}\times L^2_{\mathbb{F}}$ from [18].

Step 2: BMO *property of the solution.*

In this part we prove that $\int_0^\cdot Z^\varepsilon dW\in\mathrm{BMO}(\mathbb{P})$. Let k denote the lower bound of the uniformly bounded process Y^ε. Applying Itô's formula to $|Y^\varepsilon-k|^2$, we obtain

$$\mathbb{E}\Big[\int_\nu^T|Z_s^\varepsilon|^2ds\Big|\mathcal{F}_\nu\Big]\quad=\quad|1-k|^2-|Y_\nu^\varepsilon-k|^2-2\mathbb{E}\Big[\int_\nu^T(Y_s^\varepsilon-k)f^\epsilon(s,Y_s^\varepsilon,Z_s^\varepsilon)ds\Big|\mathcal{F}_\nu\Big]\;, \tag{4.36}$$

for any stopping times $\nu\in\mathcal{T}_{\mathbb{F}}[0,T]$, with

$$\begin{array}{rcl} f^\epsilon(t,y,z) & = & \frac{|\mu_t-\lambda_t\beta_t|^2}{|\sigma_t|^2}y-\frac{\lambda_t|\beta_t|^2}{|\sigma_t|^4}|\mu_t-\lambda_t\beta_t|^2-\lambda_t+\lambda_ty+\frac{2(\mu_t-\lambda_t\beta_t)}{|\sigma_t|^2}(\sigma_tz+\lambda_t\beta_t)\\ & & +\frac{|\sigma_tz+\lambda_t\beta_t+(\lambda_t\beta_t-\mu_t)\frac{\lambda_t|\beta_t|^2}{|\sigma_t|^2}|^2}{|\sigma_t|^2(y\vee\varepsilon)+\lambda_t|\beta_t|^2}\;, \end{array}$$

for all $(t,y,z)\in[0,T]\times\mathbb{R}\times\mathbb{R}$. We can see that

$$f^\epsilon(t,y,z)\quad\geq\quad I_t+G_ty+H_tz\;, \tag{4.37}$$

for all $(t,y,z)\in[0,T]\times\mathbb{R}\times\mathbb{R}$ where the processes I, G and H are given by

$$\left\{\begin{array}{rcl} I_t & := & -\dfrac{\lambda_t|\beta_t|^2}{|\sigma_t|^4}|\mu_t-\lambda_t\beta_t|^2-\lambda_t+2\lambda_t\beta_t\dfrac{(\mu_t-\lambda_t\beta_t)}{|\sigma_t|^2}\ ,\\ G_t & := & \dfrac{|\mu_t-\lambda_t\beta_t|^2}{|\sigma_t|^2}+\lambda_t\ ,\\ H_t & := & 2\dfrac{(\mu_t-\lambda_t\beta_t)}{\sigma_t}\ , \end{array}\right.$$

for all $t\in[0,T]$. We first notice that from **(H**S**)**, the processes I, J and K are bounded. Using (4.36) and (4.37), we get the following inequality

$$\mathbb{E}\Big[\int_\nu^T|Z_s^\varepsilon|^2ds\Big|\mathcal{F}_\nu\Big]\ \le\ |1-k|^2-2\mathbb{E}\Big[\int_\nu^T(Y_s^\varepsilon-k)(I_s+G_sY_s^\varepsilon+H_sZ_s^\varepsilon)ds\Big|\mathcal{F}_\nu\Big]\ .$$

From the inequality $2ab\le a^2+b^2$ for $a,b\ge0$, we get

$$\begin{aligned}\mathbb{E}\Big[\int_\nu^T|Z_s^\varepsilon|^2ds\Big|\mathcal{F}_\nu\Big]\ &\le\ |1-k|^2-2\mathbb{E}\Big[\int_\nu^T(Y_s^\varepsilon-k)(I_s+G_sY_s^\varepsilon)ds\Big|\mathcal{F}_\nu\Big]\\ &\quad+2\mathbb{E}\Big[\int_\nu^T|H_s|^2|Y_s^\varepsilon-k|^2ds\Big|\mathcal{F}_\nu\Big]+\frac{1}{2}\mathbb{E}\Big[\int_\nu^T|Z_s^\varepsilon|^2ds\Big|\mathcal{F}_\nu\Big]\ .\end{aligned}$$

Since I, G, H and Y^ε are uniformly bounded, we get

$$\mathbb{E}\Big[\int_\nu^T|Z_s^\varepsilon|^2ds\Big|\mathcal{F}_\nu\Big]\ \le\ C\ ,$$

for some constant C which does not depend on ν. Therefore, $\int_0^\cdot Z^\varepsilon dW\in\mathrm{BMO}(\mathbb{P})$.

Step 3: *Change of probability.*

Define the process L^ε by

$$L_t^\varepsilon\ :=\ 2\frac{(\mu_t-\lambda_t\beta_t)}{\sigma_t}+2\frac{\sigma_t\big(\lambda_t\beta_t+\frac{\lambda_t|\beta_t|^2}{|\sigma_t|^2}(\lambda_t\beta_t-\mu_t)\big)}{|\sigma_t|^2(Y_t^\varepsilon\vee\varepsilon)+\lambda_t|\beta_t|^2}+\frac{|\sigma_t|^2Z_t^\varepsilon}{|\sigma_t|^2(Y_t^\varepsilon\vee\varepsilon)+\lambda_t|\beta_t|^2}\ ,$$

for all $t\in[0,T]$. Since $Y^\varepsilon\in\mathcal{S}^\infty_{\mathbb{F}}$, $\int_0^\cdot Z^\varepsilon dW\in\mathrm{BMO}(\mathbb{P})$, we get from **(HS)** that $\int_0^\cdot L^\varepsilon dW\in\mathrm{BMO}(\mathbb{P})$. Therefore, the process $\mathcal{E}(\int_0^\cdot L_s^\varepsilon dW_s)$ is an $\mathbb{F}$-martingale from Theorem 2.3 in [17]. Applying the Girsanov theorem we get that the process $\bar W$ defined by

$$\bar W_t\ :=\ W_t+\int_0^t L_s^\varepsilon ds\ ,$$

for all $t\in[0,T]$, is a Brownian motion under the probability $\mathbb{Q}$ defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_T}\ =\ \mathcal{E}\Big(-\int_0^T L_s^\varepsilon dW_s\Big)\ .$$

We also notice that under $\mathbb{Q}$, $(Y^\varepsilon,Z^\varepsilon)$ is solution to

$$\begin{aligned}Y_t^\varepsilon\ =\ 1+\int_t^T\Big\{&\frac{\lambda_s|\beta_s|^2}{|\sigma_s|^4}|\mu_s-\lambda_s\beta_s|^2-\frac{|\mu_s-\lambda_s\beta_s|^2}{|\sigma_s|^2}Y_s^\varepsilon-2\lambda_s\beta_s\frac{(\mu_s-\lambda_s\beta_s)}{|\sigma_s|^2}+\lambda_s\\ &-\lambda_sY_s^\varepsilon-\frac{\big|\lambda_s\beta_s+(\lambda_s\beta_s-\mu_s)\frac{\lambda_s|\beta_s|^2}{|\sigma_s|^2}\big|^2}{|\sigma_s|^2(Y_s^\varepsilon\vee\varepsilon)+\lambda_s|\beta_s|^2}\Big\}ds-\int_t^T Z_s^\varepsilon d\bar W_s\ ,\quad t\in[0,T]\ .\quad(4.38)\end{aligned}$$

Step 4: *Comparison under the new probability measure* $\mathbb{Q}$.

We first notice that the generator $\bar{f}^\epsilon$ of the BSDE (4.38) admits the following lower bound

$$
\begin{aligned}
\bar{f}^\epsilon(t,y,z) \quad &\geq \quad \frac{\lambda_t|\beta_t|^2}{|\sigma_t|^4}|\mu_t-\lambda_t\beta_t|^2+\lambda_t-\lambda_t y-2\lambda_t\beta_t\frac{(\mu_t-\lambda_t\beta_t)}{|\sigma_t|^2}\\
&\quad -\frac{|\mu_t-\lambda_t\beta_t|^2}{|\sigma_t|^2}y-\frac{\left|\lambda_t\beta_t+(\lambda_t\beta_t-\mu_t)\frac{\lambda_t|\beta_t|^2}{|\sigma_t|^2}\right|^2}{\lambda_t|\beta_t|^2}\mathbb{1}_{\lambda_t\beta_t\neq 0}\\
&= \quad -\lambda_t y-\frac{|\mu_t-\lambda_t\beta_t|^2}{|\sigma_t|^2}y\ ,
\end{aligned}
$$

for all $(t,y,z)\in[0,T]\times\mathbb{R}\times\mathbb{R}$.

We now study the following BSDE

$$
\underline{Y}_t \quad = \quad 1+\int_t^T\Big[-\lambda_s-\frac{|\mu_s-\lambda_s\beta_s|^2}{|\sigma_s|^2}\Big]\underline{Y}_s ds-\int_t^T\underline{Z}_s d\bar{W}_s\ ,\quad t\in[0,T]\ . \tag{4.39}
$$

Since this BSDE is linear, it has a unique solution given by (see e.g. [8])

$$
\underline{Y}_t \quad := \quad \mathbb{E}_{\mathbb{Q}}\Big[\exp\Big(-\int_t^T\big(\lambda_s+\frac{|\mu_s-\lambda_s\beta_s|^2}{|\sigma_s|^2}\big)ds\Big)\Big|\mathcal{F}_t\Big]\ ,\quad t\in[0,T]\ .
$$

Applying Theorem 2.2 of [8] for BSDEs (4.38) and (4.39) we have

$$
Y^\epsilon_t \quad \geq \quad \underline{Y}_t\ ,\quad t\in[0,T]\ .
$$

By (4.35), we have $\varepsilon\leq\underline{Y}_t$ for any $t\in[0,T]$. Consequently, $Y^\varepsilon_t\geq\varepsilon$ for any $t\in[0,T]$, and $(Y^\varepsilon,Z^\varepsilon)$ is solution to (4.33). □

We now are able to prove that the BSDE $(\mathfrak{f},1)$ admits a solution.

Proposition 4.4. *The BSDE* (3.7) *admits a solution* $(Y,Z,U)\in\mathcal{S}^\infty_{\mathbb{G}}\times L^2_{\mathbb{G}}\times L^2(\lambda)$ *with* $Y\in\mathcal{S}^{\infty,+}_{\mathbb{G}}$.

Proof. From Theorem 4.3 and Proposition 4.3, we obtain that the BSDE (3.7) admits a solution $(Y,Z,U)\in\mathcal{S}^\infty_{\mathbb{G}}\times L^2_{\mathbb{G}}\times L^2(\lambda)$, with Y given by

$$
Y_t \quad = \quad Y^b_t\mathbb{1}_{\tau<t}+\mathbb{1}_{\tau\geq t}\ ,\quad t\in[0,T]\ .
$$

with $Y^b\in\mathcal{S}^{\infty,+}_{\mathbb{F}}$ from Proposition 4.3. Therefore $Y\in\mathcal{S}^{\infty,+}_{\mathbb{G}}$. □

4.3 Solution to the BSDE $(\mathfrak{g},H)$

We first notice that the BSDE $(\mathfrak{g},H)$ can be rewritten under the form

$$
\left\{
\begin{aligned}
d\mathcal{Y}_t \quad &= \quad \Big\{\frac{(\mu_tY_t+\sigma_tZ_t+\lambda^{\mathbb{G}}_t\beta_tU_t)(\sigma_tY_t\mathcal{Z}_t+\lambda^{\mathbb{G}}_t\beta_t(U_t+Y_t)\mathcal{U}_t)}{Y_t(|\sigma_t|^2Y_t+\lambda^{\mathbb{G}}_t|\beta_t|^2(U_t+Y_t))}-\frac{Z_t}{Y_t}\mathcal{Z}_t\\
&\qquad -\tfrac{\lambda^{\mathbb{G}}_tU_t}{Y_t}\mathcal{U}_t-\ \lambda^{\mathbb{G}}_t\mathcal{U}_t\Big\}dt+\mathcal{Z}_tdW_t+\mathcal{U}_tdN_t\ ,\quad t\in[0,T\wedge\tau]\ ,\\
\mathcal{Y}_{T\wedge\tau} \quad &= \quad H\ .
\end{aligned}
\right. \tag{4.40}
$$

Since $Y_t \mathbb{1}_{t<\tau} = Y_t^b \mathbb{1}_{t<\tau}$ and $U_t \mathbb{1}_{t\leq\tau} = (1-Y_t^b)\mathbb{1}_{t\leq\tau}$, we consider the associated decomposed BSDE in $\mathbb{F}$: find $(\mathcal{Y}^b, \mathcal{Z}^b) \in \mathcal{S}^\infty_{\mathbb{F}} \times L^2_{\mathbb{F}}$ such that

$$\left\{\begin{array}{rcl} d\mathcal{Y}_t^b & = & \Big\{\dfrac{((\mu_t-\lambda_t\beta_t)Y_t^b+\sigma_t Z_t^b+\lambda_t\beta_t)(\sigma_t Y_t^b \mathcal{Z}_t^b+\lambda_t\beta_t H_t^a-\lambda_t\beta_t\mathcal{Y}_t^b)}{Y_t^b(|\sigma_t|^2 Y_t^b+\lambda_t|\beta_t|^2)} \\ & & \quad -\frac{Z_t^b}{Y_t^b}\mathcal{Z}_t^b - \dfrac{\lambda_t}{Y_t^b}H_t^a + \dfrac{\lambda_t}{Y_t^b}\mathcal{Y}_t^b\Big\}dt + \mathcal{Z}_t^b dW_t\;, \quad t\in[0,T]\;, \\ \mathcal{Y}_T^b & = & H^b\;. \end{array}\right. \tag{4.41}$$

We notice that this BSDE has a Lipschitz generator w.r.t. the unknown $(\mathcal{Y}^b, \mathcal{Z}^b)$. However the Lipschitz coefficient depends on Z^b which is not necessarily bounded. Thus we cannot apply the existing results and have to deal with this issue.

Proposition 4.5. *The BSDE* (4.41) *admits a solution* $(\mathcal{Y}^b, \mathcal{Z}^b)$ *in* $\mathcal{S}^\infty_{\mathbb{F}} \times L^2_{\mathbb{F}}$ *with* $\int_0^\cdot \mathcal{Z}^b dW \in \mathrm{BMO}(\mathbb{P})$.

Proof. We first define the equivalent probability $\mathbb{Q}$ to $\mathbb{P}$ defined by its Radon-Nikodym density $\frac{d\mathbb{Q}}{d\mathbb{P}}\big|_{\mathcal{F}_T} = \mathcal{E}(\int_0^T \rho_t dW_t)$ where ρ is given by

$$\rho_t \;\; := \;\; \frac{Z_t^b}{Y_t^b} - \frac{\sigma_t\big((\mu_t-\lambda_t\beta_t)Y_t^b+\sigma_t Z_t^b+\lambda_t\beta_t\big)}{|\sigma_t|^2 Y_t^b+\lambda_t|\beta_t|^2}\;, \quad t\in[0,T]\;.$$

Since $\int_0^\cdot Z^b dW \in \mathrm{BMO}(\mathbb{P})$, $Y^b \in \mathcal{S}^{\infty,+}_{\mathbb{F}}$ and the coefficients μ, σ and β satisfy **(H**S**)**, it implies that $\int_0^\cdot \rho dW \in \mathrm{BMO}(\mathbb{P})$. Therefore, $\bar{W}_t := W_t - \int_0^t \rho_s ds$ is a $\mathbb{Q}$-Brownian motion. Hence, the BSDE (4.41) can be written as

$$\left\{\begin{array}{rcl} d\mathcal{Y}_t^b & = & a_t(\mathcal{Y}_t^b - H_t^a)dt + \mathcal{Z}_t^b d\bar{W}_t\;, \quad t\in[0,T]\;, \\ \mathcal{Y}_{T\wedge\tau}^b & = & H^b\;, \end{array}\right. \tag{4.42}$$

with

$$a_t \;\; := \;\; \frac{\lambda_t|\sigma_t|^2 Y_t^b - \lambda_t\beta_t((\mu_t-\lambda_t\beta_t)Y_t^b+\sigma_t Z_t^b)}{Y_t^b(|\sigma_t|^2 Y_t^b+\lambda_t|\beta_t|^2)}\;, \quad t\in[0,T]\;.$$

By definition of a we can see that $\int_0^\cdot a dW \in \mathrm{BMO}(\mathbb{P})$ since the coefficients μ, σ, β and λ are bounded, $Y^b \in \mathcal{S}^{\infty,+}_{\mathbb{F}}$ and $\int_0^\cdot Z^b dW \in \mathrm{BMO}(\mathbb{P})$. Using Theorem A.1 with $\mathbb{Q}_1 = \mathbb{P}$ and $\mathbb{Q}_2 = \mathbb{Q}$, we get $\int_0^\cdot a d\bar{W} \in \mathrm{BMO}(\mathbb{Q})$. Therefore, there exists a constant $l' \geq 0$ such that $\mathbb{E}_{\mathbb{Q}}[\int_\nu^T |a_s|^2 ds|\mathcal{F}_\nu] \leq l'$ for any $\nu \in \mathcal{T}_{\mathbb{F}}[0,T]$. We now prove that the process $\mathcal{Y}^b$ defined by

$$\mathcal{Y}_t^b \;\; := \;\; \mathbb{E}_{\mathbb{Q}}\Big[\frac{\Gamma_T}{\Gamma_t}H^b + \int_t^T \frac{\Gamma_s}{\Gamma_t}a_s H_s^a ds\Big|\mathcal{F}_t\Big]\;, \quad t\in[0,T]\;,$$

with $\Gamma_t := \exp(-\int_0^t a_s ds)$, is solution of the BSDE (4.41). We proceed in four steps.

Step 1. *Integrability of the process* Γ.

We first prove that for any $p \geq 1$ there exists a constant $C > 0$ such that the process Γ satisfies for any $t \in [0,T]$

$$\mathbb{E}_{\mathbb{Q}}\Big[\sup_{t\leq s\leq T}\Big|\frac{\Gamma_s}{\Gamma_t}\Big|^p\Big|\mathcal{F}_t\Big] \;\; \leq \;\; C\;. \tag{4.43}$$

Since $\mathbb{E}_{\mathbb{Q}}[\int_\nu^T |a_s|^2 ds|\mathcal{F}_\nu] \leq l'$ for any $\nu \in \mathcal{T}_{\mathbb{F}}[0,T]$, we get from Proposition A.1 that there exists a constant δ such that $0 < \delta < \frac{1}{l'}$ and

$$\mathbb{E}_{\mathbb{Q}}\Big[\exp\Big(\delta\int_\nu^T |a_s|^2 ds\Big)\Big|\mathcal{F}_\nu\Big] \quad \leq \quad \frac{1}{1-\delta l'} .$$

We get for any $0 \leq t \leq s \leq T$

$$\begin{aligned}\Big|\frac{\Gamma_s}{\Gamma_t}\Big|^p &\leq \exp\Big(\int_t^s \big(\delta|a_r|^2 + \frac{p^2}{4\delta}\big)dr\Big) \\ &\leq \exp\Big(\frac{p^2}{4\delta}T\Big)\exp\Big(\delta\int_0^T |a_r|^2 dr\Big) .\end{aligned}$$

Consequently, we get

$$\mathbb{E}_{\mathbb{Q}}\Big[\sup_{t\leq s\leq T}\Big|\frac{\Gamma_s}{\Gamma_t}\Big|^p\Big|\mathcal{F}_t\Big] \quad \leq \quad \exp\Big(\frac{p^2}{4\delta}T\Big)\frac{1}{1-\delta l'} .$$

Step 2. *Uniform boundedness of* $\mathcal{Y}^b$.

We now prove that $\mathcal{Y}^b \in \mathcal{S}^\infty_{\mathbb{F}}$. For that we remark that by definition of $\mathcal{Y}^b$ we have the following inequality

$$|\mathcal{Y}^b_t| \quad \leq \quad \|H^b\|_\infty \mathbb{E}_{\mathbb{Q}}\Big[\frac{\Gamma_T}{\Gamma_t}\Big|\mathcal{F}_t\Big] + \|H^a\|_{\mathcal{S}^\infty}\mathbb{E}_{\mathbb{Q}}\Big[\int_t^T |a_s|^2 ds\Big|\mathcal{F}_t\Big] + \|H^a\|_{\mathcal{S}^\infty}\mathbb{E}_{\mathbb{Q}}\Big[\int_t^T \Big|\frac{\Gamma_s}{\Gamma_t}\Big|^2 ds\Big|\mathcal{F}_t\Big] .$$

Therefore, we get that $\mathcal{Y}^b \in \mathcal{S}^\infty_{\mathbb{F}}$.

Step 3. *Dynamics of* $\mathcal{Y}^b$.

We now prove that $\mathcal{Y}^b$ satisfies (4.42). For that we introduce the $\mathbb{Q}$-martingale m defined by

$$m_t \quad := \quad \Gamma_t\mathcal{Y}^b_t + \int_0^t \Gamma_s a_s H^a_s ds , \quad t \in [0,T] .$$

We first notice that m is $\mathbb{Q}$-square integrable. Indeed, from the definition of m, there exists a constant C such that

$$\mathbb{E}_{\mathbb{Q}}\Big[|m_t|^2\Big] \quad \leq \quad C\Big(\mathbb{E}_{\mathbb{Q}}\Big[\big|\Gamma_t\mathcal{Y}^b_t\big|^2\Big] + \mathbb{E}_{\mathbb{Q}}\Big[\int_0^t \big|\Gamma_s a_s H^a_s\big|^2 ds\Big]\Big) ,$$

for all $t \in [0,T]$. Since $\mathcal{Y}^b \in \mathcal{S}^\infty_{\mathbb{F}}$, we get from (2.4) and from Cauchy-Schwarz inequality the existence of a constant C such that

$$\mathbb{E}_{\mathbb{Q}}\Big[|m_t|^2\Big] \quad \leq \quad C\Big(\mathbb{E}_{\mathbb{Q}}\Big[\big|\Gamma_t\big|^2\Big] + \sqrt{\mathbb{E}_{\mathbb{Q}}\Big[\Big(\int_0^t |a_s|^2 ds\Big)^2\Big]}\sqrt{\mathbb{E}_{\mathbb{Q}}\Big[\sup_{0\leq s\leq t}\big|\Gamma_s\big|^4\Big]}\ \Big) ,$$

for all $t \in [0,T]$. Since $\int_0^\cdot a dW \in \text{BMO}(\mathbb{P})$ we have from Theorem A.1 $\int_0^\cdot a d\bar{W} \in \text{BMO}(\mathbb{Q})$, and we get from Proposition A.1 and (4.43)

$$\mathbb{E}_{\mathbb{Q}}\Big[|m_t|^2\Big] \quad < \quad \infty , \quad t \in [0,T] .$$

Therefore, there exists a predictable process $\tilde{\mathcal{Z}}$ such that $\mathbb{E}_{\mathbb{Q}}[\int_0^T |\tilde{\mathcal{Z}}_s|^2 ds] < \infty$ and

$$\Gamma_t \mathcal{Y}_t^b + \int_0^t \Gamma_s a_s H_s^a ds \;=\; m_0 + \int_0^t \tilde{\mathcal{Z}}_s d\bar{W}_s \;, \quad t \in [0,T] \;.$$

From Itô's formula and the definition of $\mathcal{Y}_T^b$ we have

$$\mathcal{Y}_t^b \;=\; H^b - \int_t^T a_s(\mathcal{Y}_s^b - H_s^a) ds - \int_t^T \mathcal{Z}_t^b d\bar{W}_s \;, \quad t \in [0,T] \;. \tag{4.44}$$

where the process $\mathcal{Z}^b$ is defined by

$$\mathcal{Z}_t^b \;:=\; \frac{\tilde{\mathcal{Z}}_t}{\Gamma_t} \;, \quad t \in [0,T] \;.$$

We now prove that $\int_0^\cdot \mathcal{Z}^b d\bar{W} \in \mathrm{BMO}(\mathbb{Q})$. Using (4.44), there exists a constant C such that

$$\begin{aligned}\sup_{\nu \in \mathcal{T}_{\mathbb{F}}[0,T]} \mathbb{E}_{\mathbb{Q}}\Big[\int_\nu^T |\mathcal{Z}_s^b|^2 ds \Big| \mathcal{F}_\nu\Big] \;\leq\; & C\Big((\|\mathcal{Y}^b\|_{\mathcal{S}^\infty}^2 + \|H^a\|_{\mathcal{S}^\infty}^2) \sup_{\nu \in \mathcal{T}_{\mathbb{F}}[0,T]} \mathbb{E}_{\mathbb{Q}}\Big[\int_\nu^T |a_s|^2 ds \Big| \mathcal{F}_\nu\Big] \\ & + \|H^b\|_\infty^2 + \|\mathcal{Y}^b\|_{\mathcal{S}^\infty}^2 \Big) \;.\end{aligned}$$

Using $\mathcal{Y}^b \in \mathcal{S}_{\mathbb{F}}^\infty$, (2.4) and $\int_0^\cdot a d\bar{W} \in \mathrm{BMO}(\mathbb{Q})$, we get that $\int_0^\cdot \mathcal{Z}^b d\bar{W} \in \mathrm{BMO}(\mathbb{Q})$. Thus, using $\frac{d\mathbb{P}}{d\mathbb{Q}}\big|_{\mathcal{F}_T} = \mathcal{E}(-\int_0^\cdot \rho d\bar{W})_T$ and Theorem A.1 with $\mathbb{Q}_1 = \mathbb{Q}$ and $\mathbb{Q}_2 = \mathbb{P}$ we obtain that

$$\int_0^\cdot \mathcal{Z}^b dW \;=\; \int_0^\cdot \mathcal{Z}^b d\bar{W} - \langle \int_0^\cdot \mathcal{Z}^b d\bar{W}, \int_0^\cdot \rho d\bar{W} \rangle \;\in\; \mathrm{BMO}(\mathbb{P}) \;.$$

To conclude we get from (4.44) and the definition of $\bar{W}$ that $(\mathcal{Y}^b, \mathcal{Z}^b)$ is a solution to the BSDE (4.41). □

We now prove the existence of a solution to the BSDE $(\mathfrak{g}, H)$.

Proposition 4.6. *The BSDE* (3.8) *admits a solution* $(\mathcal{Y}, \mathcal{Z}, \mathcal{U}) \in \mathcal{S}_{\mathbb{G}}^\infty \times L_{\mathbb{G}}^2 \times L^2(\lambda)$.

Proof. From Theorem 4.3 and Proposition 4.5, we obtain that the BSDE (3.8) admits a solution $(\mathcal{Y}, \mathcal{Z}, \mathcal{U}) \in \mathcal{S}_{\mathbb{G}}^\infty \times L_{\mathbb{G}}^2 \times L^2(\lambda)$. □

4.4 Solution to the BSDE $(\mathfrak{h}, 0)$

We recall that the BSDE $(\mathfrak{h}, 0)$ is

$$\begin{aligned}\Upsilon_t \;=\; & \int_{t\wedge\tau}^{T\wedge\tau} \Big(|\mathcal{Z}_s|^2 Y_s + \lambda_s^{\mathbb{G}}(U_s + Y_s)|\mathcal{U}_s|^2 - \frac{|\sigma_s Y_s \mathcal{Z}_s + \lambda_s^{\mathbb{G}} \beta_s \mathcal{U}_s(U_s + Y_s)|^2}{|\sigma_s|^2 Y_s + \lambda_s^{\mathbb{G}} |\beta_s|^2 (U_s + Y_s)} \Big) ds \\ & - \int_{t\wedge\tau}^{T\wedge\tau} \Xi_s dW_s - \int_{t\wedge\tau}^{T\wedge\tau} \Theta_s dM_s \;, \quad t \in [0,T] \;.\end{aligned} \tag{4.45}$$

Using the definitions of Y, U, $\mathcal{Z}$ and $\mathcal{U}$, we therefore consider the associated decomposed BSDE in $\mathbb{F}$: find $(\Upsilon^b, \Xi^b) \in \mathcal{S}_{\mathbb{F}}^\infty \times L_{\mathbb{F}}^2$ such that

$$\begin{aligned}\Upsilon_t^b \;=\; & \int_t^T \Big(|\mathcal{Z}_s^b|^2 Y_s^b + \lambda_s |H_s^a - \mathcal{Y}_s^b|^2 - \frac{|\sigma_s Y_s^b \mathcal{Z}_s^b + \lambda_s \beta_s (H_s^a - \mathcal{Y}_s^b)|^2}{|\sigma_s|^2 Y_s^b + \lambda_s |\beta_s|^2} - \lambda_s \Upsilon_s^b \Big) ds \\ & - \int_t^T \Xi_s^b dW_s \;, \qquad t \in [0,T] \;.\end{aligned} \tag{4.46}$$

Proposition 4.7. *The BSDE* (4.46) *admits a solution* $(\Upsilon^b, \Xi^b) \in \mathcal{S}^\infty_{\mathbb{F}} \times L^2_{\mathbb{F}}$.

Proof. Denote by R the process defined by

$$R_t \;:=\; |\mathcal{Z}^b_t|^2 Y^b_t + \lambda_t |H^a_t - \mathcal{Y}^b_t|^2 - \frac{|\sigma_t Y^b_t \mathcal{Z}^b_t + \lambda_t \beta_t (H^a_t - \mathcal{Y}^b_t)|^2}{|\sigma_t|^2 Y^b_t + \lambda_t |\beta_t|^2} \,,$$

for $t \in [0,T]$. Define the process $\tilde{\Upsilon}^b$ by

$$\tilde{\Upsilon}^b_t \;:=\; \mathbb{E}\Big[\int_t^T R_s e^{-\int_0^s \lambda_u du} ds \Big| \mathcal{F}_t\Big] \,, \quad t \in [0,T] \,.$$

From $(\mathbf{H}S)$, λ is bounded, $Y^b \in \mathcal{S}^{\infty,+}_{\mathbb{F}}$, $H^a \in \mathcal{S}^\infty_{\mathbb{F}}$, $\mathcal{Y}^b \in \mathcal{S}^\infty_{\mathbb{F}}$ and $\int_0^\cdot \mathcal{Z}^b dW \in \mathrm{BMO}(\mathbb{P})$, we get from Proposition A.1 that $\tilde{\Upsilon}^b \in \mathcal{S}^\infty_{\mathbb{F}}$ and the process $\tilde{\Upsilon}^b + \int_0^\cdot R_s e^{-\int_0^s \lambda_u du} ds$ is a square integrable martingale. Hence there exists a process $\tilde{\Xi}^b \in L^2_{\mathbb{F}}$ such that

$$\tilde{\Upsilon}^b_t \;=\; \int_t^T R_s e^{-\int_0^s \lambda_u du} ds - \int_t^T \tilde{\Xi}^b_s dW_s \,, \quad t \in [0,T] \,.$$

From Itô's formula we get that the processes (Υ^b, Ξ^b) defined by

$$\Upsilon^b_t \;=\; \tilde{\Upsilon}^b_t e^{\int_0^t \lambda_s ds} \quad \text{and} \quad \Xi^b_t \;=\; \tilde{\Xi}^b_t e^{\int_0^t \lambda_s ds}$$

satisfy (4.46). Since $\tilde{\Xi}^b \in L^2_{\mathbb{F}}$ and λ is uniformly bounded we get that $\Xi^b \in L^2_{\mathbb{F}}$. Finally, since $\tilde{\Upsilon}^b \in \mathcal{S}^\infty_{\mathbb{F}}$ we get that $\Upsilon^b \in \mathcal{S}^\infty_{\mathbb{F}}$. □

Finally, we prove the existence of a solutin to the BSDE $(\mathfrak{h}, 0)$.

Proposition 4.8. *The BSDE* (3.9) *admits a solution* $(\Upsilon, \Xi, \Theta) \in \mathcal{S}^\infty_{\mathbb{G}} \times L^2_{\mathbb{G}} \times L^2(\lambda)$.

Proof. From Theorem 4.3 and Proposition 4.7, we obtain that the BSDE (3.9) admits a solution $(Y, Z, U) \in \mathcal{S}^\infty_{\mathbb{G}} \times L^2_{\mathbb{G}} \times L^2(\lambda)$. □

A Appendix

A.1 Proof of Proposition 2.1

We first suppose that X is a nonnegative $\mathcal{P}(\mathbb{G})$-measurable process. For $n \geq 1$, we define the process X^n by

$$X^n_t \;=\; X_t \wedge n \,, \quad t \in [0,T] \,.$$

Then X^n is a bounded $\mathbb{G}$-predictable process, and from Lemma 4.4 in [15], there exist a $\mathcal{P}(\mathbb{F})$-measurable process $X^{n,b}$ and a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable process $X^{n,a}$ such that

$$X^n_t \;=\; X^{n,b}_t \mathbb{1}_{t \leq \tau} + X^{n,a}_t(\tau) \mathbb{1}_{t > \tau} \,, \quad t \in [0,T] \,. \tag{A.1}$$

Since the sequence $(X^n)_n$ is nondecreasing, we can assume w.l.o.g. that the sequences $(X^{a,n})_n$ and $(X^{b,n})_n$ are also nondecreasing. Define the processes X^a and X^b by

$$X^a \;=\; \lim_{n\to\infty} X^{n,a} \quad \text{and} \quad X^b \;=\; \lim_{n\to\infty} X^{n,b} \,.$$

Then X^a is $\mathcal{P}(\mathbb{F})\otimes\mathcal{B}(\mathbb{R}_+)$-measurable and X^b is $\mathcal{P}(\mathbb{F})$-measurable and sending n to infinity in (A.1), we get

$$X_t \quad = \quad X_t^b \mathbb{1}_{t\leq\tau} + X_t^a(\tau)\mathbb{1}_{t>\tau}\ , \quad t\in[0,T]\ . \tag{A.2}$$

For a general $\mathcal{P}(\mathbb{G})$-measurable process X, we write $X = X^+ - X^-$ where $X^+ = \max(X,0)$ and $X^- = \max(-X,0)$ and we apply the previous result to the nonnegative processes X^+ and X^-. From the linear stability of the decomposition (A.2) we get the result. □

A.2 BMO Stability

Theorem A.1. *Let $\mathbb{Q}_1$ and $\mathbb{Q}_2$ be two probability measures on $(\Omega,\mathcal{G})$. Let M and N be two continuous $(\mathbb{F},\mathbb{Q}_1)$-local martingales with $N\in\mathrm{BMO}(\mathbb{Q}_1)$. Suppose that $\mathbb{Q}_1$ and $\mathbb{Q}_2$ are equivalent with $\frac{d\mathbb{Q}_2}{d\mathbb{Q}_1}\big|_{\mathcal{F}_T} = \mathcal{E}(N)_T$. If $M\in\mathrm{BMO}(\mathbb{Q}_1)$ then $M-\langle M,N\rangle\in\mathrm{BMO}(\mathbb{Q}_2)$.*

Proof. This result is a direct consequence of Theorem 3.6 in [17]. □

A.3 An estimate for conditional moments

Proposition A.1. *Let A be a continuous increasing $\mathbb{F}$-adapted process. Fix a $t\geq 0$ such that there exists a constant $C>0$ satisfying*

$$\mathbb{E}\big[A_t - A_s\big|\mathcal{F}_s\big] \quad \leq \quad C\ ,$$

for any $s\in[0,t]$. Then, we have for any $s\in[0,t]$ and any $p\geq 1$

$$\mathbb{E}\big[|A_t-A_s|^p\big|\mathcal{F}_s\big]\leq p!|C|^p$$

and

$$\mathbb{E}\Big[\exp\big(\delta(A_t-A_s)\big)\big|\mathcal{F}_s\Big] \quad \leq \quad \frac{1}{1-\delta C}\ ,$$

for any $\delta\in(0,\frac{1}{C})$.

Proof. Let A be a continuous increasing $\mathbb{F}$-adapted process satisfying $\mathbb{E}[A_t-A_s|\mathcal{F}_s]\leq C$ for any $s\in[0,t]$. We first prove by iteration that $\mathbb{E}[|A_t-A_s|^p|\mathcal{F}_s]\leq p!|C|^p$ for any $p\geq 1$.

• For $p=1$, we have by assumption $\mathbb{E}[A_t-A_s|\mathcal{F}_s]\leq C$.

• Suppose that for some $p\geq 2$, we have $\mathbb{E}[|A_t-A_s|^{p-1}|\mathcal{F}_s]\leq(p-1)!|C|^{p-1}$. Since A is a continuous increasing $\mathbb{F}$-adapted process we have

$$|A_t-A_s|^p = p\int_s^t |A_t-A_u|^{p-1}dA_u\ ,$$

for any $s\in[0,t]$. Consequently we get

$$\begin{aligned}
\mathbb{E}\big[|A_t-A_s|^p\big|\mathcal{F}_s\big] \quad &= \quad p\mathbb{E}\Big[\int_s^t |A_t-A_u|^{p-1}dA_u\Big|\mathcal{F}_s\Big]\\
&= \quad p\mathbb{E}\Big[\int_s^t \mathbb{E}\Big[|A_t-A_u|^{p-1}\Big|\mathcal{F}_u\Big]dA_u\Big|\mathcal{F}_s\Big]\\
&\leq \quad p!|C|^{p-1}\mathbb{E}[A_t-A_s|\mathcal{F}_s]\\
&\leq \quad p!|C|^p\ .
\end{aligned}$$

• Since the result holds true for $p = 1$ and for any $p \geq 2$ as soon as it holds for $p-1$, it holds for p, we get

$$\mathbb{E}\big[|A_t - A_s|^p \big| \mathcal{F}_s\big] \leq p!|C|^p ,$$

for any $p \geq 1$.

From this last inequality, we get for any $\delta \in (0, \frac{1}{C})$

$$\mathbb{E}\Big[\sum_{p\geq 0} \frac{1}{p!}|\delta|^p |A_t - A_s|^p \Big| \mathcal{F}_s\Big] \leq \sum_{p\geq 0} |\delta C|^p = \frac{1}{1-\delta C} ,$$

which is the expected result. □

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