k-tuple chromatic number of the cartesian product of graphs
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To cite this version:
Flavia Bonomo, Ivo Koch, Pablo Torres, Mario Valencia-Pabon. k-tuple chromatic number of the cartesian product of graphs. 2014. hal-01103534

HAL Id: hal-01103534
https://hal.archives-ouvertes.fr/hal-01103534
Submitted on 14 Jan 2015

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Abstract

A $k$-tuple coloring of a graph $G$ assigns a set of $k$ colors to each vertex of $G$ such that if two vertices are adjacent, the corresponding sets of colors are disjoint. The $k$-tuple chromatic number of $G$, $\chi_k(G)$, is the smallest $t$ so that there is a $k$-tuple coloring of $G$ using $t$ colors. It is well known that $\chi(G \square H) = \max\{\chi(G), \chi(H)\}$. In this paper, we show that there exist graphs $G$ and $H$ such that $\chi_k(G \square H) > \max\{\chi_k(G), \chi_k(H)\}$ for $k \geq 2$. Moreover, we also show that there exist graph families such that, for any $k \geq 1$, the $k$-tuple chromatic number of their cartesian product is equal to the maximum $k$-tuple chromatic number of its factors.

Keywords: $k$-tuple colorings, Cartesian product of graphs, Kneser graphs, Cayley graphs, Hom-idempotent graphs.

1 Introduction

A classic coloring of a graph $G$ is an assignment of colors (or natural numbers) to the vertices of $G$ such that any two adjacent vertices are assigned different colors. The smallest number $t$ such that $G$ admits a coloring with $t$ colors (a $t$-coloring) is called the chromatic number of $G$ and is denoted by $\chi(G)$. Several generalizations of the coloring problem have been introduced in the literature, in particular, cases in which each vertex is assigned not only a color but a set of colors, under different restrictions. One of these variations is the $k$-tuple coloring introduced independently by Stahl [11] and Bollobás and Thomason [3]. A $k$-tuple coloring...
coloring of a graph $G$ is an assignment of $k$ colors to each vertex in such a way that adjacent vertices are assigned distinct colors. The $k$-tuple coloring problem consists into finding the minimum number of colors in a $k$-tuple coloring of a graph $G$, which we denote by $\chi_k(G)$. The cartesian product $G \square H$ of two graphs $G$ and $H$ has vertex set $V(G) \times V(H)$, two vertices being joined by an edge whenever they have one coordinate equal and the other adjacent. This product is commutative and associative up to isomorphism. There is a simple formula expressing the chromatic number of a cartesian product in terms of its factors:

$$\chi(G \square H) = \max\{\chi(G), \chi(H)\}. \quad (1)$$

The identity (1) admits a simple proof first given by Sabidussi [10]. The Kneser graph $K(m, n)$ has as vertices all $n$-element subsets of the set $[m] = \{1, \ldots, m\}$ and an edge between two subsets if and only if they are disjoint. We will assume in the rest of this work that $m \geq 2n$, otherwise $K(m, n)$ has no edges. The Kneser graph $K(5, 2)$ is the well known Petersen Graph. Lovász [9] showed that $\chi(K(m, n)) = m - 2n + 2$. The value of the $k$-tuple chromatic number of the Kneser graph is the subject of an almost 40-year-old conjecture of Stahl [11] which asserts that: if $k = qn - r$ where $q \geq 0$ and $0 \leq r < n$, then $\chi_k(K(m, n)) = qm - 2r$. Stahl’s conjecture has been confirmed for some values of $k$, $n$ and $m$ [11, 12].

An homomorphism from a graph $G$ into a graph $H$, denoted by $G \to H$, is an edge-preserving map from $V(G)$ to $V(H)$. It is well known that an ordinary graph coloring of a graph $G$ with $m$ colors is an homomorphism from $G$ into the complete graph $K_m$. Similarly, an $n$-tuple coloring of a graph $G$ with $m$ colors is an homomorphism from $G$ into the Kneser graph $K(m, n)$.

A graph $G$ is said hom-idempotent if there is a homomorphism from $G \square G \to G$. We denote by $G \not\to H$ if there exists no homomorphism from $G$ to $H$.

The clique number of a graph $G$, denoted by $\omega(G)$, is the maximum size of a clique in $G$ (i.e., a complete subgraph of $G$). Clearly, for any graphs $G$ and $H$, we have that $\chi(G) \geq \omega(G)$ (and so, $\chi_k(G) \geq \chi_k(K_{\omega(G)}) = k\omega(G)$) and, if there is an homomorphism from $G$ to $H$ then, $\chi(G) \leq \chi(H)$ (and so, $\chi_k(G) \leq \chi_k(H)$).

A stable set $S \subseteq V$ is a subset of pairwise non adjacent vertices of $G$. The stability number of $G$, denoted by $\alpha(G)$, is the largest cardinality of a stable set in $G$. Let $m \geq 2n$. An element $i \in [m]$ is called a centre of a stable set $S$ of the Kneser graph $K(m, n)$ if it lies in each $n$-set in $S$.

**Lemma 1** (Erdős-Ko-Rado [4]). If $m > 2n$, then $\alpha(K(m, n)) = \binom{m-1}{n-1}$. An independent set of $K(m, n)$ with size $\binom{m-1}{n-1}$ has a centre $i$, for some $i \in [m]$.

**Lemma 2** (Hilton-Milner [7]). If $m \geq 2n$, then the maximum size of an stable set in $K(m, n)$ with no centre is equal to $1 + \binom{m-1}{n-1} - \binom{m-n-1}{n-1}$.

A graph $G = (V, E)$ is vertex transitive if its automorphism group acts transitively on $V$, that is, for any pair of distinct vertices of $G$ there is an automorphism mapping one to the other one. It is well known that Kneser graphs are vertex transitive graphs [5].
Lemma 3 (No-Homomorphism Lemma, Albertson-Collins [1]). Let $G, H$ be graphs such that $H$ is vertex transitive and $G \to H$. Then, $\alpha(G)/|V(G)| \geq \alpha(H)/|V(H)|$.

In this paper, we show that equality (1) does not hold in general for $k$-tuple colorings of graphs. In fact, we show that for some values of $k \geq 2$, there are Kneser graphs $K(m,n)$ for which $\chi_k(K(m,n) \Box K(m,n)) > \chi_k(K(m,n))$. Moreover, we also show that there are families of graphs for which equality (1) holds for $k$-tuple colorings of graphs for any $k \geq 1$. As far as we know, our results are the first ones concerning the $k$-tuple chromatic number of cartesian product of graphs.

2 Cartesian products of Kneser graphs

Lemma 4. Let $G$ be a graph and let $k > 0$. Then, $\chi_k(G \Box G) \leq k \chi(G)$.

Proof. Clearly, $\chi_k(G \Box G) \leq k \chi(G \Box G)$. However, by equality (1) we know that $\chi(G \Box G) = \chi(G)$, and thus the lemma holds.

Corollary 1. $\chi_k(K(m,n) \Box K(m,n)) \leq k \chi(K(m,n)) = k(m - 2n + 2)$.

Larose et al. [8] showed that no connected Kneser graph $K(m,n)$ is hom-idempotent, that is, for any $m > 2n$, there is no homomorphism from $K(m,n) \Box K(m,n)$ to $K(m,n)$.

Lemma 5 ([8]). Let $m > 2n$. Then, $K(m,n) \Box K(m,n) \not\to K(m,n)$.

Concerning the $k$-tuple chromatic number of some Kneser graphs, Stahl [11] showed the following results.

Lemma 6 ([11]). If $1 \leq k \leq n$, then $\chi_k(K(m,n)) = m - 2(n - k)$.

Lemma 7 ([11]). $\chi_k(K(2n+1,n)) = 2k + 1 + \lfloor \frac{k-1}{n} \rfloor$, for $k > 0$.

Lemma 8 ([11]). $\chi_{rn}(K(m,n)) = rm$, for $r > 0$ and $m \geq 2n$.

By using Lemma 8 we have the following result.

Lemma 9. Let $m > 2n$. Then, $\chi_{rn}(K(m,n) \Box K(m,n)) > \chi_{rn}(K(m,n))$.

Proof. By Lemma 8 when $r = 1$, we have that $\chi_{rn}(K(m,n)) = m$. If $\chi_{rn}(K(m,n) \Box K(m,n)) = m$, then there exists an homomorphism from the graph $K(m,n) \Box K(m,n)$ to $K(m,n)$ which contradicts Lemma 5.

By Lemma 6, Lemma 9 and by using Corollary 1, we have that,

Corollary 2. Let $n \geq 2$. Then, $2n+2 \leq \chi_{n}(K(2n+1,n) \Box K(2n+1,n)) \leq 3n$. In particular, when $n = 2$, we have that $\chi_{2}(K(5,2) \Box K(5,2)) = 6$.

In the case $k = 2$ we have by Lemma 9, Lemma 6 and by Corollary 1, the following result.

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Corollary 3. Let $q > 0$. Then, $q + 4 \leq \chi_2(K(2n + q, n) \Box K(2n + q, n)) \leq 2q + 4$.

By Corollary 3, notice that in the case when $k = n = 2$ and $q \geq 1$, we must have that $\chi_2(K(q + 4, 2) \Box K(q + 4, 2)) > q + 4$, otherwise there is a contradiction with Lemma 5. This provides a gap of one unity between the 2-tuple chromatic number of the graph $K(q + 4, 2) \Box K(q + 4, 2)$ and the graph $K(q + 4, 2)$. In the next Lemma 10, we will show that such a gap can be as large as desired. However, first we need to introduce the following.

It is well known that the chromatic index of a complete graph $K_{2n}$ (i.e. the minimum number of colors needed to color the edges of $K_{2n}$ such that any two incident edges be assigned different colors) on $2n$ vertices is equal to $2n - 1$ (see [2]), where each color class $i$ (i.e. the subset of pairwise non incident edges colored with color $i$) has size $n$. Therefore, using this fact, we obtain the following result.

Lemma 10. Let $q \geq 1$. Then, the set of vertices of the Kneser graph $K(2q + 4, 2)$ can be partitioned into $2q + 3$ disjoint cliques, each one with size $q + 2$.

Proof. Notice that there is a natural bijection between the vertex set of $K(2q + 4, 2)$ and the edge set of the complete graph $K_{2q+4}$ such that two edges $e, e' \in K_{2q+4}$ are non incident edges if and only if $e \cap e' = \emptyset$. Therefore, a class color of the edge-coloring of $K_{2q+4}$ represents a clique of $K(m,n)$.

Lemma 11. Let $q > 0$. Then, $\chi_2(K(2q + 4, 2) \Box K(2q + 4, 2)) \geq 2q + \lceil \frac{2q}{3} \rceil + 5$.

Proof. First, recall that a stable set $X$ in $K(2q + 4, 2)$ has size at most $2q + 3$ if $X$ has centre (see Lemma 1) and $|X| \leq 1 + (2q + 4 - 1) - (2q + 4 - 2 - 1) = 3$ if $X$ has no centre (see Lemma 2). Besides, by Lemma 10, observe that the vertex set of $K(2q + 4, 2)$ can be partitioned in $2q + 3$ sets $S_1, \ldots, S_{2q+3}$ such that each $S_i$ induces a $K_{q+2}$ for $i = 1, \ldots, 2q+3$. Consider the subgraph $H_i$ of $K(2q + 4, 2) \Box K(2q + 4, 2)$ induced by $S_i \times V(K(2q + 4, 2))$ for $i = 1, \ldots, 2q + 3$. Let $I$ be a stable set in $K(2q + 4, 2)$ and $I_i = I \cap H_i$ for $i = 1, \ldots, 2q + 3$. Then $I_i = I_i \cap \{v\} \times V(K(2q + 4, 2)))$ is a stable set in $K(2q + 4, 2)$ for each $v \in S_i$ and $i = 1, \ldots, 2q + 3$.

Now, assume w.l.o.g that $r (r \leq q + 2)$ stable sets $I_1, \ldots, I_r$ have distinct centre $j_1, \ldots, j_r$, respectively (the case when two of these stable sets have the same centre can be easily reduced to this case). Let $W$ be the set of subsets with size two of $\{j_1, \ldots, j_r\}$. Therefore, for all $m \in \{1, \ldots, r\}$, $I_m - W$ has size at most $2q + 3 - (r - 1) = 2q + 4 - r$ since each centre $j_m$ belongs to $r - 1$ elements in $W$. Besides, each element of $W$ belongs to exactly one set $I_m$ for $m \in \{1, \ldots, r\}$, since $S_j$ induces a complete subgraph. Then, $|I_1 \cup \ldots \cup I_r| \leq \left(\sum_{m=1}^{r} |I_m - W|\right) + |W| \leq r(2q + 4 - r) + \frac{r(r-1)}{2}$.

Next, each remaining stable set (if exist) $I_i^{r+1}, \ldots, I_i^{q+2}$ has no centre, then $|I_i| \leq 3$ for all $d \in \{r+1, \ldots, q+2\}$. Thus, $|I_i| \leq r(2q + 4 - r) + \frac{r(r-1)}{2} + 3(q + 2) = -\frac{r^2}{2} + r(2q + \frac{1}{2}) + 3(q + 2)$. Since the last expression is non decreasing for $r \in \{1, \ldots, q + 2\}$, we have that
\[ |I_i| \leq -\frac{(q+2)^2}{2} + (q+2)(2q + \frac{1}{2}) + 3(q+2) = (q+2)(\frac{3}{2}q + \frac{5}{2}). \]

Therefore, \( |I_i| \leq (q+2)(\frac{3}{2}q + \frac{5}{2}) \) for every \( i = 1, \ldots, 2q + 3 \). Since \( |I| = \sum_{i=1}^{2q+3} |I_i| \), it follows that \( |I| \leq (2q+3)(q+2)(\frac{3}{2}q + \frac{5}{2}) \) and thus,
\[ \alpha(K(2q+4,2) \Box K(2q+4,2)) \leq (2q+3)(q+2)(\frac{3}{2}q + \frac{5}{2}). \]

Let \( t < \frac{2q^2+18q+24}{3q+5} \). Assume that exists a 2-tuple coloring of the graph \( K(2q+4,2) \Box K(2q+4,2) \) with \( 2q + t \) colors. Therefore, there exists an homomorphism from \( K(2q+4,2) \Box K(2q+4,2) \) to \( K(2q+t,2) \).

Now, from the well known no-homomorphism Lemma 3, we have that,
\[ \alpha(K(2q+4,2) \Box K(2q+4,2)) \geq \frac{\alpha(K(2q+t,2))v(K(2q+4,2) \Box K(2q+4,2))}{|V(K(2q+t,2))|}. \]

Then, \( \alpha(K(2q+4,2) \Box K(2q+4,2)) \geq \frac{(2q+t-1)(2q+4)^2(2q+3)^2}{4(2q+t)(2q+t-1)} = \frac{(2q+4)^2(2q+3)^2}{2(2q+t)}. \)

Let us see that \( \frac{(2q+4)^2(2q+3)^2}{2(2q+t)} \) is greater than \( (2q+3)(q+2)(\frac{3}{2}q + \frac{5}{2}) \), which is a contradiction.

To this end, observe that if \( t < \frac{2q^2+18q+24}{3q+5} \), then
\[ (2q+3)(q+2)(\frac{3}{2}q + \frac{5}{2})(2q+t) = (2q+3)(q+2)((3q^2 + 5q) + (\frac{3}{2}q + \frac{5}{2})t) < (2q+3)(q+2)((3q^2 + 5q) + (q^2 + 9q + 12)) = (2q+3)(q+2)(4q^2 + 14q + 12) = \frac{(2q+4)^2(2q+3)^2}{2}. \]

Therefore, \( \chi_2(K(2q+4,2) \Box K(2q+4,2)) \geq 2q + t + 1. \)

**Claim 1.** Let \( q > 0 \) be an integer. Then, \( \frac{2q^2+18q+24}{3q+5} \) is not an integer.

**Proof.** By polynomial division, we have that \( 2q^2 + 18q + 24 = (3q + 5)(\frac{2}{3}q + \frac{44}{9}) - \frac{4}{9} \). If \( (2q^2 + 18q + 24)/(3q + 5) \) is an integer, then \( k = (\frac{2}{3}q + \frac{44}{9}) - \frac{4}{9(3q+5)} \) is also an integer. Multiplying by 9 both terms in the last equality we have, \( 6q + 44 = 9k + \frac{4}{3q+5} \). As \( q > 0 \) then, \( 0 < \frac{4}{3q+5} < 1 \) contradicting the assumption that \( k \) is an integer.

Finally, since \( \frac{2}{3}q + 4 < \frac{2q^2+18q+24}{3q+5} \) and, by previous Claim 1, \( \frac{2q^2+18q+24}{3q+5} \) is not an integer, then \( \left\lceil \frac{2}{3}q \right\rceil + 4 < \frac{2q^2+18q+24}{3q+5} \) and thus, we have finally that,
\[ 2q + \left\lceil \frac{2}{3}q \right\rceil + 5 \leq \chi_2(K(2q+4,2)). \]

As a corollary of Lemma 11 and by Corollary 1, we obtain the following result.

**Corollary 4.** \( \chi_2(K(6,2) \Box K(6,2)) = 8. \)

**Theorem 1.** Let \( k > n \) and let \( t = \chi_k(K(m,n) \Box K(m,n)) \), where \( m > 2n \). Then, either \( t > m + 2(k-n) \) or \( t < m + (k-n) \).
Proof. Suppose that \( m+(k-n) \leq t \leq m+2(k-n) \). Therefore, there exists an homomorphism \( K(m,n) \square K(m,n) \to K(t,k) \). Now, Stahl [11] showed that there is an homomorphism \( K(m,n) \to K(m-2,n-1) \) whenever \( n > 1 \) and \( m \geq 2n \). Moreover, it’s easy to see that there is an homomorphism \( K(m,n) \to K(m-1,n-1) \). By applying the former homomorphism \( t-(m+(k-n)) \) times to the graph \( K(t,k) \) we obtain an homomorphism \( K(t,k) \to K(2(m+k-n)-t, 2k+m-n-t) \). Finally, by applying \( 2k+m-t-2n \) times the latter homomorphism to the graph \( K(2(m+k-n)-t, 2k+m-n-t) \) we obtain an homomorphism \( K(2(m+k-n)-t, 2k+m-n-t) \to K(m,n) \). Therefore, by homomorphism composition, \( K(m,n) \square K(m,n) \to K(m,n) \) which contradicts Lemma 5.

We can also obtain a lower bound for the \( k \)-tuple chromatic number of the graph \( K(m,n) \square K(m,n) \) in terms of the clique number of \( K(m,n) \). In fact, notice that \( \omega(K(m,n) \square K(m,n)) = \omega(K(m,n)) = \lceil \frac{m}{n} \rceil \). Thus, we have the following result.

**Theorem 2.** Let \( k > n \). Then, \( \chi_k(K(m,n) \square K(m,n)) \geq k \omega(K(m,n)) = k \lceil \frac{m}{n} \rceil \). In particular, if \( n \) divides \( m \) then, \( \chi_k(K(m,n) \square K(m,n)) \geq m + (k-n) \frac{m}{n} \).

### 3 Cases where \( \chi_k(G \square H) = \max\{\chi_k(G), \chi_k(H)\} \)

**Theorem 3.** Let \( G \) and \( H \) be graphs such that \( \chi(G) \leq \chi(H) = \omega(H) \). Then, \( \chi_k(G \square H) = \max\{\chi_k(G), \chi_k(H)\} \).

**Proof.** Let \( t = \omega(H) \) and let \( \{h_1, \ldots, h_t\} \) be the vertex set of a maximum clique \( K_t \) in \( H \) with size \( t \). Clearly, \( \chi_k(G) \leq \chi_k(H) = \chi_k(K_t) \). Let \( \rho \) be a \( k \)-tuple coloring of \( H \) with \( \chi_k(H) \) colors. By equality (1), there exists a \( t \)-coloring \( f \) of \( G \square H \). Therefore, the assignment of the \( k \)-set \( \rho(h_f((a,b))) \) to each vertex \((a,b)\) in \( G \square H \) defines a \( k \)-tuple coloring of \( G \square H \) with \( \chi_k(K_t) \) colors.

Notice that if \( G \) and \( H \) are both bipartite, then \( \chi_k(G \square H) = \chi_k(G) = \chi_k(H) \). In the case when \( G \) is not a bipartite graph, we have the following results.

An automorphism \( \sigma \) of a graph \( G \) is called a shift of \( G \) if \( \{u, \sigma(u)\} \in E(G) \) for each \( u \in V(G) \) [8]. In other words, a shift of \( G \) maps every vertex to one of its neighbors.

**Theorem 4.** Let \( G \) be a non bipartite graph having a shift \( \sigma \in AUT(G) \), and let \( H \) be a bipartite graph. Then, \( \chi_k(G \square H) = \max\{\chi_k(G), \chi_k(H)\} \).

**Proof.** Let \( A \cup B \) be a bipartition of the vertex set of \( H \). Let \( f \) be a \( k \)-tuple coloring of \( G \) with \( \chi_k(G) \) colors. Clearly, \( \chi_k(G) \geq \chi_k(H) \). We define a \( k \)-tuple coloring \( \rho \) of \( G \square H \) with \( \chi_k(G) \) colors as follows: for any vertex \((u,v)\) of \( G \square H \) with \( u \in G \) and \( v \in H \), define \( \rho((u,v)) = f(u) \) if \( v \in A \), and \( \rho((u,v)) = f(\sigma(u)) \) if \( v \in B \).

We may also deduce the following direct result.

**Theorem 5.** Let \( G \) be an hom-idempotent graph and let \( H \) be a subgraph of \( G \). Thus, \( \chi_k(G \square H) = \max\{\chi_k(G), \chi_k(H)\} = \chi_k(G) \).
Let $A$ be a group and $S$ a subset of $A$ that is closed under inverses and does not contain the identity. The Cayley graph $\text{Cay}(A, S)$ is the graph whose vertex set is $A$, two vertices $u, v$ being joined by an edge if $u^{-1}v \in S$. If $a^{-1}Sa = S$ for all $a \in A$, then $\text{Cay}(A, S)$ is called a normal Cayley graph.

**Lemma 12** ([6]). Any normal Cayley graph is hom-idempotent.

Note that all Cayley graphs on abelian groups are normal, and thus hom-idempotents. In particular, the circulant graphs are Cayley graphs on cyclic groups (i.e., cycles, powers of cycles, complements of powers of cycles, complete graphs, etc). By Theorem 5 and Lemma 12 we have the following result.

**Theorem 6.** Let $\text{Cay}(A, S)$ be a normal Cayley graph and let $\text{Cay}(A', S')$ be a subgraph of $\text{Cay}(A, S)$, with $A' \subseteq A$ and $S' \subseteq S$. Then, $\chi_k(\text{Cay}(A, S) \boxtimes \text{Cay}(A', S')) = \max\{\chi_k(\text{Cay}(A, S)), \chi_k(\text{Cay}(A', S'))\}$.

**Definition 1.** Let $G$ be a graph with a shift $\sigma$. We define the order of $\sigma$ as the minimum integer $i$ such that $\sigma^i$ is equal to the identity permutation.

**Theorem 7.** Let $G$ be a graph with a shift $\sigma$ of minimum odd order $2s + 1$ and let $C_{2t+1}$ be a cycle graph, where $t \geq s$. Then, $\chi_k(G \boxtimes C_{2t+1}) = \max\{\chi_k(G), \chi_k(C_{2t+1})\}$.

**Proof.** Let $\{0, \ldots, 2t\}$ be the vertex set of $C_{2t+1}$, where for $0 \leq i \leq 2t$, $\{i, i + 1 \mod n\} \in E(C_{2t+1})$. Let $G_i$ be the $i$th copy of $G$ in $G \boxtimes C_{2t+1}$, that is, for each $0 \leq i \leq 2t$, $G_i = \{(g, i) : g \in G\}$. Let $f$ be a $k$-tuple coloring of $G$ with $\chi_k(G)$ colors. We define a $k$-tuple coloring of $G \boxtimes C_{2t+1}$ with $\chi_k(G)$ colors as follows: let $\sigma^0$ denotes the identity permutation of the vertices in $G$. Now, for $0 \leq i \leq 2s$, assign to each vertex $(u, i) \in G_i$ the $k$-tuple $f(\sigma^i(u))$. For $2s + 1 \leq j \leq 2t$, assign to each vertex $(u, j) \in G_j$ the $k$-tuple $f(u)$ if $j$ is odd, otherwise, assign to $(u, j)$ the $k$-tuple $f(\sigma^1(u))$. It’s not difficult to see that this is in fact a proper $k$-tuple coloring of $G \boxtimes C_{2t+1}$. \hfill $\square$

**References**


