k-tuple chromatic number of the cartesian product of graphs
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To cite this version:
Flavia Bonomo, Ivo Koch, Pablo Torres, Mario Valencia-Pabon. k-tuple chromatic number of the cartesian product of graphs. 2014. <hal-01103534>

HAL Id: hal-01103534
https://hal.archives-ouvertes.fr/hal-01103534
Submitted on 14 Jan 2015

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\textit{k}-tuple chromatic number of the cartesian product of graphs \footnote{Partially supported by MathAmSud Project 13MATH-07 (Argentina–Brazil–Chile–France), UBACyT Grant 20020130100808BA, CONICET PIP 112-200901-00178 and 112-201201-00450CO and ANPCyT PICT 2012-1324 (Argentina)}

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\begin{abstract}
A \textit{k}-tuple coloring of a graph $G$ assigns a set of $k$ colors to each vertex of $G$ such that if two vertices are adjacent, the corresponding sets of colors are disjoint. The \textit{k}-tuple chromatic number of $G$, $\chi_k(G)$, is the smallest $t$ so that there is a \textit{k}-tuple coloring of $G$ using $t$ colors. It is well known that $\chi(G \square H) = \max\{\chi(G), \chi(H)\}$. In this paper, we show that there exist graphs $G$ and $H$ such that $\chi_k(G \square H) > \max\{\chi_k(G), \chi_k(H)\}$ for $k \geq 2$. Moreover, we also show that there exist graph families such that, for any $k \geq 1$, the \textit{k}-tuple chromatic number of their cartesian product is equal to the maximum \textit{k}-tuple chromatic number of its factors.

\textbf{Keywords:} \textit{k}-tuple colorings, Cartesian product of graphs, Kneser graphs, Cayley graphs, Hom-idempotent graphs.
\end{abstract}

\section{Introduction}

A classic \textit{coloring} of a graph $G$ is an assignment of colors (or natural numbers) to the vertices of $G$ such that any two adjacent vertices are assigned different colors. The smallest number $t$ such that $G$ admits a coloring with $t$ colors (a \textit{t-coloring}) is called the \textit{chromatic number} of $G$ and is denoted by $\chi(G)$. Several generalizations of the coloring problem have been introduced in the literature, in particular, cases in which each vertex is assigned not only a color but a set of colors, under different restrictions. One of these variations is the \textit{k}-tuple coloring introduced independently by Stahl \cite{stahl} and Bollobás and Thomason \cite{bollobas_thomason}. A \textit{k}-tuple
A graph coloring of a graph $G$ is an assignment of $k$ colors to each vertex in such a way that adjacent vertices are assigned distinct colors. The $k$-tuple coloring problem consists into finding the minimum number of colors in a $k$-tuple coloring of a graph $G$, which we denote by $\chi_k(G)$. The cartesian product $G \square H$ of two graphs $G$ and $H$ has vertex set $V(G) \times V(H)$, two vertices being joined by an edge whenever they have one coordinate equal and the other adjacent. This product is commutative and associative up to isomorphism. There is a simple formula expressing the chromatic number of a cartesian product in terms of its factors:

$$\chi(G \square H) = \max\{\chi(G), \chi(H)\}. \quad (1)$$

The identity (1) admits a simple proof first given by Sabidussi [10]. The Kneser graph $K(m,n)$ has as vertices all $n$-element subsets of the set $[m] = \{1, \ldots, m\}$ and an edge between two subsets if and only if they are disjoint. We will assume in the rest of this work that $m \geq 2n$, otherwise $K(m,n)$ has no edges. The Kneser graph $K(5,2)$ is the well known Petersen Graph. Lovász [9] showed that $\chi(K(m,n)) = m - 2n + 2$. The value of the $k$-tuple chromatic number of the Kneser graph is the subject of an almost 40-year-old conjecture of Stahl [11] which asserts that: if $k = qn - r$ where $q \geq 0$ and $0 \leq r < n$, then $\chi_k(K(m,n)) = qm - 2r$. Stahl’s conjecture has been confirmed for some values of $k$, $n$ and $m$ [11, 12].

An homomorphism from a graph $G$ into a graph $H$, denoted by $G \to H$, is an edge-preserving map from $V(G)$ to $V(H)$. It is well known that an ordinary graph coloring of a graph $G$ with $m$ colors is an homomorphism from $G$ into the complete graph $K_m$. Similarly, an $n$-tuple coloring of a graph $G$ with $m$ colors is an homomorphism from $G$ into the Kneser graph $K(m,n)$.

A graph $G$ is said hom-idempotent if there is a homomorphism from $G \square G \to G$. We denote by $G \not\to H$ if there exists no homomorphism from $G$ to $H$.

The clique number of a graph $G$, denoted by $\omega(G)$, is the maximum size of a clique in $G$ (i.e., a complete subgraph of $G$). Clearly, for any graphs $G$ and $H$, we have that $\chi(G) \geq \omega(G)$ (and so, $\chi_k(G) \geq \chi_k(K_{\omega(G)}) = k\omega(G)$) and, if there is an homomorphism from $G$ to $H$ then, $\chi(G) \leq \chi(H)$ (and so, $\chi_k(G) \leq \chi_k(H)$).

A stable set $S \subseteq V$ is a subset of pairwise non adjacent vertices of $G$. The stability number of $G$, denoted by $\alpha(G)$, is the largest cardinality of a stable set in $G$. Let $m \geq 2n$. An element $i \in [m]$ is called a centre of a stable set $S$ of the Kneser graph $K(m,n)$ if it lies in each $n$-set in $S$.

**Lemma 1** (Erdős-Ko-Rado [4]). If $m > 2n$, then $\alpha(K(m,n)) = \binom{m-1}{n-1}$. An independent set of $K(m,n)$ with size $\binom{m-1}{n-1}$ has a centre $i$, for some $i \in [m]$.

**Lemma 2** (Hilton-Milner [7]). If $m \geq 2n$, then the maximum size of an stable set in $K(m,n)$ with no centre is equal to $1 + \binom{m-1}{n-1} - \binom{m-n-1}{n-1}$.

A graph $G = (V,E)$ is vertex transitive if its automorphism group acts transitively on $V$, that is, for any pair of distinct vertices of $G$ there is an automorphism mapping one to the other one. It is well known that Kneser graphs are vertex transitive graphs [5].
Lemma 3 (No-Homomorphism Lemma, Albertson-Collins [1]). Let $G, H$ be graphs such that $H$ is vertex transitive and $G \to H$. Then, $\alpha(G)/|V(G)| \geq \alpha(H)/|V(H)|$.

In this paper, we show that equality (1) does not hold in general for $k$-tuple colorings of graphs. In fact, we show that for some values of $k \geq 2$, there are Kneser graphs $K(m,n)$ for which $\chi_k(K(m,n)\Box K(m,n)) > \chi_k(K(m,n))$. Moreover, we also show that there are families of graphs for which equality (1) holds for $k$-tuple colorings of graphs for any $k \geq 1$. As far as we know, our results are the first ones concerning the $k$-tuple chromatic number of cartesian product of graphs.

2 Cartesian products of Kneser graphs

Lemma 4. Let $G$ be a graph and let $k > 0$. Then, $\chi_k(G\Box G) \leq k\chi(G)$.

Proof. Clearly, $\chi_k(G\Box G) \leq k\chi(G\Box G)$. However, by equality (1) we know that $\chi(G\Box G) = \chi(G)$, and thus the lemma holds. \qed

Corollary 1. $\chi_k(K(m,n)\Box K(m,n)) \leq k\chi(K(m,n)) = k(m-2n+2)$.

Larose et al. [8] showed that no connected Kneser graph $K(m,n)$ is hom-idempotent, that is, for any $m > 2n$, there is no homomorphism from $K(m,n)\Box K(m,n)$ to $K(m,n)$.

Lemma 5 ([8]). Let $m > 2n$. Then, $K(m,n)\Box K(m,n) \not\to K(m,n)$.

Concerning the $k$-tuple chromatic number of some Kneser graphs, Stahl [11] showed the following results.

Lemma 6 ([11]). If $1 \leq k \leq n$, then $\chi_k(K(m,n)) = m-2(n-k)$.

Lemma 7 ([11]). $\chi_k(K(2n+1,n)) = 2k + 1 + \lfloor \frac{k-1}{n} \rfloor$, for $k > 0$.

Lemma 8 ([11]). $\chi_r n(K(m,n)) = rm$, for $r > 0$ and $m \geq 2n$.

By using Lemma 8 we have the following result.

Lemma 9. Let $m > 2n$. Then, $\chi_n(K(m,n)\Box K(m,n)) > \chi_n(K(m,n))$.

Proof. By Lemma 8 when $r = 1$, we have that $\chi_n(K(m,n)) = m$. If $\chi_n(K(m,n)\Box K(m,n)) = m$, then there exists an homomorphism from the graph $K(m,n)\Box K(m,n)$ to $K(m,n)$ which contradicts Lemma 5. \qed

By Lemma 6, Lemma 9 and by using Corollary 1, we have that,

Corollary 2. Let $n \geq 2$. Then, $2n+2 \leq \chi_n(K(2n+1,n)\Box K(2n+1,n)) \leq 3n$. In particular, when $n = 2$, we have that $\chi_2(K(5,2)\Box K(5,2)) = 6$.

In the case $k = 2$ we have by Lemma 9, Lemma 6 and by Corollary 1, the following result.
Corollary 3. Let \( q > 0 \). Then, \( q + 4 \leq \chi_2(K(2n + q, n) \Box K(2n + q, n)) \leq 2q + 4 \).

By Corollary 3, notice that in the case when \( k = n = 2 \) and \( q \geq 1 \), we must have that \( \chi_2(K(q + 4, 2) \Box K(q + 4, 2)) > q + 4 \), otherwise there is a contradiction with Lemma 5. This provides a gap of one unity between the 2-tuple chromatic number of the graph \( K(q + 4, 2) \Box K(q + 4, 2) \) and the graph \( K(q + 4, 2) \). In the next Lemma 11, we will show that such a gap can be as large as desired. However, first we need to introduce the following.

It is well known that the chromatic index of a complete graph \( K_{2n} \) (i.e. the minimum number of colors needed to color the edges of \( K_{2n} \) such that any two incident edges be assigned different colors) on \( 2n \) vertices is equal to \( 2n - 1 \) (see [2]), where each color class \( i \) (i.e. the subset of pairwise non incident edges colored with color \( i \)) has size \( n \). Therefore, using this fact, we obtain the following result.

Lemma 10. Let \( q \geq 1 \). Then, the set of vertices of the Kneser graph \( K(2q + 4, 2) \) can be partitioned into \( 2q + 3 \) disjoint cliques, each one with size \( q + 2 \).

Proof. Notice that there is a natural bijection between the vertex set of \( K(2q + 4, 2) \) and the edge set of the complete graph \( \mathcal{K}_{2q+4} \) with vertex set \([2q + 4] : \) each vertex \( \{i, j\} \) in \( K(2q + 4, 2) \) is mapped to the edge \( \{i, j\} \) in \( \mathcal{K}_{2q+4} \). Now, there is a \((2q + 3)\)-edge coloring of \( \mathcal{K}_{2q+4} \) where each class color is a set of pairwise non incident edges with size \( q + 2 \). Notice that two edges \( e, e^\prime \in \mathcal{K}_{2q+4} \) are non incident edges if and only if \( e \cap e^\prime = \emptyset \). Therefore, a class color of the edge-coloring of \( \mathcal{K}_{2q+4} \) represents a clique of \( K(m, n) \).

Lemma 11. Let \( q > 0 \). Then, \( \chi_2(K(2q + 4, 2) \Box K(2q + 4, 2)) \geq 2q + \left\lceil \frac{2}{3}q \right\rceil + 5 \).

Proof. First, recall that a stable set \( X \) in \( K(2q + 4, 2) \) has size at most \( 2q + 3 \) if \( X \) has centre (see Lemma 1) and \( |X| \leq 1 + (2q + 4 - 1) - (2q + 4 - 2 - 1) = 3 \) if \( X \) has no centre (see Lemma 2). Besides, by Lemma 10, observe that the vertex set of \( K(2q + 4, 2) \) can be partitioned in \( 2q + 3 \) sets \( S_1, \ldots, S_{2q+3} \) such that each \( S_i \) induces a \( K_{q+2} \) for \( i = 1, \ldots, 2q+3 \). Consider the subgraph \( H_i \) of \( K(2q + 4, 2) \Box K(2q + 4, 2) \) induced by \( S_i \times V(K(2q + 4, 2)) \) for \( i = 1, \ldots, 2q + 3 \). Let \( I \) be a stable set in \( K(2q + 4, 2) \) and \( I_i = I \cap H_i \) for \( i = 1, \ldots, 2q + 3 \). Then \( I_i \cap (\{v\} \times V(K(2q + 4, 2))) \) is a stable set in \( K(2q + 4, 2) \) for each \( v \in S_i \) and \( i = 1, \ldots, 2q + 3 \).

Now, assume w.l.o.g. that \( r (r \leq q + 2) \) stable sets \( I_1^r, \ldots, I_r^r \) have distinct centre \( j_1, \ldots, j_r \), respectively (the case when two of these stable sets have the same centre can be easily reduced to this case). Let \( W \) be the set of subsets with size two of \( \{j_1, \ldots, j_r\} \). Therefore, for all \( m \in \{1, \ldots, r\} \), \( I_i^m - W \) has size at most \( 2q + 3 - (r - 1) = 2q + 4 - r \) since each centre \( j_m \) belongs to \( r - 1 \) elements in \( W \). Besides, each element of \( W \) belongs to exactly one set \( I_i^m \) for \( m \in \{1, \ldots, r\} \), since \( S_j \) induces a complete subgraph. Then, \( |I_i^1 \cup \ldots \cup I_i^r| \leq \sum_{m=1}^{r} |I_i^m| - |W| = r(2q + 4 - r) + \frac{r(r-1)}{2} \).

Next, each remaining stable set (if exist) \( I_i^{r+1}, \ldots, I_i^{q+2} \) has no centre, then \( |I_i| \leq 3 \) for all \( d \in \{r+1, \ldots, q+2\} \). Thus, \( |I_i| \leq r(2q + 4 - r) + \frac{r(r-1)}{2} + 3(q + 2 - r) = -\frac{r^2}{2} + r(2q + \frac{1}{2}) + 3(q + 2) \).

Since the last expression is non decreasing for \( r \in \{1, \ldots, q + 2\} \), we have that
\(|I_i| \leq -\frac{(q+2)^2}{2} + (q + 2)(2q + \frac{1}{2}) + 3(q + 2) = (q + 2)(\frac{3}{2}q + \frac{5}{2})\).

Therefore, \(|I_i| \leq (q + 2)(\frac{3}{2}q + \frac{5}{2})\) for every \(i = 1, \ldots, 2q + 3\). Since \(|I| = \sum_{i=1}^{2q+3} |I_i|\), it follows that \(|I| \leq (2q + 3)(q + 2)(\frac{3}{2}q + \frac{5}{2})\) and thus, 
\(\alpha(K(2q + 4, 2) \sqcap K(2q + 4, 2)) \leq (2q + 3)(q + 2)(\frac{3}{2}q + \frac{5}{2})\).

Let \(t < \frac{2q^2 + 18q + 24}{3q + 5}\). Assume that exists a 2-tuple coloring of the graph \(K(2q + 4, 2) \sqcap K(2q + 4, 2)\) with \(2q + t\) colors. Therefore, there exists an homomorphism from \(K(2q + 4, 2) \sqcap K(2q + 4, 2)\) to \(K(2q + t, 2)\).

Now, from the well known no-homomorphism Lemma 3, we have that,
\[\alpha(K(2q + 4, 2) \sqcap K(2q + 4, 2)) \geq \alpha(K(2q+t,2))V(K(2q+t,2))|K(2q+t,2)|\text{.}\]

Then, \(\alpha(K(2q + 4, 2) \sqcap K(2q + 4, 2)) \geq \frac{(2q+3)(q+2)(3q^2+5q)}{4(2q+t)(2q+t-1)} = \frac{(2q+4)^2(2q+3)^2}{2(2q+t)}\).

Let us see that \(\frac{(2q+4)^2(2q+3)^2}{2(2q+t)}\) is greater than \((2q+3)(q+2)(\frac{3}{2}q + \frac{5}{2})\), which is a contradiction.

To this end, observe that if \(t < \frac{2q^2+18q+24}{3q+5}\), then
\[(2q + 3)(q + 2)(\frac{3}{2}q + \frac{5}{2})(2q + t) = (2q + 3)(q + 2)((3q^2 + 5q) + (\frac{5}{2}q + \frac{5}{2})t) < (2q + 3)(q + 2)((3q^2 + 5q) + (q^2 + 9q + 12)) = (2q + 3)(q + 2)(4q^2 + 14q + 12)) = \frac{(2q+4)^2(2q+3)^2}{2}\).

Therefore, \(\chi_2(K(2q + 4, 2) \sqcap K(2q + 4, 2)) \geq 2q + t + 1\).

**Claim 1.** Let \(q > 0\) be an integer. Then, \(\frac{2q^2+18q+24}{3q+5}\) is not an integer.

**Proof.** By polynomial division, we have that \(2q^2 + 18q + 24 = (3q + 5)(\frac{2}{3}q + \frac{4}{5}) - \frac{4}{5}\). If \((2q^2 + 18q + 24)/(3q + 5)\) is an integer, then \(k = (\frac{2}{3}q + \frac{4}{5}) - \frac{4}{5(3q+5)}\) is also an integer. Multiplying by 9 both terms in the last equality we have, \(6q + 44 = 9k + \frac{4}{3q+5}\). As \(q > 0\) then, \(0 < \frac{4}{3q+5} < 1\) contradicting the assumption that \(k\) is an integer.

Finally, since \(\frac{2}{3}q + 4 < \frac{2q^2+18q+24}{3q+5}\) and, by previous Claim 1, \(\frac{2q^2+18q+24}{3q+5}\) is not an integer, then \(\lceil \frac{2}{3}q \rceil + 4 < \frac{2q^2+18q+24}{3q+5}\) and thus, we have finally that,

\[2q + \lceil \frac{2}{3}q \rceil + 5 \leq \chi_2(K(2q + 4, 2))\]

As a corollary of Lemma 11 and by Corollary 1, we obtain the following result.

**Corollary 4.** \(\chi_2(K(6, 2) \sqcap K(6, 2)) = 8\).

**Theorem 1.** Let \(k > n\) and let \(t = \chi_k(K(m, n) \sqcap K(m, n))\), where \(m > 2n\). Then, either \(t > m + 2(k - n)\) or \(t < m + (k - n)\).
Proof. Suppose that \( m+(k-n) \leq t \leq m+2(k-n) \). Therefore, there exists an homomorphism \( K(m,n) \cong K(m,n) \to K(t,k) \). Now, Stahl [11] showed that there is an homomorphism \( K(m,n) \to K(m-2,n-1) \) whenever \( n > 1 \) and \( m \geq 2n \). Moreover, it's easy to see that there is an homomorphism \( K(m,n) \to K(m-1,n-1) \). By applying the former homomorphism \( t-(m+(k-n)) \) times to the graph \( K(t,k) \) we obtain an homomorphism \( K(t,k) \to K(2(m+k-n)-t,2k+m-n-t) \). Finally, by applying \( 2k+m-t-2n \) times the latter homomorphism to the graph \( K(2(m+k-n)-t,2k+m-n-t) \) we obtain an homomorphism \( K(2(m+k-n)-t,2k+m-n-t) \to K(m,n) \). Therefore, by homomorphism composition, \( K(m,n) \cong K(m,n) \to K(m,n) \) which contradicts Lemma 5. \( \Box \)

We can also obtain a lower bound for the \( k \)-tuple chromatic number of the graph \( K(m,n) \cong K(m,n) \) in terms of the clique number of \( K(m,n) \). In fact, notice that

\[
\omega(K(m,n) \cong K(m,n)) = \omega(K(m,n)) = \lfloor \frac{m}{n} \rfloor.
\]

Thus, we have the following result.

**Theorem 2.** Let \( k > n \). Then, \( \chi_k(K(m,n) \cong K(m,n)) \geq k \omega(K(m,n)) = k \lfloor \frac{m}{n} \rfloor \). In particular, if \( n \) divides \( m \) then, \( \chi_k(K(m,n) \cong K(m,n)) \geq m + (k-n) \frac{m}{n} \).

### 3 Cases where \( \chi_k(G \cong H) = \max\{\chi_k(G), \chi_k(H)\} \)

**Theorem 3.** Let \( G \) and \( H \) be graphs such that \( \chi(G) \leq \chi(H) = \omega(H) \). Then, \( \chi_k(G \cong H) = \max\{\chi_k(G), \chi_k(H)\} \).

**Proof.** Let \( t = \omega(H) \) and let \( \{h_1, \ldots, h_t\} \) be the vertex set of a maximum clique \( K_t \) in \( H \) with size \( t \). Clearly, \( \chi_k(G) \leq \chi_k(H) = \chi_k(K_t) \). Let \( \rho \) be a \( k \)-tuple coloring of \( H \) with \( \chi_k(H) \) colors. By equality (1), there exists a \( t \)-coloring \( f \) of \( G \cong H \). Therefore, the assignment of the \( k \)-set \( \rho(f(h_{i(a,b)}) \) to each vertex \( (a,b) \) in \( G \cong H \) defines a \( k \)-tuple coloring of \( G \cong H \) with \( \chi_k(K_t) \) colors. \( \Box \)

Notice that if \( G \) and \( H \) are both bipartite, then \( \chi_k(G \cong H) = \chi_k(G) = \chi_k(H) \). In the case when \( G \) is not a bipartite graph, we have the following results.

An automorphism \( \sigma \) of a graph \( G \) is called a shift of \( G \) if \( \{u, \sigma(u)\} \in E(G) \) for each \( u \in V(G) \) [8]. In other words, a shift of \( G \) maps every vertex to one of its neighbors.

**Theorem 4.** Let \( G \) be a non bipartite graph having a shift \( \sigma \in \text{AUT}(G) \), and let \( H \) be a bipartite graph. Then, \( \chi_k(G \cong H) = \max\{\chi_k(G), \chi_k(H)\} \).

**Proof.** Let \( A \cup B \) be a bipartition of the vertex set of \( H \). Let \( f \) be a \( k \)-tuple coloring of \( G \) with \( \chi_k(G) \) colors. Clearly, \( \chi_k(G) \geq \chi_k(H) \). We define a \( k \)-tuple coloring \( \rho \) of \( G \cong H \) with \( \chi_k(G) \) colors as follows: for any vertex \( (u,v) \) of \( G \cong H \) with \( u \in G \) and \( v \in H \), define \( \rho((u,v)) = f(u) \) if \( v \in A \), and \( \rho((u,v)) = f(\sigma(u)) \) if \( v \in B \). \( \Box \)

We may also deduce the following direct result.

**Theorem 5.** Let \( G \) be an hom-idempotent graph an let \( H \) be a subgraph of \( G \). Thus, \( \chi_k(G \cong H) = \max\{\chi_k(G), \chi_k(H)\} = \chi_k(G) \).
Let $A$ be a group and $S$ a subset of $A$ that is closed under inverses and does not contain the identity. The Cayley graph $\text{Cay}(A, S)$ is the graph whose vertex set is $A$, two vertices $u, v$ being joined by an edge if $u^{-1}v \in S$. If $a^{-1}Sa = S$ for all $a \in A$, then $\text{Cay}(A, S)$ is called a normal Cayley graph.

**Lemma 12** ([6]). Any normal Cayley graph is hom-idempotent.

Note that all Cayley graphs on abelian groups are normal, and thus hom-idempotents. In particular, the circulant graphs are Cayley graphs on cyclic groups (i.e., cycles, powers of cycles, complements of powers of cycles, complete graphs, etc). By Theorem 5 and Lemma 12 we have the following result.

**Theorem 6.** Let $\text{Cay}(A, S)$ be a normal Cayley graph and let $\text{Cay}(A', S')$ be a subgraph of $\text{Cay}(A, S)$, with $A' \subseteq A$ and $S' \subseteq S$. Then, $\chi_k(\text{Cay}(A, S) \square \text{Cay}(A', S')) = \max\{\chi_k(\text{Cay}(A, S)), \chi_k(\text{Cay}(A', S'))\}$.

**Definition 1.** Let $G$ be a graph with a shift $\sigma$. We define the order of $\sigma$ as the minimum integer $i$ such that $\sigma^i$ is equal to the identity permutation.

**Theorem 7.** Let $G$ be a graph with a shift $\sigma$ of minimum odd order $2s + 1$ and let $C_{2t+1}$ be a cycle graph, where $t \geq s$. Then, $\chi_k(G \square C_{2t+1}) = \max\{\chi_k(G), \chi_k(C_{2t+1})\}$.

**Proof.** Let $\{0, \ldots, 2t\}$ be the vertex set of $C_{2t+1}$, where for $0 \leq i \leq 2t$, $\{i, i + 1 \mod n\} \in E(C_{2t+1})$. Let $G_i$ be the $i$th copy of $G$ in $G \square C_{2t+1}$, that is, for each $0 \leq i \leq 2t$, $G_i = \{(g, i) : g \in G\}$. Let $f$ be a $k$-tuple coloring of $G$ with $\chi_k(G)$ colors. We define a $k$-tuple coloring of $G \square C_{2t+1}$ with $\chi_k(G)$ colors as follows: let $\sigma^0$ denotes the identity permutation of the vertices in $G$. Now, for $0 \leq i \leq 2s$, assign to each vertex $(u, i) \in G_i$ the $k$-tuple $f(\sigma^i(u))$. For $2s + 1 \leq j \leq 2t$, assign to each vertex $(u, j) \in G_j$ the $k$-tuple $f(u)$ if $j$ is odd, otherwise, assign to $(u, j)$ the $k$-tuple $f(\sigma^1(u))$. It’s not difficult to see that this is in fact a proper $k$-tuple coloring of $G \square C_{2t+1}$. \qed

**References**


