Weak approximation of martingale representations

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Abstract

We present a systematic method for computing explicit approximations to martingale representations for a large class of Brownian functionals. The approximations are based on a notion of pathwise functional derivative and yield a consistent estimator for the integrand in the martingale representation formula for any square-integrable functional of the solution of an SDE with path-dependent coefficients. Explicit convergence rates are derived for functionals which are Lipschitz-continuous in the supremum norm. The approximation and the proof of its convergence are based on the Functional Ito calculus, and require neither the Markov property, nor any differentiability conditions on the coefficients of the stochastic differential equations involved.

Contents

1 Introduction

Let W be a standard d-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\mathcal{F}_t^W)$ its ($\mathbb{P}$-completed) natural filtration. We consider a Brownian martingale X which takes values in $\mathbb{R}^d$:

$$X(t) = X(0) + \int_0^t \sigma(u) dW(u)$$

where σ is a process adapted to $(\mathcal{F}_t^W)$ satisfying

$$\mathbb{E}\left[\int_0^T \|\sigma(t)\|^2 dt\right] < \infty \quad \text{and} \quad \det(\sigma(t)) \neq 0 \quad dt \times d\mathbb{P} - a.e$$

Then X is a square-integrable martingale with the predictable representation property [23, 32]: for any square-integrable $\mathcal{F}_T^W$-measurable random variable H, or equivalently, any square-integrable $(\mathcal{F}_t^W)$-martingale $Y(t) = \mathbb{E}[H|\mathcal{F}_t^W]$, there exists a unique $(\mathcal{F}_t^W)$-predictable process ϕ with $\mathbb{E}\left[\int_0^T \mathrm{tr}(\phi(u)^t \phi(u) d[X](u))\right] < \infty$ such that:

$$Y(T) = Y(0) + \int_0^T \phi \cdot dX, \quad i.e. \quad H = \mathbb{E}[H] + \int_0^T \phi \cdot dX \tag{1}$$

The classical proof of this representation result (see e.g. [32]) is non-constructive. However in many applications, such as stochastic control or mathematical finance, one is interested in an explicit expression for ϕ, which represents an optimal control or a hedging strategy.

Expressions for the integrand ϕ have been derived using a variety of methods and assumptions, using Markovian techniques [9, 12, 14, 21, 29], integration by parts [2] or, in the general case, using Malliavin calculus [1, 3, 19, 22, 26, 28, 17]. Some of these methods are limited to the case where X is a Markov process; others require differentiability and/or ellipticity assumptions on σ [17], differentiability assumptions, in the Fréchet or Malliavin sense, on H, or an explicit form for the density of X [2]. Almost all of these methods invariably involve an approximation step, either through the solution of an auxiliary partial differential equation (PDE) or the simulation of an auxiliary stochastic differential equation.

A systematic approach to obtaining martingale representation formulae, based on the Functional Ito calculus [11, 6, 5], has been proposed in [7], where it is shown [7, Theorem 5.9] that for any square-integrable $(\mathcal{F}_t^X)$-martingale Y,

$$\forall t \in [0, T], \quad Y(t) = Y(0) + \int_0^t \nabla_X Y \cdot dX \quad \mathbb{P}\text{−a.s.}$$

where $\nabla_X Y$ is the weak vertical derivative of Y with respect to X, constructed as an L^2 limit of pathwise directional derivatives. This approach does not rely on any Markov property nor on the Gaussian structure of the Wiener space and is thus applicable to functionals of a large class of processes.

In the present work we build on this approach to propose a general framework for computing explicit approximations to the integrand ϕ in a general setting in which X is allowed to be the solution of a stochastic differental equation (SDE) with path-dependent coefficients:

$$dX(t) = \sigma(t, X_t)dW(t) \quad X(0) = x_0 \in \mathbb{R}^d \tag{2}$$

where $X_t = X(t\wedge.)$ designates the trajectory stopped at t and $\sigma : [0,T]\times D([0,T],\mathbb{R}^d) \to \mathbb{GL}_d(\mathbb{R})$ is a Lipschitz map. For any square-integrable variable of the form $H = g(X(t), 0 \leq t \leq T)$ where $g : (D([0,T],\mathbb{R}^d), \|.\|_\infty) \to \mathbb{R}$ is a continuous functional, we construct an explicit sequence of approximations ϕ_n for the integrand ϕ in (1). These approximations are constructed as vertical derivatives, in the sense of the functional Ito calculus, of the weak Euler approximation of the martingale Y. We show that these provide explicit expressions for these approximations and analyze their convergence to the integrand ϕ. Under a Lipschitz assumption on g, we provide error estimates in L^{2p}. These approximations are easy to compute and readily integrated in commonly used numerical schemes for approximations of SDEs.

Our approach requires neither the Markov property of the underlying processes nor the differentiability of coefficients, making our approach applicable to functionals of a large class of semimartingales. By contrast to methods based on Malliavin calculus [1, 3, 19, 22, 28, 17], it does not require Malliavin differentiability of the terminal variable H nor does it involve the choice of 'Malliavin weights', a delicate step in these methods.

Ideas based on Functional Ito calculus have also been recently used by Leão and Ohashi [25] for weak approximation of Wiener functionals, using a space-filtration discretization scheme. However, unlike the approach proposed in [25], our approach is based on a Euler approximation on a fixed time grid, rather than the random time grid used in [25], which involves a sequence of first passage times. Our approach is thus much easier to implement and analyze and is readily integrated in commonly used numerical schemes for approximations of SDEs, which are typically based on fixed time grids.

Outline We first recall some key concepts and results from the Functional Itô calculus in section 2. Section 3 provides some estimates for the path-dependent SDE (2) and studies some properties of the Euler approximation for this SDE. In Section 4 we show that the weak Euler approximation (Definition 9) may be used to approximate any square-integrable martingale adapted to the filtration of X by a sequence of smooth functionals of X, in the sense of the functional Ito calculus. Moreover, we provide explicit expressions for the functional derivatives of these approximations. Section 5 analyzes the convergence of this approximation and provides error estimates in Theorem 5.1. Finally, in Section 6 we compare our approximation method with those based on Malliavin calculus.

Notations: In the sequel, we shall denote by $\mathcal{M}_{d,n}(\mathbb{R})$ the set of all $d \times n$ matrices with real coefficients. We simply denote $\mathbb{R}^d = \mathcal{M}_{d,1}(\mathbb{R})$ and $\mathcal{M}_d(\mathbb{R}) = \mathcal{M}_{d,d}(\mathbb{R})$. For $A \in \mathcal{M}_d(\mathbb{R})$, we shall denote by ${}^tA$ the transpose of A, and $\|A\| = \sqrt{\mathrm{tr}\,({}^tAA)}$ the Frobenius norm of A. For $x, y \in \mathbb{R}^d$, $x \cdot y$ is the scalar product on $\mathbb{R}^d$.

Let $T > 0$. We denote by $D([0,T],\mathbb{R}^d)$ the space of functions defined on $[0,T]$ with values in $\mathbb{R}^d$ which are right continuous with left limits (càdlàg). For a path $\omega \in D([0,T],\mathbb{R}^d)$ and $t \in [0,T]$, we denote by:

- $\omega(t)$ the value of ω at time t,

- $\omega(t-) = \lim_{s \to t, s<t} \omega(s)$ its left limit at t,
- $\omega_t = \omega(t \wedge \cdot)$ the path of ω stopped at t
- $\omega_{t-} = \omega \mathbf{1}_{[0,t)} + \omega(t-)\mathbf{1}_{[t,T]}$
- $\|\omega\|_\infty = \sup\{|\omega(t)|, t \in [0,T]\}$ the supremum norm.

We note that ω_t and ω_{t-} are elements of $D([0,T], \mathbb{R}^d)$. For a càdlàg stochastic process X, we shall similarly denote $X_t(.) = X(t \wedge .)$ and $X_{t-} = X\mathbf{1}_{[0,t)} + X(t-)\mathbf{1}_{[t,T]}$.

2 Functional Itô calculus

The Functional Itô calculus [4] is a functional calculus which extends the Ito calculus to path-dependent functionals of stochastic processes. It was first introduced in a pathwise setting [6, 5, 11] using a notion of pathwise derivative for functionals on the space of right-continuous functions with left limits, and extended in [7] to a weak calculus applicable to all square-integrable martingales, which has a natural connection to the martingale representation theorem. We recall here some key concepts and results of this approach, following [4].

Let X be the canonical process on $\Omega = D([0,T], \mathbb{R}^d)$, and $(\mathcal{F}_t^0)_{t \in [0,T]}$ be the filtration generated by X. We consider now F a functional defined on $[0,T] \times D([0,T], \mathbb{R}^d)$ with values in $\mathbb{R}$. In this paper, we are interested in one particular class of such functionals characterized by the following property: the process $t \mapsto F(t,\omega)$ defined on Ω is $(\mathcal{F}_t^0)$-adapted. Under this condition, $F(t,\cdot)$ only depends on the path stopped at t:

$$\forall \omega \in \Omega, \qquad F(t,\omega) = F(t,\omega_t). \tag{3}$$

This motivates us to consider functionals on the space of stopped paths [4]: a stopped path is an equivalence class in $[0,T] \times D([0,T], \mathbb{R}^d)$ for the following equivalence relation:

$$(t,\omega) \sim (t',\omega') \iff (t = t' \quad \text{and} \quad \omega_t = \omega'_{t'}). \tag{4}$$

The space of stopped paths is defined as the quotient of $[0,T] \times D([0,T], \mathbb{R}^d)$ by the equivalence relation (4):

$$\Lambda_T = \{(t, \omega(t \wedge \cdot)), (t,\omega) \in [0,T] \times D([0,T], \mathbb{R}^d)\} = \left([0,T] \times D([0,T], \mathbb{R}^d)\right) / \sim$$

We denote $\mathcal{W}_T$ the subset of Λ_T consisting of continuous stopped paths. We endow this set with a metric space structure by defining the following distance:

$$d_\infty((t,\omega),(t',\omega')) = \sup_{u \in [0,T]} |\omega(u \wedge t) - \omega'(u \wedge t')| + |t - t'| = \|\omega_t - \omega'_{t'}\|_\infty + |t - t'|$$

(Λ_T, d_∞) is then a complete metric space. Any functional verifying the non-anticipativity condition (3) can be equivalently viewed as a functional on $F : \Lambda_T \to \mathbb{R}$:

Definition 1. *A non-anticipative functional on $D([0,T], \mathbb{R}^d)$ is a measurable map $F\colon (\Lambda_T, d_\infty) \longrightarrow \mathbb{R}$ on the space (Λ_T, d_∞) of stopped paths.*

Using the metric structure of (Λ_T, d_∞), one can define various notions of continuity for non-anticipative functionals [6]:

Definition 2. *A non-anticipative functional F is said to be:*

- *continuous at fixed times if for any $t \in [0,T]$, $F(t,\cdot)$ is continuous with respect to the uniform norm $\|\cdot\|_\infty$ in $[0,T]$, i.e. $\forall \omega \in D([0,T],\mathbb{R}^d)$, $\forall \epsilon > 0$, $\exists \eta > 0$, $\forall \omega' \in D([0,T],\mathbb{R}^d)$,*

$$\sup |\omega - \omega'| < \eta \implies |F(t,\omega) - F(t,\omega')| < \epsilon$$

- *jointly continuous if F is continuous with respect to d_∞, i.e. $\forall (t,\omega) \in \Lambda_T$, $\forall \epsilon > 0$, $\exists \eta > 0$ such that $\forall (t',\omega') \in \Lambda_T$,*

$$d_\infty((t,\omega),(t',\omega')) < \eta \implies |F(t,\omega) - F(t',\omega')| < \epsilon$$

  We denote by $\mathbb{C}^{0,0}(\Lambda_T)$ the set of jointly continuous non-anticipative functionals.

- *left-continuous if $\forall (t,\omega) \in \Lambda_T$, $\forall \epsilon > 0$, $\exists \eta > 0$ such that $\forall (t',\omega') \in \Lambda_T$,*

$$(t' < t \quad \text{and} \quad d_\infty((t,\omega),(t',\omega')) < \eta) \implies |F(t,\omega) - F(t',\omega')| < \epsilon$$

  We denote by $\mathbb{C}_l^{0,0}(\Lambda_T)$ the set of left-continuous functionals. Similarly, we can define the set $\mathbb{C}_r^{0,0}(\Lambda_T)$ of right-continuous functionals.

We also introduce a notion of local boundedness for functionals.

Definition 3. *A non-anticipative functional F is said to be boundedness-preserving if for every compact subset K of $\mathbb{R}^d$, $\forall t_0 \in [0,T]$, $\exists C(K,t_0) > 0$ such that:*

$$\forall t \in [0,t_0], \quad \forall (t,\omega) \in \Lambda_T, \quad \omega([0,t]) \subset K \implies F(t,\omega) < C(K,t_0).$$

We denote by $\mathbb{B}(\Lambda_T)$ the set of boundedness-preserving functionals.

We now recall some notions of differentiability for functionals following [7, 4]. For $e \in \mathbb{R}^d$ and $\omega_t \in D([0,T],\mathbb{R}^d)$, we define the vertical perturbation ω_t^e of (t,ω) as the càdlàg path obtained by shifting the path by e after t

$$\omega_t^e = \omega_t + e\mathbf{1}_{[t,T]}.$$

Definition 4. *A non-anticipative functional F is said to be:*

- *horizontally differentiable at $(t,\omega) \in \Lambda_T$ if*

$$\mathcal{D}F(t,\omega) = \lim_{h \to 0^+} \frac{F(t+h,\omega) - F(t,\omega)}{h}$$

  exists. If $\mathcal{D}F(t,\omega)$ exists for all $(t,\omega) \in \Lambda_T$, then the non-anticipative functional $\mathcal{D}F$ is called the horizontal derivative of F.

- *vertically differentiable at $(t, \omega) \in \Lambda_T$ if the map:*

$$\begin{array}{ccc} \mathbb{R}^d & \longrightarrow & \mathbb{R} \\ e & \mapsto & F(t, \omega_t + e\mathbf{1}_{[t,T]}) \end{array}$$

  is differentiable at 0. Its gradient at 0 is called the vertical derivative of F at (t, ω):

$$\nabla_\omega F(t, \omega) = (\partial_i F(t, \omega), i = 1, \cdots, d) \in \mathbb{R}^d$$

  with

$$\partial_i F(t, \omega) = \lim_{h \to 0} \frac{F(t, \omega_t + he_i \mathbf{1}_{[t,T]}) - F(t, \omega_t)}{h}$$

  where $(e_i, i = 1, \cdots, d)$ is the canonical basis of $\mathbb{R}^d$. If F is vertically differentiable at all $(t, \omega) \in \Lambda_T$, $\nabla_\omega F : (t, \omega) \to \mathbb{R}^d$ defines a non-anticipative map called the vertical derivative of F.

We may repeat the same operation on $\nabla_\omega F$ and define similarly $\nabla^2_\omega F$, $\nabla^3_\omega F$, $\cdots$. This leads us to define the the following classes of smooth functionals:

Definition 5 (Smooth functionals). *We define $\mathbb{C}^{1,k}_b(\Lambda_T)$ as the set of non-anticipative functionals $F : (\Lambda_T, d_\infty) \to \mathbb{R}$ which are*

- *horizontally differentiable with $\mathcal{D}F$ continuous at fixed times;*
- *k times vertically differentiable with $\nabla^j_\omega F \in \mathbb{C}^{0,0}_l(\Lambda_T)$ for $j = 0, \cdots, k$;*
- *$\mathcal{D}F, \nabla_\omega F, \cdots, \nabla^k_\omega F \in \mathbb{B}(\Lambda_T)$.*

We denote $\mathbb{C}^{1,\infty}(\Lambda_T) = \cap_{k\geq 1}\mathbb{C}^{1,k}(\Lambda_T), \mathbb{C}^{1,\infty}_b(\Lambda_T) = \cap_{k\geq 1}\mathbb{C}^{1,k}_b(\Lambda_T)$.

Many examples of functionals may fail to be globaly smooth, but their derivatives may still be well behaved except at certain stopping times, which motivates the following definition [4]:

Definition 6. *A non-anticipative functional F is said to be locally regular of class $\mathbb{C}^{1,2}_{\text{loc}}(\Lambda_T)$ if there exists an increasing sequence $(\tau_n)_{n\geq 0}$ of stopping times with $\tau_0 = 0$ and $\tau_n \underset{n\to\infty}{\longrightarrow} \infty$, and a sequence of functionals $F_n \in \mathbb{C}^{1,2}_b(\Lambda_T)$ such that:*

$$F(t, \omega) = \sum_{n\geq 0} F_n(t, \omega)\mathbf{1}_{[\tau_n(\omega), \tau_{n+1}(\omega))}(t), \quad \forall (t, \omega) \in \Lambda_T$$

We recall now the functional Itô formula for non-anticipative functionals of a continuous semimartingale [7, Theorem 4.1]:

Proposition 2.1 ([6, 7]). *Let S be a continuous semimartingale defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For any non-anticipative functional $F \in \mathbb{C}^{1,2}_{\text{loc}}(\Lambda_T)$ and any $t \in [0, T]$, we have:*

$$F(t, S_t) - F(0, S_0) \quad = \quad \int_0^t \mathcal{D}F(u, S_u)du + \int_0^t \nabla_\omega F(u, S_u) \cdot dS(u) + \frac{1}{2}\int_0^t \text{tr}\left(\nabla^2_\omega F(u, S_u)d[S](u)\right)$$

Actually the same functional Itô formula may also be obtained for functionals whose vertical derivatives are right-continuous rather than left-continuous. We denote by $\mathbb{C}^{1,2}_{b,r}(\Lambda_T)$ the set of non-anticipative functionals F satisfying:

- F is horizontally differentiable with $\mathcal{D}F$ continuous at fixed times;
- F is twice vertically differentiable with $F \in \mathbb{C}^{0,0}_l(\Lambda_T)$ and $\nabla_\omega F, \nabla^2_\omega F \in \mathbb{C}^{0,0}_r(\Lambda_T)$;
- $\mathcal{D}F, \nabla_\omega F, \nabla^2_\omega F \in \mathbb{B}(\Lambda_T)$;

The localization is more delicate in this case, and we are not able to state a local version of the functional Itô formula by simply replacing $F_n \in \mathbb{C}^{1,2}_b(\Lambda_T)$ by $F_n \in \mathbb{C}^{1,2}_{b,r}(\Lambda_T)$ in **Definition 6** (see Remark 4.2 in [15]). However if the stopping times τ_n are deterministic, then the functional Itô formula is still valid (Proposition 2.4 and Remark 4.2 in [15]).

Definition 7. *A non-anticipative functional is said to be locally regular of class $\mathbb{C}^{1,2}_{\mathrm{loc,r}}(\Lambda_T)$ if there exists an increasing sequence $(t_n)_{n\geq 0}$ of deterministic times with $t_0 = 0$ and $t_n \underset{n\to\infty}{\longrightarrow} \infty$, and a sequence of functionals $F_n \in \mathbb{C}^{1,2}_{b,r}(\Lambda_T)$ such that:*

$$F(t,\omega) = \sum_{n\geq 0} F_n(t,\omega)\mathbf{1}_{[t_n,t_{n+1})}(t), \quad \forall (t,\omega) \in \Lambda_T$$

Proposition 2.2 ([7])**.** *Let S be a continuous semimartingale defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For any non-anticipative functional $F \in \mathbb{C}^{1,2}_{loc,r}(\Lambda_T)$ and any $t \in [0,T]$, we have:*

$$\begin{aligned} F(t,S_t) - F(0,S_0) &= \int_0^t \mathcal{D}F(u,S_u)du + \int_0^t \nabla_\omega F(u,S_u)\cdot dS(u) \\ &\quad + \frac{1}{2}\int_0^t \mathrm{tr}\left(\nabla^2_\omega F(u,S_u)d[S](u)\right) \quad \mathbb{P}\text{−a.s.} \end{aligned}$$

Finally we present briefly the martingale representation formula established in [7]. Let $(X_t)_{t\in[0,T]}$ be a continuous $\mathbb{R}^d$-valued martingale defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with absolutely continuous quadratic variation:

$$[X](t) = \int_0^t A(u)du$$

where A is a $\mathcal{M}_d(\mathbb{R})$-valued process. Denote by $(\mathcal{F}^X_t)$ the natural filtration of X and $\mathcal{C}^{1,2}_b(X)$ the set of $(\mathcal{F}^X_t)$-adapted processes Y which admit a functional representation in $\mathbb{C}^{1,2}_b(\Lambda_T)$:

$$\mathcal{C}^{1,2}_b(X) = \{Y, \exists F \in \mathbb{C}^{1,2}_b(\Lambda_T), Y(t) = F(t,X_t) \quad dt \times d\mathbb{P}\text{−a.e.}\} \tag{5}$$

If $A(t)$ is non-singular almost everywhere, i.e. $\det(A(t)) \neq 0$, $dt \times d\mathbb{P}$-a.e., then for any $Y \in \mathcal{C}^{1,2}_b(X)$, the predictable process

$$\nabla_X Y(t) = \nabla_\omega F(t,X_t)$$

is uniquely defined up to an evanescent set, independently of the choice of $F \in \mathbb{C}^{1,2}_b(\Lambda_T)$ in the representation (5). This process $\nabla_X Y$ is called the vertical derivative of Y with respect to X. For martingales which are smooth functionals of X, the operator $\nabla_X : \mathcal{C}^{1,2}_b(X) \mapsto \mathcal{C}^{0,0}_l(X)$ yields the integrand in the martingale representation theorem:

Corollary 2.1. *If $Y \in \mathcal{C}_b^{1,2}(X)$ is a square-integrable martingale, then*

$$\forall t \in [0,T], \quad Y(t) = Y(0) + \int_0^t \nabla_X Y \cdot dX \quad \mathbb{P}\text{−a.s.}$$

Consider now the case where X is a square-integrable martingale. Let $\mathcal{M}^2(X)$ be the space of square-integrable $(\mathcal{F}_t^X)$-martingales with initial value zero, equipped with the norm $\|Y\|^2 = \sqrt{\mathbb{E}|Y(T)|^2}$. Cont & Fournié [7, Theorem 5.8] show that the operator $\nabla_X : \mathcal{C}_b^{1,2}(X) \mapsto \mathcal{C}_l^{0,0}(X)$ admits a unique continuous extension to a weak derivative $\nabla_X : \mathcal{M}^2(X) \to \mathcal{L}^2(X)$ which satisfies the following martingale representation formula:

Proposition 2.3 ([7])**.** *For any square-integrable $(\mathcal{F}_t^X)$-martingale Y, we have:*

$$\forall t \in [0,T], \quad Y(t) = Y(0) + \int_0^t \nabla_X Y \cdot dX \quad \mathbb{P}\text{−a.s.}$$

This weak vertical derivative $\nabla_X Y$ coincides with the pathwise vertical derivative $\nabla_\omega F(t, X_t)$ when Y admits a locally regular functional representation, i.e. $Y(t) = F(t, X_t)$ with $F \in \mathbb{C}_{loc}^{1,2}(\Lambda_T) \cup \mathbb{C}_{loc,r}^{1,2}(\Lambda_T)$. For a general square-integrable martingale Y, the weak derivative $\nabla_X Y$ is not directly computable through a pathwise perturbation. An approximation procedure is thus necessary for computing $\nabla_X Y$. The result of [7] guarantees the existence of such approximations; in the sequel we propose explicit constructions of computable versions of such approximations.

3 Euler approximations for path-dependent SDEs

Let W be a standard d-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\mathcal{F}_t^W)$ its ($\mathbb{P}$-completed) natural filtration and denote by $\mathbb{GL}_d(\mathbb{R}$ the set of $d \times d$ non-singular real matrices. We consider the following stochastic differential equation with path-dependent coefficient (2):

$$dX(t) = \sigma(t, X_t) dW(t), \qquad X(0) = x_0 \in \mathbb{R}^d$$

where $\sigma : \Lambda_T \to \mathbb{GL}_d(\mathbb{R}$ is a non-anticipative map, assumed to be Lipschitz-continuous:

Assumption 1. $\sigma : (\Lambda_T, d_\infty) \to \mathbb{GL}_d(\mathbb{R}$ *is Lipschitz continuous:*

$$\exists \sigma_{lip} > 0, \quad \forall t, t' \in [0,T], \forall \omega, \omega' \in D([0,T], \mathbb{R}^d), \quad \|\sigma(t,\omega) - \sigma(t',\omega')\| \le \sigma_{lip} d_\infty\left((t,\omega),(t',\omega')\right).$$

Denote by $(\mathcal{F}_t^X)$ the natural filtration of X. Under Assumption 1, (2) has a unique strong solution X and $\mathcal{F}_t^X = \mathcal{F}_t^W$.

Proposition 3.1. *Under Assumption 1, there exists a unique $(\mathcal{F}_t^W)$-adapted process X satisfying* (2)*. Moreover for $p \ge 1$, we have:*

$$\mathbb{E}\left[\|X_T\|_\infty^{2p}\right] \le C(1 + |x_0|^{2p}) e^{CT} \tag{6}$$

for some constant $C = C(p, T, \sigma_{lip})$ depending on p, T and σ_{lip}.

Proof. Existence and uniqueness of a strong solution follows from [31] (Theorem 7 , Chapter 5): see [4, Section 5]. Let us prove (6). Using the Burkholder-Davis-Gundy inequality and Hölder's inequality, we have:

$$\begin{aligned}
\mathbb{E}\left[\|X_T\|_\infty^{2p}\right] &\leq C(p)\left(|x_0|^{2p} + \mathbb{E}\left[\left(\int_0^T \|\sigma(t,X_t)\|^2\,dt\right)^p\right]\right)\\
&\leq C(p,T)\left(|x_0|^{2p} + \mathbb{E}\left[\int_0^T \|\sigma(t,X_t)\|^{2p}\,dt\right]\right)\\
&\leq C(p,T)\left(|x_0|^{2p} + \mathbb{E}\left[\int_0^T \left(\|\sigma(0,\bar{0})\| + \sigma_{lip}(t+\|X_t\|_\infty)\right)^{2p}\,dt\right]\right)\\
&\leq C(p,T,\sigma_{lip})\left(|x_0|^{2p} + 1 + \int_0^T \mathbb{E}\|X_t\|_\infty^{2p}dt\right)
\end{aligned}$$

where $\bar{0}$ is the path which takes constant value 0. We conclude by Gronwall's inequality. □

In the following, we always assume that **Assumption 1** holds. The strong solution X of equation (2) is then a Brownian martingale and defines a non-anticipative functional $X : \mathcal{W}_T \to \mathbb{R}^d$ given by the Ito map associated to (2).

3.1 Euler approximations as non-anticipative functionals

We now consider an Euler approximation for the SDE (2) and study its properties as a non-anticipative functional. Let $n \in \mathbb{N}, \delta = \frac{T}{n}$. The Euler approximation ${}_nX$ of X on the grid $(t_j = j\delta, j = 0..n)$ is defined as follows:

Definition 8. *[Euler scheme] For $\omega \in D([0,T],\mathbb{R}^d)$, we denote by ${}_nX(\omega) \in D([0,T],\mathbb{R}^d)$ the piecewise constant Euler approximation for (2) computed along the path ω, defined as follows: ${}_nX(\omega)$ is constant in each interval $[t_j,t_{j+1})$, $\forall\, 0 \leq j \leq n-1$ with ${}_nX(0,\omega) = x_0$ and*

$$ {}_nX(t_{j+1},\omega) = {}_nX(t_j,\omega) + \sigma(t_j, {}_nX_{t_j}(\omega))(\omega(t_{j+1}-) - \omega(t_j-)), \quad 0 \leq j \leq n-1 \tag{7}$$

where ${}_nX_t(\omega)$ is the path of ${}_nX(\omega)$ stopped at time t, and by convention $\omega(0-) = \omega(0)$.

When computed along the path of the Brownian motion W, ${}_nX(W)$ is simply the piecewise constant Euler-Maruyama scheme [30] for the stochastic differential equation (2).

By definition, the path ${}_nX(\omega)$ depends only on a finite number of increments of ω: $\omega(t_1-) - \omega(0)$, $\cdots$, $\omega(t_n-) - \omega(t_{n-1}-)$. We can thus define an application $p : \mathcal{M}_{d,n}(\mathbb{R}) \to D([0,T],\mathbb{R}^d)$ as follows: for $y = (y_1,\cdots,y_n) \in \mathcal{M}_{d,n}(\mathbb{R})$ with each $y_l \in \mathbb{R}^d$ for $1 \leq l \leq n$,

$$p(y) = p(y_1,\cdots,y_n) = {}_nX(\omega) \tag{8}$$

with ω any path in $D([0,T],\mathbb{R}^d)$ satisfying $\omega(t_1-) - \omega(0) = y_1$, $\cdots$, $\omega(t_n-) - \omega(t_{n-1}-) = y_n$. And we note $p_t(y)$ the path of $p(y)$ stopped at time t.

The application $p : \mathcal{M}_{d,n}(\mathbb{R}) \to \big(D([0,T], \mathbb{R}^d), \|\cdot\|_\infty\big)$ is locally Lipschitz continuous as shown by the following lemma.

Lemma 3.1. *Let $y = (y_1, \cdots, y_n)$ and $y' = (y'_1, \cdots, y'_n) \in \mathcal{M}_{d,n}(\mathbb{R})$ with $y_l, y'_l \in \mathbb{R}^d$ for $1 \le l \le n$. If $\max_{1\le k\le n} |y_k - y'_k| \le \eta$, then we have:*

$$\|p(y) - p(y')\|_\infty \le C(y, \eta, \sigma_{lip}, T) \max_{1\le k\le n} |y_k - y'_k|$$

for some constant C depending only on y, η, σ_{lip} and T.

Proof. As the two paths $p(y)$ and $p(y')$ are stepwise constant by definition, it suffices to prove the inequality at times $(t_j)_{0\le j\le n}$. We prove by induction that:

$$\|p_{t_j}(y) - p_{t_j}(y')\|_\infty \le C(y, \eta, \sigma_{lip}, T) \max_{1\le k\le j} |y_k - y'_k| \tag{9}$$

with some constant C which depends only on y, η, σ_{lip} and T.

For $j = 0$, this is clearly the case as $p(y)(0) = p(y')(0) = x_0$. Assume that (9) is verified for some $0 \le j \le n-1$, consider now $\|p_{t_{j+1}}(y) - p_{t_{j+1}}(y')\|_\infty$, we have:

$$p(y)(t_{j+1}) = p(y)(t_j) + \sigma(t_j, p_{t_j}(y))y_{j+1}$$

and

$$p(y')(t_{j+1}) = p(y')(t_j) + \sigma(t_j, p_{t_j}(y'))y'_{j+1}.$$

Thus

$$\begin{aligned}
& |p(y)(t_{j+1}) - p(y')(t_{j+1})| \\
\le\ & |p(y)(t_j) - p(y')(t_j)| + \|\sigma(t_j, p_{t_j}(y))\| \cdot |y_{j+1} - y'_{j+1}| + \|\sigma(t_j, p_{t_j}(y)) - \sigma(t_j, p_{t_j}(y'))\| \cdot |y'_{j+1}| \\
\le\ & C(y, \eta, \sigma_{lip}, T) \max_{1\le k\le j} |y_k - y'_k| + \big(\|\sigma(0, \bar{0})\| + \sigma_{lip}(t_j + \|p_{t_j}(y)\|_\infty)\big) |y_{j+1} - y'_{j+1}| \\
& + \sigma_{lip} C(y, \eta, \sigma_{lip}, T) \max_{1\le k\le j} |y_k - y'_k| (|y_{j+1}| + |y_{j+1} - y'_{j+1}|) \\
\le\ & C'(y, \eta, \sigma_{lip}, T) \max_{1\le k\le j+1} |y_k - y'_k|
\end{aligned}$$

where $\bar{0}$ is the path which takes constant value 0. And consequently we have:

$$\|p_{t_{j+1}}(y) - p_{t_{j+1}}(y')\|_\infty \le C(y, \eta, \sigma_{lip}, T) \max_{1\le k\le j+1} |y_k - y'_k|$$

for some different constant C depending only on y, η, σ_{lip} and T. And we conclude by induction. □

3.2 Strong convergence

To simplify the notations, ${}_nX_T(W_T)$ will be noted simply ${}_nX_T$ in the following. The following result, which gives a uniform estimate of the discretization error, $X_T - {}_nX_T$ extends similar results known in the Markovian case [13, 30, 20] to the path-dependent SDE (2):

Proposition 3.2. *Under **Assumption 1** we have the following estimate in L^{2p} for the strong error of the piecewise constant Euler-Maruyama scheme:*

$$E\left(\sup_{s\in[0,T]} \|X(s) - {}_nX(s)\|^{2p}\right) \leq C(x_0, p, T, \sigma_{lip}) \left(\frac{1+\log n}{n}\right)^p, \quad \forall p \geq 1$$

with C a constant depending only on x_0, p, T and σ_{lip}.

Proof. The idea is to construct a 'Brownian interpolation' ${}_n\hat{X}_T$ of the Euler scheme ${}_nX_T$:

$${}_n\hat{X}(s) = x_0 + \int_0^s \sigma\left(\underline{u}, {}_nX_{\underline{u}}\right) dW(u)$$

where $\underline{u} = \left\lfloor \frac{u}{\delta} \right\rfloor \cdot \delta$ is the largest subdivision point which is smaller or equal to u.

Clearly the process ${}_n\hat{X}_T$ is a continuous martingale and $\| \sup_{s\in[0,T]} |X(s) - {}_nX(s)| \|_{2p}$ can be controlled by the sum of the two following terms:

$$\| \sup_{s\in[0,T]} |X(s) - {}_nX(s)| \|_{2p} \leq \| \sup_{s\in[0,T]} |X(s) - {}_n\hat{X}(s)| \|_{2p} + \| \sup_{s\in[0,T]} |{}_n\hat{X}(s) - {}_nX(s)| \|_{2p} \tag{10}$$

We start with the term $\| \sup_{s\in[0,T]} |X(s) - {}_n\hat{X}(s)| \|_{2p}$. Using the Burkholder-Davis-Gundy inequality and Hölder's inequality, we have

$$\begin{aligned}
\mathbb{E}\|X_T - {}_n\hat{X}_T\|_\infty^{2p} &\leq C(p)\,\mathbb{E}\left[\int_0^T \left\|\sigma(s, X_s) - \sigma(\underline{s}, {}_nX_{\underline{s}})\right\|^2 ds\right]^p \\
&\leq C(p,T)\,\mathbb{E}\left[\int_0^T \left\|\sigma(s, X_s) - \sigma(\underline{s}, {}_nX_{\underline{s}})\right\|^{2p} ds\right] \\
&\leq C(p,T,\sigma_{lip})\,\mathbb{E}\left[\int_0^T \left((s-\underline{s})^{2p} + \|X_s - {}_nX_s\|_\infty^{2p}\right) ds\right] \\
&\leq C(p,T,\sigma_{lip})\left(\frac{1}{n^{2p}} + \int_0^T \mathbb{E}\|X_s - {}_nX_s\|_\infty^{2p}\, ds\right)
\end{aligned}$$

The constants may differ from one line to another, and we have used ${}_nX_{\underline{s}} = {}_nX_s$ as ${}_nX$ is piecewise constant.

Consider now the second term $\| \sup_{s\in[0,T]} |{}_n\hat{X}(s) - {}_nX(s)| \|_{2p}$. Noting that:

$${}_n\hat{X}(s) - {}_nX(s) = {}_n\hat{X}(s) - {}_n\hat{X}(\underline{s}) = \sigma\left(\underline{s}, {}_nX_{\underline{s}}\right)(W(s) - W(\underline{s})),$$

we have

$$\|{}_n\hat{X}_T - {}_nX_T\|_\infty \leq C(\sigma_{lip}, T)(1 + \|{}_nX_T\|_\infty) \sup_{s\in[0,T]} |W(s) - W(\underline{s})|$$

and

$$\mathbb{E}\|{}_n\hat{X}_T - {}_nX_T\|_\infty^{2p} \leq C(\sigma_{lip}, T)^{2p}\,\mathbb{E}\left[(1 + \|{}_nX_T\|_\infty) \sup_{s\in[0,T]} |W(s) - W(\underline{s})|\right]^{2p}.$$

By the Cauchy-Schwarz inequality, we have:

$$\mathbb{E}\|{}_n\hat{X}_T - {}_nX_T\|_\infty^{2p} \leq C(p,\sigma_{lip},T)\left(1+\sqrt{\mathbb{E}\|{}_nX_T\|_\infty^{4p}}\right)\sqrt{\mathbb{E}\sup_{s\in[0,T]}|W(s)-W(\underline{s})|^{4p}}$$

We will make use of the following result:

$$\forall p>0, \quad \|\sup_{s\in[0,T]}|W(s)-W(\underline{s})|\|_p \leq C(W,p)\sqrt{\frac{T}{n}(1+\log n)}$$

which results from the following lemma:

Lemma 3.2. *Let $Y_1, \cdots, Y_n$ be non-negative random variables with the same distribution satisfying $\mathbb{E}\left(e^{\lambda Y_1}\right) < \infty$ for some $\lambda > 0$. Then we have:*

$$\forall p>0, \quad \|\max(Y_1,\cdots,Y_n)\|_p \leq \frac{1}{\lambda}(\log n + C(p,Y_1,\lambda))$$

We have thus:

$$\sqrt{\mathbb{E}\sup_{s\in[0,T]}|W(s)-W(\underline{s})|^{4p}} \leq C(p,T)\left(\frac{1+\log n}{n}\right)^p \tag{11}$$

Furthermore, using again the Burkholder-Davis-Gundy inequality, we have:

$$\begin{aligned}
\mathbb{E}\|{}_nX_T\|_\infty^{4p} &\leq \mathbb{E}\|{}_n\hat{X}_T\|_\infty^{4p} \\
&\leq C(p)\left(x_0^{4p} + \mathbb{E}\left(\int_0^T \left\|\sigma(\underline{s},{}_nX_{\underline{s}})\right\|^2 ds\right)^{2p}\right) \\
&\leq C(p,x_0,T)\left(1+\int_0^T \mathbb{E}\left\|\sigma(\underline{s},{}_nX_{\underline{s}})\right\|^{4p} ds\right) \\
&\leq C(p,x_0,T,\sigma_{lip})\left(1+\int_0^T \mathbb{E}\|{}_nX_s\|_\infty^{4p} ds\right)
\end{aligned}$$

We deduce from Gronwall's inequality that $\mathbb{E}\|{}_nX_T\|_\infty^{4p}$ is bounded by a constant which depends only on p, x_0, T and σ_{lip}.

Combining this result with (11), we get:

$$\mathbb{E}\|{}_n\hat{X}_T - {}_nX_T\|_\infty^{2p} \leq C(x_0,p,T,\sigma_{lip})\left(\frac{1+\log n}{n}\right)^p$$

Finally (10) becomes:

$$\begin{aligned}
&\mathbb{E}\|X_T - {}_nX_T\|_\infty^{2p} \\
\leq\ & C(p)\left(\mathbb{E}\|X_T - {}_n\hat{X}_T\|_\infty^{2p} + \mathbb{E}\|{}_n\hat{X}_T - {}_nX_T\|_\infty^{2p}\right) \\
\leq\ & C(x_0,p,T,\sigma_{lip})\left(\left(\frac{1+\log n}{n}\right)^p + \int_0^T \mathbb{E}\|X_s - {}_nX_s\|_\infty^{2p} ds\right)
\end{aligned}$$

And we conclude by Gronwall's inequality. □

Corollary 3.1. *Under Assumption 1,*

$$\forall \alpha \in [0, \frac{1}{2}), \qquad n^{\alpha} \| X_T - {}_nX_T \|_{\infty} \underset{n \to \infty}{\longrightarrow} 0, \quad \mathbb{P}\text{-a.s.}$$

Proof. Let $\alpha \in [0, \frac{1}{2})$. For a p large enough, by **Proposition 3.2**, we have:

$$\mathbb{E}\left[\sum_{n \geq 1} n^{2p\alpha} \| X_T - {}_nX_T \|_{\infty}^{2p} \right] < \infty$$

Thus

$$\sum_{n \geq 1} n^{2p\alpha} \| X_T - {}_nX_T \|_{\infty}^{2p} < \infty, \quad \mathbb{P}\text{-a.s.}$$

and

$$n^{\alpha} \| X_T - {}_nX_T \|_{\infty} \underset{n \to \infty}{\longrightarrow} 0, \quad \mathbb{P}\text{-a.s.}$$

□

4 Smooth functional approximations for martingales

Let $g : D([0,T], \mathbb{R}^d) \longrightarrow \mathbb{R}$ be a functional which satisfies the following condition:

Assumption 2. $g : (D([0,T], \mathbb{R}^d), \| \cdot \|_{\infty}) \longrightarrow \mathbb{R}$ *is continuous with polynomial growth:*

$$\exists q \in \mathbb{N}, \exists C > 0, \forall \omega \in D([0,T], \mathbb{R}^d), \quad |g(\omega)| \leq C \left(1 + \|\omega\|_{\infty}^{q}\right)$$

and consider the (square-integrable) martingale

$$Y(t) = \mathbb{E}\left[g(X_T) | \mathcal{F}_t^X \right] = \mathbb{E}\left[g(X_T) | \mathcal{F}_t^W \right].$$

Y may be represented as a non-anticipative functional of X (or W):

$$Y(t) = G(t, X_t) = F(t, W_t)$$

where the functionals F, G are square-integrable but may not have any smoothness property a priori. By **Proposition 2.3** we have:

$$g(X_T) = Y(T) = Y(t) + \int_t^T \nabla_X Y(s) \cdot dX(s) = Y(t) + \int_t^T \nabla_W Y(s) \cdot dW(s) \quad \mathbb{P}\text{-a.s.}$$

where $\nabla_X Y$ (resp. $\nabla_W Y$) is the weak vertical derivative of Y with respect to X (resp. W). The two representations are related [4, Theorem 4.19] by the equality

$${}^t(\nabla_X Y(s)) \sigma(s, X_s) = {}^t(\nabla_W Y(s))$$

outside an evanescent set. So if one of them is computable, the other one is computable as well. However in general neither G nor F is a smooth functional (for example $\in \mathbb{C}^{1,2}_{loc,r}(\Lambda_T)$) so neither of the two weak derivatives may be computed directly as a pathwise directional derivative.

The main idea is to approximate the martingale Y by a sequence of *smooth* martingales ${}_nY$ which admit a functional representation ${}_nY(s) = F_n(s, W_s)$ with $F_n \in \mathbb{C}^{1,2}_{loc,r}(\Lambda_T)$, regular enough to apply the functional Itô formula. Then by the functional Itô formula, we have:

$$\int_t^T \nabla_\omega F_n(s, W_s) \cdot dW(s) = {}_nY(T) - {}_nY(t) \underset{n\to\infty}{\longrightarrow} Y(T) - Y(t) = \int_t^T \nabla_X Y(s) \cdot dX(s)$$

One can then use the following estimator for $\nabla_X Y$:

$$Z_n(s) = {}^t\left(\sigma^{-1}(s, X_s)\right)\nabla_\omega F_n(s, W_s),$$

where the vertical derivative $\nabla_\omega F_n(s, W_s) = (\partial_i F_n(s, W_s), 1 \le i \le d)$ may be computed as a pathwise derivative

$$\partial_i F_n(s, W_s) = \lim_{h\to 0} \frac{F_n(s, W_s + h e_i \mathbf{1}_{[s,T]}) - F_n(s, W_s)}{h},$$

yielding a concrete procedure for computing the estimator.

We will show in this section that the familiar *weak Euler approximation* provides a systematic way of constructing such smooth functional approximations in the sense of Definition 7.

Define the concatenation of two càdlàg paths $\omega, \omega' \in D([0,T], \mathbb{R}^d)$ at time $s \in [0,T]$, which we note $\omega \underset{t}{\oplus} \omega'$, as the following càdlàg path on $[0,T]$:

$$\omega \underset{s}{\oplus} \omega' = \omega_s \underset{s}{\oplus} \omega' = \begin{cases} \omega(u) & u \in [0,s) \\ \omega(s) + \omega'(u) - \omega'(s) & u \in [s,T] \end{cases}$$

Observe that:

$$\forall z \in \mathbb{R}^d, \quad \omega_s^z \underset{s}{\oplus} \omega' = (\omega_s \underset{s}{\oplus} \omega') + z\mathbf{1}_{[s,T]}.$$

Definition 9 (Weak Euler approximation)**.** *We define the (level-n) weak Euler approximation of F as the functional F_n defined by*

$$F_n(s, \omega_s) = \mathbb{E}\left[g({}_nX(\omega_s \underset{s}{\oplus} W_T))\right] \tag{12}$$

Applying this functional to the path of the Wiener process W, we obtain a $(\mathcal{F}_t^W)_{t\ge 0}$-adapted process:

$${}_nY(s) = F_n(s, W_s).$$

Using independence of increments of W, we have

$${}_nY(s) = \mathbb{E}\left[g({}_nX(W_T))|\mathcal{F}_s^W\right] = \mathbb{E}\left[g({}_nX(W_s \underset{s}{\oplus} W_T))|\mathcal{F}_s^W\right] = \mathbb{E}\left[g({}_nX(W_s \underset{s}{\oplus} B_T))|\mathcal{F}_s^W\right]$$

where B is any Wiener process independent from W. In particular ${}_nY$ is a square-integrable martingale, so is weakly differentiable in the sense of [7, Theorem 5.8]. We will now show that F_n is in fact a smooth functional in the sense of Definition 7.

Theorem 4.1. *Under Assumptions 1 and 2, the functional F_n defined in* (12) *is horizontally differentiable and infinitely vertically differentiable.*

Proof. Let $(s,\omega) \in \Lambda_T$ with $t_k \le s < t_{k+1}$ for some $0 \le k \le n-1$. We start with the vertical differentiability of F_n at (s,ω), which is equivalent to the differentiability at 0 of the following map:

$$v(z) = F_n(s, \omega_s^z) = \mathbb{E}\left[g({}_nX(\omega_s^z \underset{s}{\oplus} B_T))\right], \quad z \in \mathbb{R}^d$$

The main idea of the proof is to absorb z in the density function of Gaussian variables when taking the expectation, which smoothens the dependence of v on z.

As we have already shown, ${}_nX(\omega_s^z \underset{s}{\oplus} B_T)$ depends only on $(\omega_s^z \underset{s}{\oplus} B_T)(t_1-) - (\omega_s^z \underset{s}{\oplus} B_T)(0)$, $\cdots$, $(\omega_s^z \underset{s}{\oplus} B_T)(t_n-) - (\omega_s^z \underset{s}{\oplus} B_T)(t_{n-1}-)$, which are all explicit using the definition of the concatenation. For $j < k$, we have:

$$(\omega_s^z \underset{s}{\oplus} B_T)(t_{j+1}-) - (\omega_s^z \underset{s}{\oplus} B_T)(t_j-) = \omega(t_{j+1}-) - \omega(t_j-)$$

In the case where $j = k$, we have:

$$\begin{aligned} & (\omega_s^z \underset{s}{\oplus} B_T)(t_{k+1}-) - (\omega_s^z \underset{s}{\oplus} B_T)(t_k-) \\ = \; & B(t_{k+1}) - B(s) + \omega(s) + z - \omega(t_k-) \\ = \; & B(t_{k+1}) - B(s) + z + \omega(s) - \omega(t_k-) \end{aligned}$$

And for $j > k$, we have:

$$\begin{aligned} & (\omega_s^z \underset{s}{\oplus} B_T)(t_{j+1}-) - (\omega_s^z \underset{s}{\oplus} B_T)(t_j-) \\ = \; & B(t_{j+1}) - B(s) + \omega(s) + z - (B(t_j) - B(s) + \omega(s) + z) \\ = \; & B(t_{j+1}) - B(t_j) \end{aligned}$$

Thus we have:

$$\begin{aligned} {}_nX(\omega_s^z \underset{s}{\oplus} B_T) \;=\; & p\Big(\omega(t_1-) - \omega(0), \cdots, \omega(t_k-) - \omega(t_{k-1}-), B(t_{k+1}) - B(s) + z + \omega(s) - \omega(t_k-), \\ & B(t_{k+2}) - B(t_{k+1}), \cdots, B(t_n) - B(t_{n-1})\Big) \end{aligned}$$

where $p : \mathcal{M}_{d,n}(\mathbb{R}) \to D([0,T], \mathbb{R}^d)$ is the map defined by (8).

Observe from the previous equation that, for a fixed z, the value of ${}_nX(t_{k+1}, \omega_s^z \underset{s}{\oplus} B_T)$ as a random variable depends only on a finite number of Gaussian variables: $B(t_{k+1}) - B(s)$, $B(t_{k+2}) - B(t_{k+1})$, $\cdots$, $B(t_j) - B(t_{j-1})$. Since the joint distribution of these Gaussian variables is explicit, $v(z) = \mathbb{E}\left[g({}_nX_T(\omega_s^z \underset{s}{\oplus} B_T))\right]$ can be computed explicitly as an integral in finite dimension.

Let $y = (y_1, \cdots, y_{n-k}) \in \mathcal{M}_{d,n-k}(\mathbb{R})$ with each $y_l \in \mathbb{R}^d$ for $1 \le l \le n-k$. We have:

$$\begin{aligned}
v(z) &= \mathbb{E}\left[g({}_nX_T(\omega_s^z \underset{s}{\oplus} B_T))\right] \\
&= \mathbb{E}\Big[g\Big(p(\omega(t_1-) - \omega(0), \cdots, \omega(t_k-) - \omega(t_{k-1}-), \\
&\quad B(t_{k+1}) - B(s) + z + \omega(s) - \omega(t_k-), B(t_{k+2}) - B(t_{k+1}), \cdots, B(t_n) - B(t_{n-1})\Big)\Big)\Big] \\
&= \int_{\mathbb{R}^{d\times(n-k)}} g\Big(p(\omega(t_1-) - \omega(0), \cdots, \omega(t_k-) - \omega(t_{k-1}-), y_1 + z + \omega(s) - \omega(t_k-), \\
&\quad y_2, \cdots, y_{n-k})\Big)\Phi(y_1, t_{k+1} - s) \prod_{l=2}^{n-k} \Phi(y_l, \delta) dy_1 dy_2 \cdots dy_{n-k} \\
&= \int_{\mathbb{R}^{d\times(n-k)}} g\Big(p(\omega(t_1-) - \omega(0), \cdots, \omega(t_k-) - \omega(t_{k-1}-), y_1 + \omega(s) - \omega(t_k-), \\
&\quad y_2, \cdots, y_{n-k})\Big)\Phi(y_1 - z, t_{k+1} - s) \prod_{l=2}^{n-k} \Phi(y_l, \delta) dy_1 dy_2 \cdots dy_{n-k} \qquad (13)
\end{aligned}$$

with

$$\Phi(x,t) = (2\pi t)^{-\frac{d}{2}} \exp\left(-\frac{|x|^2}{2t}\right), \quad x \in \mathbb{R}^d$$

the density function of a d-dimensional Gaussian variable with covariance matrix tI_d.

Since the only term which depends on z in the integrand of (13) is $\Phi(y_1 - z, t_{k+1} - s)$, which is a smooth function of z, thus v is differentiable at all $z \in \mathbb{R}^d$, in particular at 0. Hence F_n is vertically differentiable at $(s,\omega) \in \Lambda_T$ with: for $1 \le i \le d$,

$$\begin{aligned}
\partial_i F_n(s,\omega) &= \int_{\mathbb{R}^{d\times(n-k)}} g\Big(p(\omega(t_1-) - \omega(0), \cdots, \omega(t_k-) - \omega(t_{k-1}-), y_1 + \omega(s) - \omega(t_k-), \\
&\quad y_2, \cdots, y_{n-k})\Big)\frac{y_1 \cdot e_i}{t_{k+1} - s}\Phi(y_1, t_{k+1} - s) \prod_{l=2}^{n-k} \Phi(y_l, \delta) dy_1 dy_2 \cdots dy_{n-k} \\
&= \mathbb{E}\left[g({}_nX(\omega_s \underset{s}{\oplus} B_T))\frac{(B(t_{k+1}) - B(s)) \cdot e_i}{t_{k+1} - s}\right] \qquad (14)
\end{aligned}$$

Remark that when s tends towards t_{k+1}, $\nabla_\omega F_n(s,\omega)$ may tend to infinity because of the term $t_{k+1} - s$ in the denominator. However in the interval $[t_k, t_{k+1})$, $\nabla_\omega F_n(s,\omega)$ behaves well and is locally bounded.

Iterating this procedure, one can show that F_n is vertically differentiable to any order. For example, we have: for $z \in \mathbb{R}^d$,

$$\begin{aligned}
\partial_i F_n(s,\omega_s^z) &= \int_{\mathbb{R}^{d\times(n-k)}} g\Big(p(\omega(t_1-) - \omega(0), \cdots, \omega(t_k-) - \omega(t_{k-1}-), y_1 + z + \omega(s) - \omega(t_k-), \\
&\quad y_2, \cdots, y_{n-k})\Big)\frac{y_1 \cdot e_i}{t_{k+1} - s}\Phi(y_1, t_{k+1} - s) \prod_{l=2}^{n-k} \Phi(y_l, \delta) dy_1 dy_2 \cdots dy_{n-k}
\end{aligned}$$

Thus we have:

$$\partial_i^2 F_n(s,\omega) = \int_{\mathbb{R}^{d\times(n-k)}} g\Big(p(\omega(t_1-)-\omega(0),\cdots,\omega(t_k-)-\omega(t_{k-1}-),y_1+\omega(s)-\omega(t_k-),y_2,\cdots,y_{n-k})\Big) \left(\frac{(y_1\cdot e_i)^2}{(t_{k+1}-s)^2}-\frac{1}{t_{k+1}-s}\right)\Phi(y_1,t_{k+1}-s)\prod_{l=2}^{n-k}\Phi(y_l,\delta)dy_1dy_2\cdots dy_{n-k}$$

And for $i\neq j$:

$$\partial_{ij} F_n(s,\omega) = \int_{\mathbb{R}^{d\times(n-k)}} g\Big(p(\omega(t_1-)-\omega(0),\cdots,\omega(t_k-)-\omega(t_{k-1}-),y_1+\omega(s)-\omega(t_k-), y_2,\cdots,y_{n-k})\Big)\frac{(y_1\cdot e_i)(y_1\cdot e_j)}{(t_{k+1}-s)^2}\Phi(y_1,t_{k+1}-s)\prod_{l=2}^{n-k}\Phi(y_l,\delta)dy_1\cdots dy_{n-k}$$

The horizontal differentiability of F_n can be proved similarly. Consider the following map:

$$w(h)=F_n(s+h,\omega_s)=\mathbb{E}\left[g({}_nX(\omega_s \underset{s+h}{\oplus} B_T))\right],\quad h>0$$

The objective is to show that w is differentiable at $0+$.

We assume again that $t_k\leq s<t_{k+1}$ for some $0\leq k\leq n-1$, and we take an $h>0$ small enough such that $s+h<t_{k+1}$. Using the same argument and the fact that $\omega_s(s+h)=\omega(s)$, we have:

$${}_nX(\omega_s \underset{s+h}{\oplus} B_T) = p(\omega(t_1-)-\omega(0),\cdots,\omega(t_k-)-\omega(t_{k-1}-),B(t_{k+1})-B(s+h)+\omega(s)-\omega(t_k-), B(t_{k+2})-B(t_{k+1}),\cdots,B(t_n)-B(t_{n-1}))$$

Let $y=(y_1,\cdots,y_{n-k})\in\mathcal{M}_{d,n-k}(\mathbb{R})$ with each $y_l\in\mathbb{R}^d$ for $1\leq l\leq n-k$. We calculate explicitly $w(h)$:

$$\begin{aligned} w(h) &= \mathbb{E}\left[g({}_nX_T(\omega_s \underset{s+h}{\oplus} B_T))\right] \\ &= \mathbb{E}\Big[g\Big(p(\omega(t_1-)-\omega(0),\cdots,\omega(t_k-)-\omega(t_{k-1}-),B(t_{k+1})-B(s+h)+\omega(s)-\omega(t_k-), \\ &\quad B(t_{k+2})-B(t_{k+1}),\cdots,B(t_n)-B(t_{n-1}))\Big)\Big] \\ &= \int_{\mathbb{R}^{d\times(n-k)}} g\Big(p(\omega(t_1-)-\omega(0),\cdots,\omega(t_k-)-\omega(t_{k-1}-),y_1+\omega(s)-\omega(t_k-), \\ &\quad y_2,\cdots,y_n)\Big)\Phi(y_1,t_{k+1}-s-h)\prod_{l=2}^{n-k}\Phi(y_l,\delta)dy_1dy_2\cdots dy_{n-k} \end{aligned} \tag{15}$$

Again the only term which depends on h in the integrand of (15) is $\Phi(y_1,t_{k+1}-s-h)$, which is a smooth function of h. Therefore F_n is horizontally differentiable with:

$$\mathcal{D}F_n(s,\omega_s) = \int_{\mathbb{R}^{d\times(n-k)}} g\Big(p(\omega(t_1-)-\omega(0),\cdots,\omega(t_k-)-\omega(t_{k-1}-),y_1+\omega(s)-\omega(t_k-), y_2,\cdots,y_n)\Big)\left(\frac{d}{2(t_{k+1}-s)}-\frac{|y_1|^2}{2(t_{k+1}-s)^2}\right)\Phi(y_1,t_{k+1}-s)\prod_{l=2}^{n-k}\Phi(y_l,\delta)dy_1\cdots dy_{n-k}$$

with

$$\Phi(x,t) = (2\pi t)^{-\frac{d}{2}} \exp\left(-\frac{|x|^2}{2t}\right), \quad x \in \mathbb{R}^d$$

□

The following result shows that the functional derivatives of F_n satisfy the necessary regularity conditions for applying the functional Itô formula to F_n:

Theorem 4.2. *Under Assumptions 1 and 2, $F_n \in \mathbb{C}^{1,2}_{loc,r}(\Lambda_T)$.*

Proof. We have already shown in Theorem 4.1 that F_n is horizontally differentiable and twice vertically differentiable. Using the expressions of $\mathcal{D}F_n$, $\nabla_\omega F_n$ and $\nabla^2_\omega F_n$ obtained in the proof of 4.1 and the assumption that g has at most polynomial growth with respect to $\|\cdot\|_\infty$, we observe that in each interval $[t_k, t_{k+1})$ with $0 \le k \le n-1$, $\mathcal{D}F_n$, $\nabla_\omega F_n$ and $\nabla^2_\omega F_n$ satisfy the boundedness-preserving property. We now prove that F_n is left-continuous, $\nabla_\omega F_n$ and $\nabla^2_\omega F_n$ are right-continuous, and $\mathcal{D}F_n$ is continuous at fixed times.

Let $s \in [t_k, t_{k+1})$ for some $0 \le k \le n-1$ and $\omega \in D([0,T], \mathbb{R}^d)$. We first prove that F_n is right-continuous at (s,ω), and is jointly continuous at (s,ω) for $s \in (t_k, t_{k+1})$. By definition of joint-continuity (or right-continuous), we want to show that: $\forall \epsilon > 0, \exists \eta > 0, \forall (s',\omega') \in \Lambda_T$ (for the right-continuity, we assume in addition that $s' > s$),

$$d_\infty((s,\omega),(s',\omega')) < \eta) \Rightarrow |F_n(s,\omega) - F_n(s',\omega')| < \epsilon$$

Let $(s',\omega') \in \Lambda_T$ (with $s' > s$ for the right-continuity). We assume that $d_\infty((s,\omega),(s',\omega')) \le \eta$ with an η small enough such that $s' \in [t_k, t_{k+1})$ (this is always possible as if $s = t_k$, we are only interested in the right-continuity, thus $s' > s$). It suffices to prove that $|F_n(s,\omega) - F_n(s',\omega')| \le C(s,\omega_s,\eta)$ with $C(s,\omega_s,\eta)$ a quantity depending only on s, ω_s and η, and $C(s,\omega_s,\eta) \underset{\eta\to 0}{\longrightarrow} 0$.

We use the expression of F_n obtained in the proof of Theorem 4.1. Let $y = (y_1, \cdots, y_{n-k}) \in \mathcal{M}_{d,n-k}(\mathbb{R})$ with each $y_l \in \mathbb{R}^d$ for $1 \le l \le n-k$, we have:

$$\begin{aligned} F_n(s,\omega) \;=\; & \int_{\mathbb{R}^{d\times(n-k)}} g\Big(p(\omega(t_1-) - \omega(0), \cdots, \omega(t_k-) - \omega(t_{k-1}-), y_1 + \omega(s) - \omega(t_k-), \\ & y_2, \cdots, y_n)\Big)\Phi(y_1, t_{k+1} - s) \prod_{l=2}^{n-k} \Phi(y_l, \delta) dy_1 dy_2 \cdots dy_{n-k} \end{aligned}$$

and

$$\begin{aligned} F_n(s',\omega') \;=\; & \int_{\mathbb{R}^{d\times(n-k)}} g\Big(p(\omega'(t_1-) - \omega'(0), \cdots, \omega'(t_k-) - \omega'(t_{k-1}-), y_1 + \omega'(s') - \omega'(t_k-), \\ & y_2, \cdots, y_n)\Big)\Phi(y_1, t_{k+1} - s') \prod_{l=2}^{n-k} \Phi(y_l, \delta) dy_1 dy_2 \cdots dy_{n-k} \end{aligned}$$

with

$$\Phi(x,t) = (2\pi t)^{-\frac{d}{2}} \exp\left(-\frac{|x|^2}{2t}\right), \quad x \in \mathbb{R}^d$$

To simplify the notations, we set:

$$\tilde{p}(\omega, s, y) = p(\omega(t_1-) - \omega(0), \cdots, \omega(t_k-) - \omega(t_{k-1}-), y_1 + \omega(s) - \omega(t_k-), y_2, \cdots, y_n)$$

and

$$\tilde{p}(\omega', s', y) = p(\omega'(t_1-) - \omega'(0), \cdots, \omega'(t_k-) - \omega'(t_{k-1}-), y_1 + \omega'(s') - \omega'(t_k-), y_2, \cdots, y_n)$$

Similarly $\tilde{p}_t(\cdot)$ will be the path of $\tilde{p}(\cdot)$ stopped at time t.

As $\|\omega_s - \omega_{s'}\|_\infty \leq \eta < \delta$, by **Lemma 3.1**, we have:

$$\|\tilde{p}(\omega, s, y) - \tilde{p}(\omega', s', y)\|_\infty \leq C(\omega_s, y, \sigma_{lip}, T)\eta$$

Actually we have the following better estimate of $\|\tilde{p}(\omega, s, y) - \tilde{p}(\omega', s', y)\|_\infty$:

Lemma 4.1. *We have:*

$$\|\tilde{p}(\omega, s, y) - \tilde{p}(\omega', s', y)\|_\infty \leq C(\omega_s, \sigma_{lip}, T)\prod_{l=1}^{n-k}(1 + |y_l|\sigma_{lip})\eta$$

for some constant C which depends only on ω_s, σ_{lip} and T.

Proof. By **Lemma 3.1**, we know already that:

$$\|\tilde{p}_{t_k}(\omega, s, y) - \tilde{p}_{t_k}(\omega', s', y)\|_\infty \leq C(\omega_s, \sigma_{lip}, T)\eta$$

Now we prove by induction that, for any $k+1 \leq j \leq n$,

$$\|\tilde{p}_{t_j}(\omega, s, y) - \tilde{p}_{t_j}(\omega', s', y)\|_\infty \leq C(\omega_s, \sigma_{lip}, T)\prod_{l=1}^{j-k}(1 + |y_l|\sigma_{lip})\eta \tag{16}$$

for some constant C which depends only on ω_s, σ_{lip} and T.

Consider first the case where $j = k+1$. We have:

$$\tilde{p}(\omega, s, y)(t_{k+1}) = \tilde{p}(\omega, s, y)(t_k) + \sigma\left(t_k, \tilde{p}_{t_k}(\omega, s, y)\right)\left(\omega(s) - \omega(t_k-) + y_1\right)$$

and

$$\tilde{p}(\omega', s', y)(t_{k+1}) = \tilde{p}(\omega', s', y)(t_k) + \sigma\left(t_k, \tilde{p}_{t_k}(\omega', s', y)\right)\left(\omega'(s') - \omega'(t_k-) + y_1\right)$$

As σ is Lipschitz continuous with respect to d_∞, we have:

$$\|\sigma\left(t_k, \tilde{p}_{t_k}(\omega, s, y)\right) - \sigma\left(t_k, \tilde{p}_{t_k}(\omega', s', y)\right)\| \leq \sigma_{lip}C(\omega_s, \sigma_{lip}, T)\eta$$

In addition, we have $|\omega(s) - \omega'(s')| \leq \eta$ and $|\omega(t_k-) - \omega'(t_k-)| \leq \eta$ as $\|\omega_s - \omega'_{s'}\|_\infty \leq \eta$. Thus we have:

$$\begin{aligned}
& |\tilde{p}(\omega, s, y)(t_{k+1}) - \tilde{p}(\omega', s', y)(t_{k+1})| \\
\leq \quad & |\tilde{p}(\omega, s, y)(t_k) - \tilde{p}(\omega', s', y)(t_k)| + \|\sigma\left(t_k, \tilde{p}_{t_k}(\omega, s, y)\right)\|2\eta + \sigma_{lip}C(\omega_s, \sigma_{lip}, T)\eta \cdot (2\|\omega'_{s'}\|_\infty + |y_1|) \\
\leq \quad & C(\omega_s, \sigma_{lip}, T)\eta + C(\sigma_{lip}, T)(1 + \|\tilde{p}_{t_k}(\omega, s, y)\|_\infty)2\eta + \sigma_{lip}C(\omega_s, \sigma_{lip}, T)\eta \cdot (2\|\omega_s\|_\infty + 2\eta + |y_1|) \\
\leq \quad & C'(\omega_s, \sigma_{lip}, T)(1 + |y_1|\sigma_{lip})\eta
\end{aligned}$$

with C' a constant which depends only on ω_s, σ_{lip} and T.

Assume now that (16) holds for some $j \geq k+1$. We have:

$$\tilde{p}(\omega, s, y)(t_{j+1}) = \tilde{p}(\omega, s, y)(t_j) + \sigma\left(t_j, \tilde{p}_{t_j}(\omega, s, y)\right) y_{j-k+1}$$

and

$$\tilde{p}(\omega', s', y)(t_{j+1}) = \tilde{p}(\omega', s', y)(t_j) + \sigma\left(t_j, \tilde{p}_{t_j}(\omega', s', y)\right) y_{j-k+1}$$

Thus

$$\begin{aligned}
& |\tilde{p}(\omega, s, y)(t_{j+1}) - \tilde{p}(\omega', s', y)(t_{j+1})| \\
\leq\ & |\tilde{p}(\omega, s, y)(t_j) - \tilde{p}(\omega', s', y)(t_j)| + \|\sigma\left(t_j, \tilde{p}_{t_j}(\omega, s, y)\right) - \sigma\left(t_j, \tilde{p}_{t_j}(\omega', s', y)\right)\| \cdot |y_{j-k+1}| \\
\leq\ & C(\omega_s, \sigma_{lip}, T) \prod_{l=1}^{j-k} (1 + |y_l|\sigma_{lip})\eta \cdot (1 + |y_{j-k+1}|\sigma_{lip}) \\
\leq\ & C(\omega_s, \sigma_{lip}, T) \prod_{l=1}^{j-k+1} (1 + |y_l|\sigma_{lip})\eta
\end{aligned}$$

And we conclude by induction that

$$\|\tilde{p}(\omega, s, y) - \tilde{p}(\omega', s', y)\|_\infty \leq C(\omega_s, \sigma_{lip}, T) \prod_{l=1}^{n-k} (1 + |y_l|\sigma_{lip})\eta$$

for some constant C which depends only on ω_s, σ_{lip} and T. □

Now we can control the difference between $F_n(s, \omega)$ and $F_n(s', \omega')$.

$$\begin{aligned}
& |F_n(s, \omega) - F_n(s', \omega')| \\
\leq\ & \int_{\mathbb{R}^{d\times(n-k)}} |g(\tilde{p}(\omega, s, y))\Phi(y_1, t_{k+1} - s) - g(\tilde{p}(\omega', s', y))\Phi(y_1, t_{k+1} - s')| \prod_{l=2}^{n-k} \Phi(y_l, \delta) dy_1 dy_2 \cdots dy_{n-k} \\
\leq\ & \int_{\mathbb{R}^{d\times(n-k)}} \Big(|g(\tilde{p}(\omega, s, y)) - g(\tilde{p}(\omega', s', y))| \Phi(y_1, t_{k+1} - s') \\
& + |g(\tilde{p}(\omega, s, y))| \cdot |\Phi(y_1, t_{k+1} - s) - \Phi(y_1, t_{k+1} - s')| \Big) \times \prod_{l=2}^{n-k} \Phi(y_l, \delta) dy_1 dy_2 \cdots dy_{n-k} \qquad (17)
\end{aligned}$$

Observe that $|\Phi(y_1, t_{k+1} - s) - \Phi(y_1, t_{k+1} - s')| \leq |s - s'| \cdot \rho(y_1, \eta) \leq \rho(y_1, \eta) \cdot \eta$ with

$$\rho(y_1, \eta) = \sup_{t \in [t_{k+1} - s - \eta, \delta]} |\partial_t \Phi(y_1, t)|$$

and we have:

$$\rho(y_1, \eta) \underset{\eta \to 0}{\longrightarrow} \sup_{t \in [t_{k+1} - s, \delta]} |\partial_t \Phi(y_1, t)| = \sup_{t \in [t_{k+1} - s, \delta]} \left| \Phi(y_1, t) \left(\frac{|y_1|^2}{2t^2} - \frac{d}{2t} \right) \right| < \infty$$

So the second part of (17) can be controlled by:

$$\int_{\mathbb{R}^{d\times(n-k)}} |g(\tilde{p}(\omega, s, y))| \cdot |\Phi(y_1, t_{k+1} - s) - \Phi(y_1, t_{k+1} - s')| \prod_{l=2}^{n-k} \Phi(y_l, \delta) dy_1 dy_2 \cdots dy_{n-k} \leq C(s, \omega_s, \eta)$$

with

$$C(s,\omega_s,\eta) \underset{\eta\to 0}{\longrightarrow} 0.$$

For the first part of (17), we use the continuity of g and **Lemma 4.1**. As g is continuous at $p(\omega_s, y)$, we have:

$$|g(\tilde{p}(\omega,s,y)) - g(\tilde{p}(\omega',s',y))| \le C(s,\omega_s,y,\eta)$$

with

$$C(s,\omega_s,y,\eta) \underset{\eta\to 0}{\longrightarrow} 0$$

and

$$\Phi(y_1, t_{k+1}-s') \le \sup_{t\in[t_{k+1}-s-\eta,\delta]} \Phi(y_1,t) < \infty.$$

Thus the first part of (17) can also be bounded by $C(s,\omega_s,\eta)$ with

$$C(s,\omega_s,\eta) \underset{\eta\to 0}{\longrightarrow} 0.$$

We conclude that $|F_n(s,\omega_s) - F_n(s',\omega'_{s'})| \le C(s,\omega_s,\eta)$ with $C(s,\omega,\eta)$ depending only on s, ω_s and η, and $C(s,\omega_s,\eta) \underset{\eta\to 0}{\longrightarrow} 0$, which proves the right-continuity of F_n and the joint-continuity of F_n at all $(s,\omega)\in\Lambda_T$ for $s\neq t_k$, $0\le k\le n-1$.

The right-continuity of $\nabla_\omega F_n$, $\nabla^2_\omega F_n$ and the continuity at fixed times of $\mathcal{D}F_n$ can be deduced as above using the expressions of $\nabla_\omega F_n$, $\nabla^2_\omega F_n$ and $\mathcal{D}F_n$ obtained in Theorem 4.1. Now it remains to show that for $1\le k\le n$ and $\omega\in D([0,T],\mathbb{R}^d)$, F_n is left-continuous at (t_k,ω). Let $(s',\omega')\in\Lambda_T$ with $s'<t_k$ such that $d_\infty((t_k,\omega),(s',\omega'))\le\eta$. We choose an η small enough in order that $s'\in[t_{k-1},t_k)$, and we want to show that $|F_n(t_k,\omega)-F_n(s',\omega')|\le C(t_k,\omega_{t_k},\eta)$ for some $C(t_k,\omega_{t_k},\eta)$ depending only on t_k, ω_{t_k} and η with $C(t_k,\omega_{t_k},\eta) \underset{\eta\to 0}{\longrightarrow} 0$.

We first decompose $|F_n(t_k,\omega)-F_n(s',\omega')|$ into two terms:

$$|F_n(t_k,\omega)-F_n(s',\omega')| \le |F_n(t_k,\omega)-F_n(s',\omega_{s'})| + |F_n(s',\omega_{s'})-F_n(s',\omega')|$$

For the second part, as F_n is continuous at fixed time s' by the first part of the proof, and $\|\omega_{s'}-\omega'_{s'}\|_\infty\le\eta$, we have $|F_n(s',\omega_{s'})-F_n(s',\omega')|\le C(t_k,\omega_{t_k},\eta)$ with $C(t_k,\omega_{t_k},\eta) \underset{\eta\to 0}{\longrightarrow} 0$.

For the first part $|F_n(t_k,\omega)-F_n(s',\omega_{s'})|$, the difficulty is that s' and t_k no longer lie in the same interval, thus we need to perform one more integration for $F_n(s',\omega_{s'})$ compared to $F_n(t_k,\omega)$. Let $y=(y_1,\cdots,y_{n-k})\in\mathcal{M}_{d,n-k}(\mathbb{R})$ with each $y_l\in\mathbb{R}^d$ for $1\le l\le n-k$, and $y'\in\mathbb{R}^d$. Using again the expression of F_n we have obtained in the proof of Theorem 4.1, we have:

$$\begin{aligned} F_n(t_k,\omega) \quad &= \quad \int_{\mathbb{R}^{d\times(n-k)}} g\Big(p(\omega(t_1-)-\omega(0),\cdots,\omega(t_k-)-\omega(t_{k-1}-), y_1+\omega(t_k)-\omega(t_k-), \\ &\qquad y_2,\cdots,y_{n-k})\Big)\prod_{l=1}^{n-k}\Phi(y_l,\delta)dy_1dy_2\cdots dy_{n-k} \end{aligned}$$

and

$$
\begin{aligned}
F_n(s',\omega_{s'}) &= \int_{\mathbb{R}^{d\times(n-k+1)}} g\Big(p(\omega(t_1-)-\omega(0),\cdots,y'+\omega(s')-\omega(t_{k-1}-),y_1,\cdots,y_n)\Big) \\
&\quad \Phi(y',t_k-s')\prod_{l=1}^{n-k}\Phi(y_l,\delta)dy'dy_1\cdots dy_{n-k} \\
&= \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^{d\times(n-k)}} g(p(\omega(t_1-)-\omega(0),\cdots,y'+\omega(s')-\omega(t_{k-1}-),y_1,\cdots,y_n)) \\
&\quad \prod_{l=1}^{n-k}\Phi(y_l,\delta)dy_1\cdots dy_{n-k}\Big)\Phi(y',t_k-s')dy'
\end{aligned}
$$

with

$$
\Phi(x,t)=(2\pi t)^{-\frac{d}{2}}\exp\left(-\frac{|x|^2}{2t}\right),\quad x\in\mathbb{R}^d
$$

We now define $\zeta:\mathbb{R}^d\to\mathbb{R}$ by: for $y'\in\mathbb{R}^d$,

$$
\zeta(y')=\int_{\mathbb{R}^{d\times(n-k)}} g(p(\omega(t_1-)-\omega(0),\cdots,y'+\omega(s')-\omega(t_{k-1}-),y_1,\cdots,y_n))\prod_{l=1}^{n-k}\Phi(y_l,\delta)dy_1\cdots dy_{n-k}
$$

By **Lemma 3.1** and the continuity of g with respect to $\|\cdot\|_\infty$, the map

$$
y'\mapsto g(p(\omega(t_1-)-\omega(0),\cdots,y'+\omega(s')-\omega(t_{k-1}-),y_1,\cdots,y_n))
$$

is continuous. As g has at most polynomial growth with respect to $\|\cdot\|_\infty$, by the dominated convergence theorem, ζ is also continuous. Moreover, ζ has at most polynomial growth. And as $t_k-s'\le\eta$, we have

$$
\begin{aligned}
F_n(s',\omega_{s'}) &= \int_{\mathbb{R}^d}\zeta(y')\Phi(y',t_k-s')dy' \\
&= \int_{\mathbb{R}^d}(\zeta(y')-\zeta(0))\Phi(y',t_k-s')dy'+\zeta(0).
\end{aligned}
$$

with

$$
\left|\int_{\mathbb{R}^d}(\zeta(y')-\zeta(0))\Phi(y',t_k-s')dy'\right|\le C(t_k,\omega_{t_k},\eta)
$$

and $C(t_k,\omega_{t_k},\eta)\underset{\eta\to0}{\longrightarrow}0$.

It remains to control the difference between $F_n(t_k,\omega)$ and $\zeta(0)$. We remark that:

$$
\begin{aligned}
\zeta(0) &= \int_{\mathbb{R}^{d\times(n-k)}} g(p(\omega(t_1-)-\omega(0),\cdots,\omega(s')-\omega(t_{k-1}-),y_1,\cdots,y_n))\prod_{l=1}^{n-k}\Phi(y_l,\delta)dy_1\cdots dy_{n-k} \\
&= \mathbb{E}\left[g({}_nX_T(\omega_{s'}\underset{t_k}{\oplus}B_T))\right]=F_n(t_k,\omega_{s'})
\end{aligned}
$$

As $\|\omega_{s'}-\omega_{t_k}\|_\infty\le\|\omega_{s'}-\omega'_{s'}\|_\infty+\|\omega_{t_k}-\omega'_{s'}\|_\infty\le 2\eta$, again by the continuity of F_n at fixed time t_k established in the first part of the proof, we have:

$$
|F_n(t_k,\omega)-\zeta(0)|\le C(t_k,\omega_{t_k},\eta)
$$

with $C(t_k, \omega_{t_k}, \eta) \underset{\eta \to 0}{\longrightarrow} 0$.

We conclude that

$$|F_n(t_k, \omega) - F_n(s', \omega')| \leq C(t_k, \omega_{t_k}, \eta)$$

with $C(t_k, \omega_{t_k}, \eta) \underset{\eta \to 0}{\longrightarrow} 0$, which proves the left-continuity of F_n at (t_k, ω). □

Corollary 4.1. *Under Assumptions 1 and 2, for any $t \in [0, T)$ we have:*

$$F_n(T, W_T) - F_n(t, W_t) = \int_t^T \nabla_\omega F_n(s, W_s) \cdot dW(s), \quad \mathbb{P} - a.s. \tag{18}$$

Proof. As $F_n \in \mathbb{C}^{1,2}_{loc,r}(\Lambda_T)$, we can apply the functional Itô formula **Proposition 2.2** and we remark that the finite variation term is zero as ${}_nY(s) = F_n(s, W_s)$ is a martingale. □

Remark 4.1. *We can also verify using directly the expressions we have obtained in Theorem 4.1 for $\mathcal{D}F_n$ and $\nabla^2_\omega F_n$ that the finite variation terms in (18) cancel each other. By the functional Itô formula, the finite variation term in (18) equals to $\mathcal{D}F_n(s, W_s) + \frac{1}{2}\mathrm{tr}(\nabla^2_\omega F_n(s, W_s))$. And for $(s, \omega) \in \Lambda_T$ with $s \in [t_k, t_{k+1})$, we have:*

$$\begin{aligned}
&\mathrm{tr}\left(\nabla^2_\omega F_n(s, \omega)\right) = \sum_{i=1}^d \partial_i^2 F_n(s, \omega) \\
= \quad & \int_{\mathbb{R}^{d\times(n-k)}} g(p(\omega(t_1-) - \omega(0), \cdots, \omega(t_k-) - \omega(t_{k-1}-), y_1 + \omega(s) - \omega(t_k-), y_2, \cdots, y_n)) \\
& \sum_{i=1}^d \left(\frac{(y_1 \cdot e_i)^2}{(t_{k+1} - s)^2} - \frac{1}{t_{k+1} - s} \right) \Phi(y_1, t_{k+1} - s) \prod_{l=2}^{n-k} \Phi(y_l, \delta) dy_1 \cdots dy_{n-k} \\
= \quad & \int_{\mathbb{R}^{d\times(n-k)}} g(p(\omega(t_1-) - \omega(0), \cdots, \omega(t_k-) - \omega(t_{k-1}-), y_1 + \omega(s) - \omega(t_k-), y_2, \cdots, y_n)) \\
& \left(\frac{|y_1|^2}{(t_{k+1} - s)^2} - \frac{d}{t_{k+1} - s} \right) \Phi(y_1, t_{k+1} - s) \prod_{l=2}^{n-k} \Phi(y_l, \delta) dy_1 \cdots dy_{n-k} \\
= \quad & -2\mathcal{D}F_n(s, \omega_s)
\end{aligned}$$

which confirms that F_n is a solution of the path-dependent Kolmogorov equation [4, Sec. 5]:

$$\mathcal{D}F_n(s, W_s) + \frac{1}{2}\mathrm{tr}(\nabla^2_\omega F_n(s, W_s)) = 0.$$

5 Convergence and error analysis

In this section, we analyze the convergence rate of our approximation method. After having constructed a sequence of smooth functionals F_n (Theorem 4.1 and Theorem 4.2), we can now approximate $\nabla_X Y$ by:

$$Z_n(s) = {}^t(\sigma^{-1}(s, X_s))\nabla_\omega F_n(s, W_s)$$

which, in contrast to the weak derivative $\nabla_X Y$, is computable as a pathwise directional derivative. In practice, $\nabla_\omega F_n(s, W_s)$ can be computed numerically via a finite difference method or a Monte-Carlo method using the expression (14) of $\nabla_\omega F_n$.

For $t \in [0, T]$, the quantity we are interested in is the integral of $\nabla_X Y - Z_n$ along the path of X between t and T, i.e.

$$\int_t^T (\nabla_X Y - Z_n) \cdot dX = \int_t^T \nabla_X Y(s) \cdot dX(s) - \int_t^T \nabla_\omega F_n(s, W_s) \cdot dW(s)$$

By the martingale representation formula **Proposition 2.3** and **Corollary 4.1**, we have $\mathbb{P}$-a.s.

$$\begin{aligned} & \int_t^T (\nabla_X Y - Z_n) \cdot dX \\ = \; & Y(T) - Y(t) - ({}_nY(T) - {}_nY(t)) \\ = \; & g(X_T) - g({}_nX_T(W_T)) - \mathbb{E}\left[g(X_T) - g({}_nX_T(W_t \underset{t}{\oplus} B_T))|\mathcal{F}_t^X\right] \end{aligned}$$

where ${}_nX$ is the path of the piecewise constant Euler-Maruyama scheme defined in (7). Remark that by definition of the concatenation operation and using the fact that B and W are two independent Brownian motions, we have:

$$\begin{aligned} & \mathbb{E}\left[g({}_nX_T(W_t \underset{t}{\oplus} B_T))|\mathcal{F}_t^X\right] \\ = \; & \mathbb{E}\left[g({}_nX_T(W_t \underset{t}{\oplus} W_T))|\mathcal{F}_t^X\right] \\ = \; & \mathbb{E}\left[g({}_nX_T(W_T)|\mathcal{F}_t^X\right] \end{aligned}$$

Corollary 5.1. *Under Assumptions 1 and 2,*

$$\forall t \in [0, T], \quad \int_t^T (\nabla_X Y - Z_n) \cdot dX \underset{n \to \infty}{\longrightarrow} 0, \quad \mathbb{P}\text{-a.s.}$$

Proof. We have already shown that:

$$\int_t^T (\nabla_X Y - Z_n) \cdot dX = g(X_T) - g({}_nX_T) - \mathbb{E}\left[g(X_T) - g({}_nX_T)|\mathcal{F}_t^X\right]$$

As g is continuous with respect to $\|\cdot\|_\infty$, by **Corollary 3.1**, we have:

$$g(X_T) - g({}_nX_T) \underset{n \to \infty}{\longrightarrow} 0, \quad \mathbb{P}\text{-a.s.}$$

Moreover, g has at most polynomial growth with respect to $\|\cdot\|_\infty$, which, together with **Proposition 3.2**, ensures the uniform integrability of $g({}_nX_T)$. And thus

$$\mathbb{E}\left[g(X_T) - g({}_nX_T)|\mathcal{F}_t^X\right] \underset{n \to \infty}{\longrightarrow} 0, \quad \mathbb{P}\text{-a.s.}$$

□

Corollary 5.2. *Under Assumptions 1 and 2,*

$$\forall t \in [0,T], \qquad \left\| \int_t^T (\nabla_X Y - Z_n) \cdot dX \right\|_{2p} \underset{n\to\infty}{\longrightarrow} 0, \quad \forall p \geq 1$$

Under a slightly stronger assumption on g we can obtain a rate of convergence for our approximation:

Theorem 5.1 (Rate of convergence)**.** *Let $p \geq 1$ and assume $g : (D([0,T],\mathbb{R}^d), \|.\|_\infty) \to \mathbb{R}$ is Lipschitz-continuous:*

$$\exists g_{lip} > 0, \quad \forall \omega, \omega' \in D([0,T],\mathbb{R}^d), \quad |g(\omega) - g(\omega')| \leq g_{lip} \sup |\omega - \omega'| \, .$$

Under Assumptions 1 the L^{2p}-error of the approximation Z_n of $\nabla_X Y$ along the path of X between t and T is bounded by:

$$E\left(\left\| \int_t^T (\nabla_X Y - Z_n) \cdot dX \right\|^{2p} \right) \leq C(x_0, p, T, \sigma_{lip}, g_{lip}) \left(\frac{1 + \log n}{n} \right)^p, \quad \forall p \geq 1$$

where the constant C depends only on x_0, p, T, σ_{lip} and g_{lip}. In particular:

$$\forall \alpha \in [0, \frac{1}{2}), \qquad n^\alpha \left(\int_t^T (\nabla_X Y - Z_n) \cdot dX \right) \underset{n\to\infty}{\longrightarrow} 0, \quad \mathbb{P}\text{−a.s.}$$

Proof. This result is a consequence of Proposition 3.2 since

$$\begin{aligned}
\left\| \int_t^T (\nabla_X Y(s) - Z_n(s)) \cdot dX(s) \right\|_{2p} &\leq \|g(X_T) - g({}_nX_T)\|_{2p} + \|\mathbb{E}[g(X_T) - g({}_nX_T)|\mathcal{F}_t^X]\|_{2p} \\
&\leq 2\|g(X_T) - g({}_nX_T)\|_{2p} \\
&\leq 2g_{lip}\| \sup_{s\in[0,T]} |X(s) - {}_nX(s)| \|_{2p}.
\end{aligned}$$

□

The following example how our result may be used to construct explicit approximations with cotnroled convergence rates for conditional expectation of non-smooth functionals:

Example 5.1. *Let*

$$g(\omega) = \psi(\omega(T), \sup_{t\in[0,T]} \|\omega(t)\|)$$

where $\psi \in C^0(\mathbb{R}^d \times \mathbb{R}_+, \mathbb{R})$ is a continuous function with polynomial growth, and set $Y(t) = E[g(X_T)|\mathcal{F}_t]$. Then g satisfies Assumption 2, and our approximation method applies. Moreover, if ψ is Lipschitz-continuous, then Theorem 5.1 yields an explicit control of the approximation error with a $1/\sqrt{n}$ bound.

6 Comparison with approaches based on the Malliavin calculus

The vertical derivative $\nabla_X Y(t)$ which appears in the martingale representation formula may be viewed as a 'sensitivity' of the martingale Y to the initial condition $X(t)$. Thus, our method is related to methods previously proposed for 'sensitivity analysis' of Wiener functionals.

One can roughly classify such methods into two categories [2]: methods that differentiate paths and methods that differentiate densities. When the density of the functional is known, the sensitivity of an expectation with respect to some parameter is to differentiate directly the density function with respect to the parameter. However, as this is almost never the case in a general diffusion model, let alone a non-Markovian model, alternative methods, are used: these consist of differentiating either the functional g or the process with respect to the parameter under the expectation sign, then estimating the expectation with the Monte-Carlo method. To differentiate process, one required the existence of the so-called first variation process, which requires the regularity of the coefficients of the SDE satisfied by X.

Sensitivity estimators for non-smooth functionals may be computed using Malliavin calculus: this approach, proposed by Fournié et al. [17] and developed by Cvitanic, Ma and Zhang [8], Fournié et al. [16], Gobet and Kohatsu-Higa [18], Kohatsu-Higa and Montero [24], Davis and Johansson [10] and others, uses the Malliavin integration-by-parts formula on Wiener space in the case where g is not smooth. These methods require quite demanding regularity assumptions (differentiability and ellipticity condition on σ for example) on the coefficients of the initial SDE satisfied by X.

By contrast, the approximation method presented here allows for any continuous functional g with polynomial growth and requires only mild assumptions on σ: Lipschitz continuity and non-singularity. It is thus applicable to a wider range of examples than the Malliavin approach, while being arguably simpler from a computational viewpoint. Our method involves discretizing then differentiating, as opposed to the Malliavin approach which involves differentiating in the Malliavin sense, then discretizing the tangent process which, as argued in [2], has its computational advantages.

In our setting, we have $F_n \in \mathbb{C}^{1,2}_{loc,r}(\Lambda_T)$ which is sufficient for obtaining an approximation of martingale representations via the functional Itô formula. One can ask if the Euler approximation ${}_nX$ can also be used to obtain a Clark-Haussmann-Ocone type formula, and in this case, whether the pathwise vertical derivative $\nabla_\omega F_n(t, W_t)$ leads to the same representation as the Clark-Haussmann-Ocone formula.

For $n \in \mathbb{N}$, define $H_n = g({}_nX_T(W_T))$ with ${}_nX$ the weak piecewise constant Euler-Maruyama scheme defined by (7). By the definition of ${}_nX$, the random variable H_n actually depends only on a finite number of Gaussian variables: $W(t_1)$, $W(t_2) - W(t_1)$, $\cdots$, $W(t_n) - W(t_{n-1})$, thus it can be written as:

$$H_n = h_n(W(t_1), W(t_2) - W(t_1), \cdots, W(t_n) - W(t_{n-1}))$$

with $h_n : \mathcal{M}_{d,n}(\mathbb{R}) \to \mathbb{R}$ (h_n is actually $g \circ p$ with p defined by (8)).

Clearly if h_n is a smooth function with polynomial growth, then $H_n \in \mathbf{D}^{1,2}$ with Malliavin derivative [26]:

$$\mathbb{D}_t H_n = (\mathbb{D}_t^k H_n, 1 \le k \le d) \in \mathbb{R}^d$$

with

$$\mathbb{D}_t^k H_n = \sum_{j=0}^{n-1} \partial_{kj} h_n(W(t_1), W(t_2) - W(t_1), \cdots, W(t_n) - W(t_{n-1}))\mathbf{1}_{[t_j, t_{j+1})}(t), \quad t \in [0, T)$$

In this case, assume that $t \in [t_j, t_{j+1})$ for some $0 \le j \le n-1$, we have: for $1 \le k \le d$,

$$\begin{aligned}
& \mathbb{E}\left[\mathbb{D}_t^k H_n | \mathcal{F}_t^W\right] \\
= \; & \mathbb{E}\left[\partial_{kj} h_n(W(t_1), W(t_2) - W(t_1), \cdots, W(t_n) - W(t_{n-1})) | \mathcal{F}_t^W\right] \\
= \; & \mathbb{E}\Big[\partial_{kj} h_n(\omega(t_1), \omega(t_2) - \omega(t_1), \cdots, \omega(t_j) - \omega(t_{j-1}), W(t_{j+1}) - W(t) + \omega(t) - \omega(t_j), \\
& W(t_{j+2}) - W(t_{j+1}), \cdots, W(t_n) - W(t_{n-1}))\Big]|_{\omega_t = W_t} \\
= \; & \mathbb{E}\Big[\frac{\partial}{\partial h} h_n(\omega(t_1), \omega(t_2) - \omega(t_1), \cdots, \omega(t_j) - \omega(t_{j-1}), W(t_{j+1}) - W(t) + \omega(t) - \omega(t_j) + h e_k, \\
& W(t_{j+2}) - W(t_{j+1}), \cdots, W(t_n) - W(t_{n-1}))|_{h=0}\Big]|_{\omega_t = W_t} \\
= \; & \frac{\partial}{\partial h}\Big(\mathbb{E}\big[h_n(\omega(t_1), \omega(t_2) - \omega(t_1), \cdots, \omega(t_j) - \omega(t_{j-1}), W(t_{j+1}) - W(t) + \omega(t) - \omega(t_j) + h e_k, \\
& W(t_{j+2}) - W(t_{j+1}), \cdots, W(t_n) - W(t_{n-1}))\big]_{\omega_t = W_t}\Big)|_{h=0}
\end{aligned}$$

which is none other than

$$\partial_k F_n(t, W_t) = \lim_{h \to 0} \frac{F_n(t, W_t + h e_k \mathbf{1}_{[t,T]}) - F_n(t, W_t)}{h}.$$

So in the case where h_n are smooth, our method provides the same result as given by the Clark-Haussmann-Ocone formula applied to h_n. However, in our framework, as the functional g is only assumed to be continuous with polynomial growth, the function h_n may fail to be differentiable. So, even in the cylindrical case, it is not clear whether the random variable H_n is differentiable in the Malliavin sense, and even if it is the case, it is difficult to obtain an explicit form for $\mathbb{E}\left[\mathbb{D}_t H_n | \mathcal{F}_t^W\right]$ using the Malliavin calculus.

The reason our approximation method works even in the cases where H_n is not differentiable in the Malliavin sense is that our regularity assumptions are not on the terminal variable H_n, but on the martingale ${}_nY(t) = \mathbb{E}\left[H_n | \mathcal{F}_t^W\right]$; as shown in Section 4, ${}_nY$ is differentiable in the pathwise sense even when H_n is not differentiable in the Malliavin sense.

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