# Second geometrization: cases study

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**Abstract**

The purpose of this article is to give various discussions using 2xTAN[1] technique. This technique gives mathematical methods to study theoretically physical problems through network representations. Each example can be seen as an exercise or attempts. Sometimes, things are tested as it could be made on the board, without any preparation. Exercise are given "as is" without any second lecture. The purpose of this test article is to submit some method and technique to reviewers, to share and increase the knowledge in my research field.

# Contents



[1]xTAN is a method created by the author for extended tensorial analysis of networks. 2xTAN is for second geometrization extended tensorial analysis of networks. See http://olivier.maurice.pagesperso-orange.fr/ for more information.

# 1 Filters

A filter is a network made of two ports. One for the input and one for the output. Between these two ports, any circuit can exist.

## 1.1 Second order filters

We consider basic structures made of three branches to begin. This structure is presented figure 1.

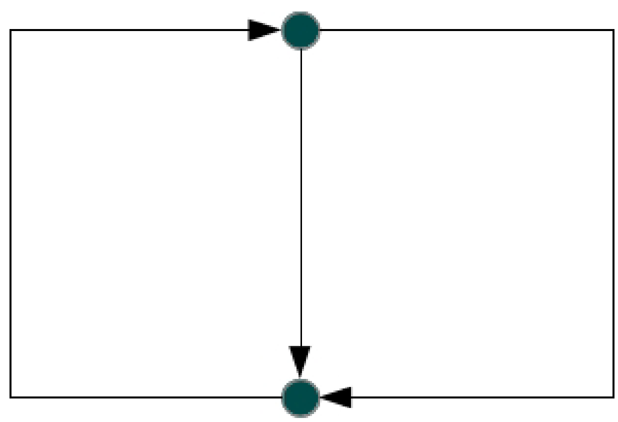

Figure 1

Each branch can wear any impedance function. Let $a$, $b$, $c$ be these three impedances. We want to study theoretically the transfer function of the filter.

We define two meshes through the connectivity with the branches:

$$C = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \tag{1}$$

From the direct summation of each function belonging to the branches, we obtain the impedance matrix following:

$$Z = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \tag{2}$$

Making $C^T ZC$ gives the impedance matrix in the mesh space (it was demonstrated that this space is the adequate one to apply geometry analysis on networks: “https://hal.archives-ouvertes.fr/hal-01079386”).

We want know to study the various behaviors of the filter depending on its functions, including the coupling one. Previous transformation leads to the matrix:

$$Z = \begin{bmatrix} a+b & -b \\ -b & b+c \end{bmatrix} \tag{3}$$

Figure 2 shows a new graph. Making same exercise, leads to the following impedance matrix:

$$Z = \begin{bmatrix} a+b & -b \\ -b & b+c \end{bmatrix} \tag{4}$$

Which is completely similar to the previous one, even if the starting matrix is not the same. There are four branches in the branch space, but the Graph characteristic stills the same (having $M$ meshes, $B$ branches, $R$ networks and $N$ nodes gives: $M = B - N + R$ meshes. In both previous cases, $M = 2$).

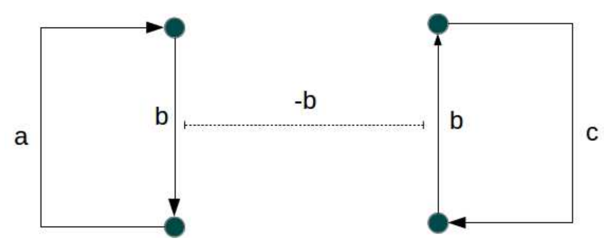


Figure 2

This new representation is easier to use. It gives the same characteristic, knowing than we cannot make a geometrical projection for dimensions less than $2^2$. It allows to change the coupling function without changing the impedance of each network, which is not the case for the structure figure 1. So, it generalizes the first structure.

#### 1.1.1 Case with no coupling function

We can change the coupling function to $\alpha$ which gives the impedance matrix:

$$Z = \begin{bmatrix} A & -\alpha \\ -\alpha & B \end{bmatrix} \tag{5}$$

with $A = a + b$, $B = b + c$, and force $\alpha = 0$. For any source vector $e_i$, the system of equations given by this graph is:

$$\begin{cases} e_1 = Ak_1 \\ e_2 = Bk_2 \end{cases} \tag{6}$$

where $k_j$ is the flux vector - current in electrical case. Now we can forget the graph and its networks, and study theoretically the flux evolution. To project the problem in a geometrical context we define a base of a parametrized surface.

[2] Because we want to make this projection in a space with at least 3 dimensions. See further how it implies dimension 2.

To do so, we need to define a third function $e_3$ in order to work at least in a three dimensions space. For example we take:

$$\left\{ \begin{array}{l} e_1 = Ak_1 \\ e_2 = Bk_2 \\ e_3 = Ck_2 \end{array} \right. \tag{7}$$

This gives the basic vectors:

$$\left\{ \begin{array}{l} \mathbf{b}_1 = \left( \frac{\partial e_1}{\partial k_1}, \frac{\partial e_2}{\partial k_1}, \frac{\partial e_3}{\partial k_1} \right) \\ \\ \mathbf{b}_2 = \left( \frac{\partial e_1}{\partial k_2}, \frac{\partial e_2}{\partial k_2}, \frac{\partial e_3}{\partial k_2} \right) \end{array} \right. \tag{8}$$

$e_i$ is considered as a vector of functions where flux $k_i$ are parameters. This leads to the vectors:

$$\left\{ \begin{array}{l} \mathbf{b}_1 = (A, 0, 0) \\ \\ \mathbf{b}_2 = (0, B, C) \end{array} \right. \tag{9}$$

and the metric:

$$G_{ij} = \langle \mathbf{b}_i, \mathbf{b}_j \rangle = \begin{bmatrix} A^2 & 0 \\ 0 & B^2 + C^2 \end{bmatrix} \tag{10}$$

As there are no coupling functions, the metric is purely diagonal. It defines all the properties of the flux space. The distance is everywhere given by: $G_{ij}k_ik_j$. To make a complete analogy with Einstein's approach, we define $k_i$ as the space axes - say $x^i$ to write:

$$ds^2 = G_{ij}x^ix^j = A^2(x^1)^2 + \left(B^2 + C^2\right)(x^2)^2 \tag{11}$$

We can define a curve attached to the mobile referential $\{\mathbf{b}_1,\ \mathbf{b}_2\}$:

$$p_i \in \gamma\,(\mathbf{p})\ \ s.t.\ \mathbf{p} = \alpha\mathbf{b}_1 + \beta\mathbf{b}_2 \tag{12}$$

Coordinates $\alpha,\ \beta$ can be associated with the parameters $x^i$ which justified the contravariant notation writing for any vector $\mathbf{p}$:

$$\mathbf{p} = x^1\mathbf{b}_1 + x^2\mathbf{b}_2 \tag{13}$$

$\mathbf{p}$ is the generalized impulsion vector defined in the mobile tangential space $TpS$ attached to the base $\mathbf{b}_1,\ \mathbf{b}_2$. $\mathbf{p}$ is given by:

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = x^1 \begin{bmatrix} A \\ 0 \\ 0 \end{bmatrix} + x^2 \begin{bmatrix} 0 \\ B \\ C \end{bmatrix} \tag{14}$$

Previous relation shows that $\mathbf{p}$ may involved the source of motion (electromotive forces). When we take a look to figure 3 where we see the basic vectors, we understand that we can define dual vectors, $\mathbf{c}^1$ and $\mathbf{c}^2$ aligned on $\mathbf{b}_1$ and $\mathbf{b}_2$. We write:

$$\begin{cases} \mathbf{c}^1 = \frac{\mathbf{b}_2 \times \mathbf{n}}{\sqrt{G}} \\ \\ \mathbf{c}^2 = \frac{\mathbf{n} \times \mathbf{b}_1}{\sqrt{G}} \end{cases} \tag{15}$$

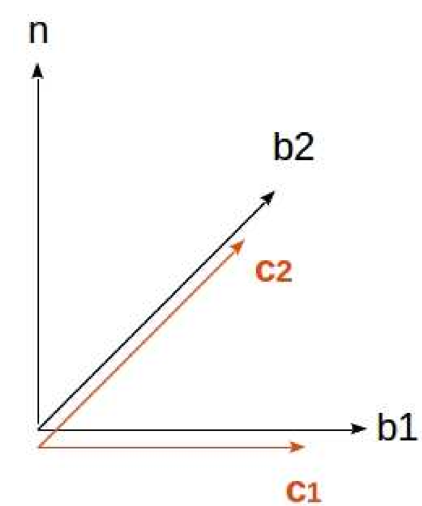


Figure 3

In our case we have $\mathbf{n} = (0, -AC, AB) / \sqrt{G}$ which comes from[3]:

$$\mathbf{n} = \frac{\mathbf{b}_1 \times \mathbf{b}_2}{\|\mathbf{b}_1 \times \mathbf{b}_2\|} \tag{16}$$

so, $\mathbf{p}$ can be developed on the dual base noting: $\mathbf{p} = p_i \mathbf{c}^i$. The sources belong to the cotangent space defined by the dual base. The components of $\mathbf{p}$ are said covariant ones.

If $p_1$ and $p_2$ are known sources and $p_3$ unknown; $C = 1$, and if $A$ and $B$ are purely real numbers. In this hypothesis the metric is constant as the current $x^i$. The elementary distance (which is in the general case an energy flux - $ds^2$ is in Volts square) is also constant. We are in a flat orthogonal space. Relation (14) gives the solution for the unknowns.

As they are constant, the derivatives of the basic vectors are equal to zero. No curvature can be found in this space.

[3] Verify that $\|\mathbf{b}_1 \times \mathbf{b}_2\| = \sqrt{G}$.

### 1.1.2 Equations coming from the lagrangian

The lagrangian is linked with energy derivatives versus variables of the chosen configuration space. Lagrange's equation deals with something like:

$$f_k = s\frac{\partial T}{\partial x^k} + \frac{\partial F}{\partial x^k} - \frac{\partial U}{\partial q^k} \tag{17}$$

where $T$ is kinetic energy, $U$ potential one, $F$ loss energies and $x = sq$, $s$ being the Laplace's operator.

for electrical circuit, each of these terms lead to potential differences. So, lagrangian $\mathcal{L}$ can be obtained using $\mathcal{L} = \sqrt{G_{ij}x^i x^j}$ with $\mathcal{L} = ds$. Solution for each flux can be obtained making:

$$p_1 = ds\frac{\partial x^1}{\partial e_1}\frac{\partial \mathcal{L}}{\partial x^1} = Ax^1 \tag{18}$$

and

$$p_2 = ds\frac{\partial x^2}{\partial e_2}\frac{\partial \mathcal{L}}{\partial x^2} = Bx^2 \tag{19}$$

Equations (14) shows that the current change with the sources, as the impedance are constants. If the source are themselves fixed, the current are fixed and the curve is reduced to two points.

**Exercise** Verify if:

$$p_3 = ds\frac{\partial x^2}{\partial e_3}\frac{\partial \mathcal{L}}{\partial x^2} = Cx^2$$

### 1.1.3 Case with no coupling function but operators for impedances A & B

The impedance matrix can include inductances and capacitances. for example:

$$Z = \begin{bmatrix} R + L\frac{d}{dt} & 0 \\ 0 & G + \frac{1}{C}\int_t dt \end{bmatrix} \tag{20}$$

$G$ and $R$ being resistances.

From this matrix definition we can construct a function $\psi$ defined by:

$$\psi(e_1, e_2, e_3) \;\; s.t. \;\; \begin{cases} e_1 = Rx^1 + L\frac{d}{dt}x^1 \\ e_2 = Gx^2 + \frac{1}{C}\int_t dtx^2 \\ e_3 = \alpha x^1 + \beta x^2 \end{cases} \tag{21}$$

To define the basic vectors, we have to compute $\partial_{x^1} L\frac{d}{dt}x^1 = 0$ or $\partial_{x^2}\frac{1}{C}\int_t dtx^2 = t/C$. But we'd like to keep the inductance in the basic vector. A solution is to write the impedance matrix using the Laplace's operator. In this case:

$$\psi\left(e_1,e_2,e_3\right)\ s.t.\ \begin{cases} e_1 = Rx^1 + Lsx^1 \\ e_2 = \frac{1}{sC}x^2 \\ e_3 = \alpha x^1 + \beta x^2 \end{cases} \tag{22}$$

In this case:

$$\mathbf{b}_1 = (R + Ls, 0, \alpha) \quad \mathbf{b}_2 = \left(0, \frac{1}{sC}, \beta\right) \tag{23}$$

the metric can be now defined by:

$$G = \begin{bmatrix} (R+Ls)^2 + \alpha^2 & \alpha\beta \\ \alpha\beta & \left(\frac{1}{sC}\right)^2 + \beta^2 \end{bmatrix} \tag{24}$$

As previously we can define a curve $\gamma(\mathbf{p}) = x^1\mathbf{b}_1 + x^2\mathbf{b}_2$, and the link with the covariant stimulus:

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} R+Ls \\ 0 \\ \alpha \end{bmatrix} x^1 + \begin{bmatrix} 0 \\ \frac{1}{sC} \\ \beta \end{bmatrix} x^2 \tag{25}$$

Now we can try to use the other approach. Writing:

$$\mathcal{L} = \left\{ \left[(R+Ls)^2 + \alpha^2\right](x^1)^2 + 2\alpha\beta x^1 x^2 + \left[\left(\frac{1}{sC}\right)^2 + \beta^2\right](x^2)^2 \right\}^{\frac{1}{2}}$$

We can compute:

$$\mathcal{L}\frac{\partial \mathcal{L}}{\partial x^1} = \frac{1}{2}\left(2\left[(R+Ls)^2 + \alpha^2\right]x^1 + 2\alpha\beta x^2\right) \tag{26}$$

and:

$$\frac{\partial}{\partial e_1}\mathcal{L}\frac{\partial \mathcal{L}}{\partial x^1}x^1 = (R+Ls)\,x^1 = p_1 \tag{27}$$

identically

$$\mathcal{L}\frac{\partial^2 \mathcal{L}}{\partial e_i \partial x^i}x^i = p_i$$

#### 1.1.4 Simple mesh but with current source in a "complete space"

Using the spaning tree on a simple mesh we obtain a system that seems like ($k^1$ is the mesh current and $J^2$ the nodes pair one):

$$\begin{cases} e_1 = z_{11}k^1 + z_{12}J^2 \\ e_2 = z_{21}k^1 + z_{22}J^2 \end{cases} \tag{28}$$

To these equations we can add a third one, linked with an available transfer function:

$$e_3 = \alpha \frac{k^1}{J^2} \tag{29}$$

$\alpha$ is a coefficient in Ampere. With the three electromotive forces we can define a function $\psi(e_1, e_2, e_3)$. We define:

$$\left\{ \begin{array}{l} \mathbf{b}_1 = \frac{\partial \psi}{\partial k^1} = \left(z_{11}, z_{21}, \alpha \frac{1}{J^2}\right) \\ \\ \mathbf{b}_2 = \frac{\partial \psi}{\partial J^2} = \left(z_{12}, z_{22}, -\alpha \frac{k^1}{(J^2)^2}\right) \end{array} \right. \tag{30}$$

This leads to the metric:

$$G = \begin{bmatrix} (z_{11})^2 + (z_{21})^2 + \left(\frac{\alpha}{J^2}\right)^2 & (z_{11}z_{12} + z_{21}z_{22}) - \alpha^2 \frac{k^1}{(J^2)^3} \\ \\ (z_{11}z_{12} + z_{21}z_{22}) - \alpha^2 \frac{k^1}{(J^2)^3} & (z_{12})^2 + (z_{22})^2 + \alpha^2 \frac{(k^1)^2}{(J^2)^4} \end{bmatrix} \tag{31}$$

We obtain a space structure similar to the one with two meshes and no nodes pair.

## 1.2 N order filters

We consider a first simple filter (figure 4).

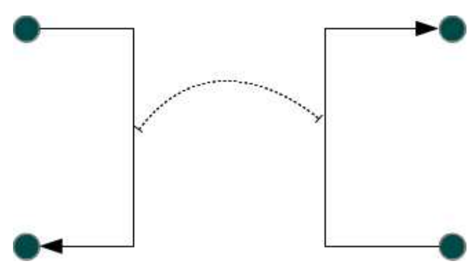

Figure 4

Whatever there is between the two ports of input and output on the filter, it leads to the next couple of equation:

$$\left\{ \begin{array}{l} e_1 = Ak^1 + hk^2 \\ e_2 = hk^1 + Bk^2 \end{array} \right. \tag{32}$$

$k^1$ and $k^2$ being the mesh currents on the two ports when they are closed by loads. the third force is given by the transfer function of the filter: $e_3 = \alpha k^1$. This leads to the base:

$$\mathbf{b}_1 = (A, h, \alpha) \quad \mathbf{b}_2 = (h, B, 0) \tag{33}$$

The metric coming from this base is:

$$G_1 = \begin{bmatrix} A^2 + h^2 + \alpha^2 & Ah + hB \\ Ah + hB & h^2 + B^2 \end{bmatrix} \tag{34}$$

We can have a second similar filter defined by:

$$\mathbf{b}_1 = (Q, g, \beta) \quad \mathbf{b}_2 = (g, W, 0) \tag{35}$$

and:

$$G_2 = \begin{bmatrix} Q^2 + g^1 + \beta^2 & Qg + gW \\ Qg + gW & g^2 + W^2 \end{bmatrix} \tag{36}$$

Making the two networks in serie is equivalent to make the direct summation of their matrices. How this acts on the metric? We call $\zeta$ the coupling function added to link the two filters. Firstly we must add the various elementary filters: $\oplus_{i=1,2} Z_i$. This can be done adding both systems:

$$\begin{cases} e_1 = Ak^1 + hk^2 + 0k^3 + 0k^4 \\ e_2 = hk^1 + Bk^2 + \alpha k^3 + 0k^4 \\ e_3 = 0k^1 + \alpha k^2 + Qk^3 + gk^4 \\ e_4 = 0k^1 + 0k^2 + gk^3 + Wk^4 \end{cases} \tag{37}$$

This kind of structure can cover all kinds of filters in fact, the coupling function $\alpha$ being any network making link between the two filters. From this four equations we obtain the base in the 4-dimension space:

$$\begin{cases} \mathbf{b}_1 = (A, h, 0, 0) \\ \mathbf{b}_2 = (h, B, \alpha, 0) \\ \mathbf{b}_3 = (0, \alpha, Q, g) \\ \mathbf{b}_4 = (0, 0, g, W) \end{cases} \tag{38}$$

Then the $\gamma = x^1\mathbf{b}_1 + x^2\mathbf{b}_2 + x^3\mathbf{b}_3 + x^4\mathbf{b}_4$ curve versus stimuli is given by:

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} A \\ h \\ 0 \\ 0 \end{bmatrix} x^1 + \begin{bmatrix} h \\ B \\ \alpha \\ 0 \end{bmatrix} x^2 + \begin{bmatrix} 0 \\ \alpha \\ Q \\ g \end{bmatrix} x^3 + \begin{bmatrix} 0 \\ 0 \\ g \\ W \end{bmatrix} x^4 \tag{39}$$

How it acts on the metric? First we calculate the metric of the coupled system:

$$G = \begin{bmatrix} A^2 + h^2 & Ah + hB & h\alpha & 0 \\ hA + Bh & h^2 + B^2 & (B + Q)\alpha & g\alpha \\ h\alpha & (B + Q)\alpha & \alpha^2 + Q^2 + g^2 & g(Q + W) \\ 0 & g\alpha & g(Q + W) & g^2 + W^2 \end{bmatrix} \tag{40}$$

We see that $G = G_1 \oplus G_2|_{\alpha,\beta=0,0} + \mu$, where $\mu$ is an interaction metric to be add in order to take into account the coupling of the two previous metrics.

## 2 Analysis of filters

Considering a low pass filter, we obtain on the base of the graph figure 1 the next relations for $\psi$:

$$\begin{cases} e_1 = Rk^1 - \frac{1}{sC}k^2 \\ e_2 = -\frac{1}{sC}k^1 + Tk^2 \\ e_3 = k^2 \end{cases} \tag{41}$$

This leads to:

$$\mathbf{b}_1 = \left(R, -\frac{1}{sC}, 0\right) \ \mathbf{b}_2 = \left(-\frac{1}{sC}, T, 1\right) \tag{42}$$

The $\gamma$ curve is:

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} R \\ -\frac{1}{sC} \\ 0 \end{bmatrix} x^1 + \begin{bmatrix} -\frac{1}{sC} \\ T \\ 1 \end{bmatrix} x^2 \tag{43}$$

Both currents $x^1$ and $x^2$ are defined by the two first equations. The third equation, knowing $x^1$ and $x^2$ gives $p_3$, the transfer function.

Let's take a look to the first equations:

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} R \\ -\frac{1}{sC} \end{bmatrix} x^1 + \begin{bmatrix} -\frac{1}{sC} \\ T \end{bmatrix} x^2 \tag{44}$$

The determinant is:

$$\Delta = RT - \left(\frac{1}{sC}\right)^2 \tag{45}$$

which leads to the admittance:

$$y = \frac{1}{\Delta} \begin{bmatrix} T & \frac{1}{sC} \\ \frac{1}{sC} & R \end{bmatrix} \tag{46}$$

We can know compute how the curve $\gamma$ goes depending here on $s$:

$$\begin{bmatrix} x^1 \\ x^2 \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} T & \frac{1}{sC} \\ \frac{1}{sC} & R \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \tag{47}$$

and

$$\begin{cases} x^1(s) = \frac{T}{\left[RT - \left(\frac{1}{sC}\right)^2\right]} p_1 + \frac{\frac{1}{sC}}{\left[RT - \left(\frac{1}{sC}\right)^2\right]} p_2 \\ \\ x^2(s) = \frac{\frac{1}{sC}}{\left[RT - \left(\frac{1}{sC}\right)^2\right]} p_1 + \frac{R}{\left[RT - \left(\frac{1}{sC}\right)^2\right]} p_2 \end{cases} \tag{48}$$

if $e_2 = 0$ (which means that $p_2 = 0$) stills:

$$\begin{cases} x^1(s) = \frac{T}{\left[RT - \left(\frac{1}{sC}\right)^2\right]} p_1 \\ \\ x^2(s) = \frac{1}{[RTs^2C^2 - 1]} p_1 \end{cases} \tag{49}$$

Figure 5 shows the curve obtained.

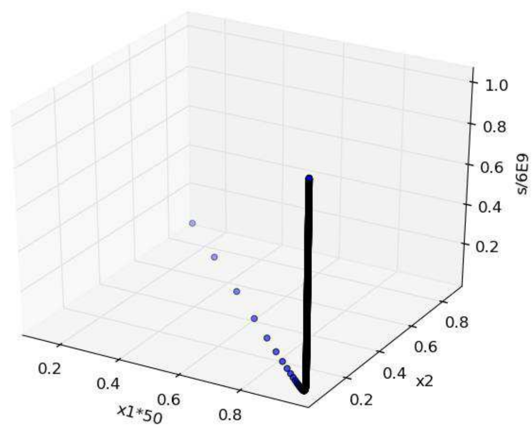


Figure 5

The curve obtained shows that the triplet $x_1, x_2, s$ (drawn with some factors but without changing the meanings) can be for example (low value, high value, low value) or (high value, low value, high or low value). When the frequencies are low, the output current is high while the output current is low at high frequencies. It means that the circuit is a low pass filter.

We may now trace the $\gamma$ curve. Another way to see the parametrized surface is to replace both $x^1$ and $x^2$ with all possible values (in given domains), and that for each frequency value. Figure 6 shows the curve obtained for frequencies from 1 to 100 MHz.

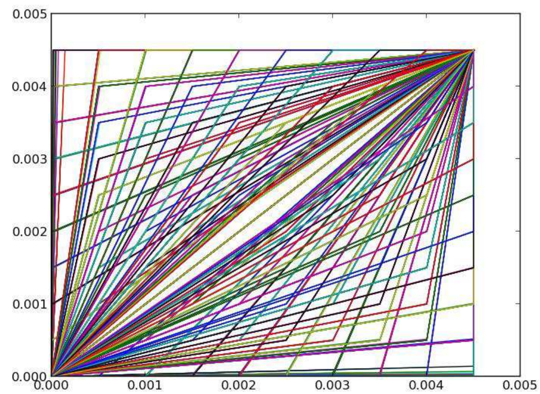


Figure 6

The surface shows that for various values of current and frequency, the stimulus $p_1$ and $p_2$ cover a squared surface that could be drawn continuously with more samples. The intersection between a given value for $p_1$ with $p_2 = 0$ can lead to the current solution. the symetry of the figure indicates that the circuit itself is symetric. No difference comes from both parts of applied stimulus.

# 3 Guided waves

Studying guided waves may lead to very interesting interpretations, as the linked equations can be applied to many physics. To do that, we use generalized Branin's model firstly created for lines. Branin's basic circuit is given figure 7.

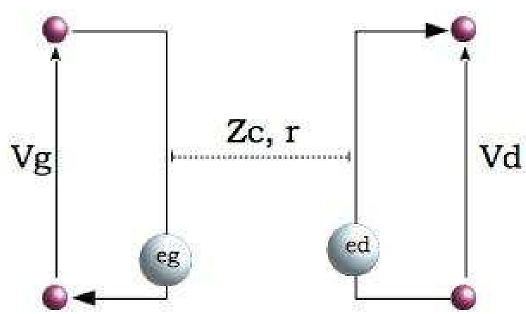


Figure 7

Left voltage is defined by : $e_g = \left(V_d - Z_c i^d\right) e^{-sr}$. Right one by : $e_d = \left(V_g + Z_c i^g\right) e^{-sr}$. By replacing $V_g$ and $V_d$ in the equations and defining $Z_c$ and

$\tau$ meanings, Branin's model can be applied to many problems, lines, guided waves, antennas, etc.

In a basic case where this model is connected to a generator on the left ($R_0$, $E_0$) and a load on the right $R_L$, Branin's equations becomes in the mesh space (mesh one on the left, two on the right):

$$\left\{ \begin{array}{l} e_g = (R_L - Z_c)\, i_2 e^{-sr} \\ e_d = E_0 e^{-sr} + (Z_c - R_0)\, i_1 e^{-sr} \end{array} \right. \tag{50}$$

Which leads to the following network equations:

$$\left\{ \begin{array}{l} e_1 = E_0 = (R_0 + Z_c)\, i_1 + (R_L - Z_c)\, i_2 e^{-sr} \\ e_2 = E_0 e^{-sr} = (Z_c - R_0)\, i_1 e^{-sr} + (Z_c + R_L)\, i_2 \end{array} \right. \tag{51}$$

A transfer function can be defined giving the ratio between the input and output voltages:

$$f = \frac{RLi_2}{E_0 - R_0 i_1} \tag{52}$$

So, a $\psi$ function can be defined by the triplet $(e_1, e_2, f)$. It leads to the following base:

$$\left\{ \begin{array}{l} \mathbf{b}_1 = \frac{\partial \psi}{\partial i_1} = \left( (R_0 + Z_c)\, , (Z_c - R_0)\, e^{-sr}, \frac{RLi_2 R_0}{[E_0 - R_0 i_1]^2} \right) \\ \\ \mathbf{b}_2 = \frac{\partial \psi}{\partial i_2} = \left( (R_L - Z_c)\, e^{-sr}, (Z_c + R_L)\, , \frac{RL}{E_0 - R_0 i_1} \right) \end{array} \right. \tag{53}$$

The basic vectors lead to the following metric:

$$\left\{ \begin{array}{l} G_{11} = (R_0 + Z_c)^2 + (Z_c - R_0)^2 e^{-2sr} + \left( \frac{RLi_2 R_0}{[E_0 - R_0 i_1]^2} \right)^2 \\ \\ G_{12} = (R_0 + Z_c)(R_L - Z_c)\, e^{-sr} + \frac{R^2 L^2 R_0 i_2}{[E_0 - R_0 i_1]^3} + (Z_c - R_0)(Z_c + R_L)\, e^{-sr} \\ \\ G_{21} = G_{12} \\ \\ G_{22} = (R_L - Z_c)^2 e^{-2sr} + \left( \frac{RL}{E_0 - R_0 i_1} \right)^2 + (Z_c + R_L)^2 \end{array} \right. \tag{54}$$

Can we solve the system equations ?

$$\mathcal{L} \frac{\partial^2 \mathcal{L}}{\partial e_i \partial x^i} x^i = p_i$$

We can write first the $\gamma$ projection:

$$\left[ \begin{array}{c} p_1 \\ p_2 \end{array} \right] = \left[ \begin{array}{c} (R_0 + Z_c) \\ (Z_c - R_0)\, e^{-sr} \end{array} \right] x^1 + \left[ \begin{array}{c} (R_L - Z_c)\, e^{-sr} \\ (Z_c + R_L) \end{array} \right] x^2 \tag{55}$$

What we see is that the curvature if there is any, exists only through the third component coming from the transfer function. But this function is an arbitratry one, not linked intrinsically with the network. So, looking at the $\gamma$ expression we don't see what we could call an intrinsic curvature, the only one which interest us. This suggest us, at least to care with the transfer function dimension. So, as a general method we should kept $f$ as a global function for third component of basic vectors before to define it. This gives this time the metric:

$$\begin{cases} G_{11} = (R_0 + Z_c)^2 + (Z_c - R_0)^2 e^{-2sr} + f^2 \\ G_{12} = (R_0 + Z_c)(R_L - Z_c) e^{-sr} + (Z_c - R_0)(Z_c + R_L) e^{-sr} + f^2 \\ G_{21} = G_{12} \\ G_{22} = (R_L - Z_c)^2 e^{-2sr} + (Z_c + R_L)^2 + f^2 \end{cases} \tag{56}$$

What is done classically is to write equation (55) matricially and to inverse the tensor multiplied by the flux vector components. This is similar to solve:

$$x^i = \left[\mathcal{L}\frac{\partial^2 \mathcal{L}}{\partial e_i \partial x^i}\right]^{-1} p_i \tag{57}$$

we define:

$$y = \left[\mathcal{L}\frac{\partial^2 \mathcal{L}}{\partial e_i \partial x^i}\right]^{-1} \tag{58}$$

We find Kron's equations, classically obtained in the tensorial analysis of network. But the metric is this time symetric and can lead to the admittance solution (or similar one like least action approach once defined the lagragian $\mathcal{L}$).

In general for real cases, the $\psi$ function is not linear. System (51) can be written now differently:

$$\begin{cases} \psi_1(i_1, i_2) = (R_0 + Z_c)\, i_1 + (R_L - Z_c)\, i_2 e^{-sr} - E_0 = 0 \\ \psi_2(i_1, i_2) = (Z_c - R_0)\, i_1 e^{-sr} + (Z_c + R_L)\, i_2 - E_0 e^{-sr} = 0 \end{cases} \tag{59}$$

Under this writing, the basic vectors lead to the Jacobian matrix $W$:

$$\psi'(i) = W(i) = \begin{bmatrix} \frac{\partial \psi_1}{\partial i_1} & \frac{\partial \psi_1}{\partial i_2} \\ \frac{\partial \psi_2}{\partial i_1} & \frac{\partial \psi_2}{\partial i_2} \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} \tag{60}$$

It means that the Jacobian matrix is the covector of the basic vectors. The system can be written $\psi(i) = 0$.

To solve it we use a Newton's method. Imagine that we find a $p^{eme}$ approximation: $i^{(p)} = \left(i_1^{(p)}, i_2^{(p)}, \ldots, i_n(p)\right)$. The exact solution can then be written:

$i = i^{(p)} + \epsilon^{(p)}$ where $\epsilon$ is a corrective factor ($\epsilon^{(p)} = \left(\epsilon_1^{(p)}, \epsilon_2^{(p)}, \ldots, \epsilon_n(p)\right)$). Using this expression in the system equation we obtain: $\psi\left(i^{(p)} + \epsilon^{(p)}\right) = 0$. Starting from that, we can develop the equation:

$$\psi\left(i^{(p)} + \epsilon^{(p)}\right) = \psi\left(i^{(p)}\right) + \psi'\left(i^{(p)}\right)\epsilon^{(p)} = 0 \tag{61}$$

which gives:

$$\psi\left(i^{(p)}\right) + W\left(i^{(p)}\right)\epsilon^{(p)} = 0 \tag{62}$$

in other words:

$$\psi\left(i^{(p)}\right) = -\left[\begin{array}{cc} \mathbf{b}_1 & \mathbf{b}_2 \end{array}\right]\epsilon^{(p)} \tag{63}$$

The corrective factor vector is so the coordinates of the $\gamma$ projection in the mobile space. Their values can be solved through:

$$\epsilon^{(p)} = -W^{-1}\left(i^{(p)}\right)\psi\left(i^{(p)}\right) = \Delta(p)i^{(p)} \tag{64}$$

or:

$$i^{(p+1)} = i^{(p)} - W^{-1}\left(i^{(p)}\right)\psi\left(i^{(p)}\right) \tag{65}$$

The vector $i^{(p)}$ can be seen as the original impulsion. We write tensorially the variation value:

$$W_{\alpha\beta}\left[i^{(q)}, t\right]\epsilon^{\beta}[q, t] = \psi_{\alpha}\left[i^{(q)}, t\right] \tag{66}$$

$t$ being the time when the calculation is made and $q$ the approximation order. By definition:

$$W_{\alpha\beta} = \frac{\partial\psi_{\alpha}}{\partial i_{\beta}} \tag{67}$$

At each time step, equation (55) is the equation of the problem. To solve it, we use Newton's method expressed in (66).

# 4 Another method to define the least action

Consider next system:

$$\left\{\begin{array}{l} e_1 = Ri_1 + Lpi_2 \\ e_2 = yi_1 + zi_2 \\ e_3 = ui_1 - vi_2 \end{array}\right. \tag{68}$$

It creates the following basis:

$$\mathbf{b}_1 = \begin{bmatrix} R \\ y \\ u \end{bmatrix} \quad \mathbf{b}_2 = \begin{bmatrix} Lp \\ z \\ -v \end{bmatrix} \tag{69}$$

and the metric:

$$G = \begin{bmatrix} R^2 + y^2 + u^2 & RLp + yz - uv \\ RLp + yz - uv & L^2p^2 + z^2 + v^2 \end{bmatrix} \tag{70}$$

Equation (68) can be written as: $e_\alpha = \psi_\alpha (i_1, i_2)$. Making the identification between $i_\alpha$ and $x^\alpha$ this is equivalent to $e_\alpha - \psi_\alpha \left(x^\beta\right) = 0$.

The $e_\alpha$ can be the component of an impulse covector $\mathbf{p}$.

How objective is to solve equation $e_\alpha - \psi_\alpha \left(x^\beta\right) = \epsilon$ with $\epsilon \to 0$. It is equivalent to search for:

$$\mathcal{A}_\alpha = \int_t dt \left[e_\alpha - \psi_\alpha \left(x^\beta\right)\right]^2 \to 0 \tag{71}$$

and leading $\mathcal{A}_\alpha$ as low as possible, $\forall\alpha$.

If $\psi_1 \left(x^\beta\right) = Ax^1 + Bx^2$ and $\psi_2 \left(x^\beta\right) = Cx^1 + Dx^2$ we obtain:

$$\begin{aligned} \mathcal{A}_1 &= \textstyle\int_t dt \left[e_1 - \psi_1 \left(x^\beta\right)\right]^2 \\ \mathcal{A}_1 &= (e_1)^2 + A^2(x^1)^2 + B^2(x^2)^2 + 2ABx^1x^2 - 2e_1 \left(Ax^1 + Bx^2\right) \end{aligned} \tag{72}$$

and

$$\begin{aligned} \mathcal{A}_2 &= \textstyle\int_t dt \left[e_2 - \psi_2 \left(x^\beta\right)\right]^2 \\ \mathcal{A}_2 &= (e_2)^2 + C^2(x^1)^2 + D^2(x^2)^2 + 2CDx^1x^2 - 2e_1 \left(Cx^1 + Dx^2\right) \end{aligned} \tag{73}$$

In that case the metric is:

$$G = \begin{bmatrix} A^2 + C^2 & AB + CD \\ AB + CD & B^2 + D^2 \end{bmatrix} \tag{74}$$

We see that:

$$\frac{1}{2} \sum_\alpha \frac{\partial A_\alpha}{\partial x^m} = \int_t dt \sum_\alpha (G_{\alpha m} x^m - \mathbf{p} \cdot \mathbf{b}_\alpha) = 0 \tag{75}$$

which is the fundamental equation of the graph. It makes the link between the impulse transmitted to the mobile repair and its action on the flux through the metric. This approach can be used in general cases to justify the metric and the base vectors. It finally leads to the classical system of equations given by the Kron's method. But for spaces with curvature, it may give another point of view on the metric meaning.

## 5 A circuit with ferrite and diode

Ferrites are materials which properties depends on current amplitudes. Diodes are non linear components. So the combination of both ferrite and diode is a complicated and interesting problem.

We consider next circuit:

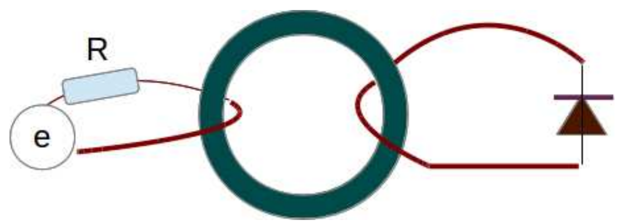


Figure 8

Impedance tensor of such a circuit is given by:

$$g = \begin{bmatrix} R + (L_0 + S_u)\, p & -Mp \\ -Mp & (L_0 + S_u)\, p + Z_d \end{bmatrix} \tag{76}$$

$L_0$ is the inductance linked with the closed circulation of the wires. $S_u$ is the inductance of the ferrite material. $Z_d$ is the impedance operator of the diode. $M$ is the mutual inductance passing through the material. We define:

$$Z_d = exp\left(-\left[\frac{v_d - 1}{2}\right]^2\right) exp\left(-\left[\frac{i_d - 1}{2}\right]^2\right) 10^6 + exp\left(-\left[\frac{v_d + 1}{-2}\right]^2\right) exp\left(-\left[\frac{i_d + 1}{-2}\right]^2\right) 10^{-3} \tag{77}$$

and:

$$S_u = \left(\frac{\beta}{1 + \frac{i}{i_s}}\right),\ M = \alpha S_u \tag{78}$$

$i_s$ is a saturation current threshold.

$g_n\left(i_q\right)$ gives the function vector for the intrinsic part:

$$\begin{cases} g_1\left(i_1, i_2\right) = Ri_1 + L_0 p i_1 + \left(\frac{\beta}{1+\frac{i_1+i_2}{i_s}}\right) p i_1 - \alpha\left(\frac{\beta}{1+\frac{i_1+i_2}{i_s}}\right) p i_2 \\ g_2\left(i_1, i_2\right) = -\alpha\left(\frac{\beta}{1+\frac{i_1+i_2}{i_s}}\right) p i_1 + L_0 p i_2 + \left(\frac{\beta}{1+\frac{i_1+i_2}{i_s}}\right) p i_2 + Z_d i_2 \end{cases} \tag{79}$$

Last function can be linked with thermal description:

$$g_3\left(i_1, i_2\right) = R\left(i_1\right)^2 \tag{80}$$

We can now calculate the basic vectors:

$$\mathbf{b}_1 = \begin{bmatrix} R + L_0 p - \frac{\beta (i_s)^{-1}}{\left(1+\frac{i_1+i_2}{i_s}\right)^2} p i_1 + \left(\frac{\beta}{1+\frac{i_1+i_2}{i_s}}\right) p + \alpha\beta \frac{(i_s)^{-1}}{\left(1+\frac{i_1+i_2}{i_s}\right)^2} p i_2 \\ \alpha\beta \frac{(i_s)^{-1}}{\left(1+\frac{i_1+i_2}{i_s}\right)^2} p i_1 - \alpha \left(\frac{\beta}{1+\frac{i_1+i_2}{i_s}}\right) p - \frac{\beta (i_s)^{-1}}{\left(1+\frac{i_1+i_2}{i_s}\right)^2} p i_2 \\ 2 R i_1 \end{bmatrix} \quad (81)$$

and:

$$\mathbf{b}_2 = \begin{bmatrix} -\left(\frac{\beta (i_s)^{-1}}{\left[1+\frac{i_1+i_2}{i_s}\right]^2}\right) p i_1 - \alpha \left\{ \left(\frac{\beta (i_s)^{-1}}{\left[1+\frac{i_1+i_2}{i_s}\right]^2}\right) p i_2 + p \left(\frac{\beta}{1+\frac{i_1+i_2}{i_s}}\right) \right\} \\ L_0 p + Z_d - \alpha \left(\frac{\beta (i_s)^{-1}}{\left[1+\frac{i_1+i_2}{i_s}\right]^2}\right) p i_1 + \left\{ \left(\frac{\beta (i_s)^{-1}}{\left[1+\frac{i_1+i_2}{i_s}\right]^2}\right) p i_2 + p \left(\frac{\beta}{1+\frac{i_1+i_2}{i_s}}\right) \right\} \\ 0 \end{bmatrix} \quad (82)$$

It's clear that derivation of $\mathbf{b}_q$ versus $i_m$ is not zero. But this case is quite complicated to write. We may continue with a simplest example. We consider the system:

$$\begin{cases} \psi_1 (i_1, i_2) = (R + L_0 s) i_1 + M_0 s (1 + \alpha i_1) i_2 \\ \psi_2 (i_1, i_2) = M_0 s (1 + \alpha i_2) i_1 + Q s i_2 \\ \psi_3 (i_1, i_2) = \beta (-i_1 + i_2) \end{cases} \quad (83)$$

It makes the base:

$$\mathbf{b}_1 = \begin{bmatrix} (R + L_0 s) + \alpha M_0 s i_2 \\ M_0 s (1 + \alpha) i_2 \\ -\beta \end{bmatrix} \quad \mathbf{b}_2 = \begin{bmatrix} M_0 s (1 + \alpha i_1) \\ M_0 s \alpha i_1 + Q s \\ \beta \end{bmatrix} \quad (84)$$

as a consequence leads to the metric:

$$\begin{cases} G_{11} = \left[(R + L_0 s) + \alpha M_0 s i_2\right]^2 + \left[M_0 s (1 + \alpha) i_2\right]^2 + \beta^2 \\ G_{12} = \left[(R + L_0 s) + \alpha M_0 s i_2\right] \left[M_0 s (1 + \alpha i_1)\right] + \left[M_0 s (1 + \alpha) i_2\right] \left[M_0 s \alpha i_1 + Q s\right] - \beta^2 \\ G_{21} = \left[(R + L_0 s) + \alpha M_0 s i_2\right] \left[M_0 s (1 + \alpha i_1)\right] + \left[M_0 s (1 + \alpha) i_2\right] \left[M_0 s \alpha i_1 + Q s\right] - \beta^2 \\ G_{22} = \left[M_0 s (1 + \alpha i_1)\right]^2 + \left[M_0 s \alpha i_1 + Q s\right]^2 + \beta^2 \end{cases} \quad (85)$$

We can determine the normal vector:

$$\mathbf{n} = \frac{\mathbf{b}_1 \times \mathbf{b}_2}{||\mathbf{b}_1 \times \mathbf{b}_2||} \tag{86}$$

This allows to define the dual base:

$$\mathbf{c}^1 = \frac{1}{\sqrt{G}} (\mathbf{b}_2 \times \mathbf{n}) \quad \mathbf{c}^2 = \frac{1}{\sqrt{G}} (\mathbf{n} \times \mathbf{b}_1) \quad \mathbf{n}' \tag{87}$$

This dual base allows to define the impulse covector:

$$\mathbf{p} = e_1 \mathbf{c}^1 + e_2 \mathbf{c}^2 \tag{88}$$

Now the system of equations that solve the problem each time step is:

$$\sum_\alpha G_{\alpha m} x^m = \sum_\alpha \mathbf{p} \cdot \mathbf{b}_\alpha \tag{89}$$

this gives after variables separation:

$$\begin{cases} \left[(R + L_0 s) + \alpha M_0 s i_2\right] x^1 + \left[M_0 s (1 + \alpha i_1)\right] x^2 = e_1 \\ \\ \left[M_0 s (1 + \alpha) i_2\right] x^1 + \left[M_0 s \alpha i_1 + Q s\right] x^2 = e_2 \end{cases} \tag{90}$$

The basic vectors can be derived. We obtain:

$$\begin{array}{cc} \mathbf{b}_{11} = \frac{\partial \mathbf{b}_1}{\partial i_1} = 0 & \mathbf{b}_{12} = \frac{\partial \mathbf{b}_1}{\partial i_2} = \begin{bmatrix} \alpha M_0 s \\ M_0 s (1 + \alpha) \\ 0 \end{bmatrix} \\ \mathbf{b}_{21} = \frac{\partial \mathbf{b}_2}{\partial i_1} = \begin{bmatrix} M_0 s \alpha \\ M_0 s \alpha \\ 0 \end{bmatrix} & \mathbf{b}_{22} = \frac{\partial \mathbf{b}_2}{\partial i_2} = 0 \end{array} \tag{91}$$

To solve each step equation (90), we define the jacobian (obtained from the reduced base and the intrinsic metric):

$$W = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} \tag{92}$$

after what we have to solve (65) :

$$i^{(p+1)} = i^{(p)} - W^{-1}\left(i^{(p)}\right) \psi \left(i^{(p)}\right)$$

At initial condition, $i_1 = 0$ and $i_2 = 0$. In that case we see that: $W = \begin{bmatrix} \mathbf{b}_1 (i_u = 0, \forall u) & \mathbf{b}_2 (i_u = 0, \forall u) \end{bmatrix}$. Writing $\mathbf{b}_v^0 = \mathbf{b}_v (i_u = 0, \forall u)$, we have at the next steps:

$$W = \begin{bmatrix} \mathbf{b}_1^0 + \mathbf{b}_{1u} i_u & \mathbf{b}_2^0 + \mathbf{b}_{2u} i_u \end{bmatrix} \tag{93}$$

and (65) becomes:

$$x^u_{(p+1)} = x^u_{(p)} - \bar{W}^{uv}\left(i_{(u,p)}\right) \psi_v \left(i_{(u,p)}\right) \tag{94}$$

with: $i_{(u,p=0,t)} = x^u_{(p,t-dt)}$ and $\bar{W} = W^{-1}$. Remember that in that case:

$$\psi_\alpha = \tilde{G}_{\alpha m} x^m - e_\alpha$$

$\tilde{G}_{\alpha m}$ being the root metric, directly obtained through the projection:

$$\tilde{G}_{\alpha m} x^m = (\mathbf{b}_m x^m)_\alpha$$

($\alpha$ component of each basic vector time the variable). So (94) can be written:

$$\delta x^u_{(p)} = -\left[\mathbf{b}^0_v + \mathbf{b}_{vu} i_u\right]^{-1} \psi_v \tag{95}$$

The derivative covector of the basis vector can be developed in the adapted base writing:

$$\mathbf{b}_{uv} = \Gamma^1_{uv}\mathbf{b}_1 + \Gamma^2_{uv}\mathbf{b}_2 = h_{uv}\mathbf{n} \tag{96}$$

So that each step:

$$\delta x^u_{(p)} = -\left(\mathbf{b}^0_v + \left[\Gamma^1_{uv}\mathbf{b}_1 + \Gamma^2_{uv}\mathbf{b}_2 + h_{uv}\mathbf{n}\right] i_{(u,p)}\right)^{-1} \psi_v \left(i_{(u,p)}\right) \tag{97}$$

$\Gamma$ are the Christoffel's coefficients giving information on the tangential geometry of the surface and $h$ the second fundamental form giving information on the normal variation to the surface.

This last equation shows the influence of base vectors depending on the parameters, on each time computation of the variables. Note that, as the basic vector can be written $\mathbf{b}_u = \mathbf{b}^0_u + \mathbf{b}_{uv} i_v$, by changing the $\mathbf{b}$ vectors by this expression in the development of the metric, it makes appearing Christoffel's coefficients in the metric components. This makes in evidence the kind of metric created by such basic vectors. Remember: $\Gamma^k_{uv} = \langle \mathbf{c}_k, \mathbf{b}_{uv}\rangle$ and $h_{uv} = \langle \mathbf{b}_{uv}, \mathbf{n}\rangle$.

## 6 Conclusion

The second geometrization makes a little more complicated the writing of the equations classically directly obtained in the Kron's method. But it gives a depper physical meaning showing how act the various elements seen in the start function. It gives too a robust and clear interpretation of the metric meaning, even with non linear networks. It gives a clear influence of the Christoffel's symbols and how they act when they exist. Many works still to be done in order to explore more completely the advantages of this approach. Previous cases show that fastly, complex expressions are obtained. So, contrary to the Kron's method, it don't seem easy to develop a software using these developments. But to study theoretically the systems under their networks representation, the approach seems a good way to understand the complexity of the system behaviors, in the various cases encountered. The fact to transform the star non linear function into basic vectors of a parametrized surface gives the opportunity to define by a best way the transformation that can be applied on the system through its evolution. In this new approach, to transform a system can be translated in changing the basic vectors, which define the metric. In the 2xTAN method context[4] this writing is very interesting, giving more meaning to the transformation group than when they are directly applied on the impedance tensor of the Kron's method.

The real world is non linear and multiphysic. It's time know to develop a method based on this fundamental hypothesis. 2xTAN method used under 2° geometrization Kron's method is one solution I submit to critics. One proposal to model complex systems.

[4] http://www.theses.fr/2013LIMO4048