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SUPERSYMMETRIC RIGIDITY OF ASYMPTOTICALLY LOCALLY HYPERBOLIC MANIFOLDS

OUSSAMA HIJAZI AND SEBASTIÁN MONTIEL

ABSTRACT. Let (M, g) be an asymptotically locally hyperbolic (ALH) manifold which is the interior of a conformally compact manifold and $(\partial M, [\gamma])$ its conformal infinity. Suppose that the Ricci tensor of (M, g) dominates that of the hyperbolic space and that its scalar curvature satisfies a certain decay condition at infinity. If the Yamabe invariant of $(\partial M, [\gamma])$ is non-negative, we prove that there exists a conformal metric on M with non-negative scalar curvature and whose boundary ∂M has either positive or zero constant inner mean curvature. In the spin case, we make use of a previous estimate obtained by X. Zhang and the authors for the Dirac operator of the induced metric on ∂M . As a consequence, we generalize and simplify the proof of the result by L. Andersson and M. Dahl in [AD] about the rigidity of the hyperbolic space when the prescribed conformal infinity ∂M is a round sphere. We also provide non-existence results for conformally compact ALH spin metrics when ∂M is conformal to a Riemannian manifold with special holonomy.

1. INTRODUCTION

In this paper, we study the rigidity of asymptotically locally hyperbolic (ALH) manifolds in the more general setting of conformally compact manifolds by prescribing their conformal infinities. In the Lorentzian setting, conformal compactifications were introduced by R. Penrose [P] in order to analyze gravitational fields at the lightlike infinity. During the last decades, due to the important role that they play in the so-called Anti-de Sitter/Conformal Field Theory (AdS/CFT)

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correspondence (see, among others, [Bi, An1, An2]), the analogous setting for Riemannian manifolds has attracted a great deal of interest in both physical and mathematical realms. As for the mathematical aspects, the importance of studying conformally compact manifolds in conformal geometry appeared for the first time in the seminal paper [FG] by C. Fefferman and C. Graham.

In short, a conformally compact manifold is a connected complete Riemannian manifold whose metric extends conformally to a compact manifold with boundary whose interior is the original manifold. So, through this extended conformal metric, the corresponding original metric determines a conformal structure on the boundary, which is usually called the conformal infinity or the boundary at infinity. A particularly interesting class of conformally compact manifolds are the Poincaré-Einstein (PE) spaces, characterized by the fact that the manifold is Einstein (necessarily with negative scalar curvature). In this setting, L. Andersson and M. Dahl [AD, Theorem 5.2] proved that, if the manifold is spin and its conformal infinity is an n -dimensional round sphere, then its only possible compact conformal filling is the $(n + 1)$ -dimensional hyperbolic disc (see [Q] for a partial attempt to drop the spin assumption and [Wa2, Theorem 1] for an alternative approach). Note that, recently, L. Andersson, M. Cai and G. Galloway have obtained in [ACG] a uniqueness result for the AdS spacetime without the spin hypothesis by requiring a lower bound for the scalar curvature. A later result by M. Anderson ([An1, Theorem 3.3] and [An3, Theorem 2.5]) may be seen as a generalization of the uniqueness of the hyperbolic space. Indeed, he showed that each conformal transformation on the boundary at infinity of a PE manifold comes from an isometry on the bulk manifold. Hence, when the boundary is maximally symmetric (a conformal round sphere), the original manifold is maximally symmetric as well (the hyperbolic space).

In this paper, we give a *supersymmetric* version of the above result by showing that each twistor spinor on the conformal infinity comes from an imaginary Killing spinor on the bulk manifold (see Theorem 6 and Remark 5), that we obviously suppose to have a spin structure. It is important to note that these results work only when we consider, on the conformal infinity, the spin structure induced from that on the bulk manifold (we will discuss it in depth in Remarks 7, 8 and 9). For this, we remove the Einstein assumption on the manifold and only assume that the Ricci tensor dominates that of the hyperbolic space and that the scalar curvature satisfies a suitable decay condition at infinity. So,

our results are valid not only for PE spaces, but also for the so-called *weakly Poincaré-Einstein* (WPE) manifolds (see [MP, end of Section 2] and [GQ, Remark 1.3, 1])). They have as a consequence an improvement of the uniqueness result by L. Andersson and M. Dahl (Corollary 7) as well as some new non-existence results when the prescribed infinity is conformal to a non-spherical compact spin manifold admitting non-trivial real Killing (Corollary 8) or parallel (Corollary 9) spinors. In the last section (Section 5), we will see that our results strongly invite to consider ALH spin manifolds with some kind of singularities. It is worth mentioning that X. Dai has also dealt in [D1, D2] with the supersymmetric (SUSY) compactifications in a different setting.

With respect to the techniques used to obtain such results, in the spirit of the famous positive mass theorem, L. Andersson and M. Dahl defined (see [AD]) a notion of *mass* for AH (that is, ALH with spherical infinity) manifolds whose vanishing characterizes the hyperbolic space. Then they proved that this mass is zero for a PE manifold whose boundary at infinity is a round sphere. On the other hand, in [Q] J. Qing used the existence of certain positive eigenfunctions of the Laplacian for each PE manifold, established by J. Lee in [L]. He used these functions to conformally deform the metric in order to get a complete non-compact asymptotically Euclidean Riemannian manifold with boundary, with non-negative integrable scalar curvature. Then he applied the positive mass theorem to the double manifold obtained by gluing through their boundaries two copies of such Asymptotically Euclidean manifolds with boundary, proving that its ADS mass is zero. In this way, he could drop the spin assumption for dimensions between 3 and 6.

In this paper, we will work with ALH manifolds, a notion which is somewhat weaker than that of AH manifold used in [AD, Q]. Indeed, we will weaken the PE condition to the WPE one. Note that, we will still use the eigenfunctions discovered by J. Lee, but we will modify the arguments by J. Qing to get a *total* compact conformal filling of the original manifold with non-negative scalar curvature and boundary at infinity with constant positive or zero mean curvature, according to whether the Yamabe invariant of this boundary is positive or zero. In this situation, we apply a previous result by X. Zhang and the authors regarding the first eigenspace of the Dirac operator of a certain metric on the conformal infinity to obtain our main unicity result and non-existence results stated in Theorem 6 and Corollaries 7, 8 and 9.

2. CONFORMALLY COMPACT RIEMANNIAN MANIFOLDS

In the following, all definitions that we will set up are well-known see for example [An2, An3, AD, CH, L, Q, Wa1, Wa2]. Anyway, we will insist on certain issues for the sake of clarity.

Let $\overline{M}$ be a (connected) compact $(n+1)$ -manifold with (non-necessarily connected for the moment) boundary and $n \geq 2$. We will denote by M its interior and by ∂M its boundary. Suppose that g is a smooth Riemannian metric on the interior $M = \overline{M} - \partial M$ of $\overline{M}$. A *defining function* of the boundary is a non-negative function $\rho \in C^1(\overline{M})$ such that $\rho^{-1}(\{0\}) = \partial M$ and $d\rho \neq 0$ everywhere on ∂M . The open Riemannian manifold (M, g) is said to be *conformally compact* of order $C^{m,\alpha}$ if, for some (and hence for all) smooth defining function ρ , the smooth conformal metric $\rho^2 g$ on M extends to a $C^{m,\alpha}$ Riemannian metric $\overline{g}$ on $\overline{M}$. Of course, in such a case (M, g) must be complete and of infinite volume. It is also obvious that there are many different defining functions for ∂M , but all the corresponding extended metrics $\overline{g} = \rho^2 g$ will have conformally equivalent restrictions on the boundary ∂M . Then, if $\gamma = \overline{g}|_{\partial M}$, the conformal manifold $(\partial M, [\gamma])$ is well defined, it depends only on (M, g) and is called the *conformal infinity*. Throughout this paper, we will say that each of these $(\overline{M}, \overline{g})$ is a *compact conformal filling* of $(\partial M, [\gamma])$. A considerable progress on the topological and analytical properties of the map $(M, g) \mapsto (\partial M, [\gamma])$ has been obtained by C. Graham, J. Lee and M. Anderson (see among others [GL, L, An2, An3]), particularly when the metric g is Einstein.

It is clear that the main example of a conformally compact Riemannian manifold is the Poincaré hyperbolic space $\left(B^{n+1}, \frac{4|dx|^2}{(1-|x|^2)^2}\right)$, normalized so that its constant sectional curvature is -1 , where $B^{n+1} = \{x \in \mathbb{R}^{n+2} \mid |x| < 1\}$ is the $(n+1)$ -dimensional Euclidean unit open ball. Hence $\frac{1-|x|^2}{2}$ is a defining function for the boundary $\partial B^{n+1} = \mathbb{S}^n$ and the corresponding conformal metric is the Euclidean metric $|dx|^2$, which smoothly extends to the closed unit ball, being its restriction to the sphere $\mathbb{S}^n$, the round metric of constant sectional curvature 1. Hence the conformal infinity of the hyperbolic space is $\mathbb{S}^n$, endowed with its standard conformal structure.

In a certain sense, all conformally compact Riemannian manifolds of order at least C^2 locally resemble to the Cartan-Hadamard spaces near their conformal infinities. In fact, using the transformation properties

of the Riemannian curvature tensors under conformal changes of metrics (see, for instance, [Be, p. 59] or [An3, Appendix]), it can be easily seen that all the sectional curvatures K_g of (M, g) uniformly approach $-|\bar{\nabla}\rho|^2$ as $\rho \rightarrow 0$. Here $\bar{\nabla}$ is the gradient operator corresponding to the metric $\bar{g}$ and the norm is taken wrt the same metric as that of the gradient. So, it is clear that the quantity $|\bar{\nabla}\rho|$ restricted to ∂M depends only on the original metric g . Thus, *conformally compact manifolds of order at least C^2 are asymptotically negatively curved*. This fact makes the following definition natural: we will say that a conformally compact Riemannian manifold (M, g) is *asymptotically locally (weakly, according to [Wa1]) hyperbolic (ALH in short)* when $|\bar{\nabla}\rho|_{\partial M}$ is constant, that we may normalize to be $|\bar{\nabla}\rho|_{\partial M} = 1$. For this class of manifolds we have $K_g \rightarrow -1$ near infinity. As a consequence, the Ricci tensor satisfies $\text{Ric}_g \rightarrow -ng$, that is, the manifold seems to be Einstein with Ricci curvature $-n$ when one moves towards infinity. Hence, the scalar curvature $R_g \rightarrow -n(n+1)$.

Conversely, from the transformation rules of the Ricci tensor and the scalar curvature of conformal metrics, it can be seen that any of these asymptotical behaviours for K_g , Ric_g or R_g implies that $|\bar{\nabla}\rho|_{\partial M} = 1$. This means that *a C^2 conformally compact Riemannian manifold is ALH if and only if it is asymptotically Einstein, that is, $\text{Ric}_g + ng \rightarrow 0$ uniformly, or it has asymptotical constant scalar curvature, that is, $R_g + n(n+1) \rightarrow 0$ uniformly as $\rho \rightarrow 0$, and all these geometrical conditions are reflected into the analytical condition $|\bar{\nabla}\rho|_{\partial M} = 1$, which is independent of the chosen defining function ρ . In particular, this occurs when the manifold (M, g) is supposed to be Einstein. In this case, (M, g) is often called a Poincaré-Einstein manifold (in short, a PE manifold) and, in this case, we will have $\text{Ric}_g = -ng$. The weaker condition on the constant scalar curvature $R + n(n+1) = 0$ implies that (M, g) is an ALH manifold as well.*

In the general non-necessarily Einstein ALH case, if we assume (M, g) to be conformally compact of order at least $C^{3,\alpha}$, we can modify any given smooth defining function ρ to get another one $r \in C^{2,\alpha}(\bar{M})$ such that the corresponding extended conformal metric $\bar{g} = r^2g$ is of class $C^{2,\alpha}$ and $|\bar{\nabla}r| \equiv 1$ in a collar neighbourhood of the conformal infinity ∂M . In other words, the conformal metric $\bar{g} = dr^2 + g_r$ extended on $\bar{M}$ is locally expressed taking the $\bar{g}$ -distance to the boundary ∂M as a first coordinate. This means that, near infinity, the metric g of M can

be written as

$$(1) \quad g = \frac{1}{r^2}(dr^2 + g_r) = \frac{1}{r^2}(dr^2 + g^{(0)} + rg^{(1)} + r^2g^{(2)} + \mathcal{O}(r^3)),$$

since, the extended metric $\bar{g}$ being $C^{2,\alpha}$, it admits a second order Taylor expansion in powers of r , where these $g^{(i)}$ and $\mathcal{O}(r^3)$ are symmetric two-tensors on ∂M . The coefficients $g^{(i)}$ are computable from the iterated Lie derivatives of the extended metric:

$$(2) \quad g^{(i)} = \frac{1}{i!} \mathcal{L}_{\bar{\nabla}r}^{(i)} \bar{g} \Big|_{r=0}.$$

So, $g^{(0)} = \gamma = \bar{g}|_{\partial M}$ and $g^{(i)}$ is of class $C^{2-i,\alpha}$ for $i = 0, 1, 2$ (for the regularity of g_r with respect to r when the metric γ induced on the conformal infinity is supposed to be smooth, see [CDLS]).

Lemma 1. [An2, Section 3], [AD, Lemma 5.4], [GL, Lemma 5.2], [L, Lemma 5.1], [Wa1, Lemma 2.2] *Let (M, g) be an ALH manifold of class $C^{m,\alpha}$, $m \geq 3$. For each choice of representative γ in the conformal class $[\gamma]$ on ∂M , there exists a unique defining function $r \in C^{m-1,\alpha}(\bar{M})$ such that the extended conformal metric $\bar{g} = r^2g$ is of class $C^{m-1,\alpha}$, $|\bar{\nabla}r| \equiv 1$ in a collar neighbourhood of ∂M and we have the expansion*

$$\bar{g} = dr^2 + \gamma + rg^{(1)} + r^2g^{(2)} + \mathcal{O}(r^3),$$

as in (1) and (2).

We will say that such a function r is the *geodesic defining function* associated with the metric γ in the conformal class induced on ∂M . The existence of geodesic defining functions on a ALH manifold is a key point to obtain the connectedness of its boundary when one has both a lower estimate on the Ricci tensor and a decay condition on the scalar curvature, along with the non-negativity of the conformal structure at infinity. This connectedness, that we will need later, was proved by E. Witten and S.-T. Yau for positive conformal infinities and by M. Cai and G. Galloway for null conformal infinities.

Theorem 2. ([WY, CG], [An1, Theorem 4.1]) *Let (M, g) be a conformally compact Riemannian manifold whose Ricci tensor and scalar curvature satisfy*

$$\text{Ric}_g + ng \geq 0, \quad R_g + n(n+1) = o(r^2),$$

where r is a geodesic defining function. Suppose that the conformal infinity $(\partial M, [\gamma])$ has a connected component with non-negative Yamabe invariant. Then ∂M is connected.

Remark 1. It is clear that the second hypothesis on the scalar curvature is a slight hardening of the ALH condition. Note as well that the two conditions on the Ricci tensor and the scalar curvature are obviously satisfied when (M, g) is a PE space. Moreover, from now on, we will not focus on the regularity of conformal fillings: one can assume that our conformally compact manifolds are C^∞ , although much less is needed to get the results. For example, for Theorem 2 it suffices either that $\bar{g}$ is of class C^2 with a C^∞ induced metric $\gamma = \bar{g}|_{\partial M}$ on the infinity (see [CDLS]) or that (M, g) is of class $C^{3,1}$.

Remark 2. Theorem 2 is a crucial fact from the physical point of view of the AdS/CFT theory. In fact, a conformal infinity with more than one connected component would hardly permit to justify some kind of coupling between the conformal theories on each one of them. The result was originally proved by S.-T Yau and E. Witten [WY] in the case of conformal infinity with positive Yamabe invariant when the bulk manifold is a PE space. M. Cai and G. Galloway [CG] realized that, in fact the result is valid under the weaker hypotheses $\text{Ric}_g + ng \geq 0$ and $|\text{Ric}_g + ng| = o(r^2)$ and completed it by adding the case of zero Yamabe invariant. Note that, in presence of the lower bound on the Ricci tensor, the decay hypotheses $|\text{Ric}_g + ng| = o(r^2)$ and $R_g + n(n+1) = o(r^2)$ are equivalent.

Remark 3. It is not too difficult to check that the first and second order decay conditions on the Ricci tensor and the scalar curvature of the ALH manifold (M, g) can be translated into precise respective behaviours of the coefficients $g^{(1)}$ and $g^{(2)}$ in the expansion (1) of the extended metric $\bar{g}$ in powers of r . First, it is obvious from definitions (2) that

$$g^{(1)} = \mathcal{L}_{\bar{\nabla}r} \bar{g}|_{r=0} = -2\bar{\sigma}|_{\partial M}, \quad g^{(2)} = \frac{1}{2} \mathcal{L}_{\bar{\nabla}r} \bar{g}|_{r=0} = -(\bar{\nabla}_{\bar{\nabla}r} \bar{\sigma})|_{\partial M},$$

where, $\bar{\sigma}$ is the second fundamental form of the level hypersurfaces of r with respect to the choice of the inner unit normal $\bar{\nabla}r$ and to the metric $\bar{g}$ on $\bar{M}$. Using these expressions, we see that

$$|\text{Ric}_g + ng| = o(r) \iff g^{(1)} = 0$$

and, as a consequence,

$$\left. \begin{array}{l} \text{Ric}_g + ng \geq 0, \\ R_g + n(n+1) = o(r) \end{array} \right\} \implies g^{(1)} = 0.$$

Analogously, with a little more extra effort, we prove that

$$|\text{Ric}_g + ng| = o(r^2) \iff \begin{cases} g^{(1)} = 0, \\ g^{(2)} = -P_\gamma, \end{cases}$$

where P_γ is the Schouten tensor of γ , given by

$$P_\gamma = \frac{1}{n-2} \left(\text{Ric}_\gamma - \frac{R_\gamma}{2(n-1)} \gamma \right)$$

for $n > 2$ and is a symmetric 2-tensor on ∂M satisfying

$$\text{trace}_\gamma P_\gamma = \frac{R_\gamma}{2}, \quad \delta P_\gamma = \nabla_\gamma R_\gamma.$$

As a consequence, we have

$$\left. \begin{array}{l} \text{Ric}_g + ng \geq 0, \\ R_g + n(n+1) = o(r^2) \end{array} \right\} \implies \left\{ \begin{array}{l} g^{(1)} = 0, \\ g^{(2)} = -P_\gamma, \end{array} \right.$$

and so the hypotheses of Theorem 2 imply the following expansion of the metric g in terms of powers of r :

$$\left. \begin{array}{l} \text{Ric}_g + ng \geq 0, \\ R_g + n(n+1) = o(r^2) \end{array} \right\} \implies g = \frac{1}{r^2} (dr^2 + \gamma - r^2 P_\gamma + \mathcal{O}(r^3)),$$

that is, they imply that the ALH manifold (M, g) is a *weakly Poincaré-Einstein* manifold ([MP, end of Section 2] and [GQ, Remark 1.3, 1]).

3. DEFINING FUNCTIONS FROM EIGENFUNCTIONS

It is an obvious remark that when we consider the hyperbolic space $\mathbb{H}^{n+1}$ of curvature -1 as one of the two connected components of the hyperquadric in the Lorentz-Minkowski spacetime $\mathbb{R}_1^{n+2}$ consisting of all its unit timelike vectors, that is,

$$\mathbb{H}^{n+1} = \{p \in \mathbb{R}_1^{n+2} \mid |p|^2 = -1, p_0 > 0\},$$

its usual compactifications (as the Poincaré disc or as the hemisphere, and even its *partial* compactification as the Poincaré half-space) are all obtained through defining functions which come from eigenfunctions of the (positive) Laplacian associated with the eigenvalue $-(n+1)$. Indeed, it is a well-known fact that, for each fixed $a \in \mathbb{R}_1^{n+2}$,

$$\Delta \langle p, a \rangle = (n+1) \langle p, a \rangle, \quad \forall p \in \mathbb{H}^{n+1},$$

and that these *height* functions are all the possible corresponding eigenfunctions. Moreover, $\langle p, a \rangle \geq 1$ when a is timelike, $\langle p, a \rangle > 0$ when a is lightlike and $\langle p, a \rangle$ changes sign when a is spacelike. This is why one only uses eigenfunctions $\langle p, a \rangle$ with a timelike or lightlike a in order to get these partial or total compactifications.

To study the continuous spectrum of general PE manifolds (M, g) , J. Lee proved in [L] the existence of positive solutions of the eigenvalue equation $\Delta u = (n+1)u$ with a prescribed growth at infinity ∂M .

Some years later, J. Qing used in [Q] these particular eigenfunctions as defining functions to *partially* compactify (M, g) and give a beautiful proof, based on the positive mass theorem, of the rigidity of the hyperbolic space among all the PE manifolds with prescribed conformal infinity a round sphere. In fact, he was seeking to drop the spin assumption in the corresponding result by L. Andersson and M. Dahl ([AD]). Our strategy in this paper will be, in the spin case, to build slightly modified defining functions from these same eigenfunctions in order to *totally* compactify the manifold and apply, to the conformal infinity ∂M of the compactification $(\overline{M}, \overline{g})$, a previous estimate by X. Zhang and ourselves (see [HMZ1, HMZ2]) for the spectrum of the Dirac operator on spin manifolds which bound compact domains.

First, in the following proposition, we will extend the existence results of eigenfunctions obtained in [L, Section 5] for PE manifolds to non-necessarily Einstein ALH manifolds whose scalar curvature decays a bit more quickly than usual.

Proposition 3. (Cfr. [L, Proposition 4.1, Lemmae 5.1 and 5.2] and [GQ, Lemma 2.1, Remark below]) *Let (M, g) be an $(n+1)$ -dimensional conformally compact manifold and r a geodesic defining function associated with a choice of a representative on the conformal class $[\gamma]$ induced on ∂M , according to Lemma 1. If the Ricci tensor and the scalar curvature of (M, g) satisfy*

$$\text{Ric}_g + ng \geq 0, \quad R_g + n(n+1) = o(r^2),$$

then there exists a unique positive function $u \in C^\infty(M)$ such that $\Delta u = (n+1)u$ and $u - \frac{1}{r}$ is a bounded function. Moreover, one has

$$(3) \quad u = \frac{1}{r} + hr,$$

for a function $h \in C^0(\overline{M}) \cap C^{2,\alpha}(M)$ such that

$$(4) \quad h|_{\partial M} = \frac{R_\gamma}{4n(n-1)}, \quad |\nabla h| = O(r^{\frac{\alpha}{2}}),$$

where $\alpha > 0$ and R_γ is the scalar curvature of the chosen representative γ on the conformal class induced on ∂M .

Remark 4. The hypothesis on the scalar curvature, as in Theorem 2, is slightly stronger than the ALH condition $R_g + n(n+1) \rightarrow 0$. Note that also the two conditions in the statement on the Ricci tensor and the scalar curvature, as we saw in Remark 3, imply that (M, g) is a WPE manifold. In particular, these conditions are automatically satisfied when (M, g) is a PE space.

Proof. This result simply combines some assertions from Proposition 4.1, Lemma 5.1 (see Lemma 1 of this paper) and Lemma 5.2 in [L]. That proposition and the first of the two lemmata only require (M, g) to be ALH, but this condition is satisfied by (M, g) as we have just pointed out in Remark 4. The proof of the second of these lemmata definitely uses the hypothesis that (M, g) is Einstein. However, as it was explicitly settled in [GQ, remark after Lemma 2.1] and one can directly check, the proof of [L, Lemma 5.2] still works for WPE manifolds with $\text{Ric}_g + ng \geq 0$. So, it suffices to take into account Remark 4 above.

q.e.d.

The idea of using the eigenfunctions provided by Proposition 3 as defining functions on a given conformally compact Einstein space (M, g) is due to J. Qing (see [Q]), although he conformally modifies the original complete manifold through these eigenfunctions without actually compactifying it. Instead, he gets a *partial* compactification, that is, a conformal complete Riemannian manifold $(\overline{M}, \overline{g})$ whose boundary ∂M is diffeomorphic to the Euclidean space $\mathbb{R}^n$ and such that the Riemannian manifold constructed by doubling $(\overline{M}, \overline{g})$ along this boundary is an Asymptotically Euclidean manifold without boundary and with non-negative integrable scalar curvature. Then a suitable use of the positive mass theorem allows him to conclude. We will proceed in a different way in order to *totally* compactify (M, g) through defining functions obtained by slightly modifying the eigenfunctions given by Proposition 3.

Theorem 4. *Let (M, g) be a conformally compact manifold such that the Ricci tensor and the scalar curvature satisfy*

$$\text{Ric}_g + ng \geq 0, \quad R_g + n(n+1) = o(r^2),$$

where r is a geodesic defining function. Suppose that its conformal infinity ∂M has non-negative Yamabe invariant. Then there exists a defining function ρ^ for (M, g) such that, if g^* is the extension of $(\rho^*)^2 g$ to $\overline{M}$,*

$$R_{g^*} \geq 0, \quad H^* = \varepsilon,$$

where R_{g^} is the scalar curvature of the compact Riemannian manifold $(\overline{M}, g^*)$, H^* is the (inner) mean curvature of the conformal infinity ∂M as a hypersurface of $(\overline{M}, g^*)$. Here $\varepsilon = 1$ or $\varepsilon = 0$ depending on the Yamabe invariant of ∂M being positive or zero.*

Proof. First, note that ∂M is connected by Theorem 2. By the solution to the Yamabe problem [Sc], there exists a metric γ on the conformal

class $[\gamma]$ induced on ∂M with constant scalar curvature $R_\gamma = n(n-1)\varepsilon^2$, where ε is either 1 or 0, depending on whether the Yamabe invariant of $(\partial M, [\gamma])$ is positive or zero. Then, if r is the geodesic defining function provided by Lemma 1 and u is the eigenfunction associated with this r by Proposition 3, we define the function ρ^* by

$$\rho^* = \frac{1}{u + \varepsilon}.$$

Since u is a positive smooth function on M , we have that $\rho^* = \frac{1}{u + \varepsilon}$ is another positive smooth function on M . Moreover, using the relation between the eigenfunction u and the geodesic defining function r provided by Proposition 3, we have

$$\rho^* = \frac{r}{1 + \varepsilon r + hr^2}$$

and so $\rho^* \in C^\infty(M)$ extends to a $C^{2,\beta}$ function on $\overline{M}$ since the function hr^2 extends to $C^{2,\beta}(\overline{M})$. It is immediate to see that $(\rho^*)^{-1}(\{0\}) = r^{-1}(\{0\}) = \partial M$ and $d\rho^*_{|\partial M} = dr_{|\partial M}$. So, ρ^* is a defining function for (M, g) . Moreover the metric

$$g^* = (\rho^*)^2 g = \frac{1}{(1 + \varepsilon r + hr^2)^2} r^2 g$$

defined on M extends to a $C^{2,\beta}$ metric on $\overline{M}$. This metric g^* , restricted to ∂M , gives

$$g^*_{|\partial M} = (r^2 g)_{|\partial M} = (\rho^2 g)_{|\partial M} = \gamma.$$

Hence $(\overline{M}, g^* = (\rho^*)^2 g)$ is a $C^{2,\beta}$ compactification of the ALH manifold (M, g) such that the Riemannian metric induced on the conformal infinity ∂M is just the metric γ with constant scalar curvature $n(n-1)\varepsilon^2$. (In fact, in the Einstein case, if the boundary metric γ were smooth, a suitable application of the definitive regularity result [CDLS, Theorem A] would increase the regularity level of g^* . The same would occur if we imposed a high regularity on the extended metric $\overline{g}$.)

Now, rewriting the conformally related metrics g and g^* as

$$g = (\rho^*)^{-2} g^* = (u + \varepsilon)^2 g^*,$$

the relation between the Ricci tensors Ric_{g^*} and Ric_g of the conformal metrics g^* and g on the open manifold M (see, for example, [Be, p. 59] or [An3, Appendix]), is given by

$$\text{Ric}_{g^*} = \text{Ric}_g + (n-1) \frac{\nabla du}{u + \varepsilon} + \frac{\Delta u}{u + \varepsilon} g - n \frac{|\nabla u|^2}{(u + \varepsilon)^2} g.$$

Taking traces with respect to g and multiplying by $(u + \varepsilon)^2$, we obtain the corresponding relation for their scalar curvatures

$$R_{g^*} = (u + \varepsilon)^2 R_g + 2n(u + \varepsilon)\Delta u - n(n + 1)|\nabla u|^2.$$

Since we are assuming $\text{Ric}_g + ng \geq 0$, we have $R_g + n(n + 1) \geq 0$. Moreover the function u is an eigenfunction of Δ associated with the eigenvalue $-(n + 1)$. Thus, we finally get the following lower bound for the scalar curvature R_{g^*} of $(\overline{M}, g^*)$

$$(5) \quad R_{g^*} \geq n(n + 1) (u^2 - |\nabla u|^2 - \varepsilon^2).$$

Now, our strategy to show that $R_{g^*} \geq 0$ is to observe that the smooth function on the rhs of (5) is superharmonic on M , extends continuously to $\overline{M}$ and vanishes at ∂M . The fact that it is superharmonic was already observed in [L, Proposition 4.2], although we reproved it here because its proof is straightforward and we will need the corresponding computation below. Taking into account the Bochner formula for the Laplacian of the squared length of a gradient, we have

$$\frac{1}{2}\Delta (u^2 - |\nabla u|^2) = |\nabla u|^2 + u\Delta u - |\nabla du|^2 - g(\nabla u, \nabla \Delta u) - \text{Ric}_g(\nabla u, \nabla u).$$

Since $\Delta u = (n + 1)u$ and $\text{Ric}_g + ng \geq 0$, we obtain

$$(6) \quad \frac{1}{2}\Delta (u^2 - |\nabla u|^2) \leq (n + 1)u^2 - |\nabla du|^2 = -|\nabla du - u g|^2 \leq 0.$$

Now, we want to study the asymptotical behaviour of the superharmonic function. For this, we use the expression (3) of the eigenfunction u in terms of the geodesic defining function r . Then

$$\nabla u = \left(h - \frac{1}{r^2}\right) \nabla r + r \nabla h.$$

Taking squared norms with respect to the metric g , we get

$$|\nabla u|^2 = \left(h - \frac{1}{r^2}\right)^2 |\nabla r|^2 + r^2 |\nabla h|^2 + 2r \left(h - \frac{1}{r^2}\right) g(\nabla r, \nabla h).$$

But we know that $|\nabla r|^2 = r^2$ near the conformal infinity due to the geodesic character of the defining function r . Putting this into the last equation and using (3) again, we have

$$u^2 - |\nabla u|^2 = 4h - r^2 |\nabla h|^2 - 2r \left(h - \frac{1}{r^2}\right) g(\nabla r, \nabla h),$$

which is valid in a collar neighbourhood of ∂M . From (4), we know that h extends to $C^0(\overline{M})$ and that $|\nabla h| = O(r^\beta)$. So, the third term

on the right hand side of the above equation satisfies

$$\left| r \left(h - \frac{1}{r^2} \right) g(\nabla r, \nabla h) \right| \leq |hr^2 - 1| |\nabla h| = O(r^\beta)$$

as a consequence of the Schwarz inequality for g and the fact that $|\nabla r| = r$ in a collar neighbourhood of ∂M . From this inequality and the previous equality, taking limits when $r \rightarrow 0$, we conclude that

$$(u^2 - |\nabla u|^2 - \varepsilon^2)|_{\partial M} = 4h|_{\partial M} - \varepsilon^2 = \frac{R_\gamma}{n(n-1)} - \varepsilon^2 = 0,$$

where we have used again (4) and the choice of ε . As a consequence, we see that $u^2 - |\nabla u|^2 - \varepsilon^2$ is a smooth superharmonic function on M which admits a continuous extension to $\overline{M}$ and vanishes everywhere along ∂M . The strong minimum principle implies that

$$(7) \quad u^2 - |\nabla u|^2 - \varepsilon^2 \geq 0 \quad \text{on} \quad \overline{M}.$$

This fact together with (5) imply that the scalar curvature R_{g^*} of the compact Riemannian manifold with boundary $(\overline{M}, g^*)$ is non-negative, as asserted.

To finish the proof, it remains to compute the mean curvature H^* of the conformal infinity ∂M as a hypersurface of the compactified Riemannian manifold $(\overline{M}, g^* = (\rho^*)^2 g)$. Observe that, by definition,

$$g^* = (\rho^*)^2 g = \left(\frac{r}{1 + \varepsilon r + hr^2} \right)^2 g = \left(\frac{1}{1 + \varepsilon r + hr^2} \right)^2 \overline{g}$$

where $\overline{g} = r^2 g$ is the extended metric on $\overline{M}$ corresponding to the geodesic defining function r . First of all, observe that the mean curvature $\overline{H}$ of the conformal infinity ∂M as a hypersurface of $(\overline{M}, \overline{g})$ vanishes (in fact, it is a totally geodesic hypersurface). For this, consider the following relation between the Ricci tensors $\text{Ric}_{\overline{g}}$ and Ric_g of the conformally related metrics $\overline{g}$ and g on the open manifold M (see again [Be, p. 59] or [An3, Appendix]):

$$\text{Ric}_g = \text{Ric}_{\overline{g}} + (n-1) \frac{\overline{\nabla} dr}{r} + \frac{\overline{\Delta} r}{r} \overline{g} - n \frac{|\overline{\nabla} r|^2}{r^2} \overline{g},$$

where $\overline{\nabla}$ and $\overline{\Delta}$ are respectively the covariant derivative and the Laplacian operator corresponding to the metric $\overline{g}$. Multiplying by r and considering that $|\overline{\nabla} r|^2 = 1$, we obtain

$$(8) \quad r(\text{Ric}_g + ng) = r\text{Ric}_{\overline{g}} + (n-1)\overline{\nabla} dr + (\overline{\Delta} r)\overline{g}.$$

Taking traces with respect to $\bar{g}$, we have

$$(9) \quad \frac{1}{r}(R_g + n(n+1)) = rR_{\bar{g}} + 2n\bar{\Delta}r.$$

From equality (9) and the hypothesis $R_g + n(n+1) = o(r^2)$, it follows

$$(10) \quad \bar{\Delta}r|_{\partial M} = 0.$$

Now, take into account that the Hessian $\bar{\nabla}dr$, restricted to directions orthogonal to the gradient $\bar{\nabla}r$, is the opposite of the second fundamental form $\bar{\sigma}$ of the level hypersurfaces $r = r_0$ with respect to the choice of the inner unit normal $\bar{N} = \bar{\nabla}r$ and to the metric $\bar{g}$ on $\bar{M}$. Moreover, taking derivatives in the equality $|\bar{\nabla}r|^2 = 1$ with respect to a vector field X tangent to M , we obtain $(\bar{\nabla}dr)(\bar{\nabla}r, X) = 0$. In particular,

$$(\bar{\nabla}dr)(\bar{\nabla}r, \bar{\nabla}r) = 0.$$

Hence, we see that $\bar{H} = -\frac{1}{n}\bar{\Delta}r$ is the mean curvature function of these level hypersurfaces. Using (10), we have

$$\bar{H}|_{\partial M} = 0,$$

as wanted.

Finally, in order to compute H^* , it suffices to use the well-known relation between the two mean curvatures of a hypersurface corresponding to two conformal metrics on the ambient space (see, for instance, [E] or [HMZ2, (4.4)]):

$$H^* = \frac{1}{R}(\bar{H} - \bar{g}(\bar{\nabla} \log R, \bar{N})) = -\frac{1}{R^2} \bar{g}(\bar{\nabla} R, \bar{N}),$$

where $R = \frac{1}{1+\varepsilon r+hr^2}$ and $\bar{N}$ is the inner unit normal along ∂M with respect to the metric $\bar{g}$, which can be also written as $\bar{\nabla}r$. Since we have

$$\bar{\nabla}R|_{\partial M} = -\left. \frac{\varepsilon \bar{\nabla}r + \bar{\nabla}(r^2 h)}{(1 + \varepsilon r + hr^2)^2} \right|_{\partial M} = \varepsilon \bar{\nabla}r|_{\partial M}.$$

Thus, we finally obtain

$$(H^*)|_{\partial M} = \varepsilon \bar{g}(\bar{\nabla}r, \bar{\nabla}r)|_{\partial M} = \varepsilon.$$

This equality finishes the proof.

q.e.d.

As a first consequence, we have that the assumption on the Yamabe invariant of the conformal infinity of (M, g) implies certain constraints on the relative Yamabe invariant (see [E] for a definition) of its compactifications.

Corollary 5. *Let (M, g) be an $(n+1)$ -dimensional conformally compact manifold whose Ricci and scalar curvatures satisfy*

$$\text{Ric}_g + ng \geq 0, \quad R_g + n(n+1) = o(r^2)$$

for any geodesic defining function r . Let $(\overline{M}, \overline{g})$ be any of its compact conformal fillings. If the Yamabe invariant of the conformal infinity $(\partial M, [\overline{g}|_{\partial M}])$ is positive (resp. zero), then the relative Yamabe invariant of the Riemannian manifold with boundary $(\overline{M}, \overline{g})$ is also positive (resp. non-negative).

Proof. In the context of the Yamabe problem for compact manifolds with boundary, J. Escobar introduced in [E] the following eigenvalue problem

$$\begin{cases} -\frac{4n}{n-1}\overline{\Delta}f + R_{\overline{g}}f = 0, & \text{on } \overline{M}, \\ -\frac{2}{n-1}\overline{g}(\overline{\nabla}f, \overline{N}) + \overline{H}f = \overline{\nu}f, & \text{along } \partial M, \end{cases}$$

for functions $f \in C^1(\overline{M})$. J. Escobar proved that the sign of the first eigenvalue $\overline{\nu}_1$ (if finite) of this problem is a conformal invariant of the metric $\overline{g}$ and that it coincides with that of the relative Yamabe invariant of the manifold with boundary $\overline{M}$, whose value is a conformal invariant (see [HMZ2] for a relation between ν_1 and the Dirac operator of the boundary). Under our hypotheses, we can apply Theorem 4 and dispose of the metric g^* on $\overline{M}$ conformal to $\overline{g}$ and such that $R_{g^*} \geq 0$ on $\overline{M}$ and $H^* = \varepsilon$ along ∂M , where $\varepsilon = 1$ (resp. 0). Since the metrics $\overline{g}$ and g^* are conformal we can compute ν_1^* in order to know the sign of $\overline{\nu}_1$. To do this, take a non trivial function $f \in C^1(\overline{M})$. We have

$$\int_{\overline{M}} \left(\frac{2}{n-1} |\nabla^* f|^2 + \frac{1}{2n} R_{g^*} f^2 \right) + \int_{\partial M} H^* f^2 \geq \varepsilon \int_{\partial M} f^2.$$

Hence, by the variational characterization of the eigenvalue ν_1^* , we have $\nu_1^* \geq \varepsilon$ and so ν_1^* (and hence $\overline{\nu}_1$) is positive when $\varepsilon = 1$ and non-negative, when $\varepsilon = 0$.

q.e.d.

4. CONFORMAL INFINITIES ADMITTING A TWISTOR SPINOR

Suppose now that the ALH Riemannian manifold (M, g) admits a compactification $\overline{M}$ which is a spin manifold. Then we will say that (M, g) is a *ALH spin manifold*. Fix a spin structure on $\overline{M}$. Then, since the conformal infinity ∂M is always an orientable hypersurface (recall that the gradient of a geodesic defining function provides a global unit normal field), we have that ∂M is also a spin manifold and an

induced spin structure on the infinity is inherited from that fixed on $\overline{M}$. Moreover, for each Riemannian metric $\overline{g}$ on $\overline{M}$ we have an associated spinor bundle $(\mathbb{S}\overline{M}, \overline{\nabla}, \overline{\mathfrak{c}})$, where $\overline{\nabla}$ is the spin Levi-Civita connection and $\overline{\mathfrak{c}}$ is the Clifford multiplication (for generalities on spin structures see any of [BFGK, BHMMM, Fr, Gi, LM]). It is a well-known fact that the restriction to the hypersurface ∂M of the spinor bundle $\mathbb{S}\overline{M}$ can be identified with one or two copies of the spinor bundle corresponding to the induced spin structure and the induced Riemannian metric $\gamma = \overline{g}|_{\partial M}$ according to the parity of the dimension n . More precisely, if φ is a section of the restricted bundle $\mathbb{S}\overline{M}|_{\partial M}$, we consider the new Clifford multiplication

$$\mathfrak{c}^{\partial M}(X)\varphi = \overline{\mathfrak{c}}(X)\overline{\mathfrak{c}}(N)\varphi.$$

and the new connection

$$\nabla_X^{\partial M}\varphi = \overline{\nabla}_X\varphi - \frac{1}{2}\overline{\mathfrak{c}}(\overline{A}X)\overline{\mathfrak{c}}(\overline{N})\varphi = \overline{\nabla}_X\varphi - \frac{1}{2}\mathfrak{c}^{\partial M}(\overline{A}X)\varphi, \quad \forall X \in \Gamma(T\partial M),$$

where $\overline{N}$ is the (inner) unit normal field along ∂M and $\overline{A}$ is its corresponding shape operator. Then, we have an isomorphism

$$(\mathbb{S}\overline{M}|_{\partial M}, \nabla^{\partial M}, \mathfrak{c}^{\partial M}) \cong \begin{cases} (\mathbb{S}\partial M, \nabla, \mathfrak{c}), & \text{if } n \text{ is even} \\ (\mathbb{S}\partial M, \nabla, \mathfrak{c}) \oplus (\mathbb{S}\partial M, \nabla, -\mathfrak{c}), & \text{if } n \text{ is odd,} \end{cases}$$

where $(\mathbb{S}\partial M, \nabla, \mathfrak{c})$ is the spinor bundle corresponding to the spin structure and to the Riemannian metric induced on ∂M (for this relationship between the spinor bundles on a hypersurface and on its ambient space, see, for instance, [Bä3, Fr, HM, HMZ1, HMZ2]). Due to this identification we can say that each spinor field on $\overline{M}$ determines by restriction a spinor field on the boundary ∂M and we can talk about possible extensions to $\overline{M}$ of the spinor fields defined on ∂M .

Let $\varphi \in \Gamma(\mathbb{S}\partial M)$ be a spinor field on the conformal boundary of a conformally compact spin Riemannian manifold (M, g) , where the spinor bundle $\mathbb{S}\partial M$ is referred to the induced metric $\gamma = \overline{g}|_{\partial M}$ from a conformally extended metric $\overline{g}$ on $\overline{M}$. We say that φ is a *twistor spinor* if it satisfies the first order equation

$$P_X\varphi = \nabla_X\varphi + \frac{1}{n}\mathfrak{c}(X)D\varphi = 0, \quad \forall X \in \Gamma(T\partial M),$$

where P is said to be the *Penrose or twistor operator* and D is the well-known Dirac operator locally defined by

$$D\varphi = \sum_{i=1}^n \mathfrak{c}(e_i)\nabla_{e_i}\varphi,$$

where $\{e_1, \dots, e_n\}$ is a local orthonormal frame on ∂M . Of course, these definitions are usually made for general spin manifolds, not necessarily boundaries of compactifications. For the main properties and statements about twistor spinors the reader is invited to refer to [BFGK, BHMMM, CGLS, Gi]. We limit ourselves to point out that, if φ is a twistor spinor on ∂M and we consider a conformal change of the metric $\gamma^* = h^2\gamma$ on ∂M , then the spinor field $\varphi^* = h^{\frac{1}{2}}\varphi$ is a twistor spinor with respect to the new metric γ^* (see [Li], [Gi, Proposition A.2.1]). Thus, in general, the existence of a non-trivial twistor spinor as well as the dimension of the space of twistor spinors on a given spin Riemannian manifold do not depend on the metric, but only on its conformal class. So, in particular, *the existence of twistor spinors and the dimension of the corresponding space on the conformal boundary $(\partial M, [\gamma])$ of an ALH spin manifold (M, g) do not depend on the conformal compactification $\bar{g}$ of the metric.*

When a twistor spinor φ is also an eigenspinor for the Dirac operator, that is, when $D\varphi = \lambda\varphi$ for some $\lambda \in \mathbb{C}$ (note that, if the manifold is compact, λ must be real), it satisfies the overdetermined system

$$\nabla_X \varphi + \alpha \mathfrak{c}(X)\varphi = 0, \quad \alpha \in \mathbb{C},$$

and we say that φ is a *Killing spinor* when $\alpha \in \mathbb{C}^*$. Of course, if the same condition is satisfied with $\alpha = 0$, we say that φ is a *parallel spinor* (we refer again to [BFGK, BHMMM, CGLS, Gi]). It can be shown that α has to be either a real or a purely imaginary number. So, we will talk about *either real or imaginary Killing spinors* according to either $\alpha \in \mathbb{R}^*$ or $\alpha \in i\mathbb{R}^*$. The existence of parallel or Killing spinors imposes strong restrictions on the geometry of the manifold and on its holonomy. Such manifolds have to be Einstein with scalar curvature $R = 4n(n-1)\alpha^2$.

Indeed, M. Wang, H. Baum and C. Bär ([W1, W2, Bm, B1, B2]) classified some types of spin Riemannian manifolds admitting non-trivial parallel, imaginary Killing or real Killing spinors, respectively. When the considered spin Riemannian manifold is compact, since the eigenvalues of its Dirac operator have to be real, Killing spinors must be real and, moreover, as it was shown by T. Friedrich [BFGK, Corollary 1, Theorem 9], they are eigenspinors corresponding to the eigenvalues with the least absolute value $\pm \sqrt{\frac{nR}{4(n-1)}}$. This quick review about spin structures, twistor spinors and Killing spinors allows to set up our first rigidity result.

Theorem 6. *Let (M, g) be an $(n+1)$ -dimensional conformally compact spin manifold such that its Ricci and scalar curvatures satisfy*

$$\text{Ric}_g + ng \geq 0, \quad R_g + n(n+1) = o(r^2)$$

for a geodesic defining function r and that its conformal infinity ∂M admits a non-trivial twistor spinor, where we consider on ∂M the spin structure induced from that of M . Then (M, g) is isometric to the hyperbolic space $\mathbb{H}^{n+1}$ of constant sectional curvature -1 (and so ∂M is the conformal round sphere).

Proof. Let $(\overline{M}, \overline{g})$ be a conformal spin compactification of (M, g) . Take a non-trivial twistor spinor $\varphi \in \Gamma(\mathbb{S}\partial M)$ on the conformal infinity. Since ∂M is compact, we can use a result due to Lichnerowicz ([Li], see [BFGK, Theorem 10]) asserting that there is a metric $\gamma^* = h^2\gamma$, $h \in C^\infty(\partial M)$, $h > 0$, in the conformal class of $\gamma = \overline{g}|_{\partial M}$ with constant scalar curvature $R_{\gamma^*} = n(n-1)\varepsilon^2$, where ε is 0 or 1, and such that $\varphi^* = h^{\frac{1}{2}}\varphi$ is either a sum of two Killing spinors which are eigenvalues for the Dirac operator D^* corresponding to the eigenvalues $\pm\sqrt{\frac{nR_{\gamma^*}}{4(n-1)}} = \pm\frac{n}{2}$ when $\varepsilon = 1$, or it is parallel when $\varepsilon = 0$.

That is, we consider a conformal change of the metric on ∂M in order to have either positive constant scalar curvature and a non-trivial real Killing spinor, namely φ_1^* , or identically zero scalar curvature and a non-trivial parallel spinor, namely φ_0^* . In particular, it is clear that the Yamabe invariant of $(\partial M, [\gamma])$ is non-negative and we can apply Theorem 4 in this situation. So we have a metric g^* on the spin compactification $\overline{M}$ such that $R_{g^*} \geq 0$, $H^* = \varepsilon$ and $(g^*)|_{\partial M} = \gamma^*$. Under these conditions, we make use of a lower bound for the spectrum of the Dirac operator D^* obtained by X. Zhang and the authors in [HMZ1] (see [Gi, Theorem 3.7.1]). This result asserts that, if $\lambda_1(D^*)$ stands for the eigenvalue of D^* with the lowest absolute value, then $|\lambda_1(D^*)| \geq \frac{n}{2}\varepsilon$ and, if the equality holds, then the eigenspace associated with $\lambda_1(D^*)$ is built from parallel spinor fields (with respect to the metric g^*) on $\overline{M}$.

But we know that there exists on ∂M a non-trivial Killing or parallel spinor φ_ε^* and so it must be an eigenspinor for the Dirac operator. More precisely, $D^*\varphi_\varepsilon^* = \pm\frac{n}{2}\varepsilon\varphi_\varepsilon^*$. Then the equality $|\lambda_1(D^*)| = \frac{n}{2}\varepsilon$ holds and so φ_ε^* comes from a spinor field $\Psi \in \mathbb{S}\overline{M}$ parallel with respect to the metric g^* . Note that Ψ has to be a non-trivial parallel spinor because its restriction to ∂M is non-trivial. It was shown by Hitchin in [Hit] (see also [BFGK, Chapter 6]) that the existence of a non-zero parallel spinor forces the Ricci tensor to vanish everywhere. Then $\text{Ric}_{g^*} = 0$ on

$\overline{M}$ and so $R_{g^*} = 0$ as well. Then, from (5), (6) and (7), and the proof of Theorem 4, we conclude

$$(11) \quad \nabla du = ug, \quad u^2 - |\nabla u|^2 - \varepsilon^2 = 0,$$

where $u \in C^\infty(M)$ is the positive eigenfunction provided by Proposition 3. Hence the complete manifold (M, g) admits a non-trivial (in fact, positive) solution u to the Obata type equation $\nabla du = ug$. Theorem C in [K] finishes the proof if the function u has a critical point.

Suppose on the contrary that u has no critical point on M . We can normalize the gradient ∇u to obtain a global unit vector field $X = \frac{\nabla u}{|\nabla u|}$ on M satisfying $\nabla_X X = 0$ as a consequence of (11). Hence the integral curves of X are geodesics and are defined on the whole real line. Take a positive number $a \in u(M)$ in the image of u . Then $P = u^{-1}(\{a\})$ is a closed hypersurface in M and so compact because u tends to $+\infty$ when one approaches ∂M (see (3)).

Let $\Phi : M \times \mathbb{R} \rightarrow M$ be the flow of X . From the previous considerations it follows that the restriction $\Phi : P \times \mathbb{R} \rightarrow M$ is a diffeomorphism with

$$\Phi(p, s) = \gamma_p(s), \quad \forall p \in P, \forall s \in \mathbb{R}$$

and where $\gamma_p : \mathbb{R} \rightarrow M$ is the integral (geodesic) curve of X with initial condition $\gamma_p(0) = p$. In particular, P must be connected. Moreover, if $Y \in \Gamma(TP)$ is a vector field tangent to P , we have $Y \cdot u = g(\nabla u, Y) = 0$, since the gradient ∇u is orthogonal to the level hypersurfaces. This means that $u(\Phi(p, s))$ depends only on s . On the other hand, equation (11) implies

$$(u \circ \gamma_p)''(s) = u \circ \gamma_p(s), \quad (u \circ \gamma_p(s))^2 - ((u \circ \gamma_p)'(s))^2 - \varepsilon^2 = 0,$$

for each $p \in P$ and $s \in \mathbb{R}$. It follows that

$$u(\gamma_p(s)) = a \cosh s + b \sinh s, \quad b \in \mathbb{R}, \quad \text{with} \quad a^2 = b^2 + \varepsilon^2.$$

Since we are assuming that u has no critical point on M and $u(\gamma_p(s))$ only depends on s , we deduce

$$(u \circ \gamma_p)'(s) = a \sinh s + b \cosh s \neq 0, \quad \forall s \in \mathbb{R},$$

and this is equivalent to the inequality $|b| \geq a$. It turns out that $|b| = a > 0$ and $\varepsilon = 0$ and so, reversing the parameter s if necessary, we obtain

$$u(\gamma_p(s)) = ae^s, \quad \forall p \in P, \forall s \in \mathbb{R}.$$

But, from (3), we know that $u(\gamma_p(s)) \rightarrow +\infty$ when $\gamma_p(s)$ approaches ∂M . Hence

$$\exists \lim_{s \rightarrow -\infty} \gamma_p(s) = \gamma_p(-\infty) \in M \quad \text{and} \quad u(\gamma_p(-\infty)) = 0,$$

which is a contradiction because u is positive. This finishes the proof.
q.e.d.

Remark 5. A careful analysis of the proof of Theorem 6, allows one to show that, for each twistor spinor on the conformal infinity, there exists an imaginary Killing spinor on the bulk manifold (M, g) . In fact, we proved that there exists a parallel spinor $\Psi \in \Gamma(\mathbb{S}\overline{M})$ on the compact conformal filling $(\overline{M}, g^*)$. Now, the same construction which allows one to pass from parallel Euclidean spinors to imaginary hyperbolic Killing spinors by using the conformal factor between the Euclidean and the hyperbolic metrics on the disc B^{n+1} (see [BFGK, Gi]) can be used here in order to build from Ψ an imaginary Killing spinor on (M, g) .

This is an analogue to the result of the extension of conformal transformations on ∂M to isometries on M by M. Anderson (see [An1, Theorem 3.3] and [An3, Theorem 2.5]), although in this supersymmetric setup our assumptions on the Ricci curvature are weaker and the conclusion is stronger: the presence of only one twistor spinor on the boundary at infinity forces the bulk manifold to be maximally symmetric. We will point out that this occurs because we prevent conformally compact manifolds (M, g) to possess singularities. Anyway, from the existence of this imaginary Killing spinor on (M, g) and the H. Baum classification given in [Bm] we could finish the proof, but other considerations have led us to prefer the approach we followed.

Due to its simple connectedness, the sphere $\mathbb{S}^n$ has a unique spin structure. Moreover, the spinor bundle corresponding to this structure and to the round Riemannian metric has a $2^{\lfloor \frac{n}{2} \rfloor}$ -dimensional space of real Killing spinors with $\alpha = \frac{1}{2}$ and another one with the same dimension with $\alpha = -\frac{1}{2}$. Thus, it follows that $\mathbb{S}^n$ endowed with its standard conformal structure admits always a twistor spinor, in fact, a maximal number of $2^{\lfloor \frac{n}{2} \rfloor + 1}$ linearly independent twistor spinors (see [BFGK, p. 37], [Gi, Examples A.1.3.2]). In this way we see that Theorem 6 simplifies the known proofs of the rigidity result by L. Andersson and M. Dahl and generalizes the result itself.

Corollary 7. *Let (M, g) be a conformally compact spin manifold of dimension $n + 1$ whose Ricci tensor and scalar curvature satisfy*

$$\text{Ric}_g + ng \geq 0, \quad R_g + n(n + 1) = o(r^2)$$

for a geodesic defining function r and whose infinity is the sphere $\mathbb{S}^n$ endowed with its round conformal structure. Then (M, g) is isometric to the hyperbolic space $\mathbb{H}^{n+1}$ with constant curvature -1 .

Remark 6. Corollary 7 can be compared with the rigidity results [AD, Theorem 5.2], [Q, Theorem 1.1], [Wa2, Theorem 1], [An1, Theorem 3.3] and [An3, Theorem 2.5]. All of them require (M, g) to be a PE space. Instead, we only impose some weaker estimates implying that (M, g) is a WPE manifold. Moreover, the notions of asymptotical hyperbolicity and conformal roundness of the infinity are slightly different and stronger than ours in each case. The results by M. Anderson and J. Qing are the only ones which do not use the spinorial machinery.

Remark 7. It is worth noting that, in Theorem 6, the involved spin structures are crucial on several levels. According to our definition of conformally compact *spin* manifolds, the compact conformal fillings $\overline{M}$ that we consider must be spin compact manifolds with boundary. So the corresponding conformal infinities ∂M are spin as well and, when these ∂M are not simply-connected (thus they can have more than one spin structure), we always refer to the one *induced* from that of $\overline{M}$. Then, Theorem 6 above applies only when this induced spin structure has a non-trivial twistor spinor. Note that, in [AD, Section 6], when such a spinor field exists the induced spin structure is called *trivial spin structure*.

Besides these considerations about spin structures, it is important to note as well that once the spin structure is fixed, the existence of real Killing spinors also depends on the Riemannian metric. For example, in Corollary 7 we did not refer at all to the spin structure that we consider on the spherical boundary at infinity since $\mathbb{S}^n$ supports a unique spin structure, being simply connected. So, the induced spin structure from any spin conformal compactification has to be this one. Instead, it is necessary to assume that the conformal structure is the *round* one. In fact, for each (non necessarily round) conformal structure on $\mathbb{S}^3$, in the same component that the round one, there exists a conformal Einstein compactification with topology B^4 (see [An2, p. 6] or [An2, Theorem B], for instance).

If the prescribed conformal structure is not the round one, then Corollary 7 cannot apply because the associated conformal structures do not support twistor spinors and indeed it is obvious that the Einstein metric on the compactification supplied by the M. Anderson theorems cannot be the hyperbolic one. Moreover, some non-round conformal structures on the sphere $\mathbb{S}^3$ admit compact conformal fillings which are not topologically the ball B^4 . More interestingly, there are manifolds which do not admit any spin structure. For example, if we think of $\mathbb{S}^3$ as a geodesic sphere in the complex projective plane $\mathbb{C}P^2$ endowed with its usual Fubini-Study metric and denote by B^4 its inner open geodesic ball, the exterior $\mathbb{C}P^2 - \overline{B}^4$ can be endowed with an Einstein metric with negative scalar curvature belonging to the so-called AdS Taub-bolt family. This AdS Taub-bolt space gives a compact conformal filling of the 3-sphere and the conformal structure induced on this 3-sphere at infinity is a Berger metric on $\mathbb{S}^3$ (see [An1, p. 12, Table]). However, since this compactification is topologically $\mathbb{C}P^2 - B^4$, it does not admit any spin structure (see for more details [CEJM, p. 4, note 4] or [AC, p. 2059, note 7]).

Besides the uniqueness of the hyperbolic space when the prescribed conformal infinity is a round sphere, Theorem 6 provides a non-existence result when the infinity is conformal to a non-spherical compact spin manifold carrying non-trivial real Killing spinors. In fact, the result by Lichnerowicz ([Li], [BFGK, Theorem 10]) mentioned above implies that the conformal structure of such a manifold must support non-trivial twistor spinors. In the simply-connected case, C. Bär determined in [Bä1] all these spin manifolds (see also [A] and [Bä2, Theorem 3] for non-simply-connected space forms). Using his classification, we obtain the following assertion.

Corollary 8. *There is no conformally compact spin manifolds (M, g) with*

$$\text{Ric}_g + ng \geq 0, \quad R_g + n(n+1) = o(r^2),$$

and boundaries at infinity conformal to non-spherical compact spin manifolds whose induced spin structure admits a non-trivial real Killing spinor. In the simply-connected case, they are Einstein-Sasaki manifolds, 3-Sasaki manifolds, nearly-Kähler non-Kähler 6-manifolds and 7-manifolds carrying nice 3-forms. In the non-simply-connected case, they include, for example, all the round quotients $\mathbb{S}^3/\Gamma$, where $\Gamma \subset \mathbb{S}^3$ is any of its finite subgroups, and real projective spaces $\mathbb{R}P^n$ with dimensions $n = 8k + 3$ or $n = 8k + 7$, $k \geq 0$.

Remark 8. As in Remark 7, in Corollary 8 it is important to pay attention to the involved spin structures. For instance, the quotient $\mathbb{S}^3/\mathbb{Z}_3$ has fundamental group $\mathbb{Z}_3$, then $H^1(\mathbb{S}^3/\mathbb{Z}_3, \mathbb{Z}_2) \cong \text{Hom}(\mathbb{Z}_3, \mathbb{Z}_2) = \{0\}$ and so this 3-manifold has a unique spin structure. Endowed with the round metric, it admits non-trivial Killing spinors ([Bä2, Theorem 3]). According to Corollary 8, it has no conformal Einstein *spin* compactifications. Yet, if E_3 denotes the complex line bundle over $\mathbb{S}^2$ with Chern number 3, there is an AdS Taub-bolt metric on E_3 whose conformal infinity is the round $\mathbb{S}^3/\mathbb{Z}_3$. We already mentioned in Remark 7 that the family of AdS Taub-bolt spaces do not include spin manifolds ([AC, CEJM], [An1, p. 12, Table]). Note that, this same space $\mathbb{S}^3/\mathbb{Z}_3$ admits an Einstein spin compact conformal filling, namely the Calderbank-Singer space X_3 , a resolution of the orbifold $\mathbb{C}^2/\mathbb{Z}_3$, but in this case the inherited conformal structure is not the round one (see [An1, Remark 3.4 i])).

On the other hand, real projective spaces $\mathbb{R}P^n$ with $n = 8k + 3$ or $n = 8k + 7$, $k \geq 0$, endowed with any of their two spin structures (note that $H^1(\mathbb{R}P^n, \mathbb{Z}_2) \cong \mathbb{Z}_2$), and with the round metric, carry non-trivial Killing spinors (see [Bä2, Theorem 3], [Gi, p. 133]). By Corollary 8 they have no conformal Einstein spin compactifications. However, the AdS Taub-bolt space E_2 is a non-spin compactification of the round $\mathbb{R}P^3$ and the Calderbank-Singer space X_2 is a spin compactification of a non-round $\mathbb{R}P^3$.

Theorem 6 also provides a non-existence result when the boundary at infinity is conformal to a compact spin manifold admitting non-trivial parallel spinors. In fact, the conformal covariance of the twistor operator implies that, for a fixed spin structure, the existence of twistor spinors for a given conformal structure implies the existence of parallel spinors with respect to a (Ricci-flat) metric in this conformal class. Taking into account that the product of spin manifolds with non-trivial parallel spinors is also another spin manifold of this same type, and that M. Wang determined in [W1, W2] all irreducible spin manifolds carrying parallel spinors (see also [Gi, Theorem A.4.2]), we obtain the following:

Corollary 9. *There is no conformally compact spin manifolds (M, g) with*

$$\text{Ric}_g + ng \geq 0, \quad R_g + n(n+1) = o(r^2),$$

and boundaries at infinity conformal to compact spin manifolds admitting a non-trivial parallel spinor. In the simply-connected case, they are just Calabi-Yau manifolds, hyper-Kähler manifolds, G_2 7-manifolds,

Spin(7) 8-manifolds and all the Riemannian products made from them. In the non-simply-connected case, they include, for example, all flat tori $\mathbb{T}^n$ with the trivial spin structure and all Riemannian products of trivial flat tori $\mathbb{T}^k$, $1 \leq k \leq n - 2$, with the examples of simply-connected manifolds.

Remark 9. As in Remark 8, spin structures considerations are essential. In fact, it is well-known that the so-called family of AdS toroidal black hole metrics (see [BMN] or [An1, Example 2.2, Remark 3.4 ii)]), constructed on the solid $(n + 1)$ -dimensional torus $B^2 \times \mathbb{T}^{n-1}$, are the unique compact conformal fillings of the flat torus $\mathbb{T}^n$. They are clearly spin compactifications, but the spin structure inherited on the boundary at infinity is not the trivial one, since they are constructed by gluing on a 2-disc onto a simple closed geodesic of the flat $\mathbb{T}^n$.

5. SUPERSYMMETRIES ON THE BOUNDARY AND SPIN CONFORMAL COMPACTIFICATIONS WITH SINGULARITIES

If we compare Theorem 3.3 in [An1] (see also Theorem 2.5 in [An3]) with the main Theorem 6 of this paper, we see that, in the spirit of the AdS/CFT correspondence, all the conformal symmetries on the boundary of a conformally compact Einstein manifold come from Riemannian symmetries of its compact conformal fillings. However, when we require these compactifications to be spin in order to have supergravity models, the conformal supersymmetries on the boundary at infinity yield to non-existence of spin compactifications, apart from the spherical case. Yet, the proof of Theorem 6 can be read as a supersymmetric version of Anderson's result under weaker conditions on the Ricci curvature.

Indeed, its proof shows that each twistor spinor on the boundary comes from an imaginary Killing spinor on the bulk manifold (see Remark 5). But a careful reading of this proof makes also manifest that, if we have conformal boundaries with a unique component and exclude conical or cusp singularities, the only compactifications supporting these imaginary Killing spinors are hyperbolic spaces. This is why we finish this paper with some examples showing that certain boundaries at infinity carrying twistor spinors admit supergravity compact conformal fillings provided that we allow non-connectedness at the infinity and singularities.

We could say that conformal supersymmetries on a non-spherical conformal boundary yield supergravity compactifications with hyperbolic conical or cusps singularities or with two copies of the prescribed

boundary at infinity. Mathematically, these examples take the form of warped products and they are frequently utilized (see [AC, BMN]) in physics. On the other hand, a strong necessity of considering conformally compact Einstein metrics with singularities arises as well when one tries to understand the manifold structure of the space of these metrics with a given topology (see, for example, [An1, (3.6)]). In a forthcoming paper, we will study the uniqueness of the spin compactifications given by the examples below by adapting our methods in the presence of singularities.

Example 1. Consider again our model of conformally compact manifold, the Poincaré hyperbolic ball $\left(B^{n+1}, \frac{4|dx|^2}{(1-|x|^2)^2}\right)$. Using polar coordinates $x = rp$, with $r \in]0, 1]$ and $p \in \mathbb{S}^n$, we have that the hyperbolic metric of constant sectional curvature -1 takes the form

$$g = \left(\frac{2}{1-r^2}\right)^2 (dr^2 + r^2\gamma_{\mathbb{S}^n}),$$

where $\gamma_{\mathbb{S}^n}$ stands for the unit round metric on the sphere. If we make the change of variables given by $s = \ln \frac{1+r}{1-r} \in \mathbb{R}^+$, we obtain

$$g = ds^2 + (\sinh^2 s)\gamma_{\mathbb{S}^n}.$$

These two expressions for the Poincaré metric are valid only on the punctured ball $B^{n+1} - \{0\} \cong]0, 1[\times \mathbb{S}^n \cong \mathbb{R}^+ \times \mathbb{S}^n$, although they are smoothly extendible to the origin. This latter is an example of the so-called *warped Riemannian products* (see, for instance, [Be, O’N, K]). In general, if $I \subset \mathbb{R}$ is an open interval, (P, γ) a Riemannian n -manifold and $f \in C^\infty(I)$ is a positive function, we will say that the $(n+1)$ -dimensional Riemannian manifold $(I \times P, g = ds^2 + f(s)^2\gamma)$ is the product of I and P warped by means of the function f . If we only consider, for the sake of simplicity, conformally compact Einstein manifolds, using the form of the Ricci tensor on a warped product (see [K, Lemma 4]), we must restrict ourselves to warping functions f satisfying $f'' - f = 0$. With this choice, we ensure that $\text{Ric}_g(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}) = -n$ at each point of $I \times P$. Taking now into account the values of Ric_g on the directions orthogonal to the vector field $\frac{\partial}{\partial s}$, that is, directions tangent to P , we conclude that there are essentially three types of warped products which eventually may produce Einstein manifolds with scalar curvature $-n(n+1)$. The first one is the *hyperbolic cone* on a given compact Riemannian manifold (P, γ) , given by

$$(\mathbb{R}^+ \times P, g = ds^2 + (\sinh^2 s)\gamma).$$

It is immediate to see again from [K, Lemma 4], for instance, that we have for the directions tangent to this cone and perpendicular to the radial direction

$$\text{Ric}_g + ng = \text{Ric}_\gamma - (n-1)\gamma.$$

So, we will assume that (P, γ) is an Einstein manifold with scalar curvature $n(n-1)$. We can also see in [K] that the smooth function defined on $I \times P$ by $u(s, p) = \cosh s$, for each $s \in \mathbb{R}^+$ and $p \in P$, is a solution to the Obata type equation $\nabla du = ug$. Defining a new variable $t \in]0, 1]$ by $t = \tanh \frac{s}{2}$, we obtain

$$\bar{g} = \left(\frac{1}{1 + \cosh s} \right)^2 g = dt^2 + t^2 \gamma.$$

The conical metric $dt^2 + t^2 \gamma$ obviously extends to $[0, 1] \times P$, a compact manifold with boundary $\{1\} \times P \cong P$ and a conical singularity at $t = 0$. This singularity is removable if and only if (P, γ) is the round unit n -sphere and, in this case, the corresponding hyperbolic cone is nothing but the $(n+1)$ -dimensional hyperbolic space (see [Be, p. 269, Lemma 9.114]).

Example 2. The second type of warped product relevant for our purposes is the so-called *hyperbolic cusp* on a compact Riemannian manifold (P, γ) , given by

$$(\mathbb{R} \times P, g = ds^2 + e^{2s} \gamma).$$

For these cusps, the Ricci curvature is also $-n$ along the radial direction $\frac{\partial}{\partial s}$ and, for orthogonal directions tangent to P , we have

$$\text{Ric}_g + ng = \text{Ric}_\gamma.$$

Then, we will suppose that in this case (P, γ) is Ricci-flat. As in Example 1, we can see in [K] that $u(s, p) = e^s$ is a solution of $\nabla du = ug$. Defining a new variable $t \in \mathbb{R}^+$ as $t = e^s$, we obtain

$$\bar{g} = \frac{1}{e^{2s}} g = dt^2 + \gamma.$$

The cylindrical metric $dt^2 + \gamma$ clearly extends to $[0, +\infty[\times P$, which is a non-compact manifold with boundary $\{0\} \times P \cong P$ and one cylindrical end. So, the hyperbolic cusp has $(P, [\gamma])$ as a conformal infinity at $s = +\infty$, but it is not compactifiable at $s = -\infty$. Of course, the cusp singularity is always unremovable, although it has finite volume.

Example 3. The third and last case of warped product that we consider is of the form

$$(\mathbb{R} \times P, g = ds^2 + (\cosh^2 s)\gamma),$$

which also satisfies $\text{Ric}_g(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}) = -n$ and

$$\text{Ric}_g + ng = \text{Ric}_\gamma + (n-1)\gamma$$

for the directions tangent to P . Hence, to get an Einstein space, we must impose (P, γ) to be Einstein with scalar curvature $-n(n-1)$. In this case, the function $u(s, p) = \sinh s$ is a solution of $\nabla du = ug$, but it is not a positive function. In order to compactify, we define a new variable $t \in]0, \pi[$ by the relation $t = 2 \arctan e^s$. Then

$$\bar{g} = \left(\frac{1}{\cosh s} \right)^2 g = dt^2 + \gamma,$$

which is smoothly extendable to the compact manifold $[0, \pi] \times P$. Hence, its conformal infinity consists of two copies of $(P, [\gamma])$. Thus we obtain an example of compact conformal filling with two connected components at the conformal boundary (a *wormhole* in the physical jargon). Note that the Yamabe invariant for these two components is negative and that the existence of real Killing or parallel spinors implies non-negative constant scalar curvature. So, neither Theorem 2 nor Theorem 6 can be applied in this situation.

In the three examples, if we choose the n -dimensional compact Riemannian manifold (P, γ) to be spin and allow it to carry a non-trivial real Killing spinor, a parallel spinor or an imaginary Killing spinor (for a complete non-compact P), then the corresponding conical singularity, cusp or wormhole will be an $(n+1)$ -dimensional spin manifold carrying a non-trivial imaginary Killing spinor. We expect to characterize these three examples as the only total or partial conformal spin compactifications for a given infinity supporting an infinitesimal supersymmetry.

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