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Abstract

We investigate here the nonlinear elliptic equations $-\Delta u = |x|^\alpha e^u$ and $-\Delta u = |x|^\alpha |u|^{p-1}u$ with $\alpha > -2$, $p > 1$ and $N \geq 2$. In particular, we prove some Liouville type theorems for weak solutions with finite Morse index in the low dimensional Euclidean spaces or half spaces.

Keywords: Liouville theorem, Hénon equation, stability, finite Morse index solution
MSC: 35J60, 35B45.

1 Introduction

In this paper, we consider two Hénon type elliptic equations as

$$-\Delta u = |x|^\alpha e^u \quad \text{in} \quad \Omega \subset \mathbb{R}^N, \tag{1.1}$$

and

$$-\Delta u = |x|^\alpha |u|^{p-1}u \quad \text{in} \quad \Omega \subset \mathbb{R}^N, \tag{1.2}$$

where $\alpha > -2$, $p > 1$ and $N \geq 2$.

The study of stable solutions in the autonomous case, i.e. when $\alpha = 0$ has been studied recently, Farina classified completely in [11] all finite Morse index classical solutions of (1.2) in $\mathbb{R}^N$ for $1 < p < p_{JL}$, where $p_{JL} = p(N, 0)$ stands for the Joseph-Lundgren exponent (see [14] and Theorem 1.1 below). More precisely, the equation (1.2) with $\alpha = 0$ admits nontrivial classical solutions in $\mathbb{R}^N$ which are stable outside a compact set, if and only if $N \geq 3$, $p = \frac{N+2}{N-2}$; or $N \geq 11$ and $p \geq p_{JL}$.

For the exponential case, it is shown by Farina in [12] that $\Delta u + e^u = 0$ has no stable classical solution in $\mathbb{R}^N$ for $2 \leq N \leq 9$. He proved also that any classical solution of $\Delta u + e^u = 0$ with finite Morse index in $\mathbb{R}^2$ verifies $e^u \in L^1(\mathbb{R}^2)$, so it must be a solution classified by Chen & Li [2], that is

$$u(x) = \ln\left[\frac{32\lambda^2}{(4 + \lambda^2|x - x_0|^2)^2}\right] \quad \text{with} \quad \lambda > 0, \; x_0 \in \mathbb{R}^2.$$

Finally, when $N \geq 3$, Dancer & Farina proved in [5] that the equation (1.1) with $\alpha = 0$ admits classical entire solutions which are stable outside a compact set, if and only if $N \geq 10$.

A natural question is to ask if similar results can be observed for the nonautonomous case, i.e. when $\alpha \neq 0$. The equation (1.2) has been considered by Dancer, Du & Guo in [4], they proved

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Theorem 1.1 *Let $\alpha > -2$ and $u \in H^1_{loc} \cap L^\infty_{loc}(\mathbb{R}^N)$ be a stable solution of* (1.2) *in $\mathbb{R}^N$ with $1 < p < p(N, \alpha)$, where*

$$p(N,\alpha) = \begin{cases} \infty, & \text{if } N \le 10 + 4\alpha \\ \dfrac{(N-2)^2 - 2(\alpha+2)(\alpha+N) + 2\sqrt{(\alpha+2)^3(\alpha+2N-2)}}{(N-2)(N-4\alpha-10)}, & \text{if } N > 10 + 4\alpha. \end{cases}$$

Then $u \equiv 0$. On the other hand, for $N > 10 + 4\alpha$ and $p \ge p(N, \alpha)$, (1.2) *admits a family of stable positive radial solutions in $\mathbb{R}^N$.*

Dancer, Du & Guo have also studied *positive* solutions either in a punctured domain $\Omega \setminus \{0\}$ or in an exterior domain $\mathbb{R}^N \setminus \Omega$, they obtained interesting results on asymptotic behavior when $|x|$ tends to 0 or ∞, for solutions which are stable respectively near the origin or outside a compact set. Then they used these estimates and [1] to get some classification results.

In [10], Esposito studied the stability of solutions to the Hénon type equations (1.1), (1.2) in $\mathbb{R}^N$ and proved some Liouville results with $\alpha \ge 0$ and respectively bounded e^u or $|u|$. It is also worthy to mention that in [9, 10, 7], positive entire solutions of $\Delta u = |x|^\alpha u^{-p}$ $(p > 0)$, with finite Morse index or stable in an exterior domain are classified.

Although we borrow many ideas from the previous works, we try to handle the problems in more general setting by three folds.

- In all previous works for (1.1) and (1.2), the authors considered solutions with locally or globally bounded e^u or $|u|$. Here we deal with weak solutions which are not supposed *a priori* to be locally upper or lower bounded. For example, if $\alpha < 0$ and $0 \in \Omega$, any weak solution of (1.1) or (1.2) cannot be a classical solution, due to the singularity at the origin.

- We work with general $\alpha > -2$, and classify not only stable solution but also entire solution stable out of a compact set for (1.1), or with finite Morse index for (1.2) under suitable conditions on p, which were not considered in [4, 10].

- For the equation (1.2), we do not impose any sign condition for u and prove the fast decay behavior near 0 (resp. ∞) for weak solutions which are stable near the origin (resp. outside a compact set) with suitable exponent p. Finally we consider also finite Morse index solutions in the half space $\mathbb{R}^N_+$.

In order to state our results more accurately, let us precise the meaning of weak solution and recall some basic notions. For simplicity, we assume always that Ω is a regular domain in $\mathbb{R}^N$ and $f(x, \cdot) : \mathbb{R} \to \mathbb{R}$ is a C^1 function for almost every $x \in \Omega$.

Definition 1.2 *We say that u is a weak solution of $-\Delta u = f(x, u)$ in domain $\Omega \subset \mathbb{R}^N$ (bounded or not), if $u \in H^1_{loc}(\Omega)$ verifies $f(x, u) \in L^1_{loc}(\Omega)$ and*

$$\int_\Omega \Big[\nabla u \cdot \nabla \psi - f(x,u)\psi\Big] dx = 0, \quad \forall\, \psi \in C^1_c(\Omega). \tag{1.3}$$

Here and in the following $C^k_c(\Omega)$ denotes the set of C^k functions with compact support in Ω.

- Let u be a weak solution of $-\Delta u = f(x, u)$ in Ω. We say that u is stable if $\partial_u f(x, u) \in L^1_{loc}(\Omega)$ and

$$Q_u(\psi) := \int_\Omega \Big[|\nabla \psi|^2 - \partial_u f(x,u)\psi^2\Big] dx \ge 0 \quad \text{for all } \psi \in C^1_c(\Omega). \tag{1.4}$$

  Remark that when $\partial_u f(x, u) \ge 0$, (1.4) holds for any $\psi \in H^1_0(\Omega)$ by density argument. It is the case for the equations (1.1) and (1.2).

- The Morse index of a solution u, $\mathrm{ind}(u)$ is defined as the maximal dimension of all subspaces X of $C_c^1(\Omega)$ such that $Q_u(\psi) < 0$ for any $\psi \in X\backslash\{0\}$. Readily u is stable if and only if $\mathrm{ind}(u) = 0$.

- A weak solution u of $-\Delta u = f(x, u)$ in Ω is said to be stable outside a compact set $\mathcal{K}$, if $Q_u(\psi) \geq 0$ for any $\psi \in C_c^1(\Omega\backslash\mathcal{K})$. Recall that any finite Morse index solution u is stable outside a compact set $\mathcal{K} \subset \Omega$. Indeed, for $\ell = \mathrm{ind}(u) \geq 0$, there exists $X = \mathrm{span}\{\varphi_1, \cdots, \varphi_\ell\} \subset C_c^1(\Omega)$ such that $Q_u(\varphi) < 0$ for any $\varphi \in X\backslash\{0\}$, so $Q_u(\psi) \geq 0$ for all $\psi \in C_c^1(\Omega\backslash\mathcal{K})$, where $\mathcal{K} = \bigcup_j \mathrm{supp}(\varphi_j)$.

We should mention that when $\alpha = 0$, it is proved in [6] that any weak solution of (1.2) with finite Morse index and $p < p_{JL}$ is indeed in $C^2(\Omega)$, therefore all results for the special case with $\alpha = 0$ in (1.2) are well known thanks to [11]. Moreover, many other interesting results for solutions stable outside a compact set can be found in [11] for the autonomous case of equation (1.2).

So our work concerns only the *nonautonomous* case $\alpha \neq 0$, which is different. Furthermore, the restriction on $\alpha > -2$ is necessary, seeing the following nonexistence result.

Proposition 1.3 *For $\alpha \leq -2$,* (1.1) *admits no weak solution for any domain $\Omega \subset \mathbb{R}^N$ containing* 0*.*

It was proved in [4] that when $\alpha \leq -2$, (1.2) admits no *positive* solution over any punctured domain $B(0, R)\backslash\{0\}$. Here and after, $B(y, r)$ denotes the ball of center y and radius $r > 0$ in $\mathbb{R}^N$. However, as far as we are aware, it is not known if the condition $\alpha > -2$ is necessary to have a sign-changing weak solution to (1.2) around the origin.

From now on, we assume that $\alpha > -2$. Our main objective is to classify weak solutions of (1.1) or (1.2) in $\mathbb{R}^N$, which are stable outside a compact set or with finite Morse index.

Theorem 1.4 *Let $\alpha > -2$ and $\Omega = \mathbb{R}^N$. For $2 \leq N < 10 + 4\alpha$, there is no weak stable solution of* (1.1)*.*

Theorem 1.5 *Let $\alpha > -2$, $\Omega = \mathbb{R}^N$ and $2 < N < 10 + 4\alpha^-$ where $\alpha^- = \min(\alpha, 0)$. Then* (1.1) *has no weak solution which is stable outside a compact set. In particular, any weak solution of* (1.1) *in $\mathbb{R}^N$ has infinite Morse index if $2 < N < 10 + 4\alpha^-$.*

Theorem 1.4 is sharp. Indeed, for $N \geq 10 + 4\alpha$ and $\alpha > -2$, (1.1) possesses radial stable weak solutions in $\mathbb{R}^N$,

$$U(x) = -(2 + \alpha) \ln |x| + \ln[(2 + \alpha)(N - 2)]$$

where $|x|$ denotes the Euclidean norm. The stability of U is a direct consequence of the classical Hardy inequality,

$$\int_{\mathbb{R}^N} |\nabla\psi|^2 dx \geq \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{\psi^2}{|x|^2} dx, \quad \forall\, \psi \in H^1(\mathbb{R}^N). \tag{1.5}$$

But we do not know if Theorem 1.5 holds under the assumptions of Theorem 1.4, i.e. when $\alpha > 0$ and $10 \leq N < 10 + 4\alpha$.

When $N = 2$ and $\alpha > -2$, it is not difficult to see that any finite Morse index solution is indeed an energy solution, that is $|x|^\alpha e^u \in L^1(\mathbb{R}^2)$; Prajapat & Tarantello [16] have already classified all such solutions.

Theorem 1.6 *Let $\alpha > -2$, $\Omega = \mathbb{R}^2$ and u be a weak solution of* (1.1) *which is stable outside a compact set, then*

$$\int_{\mathbb{R}^2} |x|^\alpha e^u dx = 4\pi(\alpha + 2). \tag{1.6}$$

Furthermore, if $\alpha \notin 2\mathbb{N}$, we have $u(x) = U_*(\varepsilon x) + (\alpha + 2) \ln \varepsilon$ where $\varepsilon > 0$ and

$$U_*(x) = \ln 2 + 2 \ln \frac{\alpha + 2}{1 + |x|^{\alpha+2}}.$$

If $\alpha \in 2\mathbb{N}$, let θ be the angle of x in polar coordinates, $k = \frac{\alpha+2}{2}$ and

$$U_{**}(x) = 2 \ln \frac{2k}{1 + |x|^{2k} - 2|x|^k \cos(k\theta - \theta_0) \tanh \xi} + \ln \frac{2}{\cosh^2 \xi}$$

with $\xi, \theta_0 \in \mathbb{R}$. We have then $u(x) = U_{**}(\varepsilon x) + (\alpha + 2) \ln \varepsilon$ where $\varepsilon > 0$.

For equation (1.2), we have similar Liouville type results.

Theorem 1.7 *Let $\alpha > -2$, $N \geq 2$. The result of Theorem 1.1 holds true for weak solution of* (1.2) *in $\mathbb{R}^N$. Moreover, let u be a weak solution of* (1.2) *in $\mathbb{R}^N$ with finite Morse index. Assume that*

$$1 < p < p(N, \alpha^-) \quad \text{and} \quad p \neq \frac{N + 2 + 2\alpha}{N - 2}, \tag{1.7}$$

then $u \equiv 0$.

In [4, Theorems 1.3-1.4], Dancer, Du & Guo have considered *positive* solutions to (1.2) with finite Morse index, in punctured domains or in exterior domains under similar conditions on p. Using results in [1], it was proved that either these solutions are of fast decay, or up to a suitable scaling, they converge uniformly to a positive function on S^{N-1}, but they did not consider the Liouville type result for entire solutions of (1.2) in $\mathbb{R}^N$ with finite Morse index.

Theorem 1.7 is sharp for $\alpha \in (-2, 0]$. However, we don't know if it holds true for $\alpha > 0$, $p(N, 0) \leq p < p(N, \alpha)$ and $N > 10$. On the other hand, let $N \geq 3$, $p = \frac{N+2+2\alpha}{N-2}$ be the critical exponent and $\alpha > -2$, it is well known that (1.2) possesses radial positive entire solutions, given by

$$V(x) = \lambda^{\frac{N-2}{2}} \left(\frac{\sqrt{(N + \alpha)(N - 2)}}{1 + \lambda^{2+\alpha}|x|^{2+\alpha}} \right)^{\frac{N-2}{2+\alpha}}, \quad \lambda > 0. \tag{1.8}$$

We see that $|x|^\alpha V(x)^{p-1} = O(|x|^{-4-\alpha})$ when $|x| \to \infty$. As $\alpha > -2$, V is clearly stable outside a compact set of $\mathbb{R}^N$ by Hardy's inequality (1.5).

Finally we consider the half-space problem of (1.2) with the Dirichlet boundary condition.

Theorem 1.8 *Let $\alpha > -2$ and u be a weak solution of*

$$-\Delta u = |x|^\alpha |u|^{p-1} u \;\; \text{in} \;\; \mathbb{R}^N_+ = \{x \in \mathbb{R}^N, x_N > 0\}, \quad u = 0 \;\; \text{on} \;\; \partial\mathbb{R}^N_+.$$

Suppose that u has finite Morse index in $\mathbb{R}^N_+$. Then $u \equiv 0$ under the assumption (1.7).

Our proofs are based on some *a priori* estimations using the stability condition (1.4), in the spirit of Proposition 4 in [11] (see also Proposition 1.7 of [4] and Proposition 5 of [12]), but we need to be more careful with the weak solutions since they are not supposed to be bounded *a priori*. Another important ingredient to handle (1.2) is the asymptotic behavior of solutions near the origin and near infinity.

Theorem 1.9 *Let u be a weak solution of (1.2) in Ω containing 0. Suppose that u has finite Morse index and p satisfies*

$$1 < p < p(N, \alpha^-).$$

Then $u \in C(\Omega) \cap C^2(\Omega \setminus \{0\})$ and the following fast decay estimate holds:

$$\lim_{|x|\to 0} |x|^{1+\frac{2+\alpha}{p-1}} |\nabla u(x)| = 0. \tag{1.9}$$

For the fast decay estimate as $|x|$ goes to ∞, we need more restriction on the exponent p.

Theorem 1.10 *Suppose that u is a weak stable solution of (1.2) in $\mathbb{R}^N \setminus \mathcal{K}$ where $\mathcal{K} \subset \mathbb{R}^N$ is a compact set. Assume that $\alpha > -2$ and p satisfies*

$$\underline{p}(N, \alpha) < p < p(N, \alpha^-).$$

where

$$\underline{p}(N, \alpha) = \frac{(N-2)^2 - 2(\alpha+2)(\alpha+N) - 2\sqrt{(\alpha+2)^3(\alpha+2N-2)}}{(N-2)(N-4\alpha-10)}, \quad \textit{for all } N \geq 2.$$

Then we have

$$\lim_{|x|\to\infty} |x|^{\frac{2+\alpha}{p-1}} |u(x)| = \lim_{|x|\to\infty} |x|^{1+\frac{2+\alpha}{p-1}} |\nabla u(x)| = 0. \tag{1.10}$$

The estimates (1.10) were proved in [4, Theorem 1.6] only for positive solutions. We note that the Harnack type argument used in [4] cannot work with sign-changing solutions.

We will deal with equations (1.1), (1.2) respectively in sections 2 and 3, some further remarks and open questions are exposed in section 4. In the following, C or C_i denotes some generic positive constant.

2 Hénon equation with exponential nonlinearity

Consider $-\Delta u = |x|^\alpha e^u$. First we show the necessity to working with $\alpha > -2$ and a quick proof of Theorem 1.6, then we prove Theorems 1.4 and 1.5.

2.1 Nonexistence of weak solution for $\alpha \leq -2$

Arguing by contradiction, assume that $\alpha \leq -2$ and u is a weak solution of (1.1) with $0 \in \Omega$. Let $B(0, R) \subset \Omega$. Define v to be the average of u over spheres centered at the origin, i.e.

$$v(r) := \overline{u(r, \theta)} = \frac{1}{\omega_N} \int_{S^{N-1}} u(r, \theta) d\theta \quad \text{with } \omega_N = |S^{N-1}|.$$

By Jensen's inequality, there holds

$$-\Delta v = \overline{|x|^\alpha e^u} = r^\alpha \overline{e^u} \geq r^\alpha e^v,$$

which means that

$$-\frac{1}{r^{N-1}} (r^{N-1} v')' \geq r^\alpha e^v. \tag{2.1}$$

Note that

$$-r^{N-1} \omega_N v'(r) = -\int_{\partial B(0,r)} \frac{\partial v}{\partial \nu} d\sigma = -\int_{B(0,r)} \Delta v dx = \int_{B(0,r)} |x|^\alpha e^u dx > 0,$$

thus $v'(r) < 0$ for any $r \in (0, R)$ and v is decreasing in $(0, R)$. As $r^\alpha e^v \in L^1_{loc}(\Omega)$ and $0 \in \Omega$, we must have $\alpha > -N$. Furthermore, integrate (2.1) in $[r_1, r]$ with $0 < r_1 < r < R$,

$$-r^{N-1}v'(r) \geq \int_{r_1}^{r} s^{N-1+\alpha}e^{v(s)}ds - r_1^{N-1}v'(r_1) \geq \int_{r_1}^{r} s^{N-1+\alpha}e^{v(s)}ds.$$

Tending r_1 to 0, we get

$$-r^{N-1}v'(r) \geq \int_0^r s^{N-1+\alpha}e^{v(s)}ds \geq \frac{r^{N+\alpha}e^{v(r)}}{N+\alpha}. \tag{2.2}$$

Hence $-e^{-v}v' \geq Cr^{1+\alpha}$, which yields

$$e^{-v(r)} > e^{-v(r)} - e^{-v(r_1)} \geq C\int_{r_1}^{r} s^{1+\alpha}ds \quad \forall\, 0 < r_1 < r.$$

As $\alpha \leq -2$, a contradiction occurs by tending r_1 to 0 since the last term goes to ∞. So Proposition 1.3 is proved. $\square$

Remark 2.1 *We can remark that the above proof works for $-\Delta u = |x|^\alpha g(u)$ with a positive, convex and nondecreasing nonlinearity g.*

2.2 Two dimensional case

Here we prove Theorem 1.6. Assume that u is stable outside $B(0, R_0)$. Fix $\phi \in C_c^\infty(\mathbb{R})$ verifying $\phi(t) = 1$ if $|t| \leq 1$ and $\phi(t) = 0$ if $|t| \geq 2$. For any $R \geq 4R_0$, let $\psi_R(x) = \phi(R^{-1}|x|) - \phi(R_0^{-1}|x|)$, then $\mathrm{supp}(\psi_R) \subset B(0, R_0)^c$, ψ_R is fixed in $B(0, 2R_0)$ and

$$\psi_R \equiv 1 \text{ in } B(0, R) \setminus B(0, 2R_0), \quad |\nabla\psi_R(x)| \leq CR^{-1} \text{ in } B(0, R)^c.$$

Using ψ_R as the test test function in (1.4), there holds

$$\begin{aligned}\int_{B(0,R)\setminus B(0,2R_0)} |x|^\alpha e^u dx &\leq \int_{\mathbb{R}^2} |x|^\alpha e^u \psi_R^2 dx\\ &\leq \int_{\mathbb{R}^2} |\nabla\psi_R|^2 dx\\ &\leq \int_{B(0,2R_0)} |\nabla\psi_R|^2 dx + \int_{B(0,2R)\setminus B(0,R)} |\nabla\psi_R|^2 dx \leq C\end{aligned}$$

Tending R to ∞, we get $|x|^\alpha e^u \in L^1(\mathbb{R}^2)$. All conclusions follow straightforwardly from Theorem 1.1 in [16]. $\square$

2.3 Main technical tool

As already mentioned, our proof of Liouville type results is based on the estimate $Q_u(\psi) \geq 0$ with suitable test function. In fact, we have the following estimate which is an extension of result in [12].

Proposition 2.2 *Let Ω be a domain (bounded or not) in $\mathbb{R}^N$, $N \geq 2$. Let u be a weak and stable solution of (1.1) with $\alpha > -2$. Then for any integer $m \geq 5$ and any $\beta \in (0, 2)$, there exists $C > 0$ depending on m, α and β such that*

$$\int_\Omega |x|^\alpha e^{(2\beta+1)u}\psi^{2m}dx \leq C\int_\Omega |x|^{-2\beta\alpha}\left(|\nabla\psi|^2 + |\psi||\Delta\psi|\right)^{2\beta+1}dx, \tag{2.3}$$

for all functions $\psi \in C_c^\infty(\Omega)$ verifying $\|\psi\|_\infty \leq 1$.

Proof. We use some ideas in [12, 3], but we need to pay more attention with weak solution. As u is not assumed to be bounded, $e^{\beta u}\varphi$ is not, *a priori*, a licit test function (for any $\beta > 0$), even with $\varphi \in C_c^\infty(\Omega)$. Our idea is to consider suitable truncations of $e^{\beta u}$ and proceed as in [3]. Let $\beta > 0$, $k \in \mathbb{N}$, $k \geq \beta^{-1}$ and

$$\zeta_k(t) = \begin{cases} e^{\beta t}, & \text{if } t \leq k \\ \frac{e^{\beta k}}{k} t, & \text{if } t \geq k. \end{cases}$$

We choose also another Lipschitz function η_k and a differentiable function ξ_k such that $\eta_k' = \zeta_k'^2$, $\xi_k' = \eta_k$ in $\mathbb{R}$. Let

$$\eta_k(t) = \begin{cases} \frac{\beta}{2} e^{2\beta t}, & \text{if } t \leq k \\ \frac{e^{2\beta k}}{k^2}(t-k) + \frac{\beta}{2} e^{2\beta k}, & \text{if } t \geq k \end{cases}$$

and

$$\xi_k(t) = \begin{cases} \frac{e^{2\beta t}}{4}, & \text{if } t \leq k \\ \frac{e^{2\beta k}}{2k^2}(t-k)^2 + \frac{\beta}{2} e^{2\beta k}(t-k) + \frac{e^{2\beta k}}{4}, & \text{if } t \geq k. \end{cases}$$

Since $u \in H^1_{loc}(\Omega)$, clearly $\zeta_k(u), \eta_k(u) \in H^1_{loc}(\Omega)$ for any $k \in \mathbb{N}$.

Applying now (1.4) with the test function $\zeta_k(u)\varphi$, where $\varphi \in C_c^\infty(\Omega)$, we get

$$\begin{aligned} \int_\Omega |x|^\alpha e^u \zeta_k^2(u)\varphi^2 dx &\leq \int_\Omega |\nabla(\zeta_k(u)\varphi)|^2 dx \\ &= \int_\Omega |\nabla(\zeta_k(u))|^2 \varphi^2 dx + \int_\Omega \zeta_k^2(u)|\nabla\varphi|^2 dx - \int_\Omega \frac{\zeta_k^2(u)}{2}\Delta(\varphi^2)dx. \end{aligned}$$

So $|x|^\alpha e^u \zeta_k^2(u) \in L^1_{loc}(\Omega)$. On the other hand,

$$\begin{aligned} \int_\Omega |\nabla(\zeta_k(u))|^2\varphi^2 dx = \int_\Omega \zeta_k'^2(u)|\nabla u|^2\varphi^2 dx &= \int_\Omega (\nabla\eta_k(u)\cdot\nabla u)\varphi^2 dx \\ &= \int_\Omega \nabla u \cdot \nabla\left(\eta_k(u)\varphi^2\right) dx - \int_\Omega \eta_k(u)\nabla u\cdot\nabla(\varphi^2)dx \\ &= \int_\Omega |x|^\alpha e^u \eta_k(u)\varphi^2 dx - \int_\Omega \eta_k(u)\nabla u\cdot\nabla(\varphi^2)dx \\ &= \int_\Omega |x|^\alpha e^u \eta_k(u)\varphi^2 dx + \int_\Omega \xi_k(u)\Delta(\varphi^2)dx. \end{aligned}$$

For the third line, we used the following argument: as $|x|^\alpha e^u > 0$, (1.3) is valid for any $\psi \in H^1_0(\Omega)$ by density argument. The above estimates imply then

$$\begin{aligned} &\int_\Omega |x|^\alpha e^u \zeta_k^2(u)\varphi^2 dx \\ &\leq \int_\Omega |x|^\alpha e^u \eta_k(u)\varphi^2 dx + \int_\Omega \zeta_k^2(u)|\nabla\varphi|^2 dx + \int_\Omega \left[\xi_k(u) - \frac{\zeta_k^2(u)}{2}\right]\Delta(\varphi^2)dx. \end{aligned}$$

Moreover, direct calculation shows that

$$\eta_k(t) \leq \left(\frac{\beta}{2} + \frac{1}{4k}\right)\zeta_k^2(t) \quad \text{in } \mathbb{R}.$$

Therefore

$$\left(1 - \frac{\beta}{2} - \frac{1}{4k}\right)\int_\Omega |x|^\alpha e^u \zeta_k^2(u)\varphi^2 dx \leq \int_\Omega \zeta_k^2(u)|\nabla\varphi|^2 dx + \int_\Omega \left[\xi_k(u) - \frac{\zeta_k^2(u)}{2}\right]\Delta(\varphi^2)dx. \quad (2.4)$$

Set now $\varphi = \psi^m$ with $\psi \in C_c^\infty(\Omega)$ satisfying $\|\psi\|_\infty \le 1$ and $m \in \mathbb{N}^*$. Using $\zeta_k(u) \le e^{\beta u}$ and Hölder's inequality, there holds

$$\begin{aligned}
\int_\Omega \zeta_k^2(u)|\nabla\varphi|^2 dx &= m^2 \int_\Omega \zeta_k^2(u)\psi^{2(m-1)}|\nabla\psi|^2 dx \\
&\le C \int_\Omega \left[e^u \zeta_k^2(u)\right]^{\frac{2\beta}{2\beta+1}} \psi^{2(m-1)}|\nabla\psi|^2 dx \\
&\le C \left(\int_\Omega |x|^\alpha e^u \zeta_k^2(u)|\psi|^{\frac{(m-1)(2\beta+1)}{\beta}} dx\right)^{\frac{2\beta}{2\beta+1}} \left(\int_\Omega |x|^{-2\beta\alpha}|\nabla\psi|^{4\beta+2} dx\right)^{\frac{1}{2\beta+1}}.
\end{aligned}$$

Take $m \ge 5$ so that $(m-1)(2\beta+1) \ge 2m\beta$ for any $\beta \in (0,2)$. As $\|\psi\|_\infty \le 1$, we obtain

$$\int_\Omega \zeta_k^2(u)|\nabla\varphi|^2 dx \le C \left(\int_\Omega |x|^\alpha e^u \zeta_k^2(u)\psi^{2m} dx\right)^{\frac{2\beta}{2\beta+1}} \left(\int_\Omega |x|^{-2\beta\alpha}|\nabla\psi|^{4\beta+2} dx\right)^{\frac{1}{2\beta+1}}. \tag{2.5}$$

Furthermore, for any $\beta \in (0,2)$ there exists $C > 0$ depending only on β such that

$$e^{\frac{2\beta}{2\beta+1}u} \ge C e^{\frac{2\beta}{2\beta+1}k}\left[1 + (u-k)^2\right] \quad \forall\, u \ge k,$$

because $e^t \ge 1 + t + \frac{t^2}{2}$ for $t \ge 0$. We deduce then for any $\beta \in (0,2)$, there exists $C > 0$ (independent of $k \in \mathbb{N}^*$) satisfying

$$\left|\xi_k(u) - \frac{\zeta_k^2(u)}{2}\right| \le C\left[e^u \zeta_k^2(u)\right]^{\frac{2\beta}{2\beta+1}} \quad \text{in } \mathbb{R}.$$

As $\Delta(\varphi^2) = 2m\psi^{2m-1}\Delta\psi + 2m(2m-1)\psi^{2m-2}|\nabla\psi|^2$, proceeding as above (see also [11]), fix $m \ge 5$ and applying once again Hölder's inequality, we get

$$\begin{aligned}
&\left|\int_\Omega \left[\xi_k(u) - \frac{\zeta_k^2(u)}{2}\right]\Delta(\varphi^2) dx\right| \\
&\le C\left(\int_\Omega |x|^\alpha e^u \zeta_k^2(u)\psi^{2m} dx\right)^{\frac{2\beta}{2\beta+1}} \left[\int_\Omega |x|^{-2\beta\alpha}\left(|\nabla\psi|^2 + |\psi||\Delta\psi|\right)^{2\beta+1} dx\right]^{\frac{1}{2\beta+1}}.
\end{aligned} \tag{2.6}$$

Combining (2.4)-(2.6),

$$\begin{aligned}
&\left(1 - \frac{\beta}{2} - \frac{1}{4k}\right)\int_\Omega |x|^\alpha e^u \zeta_k^2(u)\psi^{2m} dx \\
&\le C\left(\int_\Omega |x|^\alpha e^u \zeta_k^2(u)\psi^{2m} dx\right)^{\frac{2\beta}{2\beta+1}} \left[\int_\Omega |x|^{-2\beta\alpha}\left(|\nabla\psi|^2 + |\psi||\Delta\psi|\right)^{2\beta+1} dx\right]^{\frac{1}{2\beta+1}},
\end{aligned}$$

which means that there exists $C > 0$ independent of k such that

$$\int_\Omega |x|^\alpha e^u \zeta_k^2(u)\psi^{2m} dx \le C\int_\Omega |x|^{-2\beta\alpha}\left(|\nabla\psi|^2 + |\psi||\Delta\psi|\right)^{2\beta+1} dx,$$

provided $1 - \frac{\beta}{2} - \frac{1}{4k} > \delta > 0$. Fix $\beta \in (0,2)$, tending $k \to \infty$, the proof of (2.3) is completed by the monotone convergence Theorem. $\square$

Remark 2.3 *In [19], the author considered the regularity of weak stable solutions for* (1.1) *with* $\alpha = 0$. *He obtained higher integrability similar to* (2.3) *by using the simple cut-off function* $u_k = \min(k, u)$. *However, several arguments as the iteration process in Step 3 of the proof for [19, Lemma 2.1], are not valid for the nonautonomous equation.*

2.4 Proof of Theorem 1.4

Suppose that (1.1) admits a weak and stable solution u with $\Omega = \mathbb{R}^N$ and $N < 10 + 4\alpha$. Fix $m \geq 5$ and choose $\beta \in (0, 2)$ such that $N - 2(2\beta + 1) - 2\beta\alpha < 0$.

For every $R > 0$, consider the function $\phi_R(x) = \phi\left(R^{-1}|x|\right)$, where $\phi \in C_c^\infty(\mathbb{R})$ satisfies $0 \leq \phi \leq 1$ in $\mathbb{R}$, $\phi(t) = 1$ if $|t| \leq 1$ and $\phi(t) = 0$ if $|t| \geq 2$. Applying Proposition 2.2 with $\psi = \phi_R$,

$$\int_{B(0,R)} |x|^\alpha e^{(2\beta+1)u} dx \leq \int_{\mathbb{R}^N} |x|^\alpha e^{(2\beta+1)u} \phi_R^{2m} dx \leq C R^{N-2(2\beta+1)-2\beta\alpha}, \quad \forall\, R > 0.$$

Taking $R \to \infty$, we have

$$\int_{\mathbb{R}^N} |x|^\alpha e^{(2\beta+1)u} dx = 0,$$

which is impossible, so we are done. □

2.5 Proof of Theorem 1.5

We argue always by contradiction. Suppose that (1.1) admits a weak solution u which is stable outside a compact set $\mathcal{K}$ in $\mathbb{R}^N$. There exists $R_0 > 0$ such that $\mathcal{K} \subset B(0, R_0)$, therefore we can apply Proposition 2.2 with $\Omega = \mathbb{R}^N \backslash \overline{B(0, R_0)}$. We claim:

- For any $\beta \in (0, 2)$ and any $R > 2R_0$, it holds

$$\int_{2R_0<|x|<R} |x|^\alpha e^{(2\beta+1)u} dx \leq A + BR^{N-2(2\beta+1)-2\beta\alpha} \tag{2.7}$$

  where $A, B > 0$ are independent of R.

- For any $\beta \in (0, 2)$ and any $B(y, 2r) \subset \mathbb{R}^N \setminus \overline{B(0, R_0)}$, it holds

$$\int_{B(y,r)} |x|^\alpha e^{(2\beta+1)u} dx \leq C r^{N-2(2\beta+1)-2\beta\alpha}, \tag{2.8}$$

  where $C > 0$ is independent of r and y.

For $R > 2R_0$, let ϕ and ϕ_R be as in the previous proof and define $\psi_R = \phi_R - \phi_{R_0}$. Notice that $\psi_R \in C_c^\infty(\mathbb{R}^N \backslash \mathcal{K})$ and $0 \leq \psi_R \leq 1$ in $\mathbb{R}^N$ and ψ_R is a fixed function η_0 in $B(0, 2R_0)$. Hence Proposition 2.2 (applied with $m = 5$ and ψ_R) implies that for all $R > 2R_0$,

$$\begin{aligned}
\int_{\{2R_0<|x|<R\}} |x|^\alpha e^{(2\beta+1)u} dx &\leq \int_{\mathbb{R}^N\backslash\mathcal{K}} |x|^\alpha e^{(2\beta+1)u} \psi_R^2 dx \\
&\leq C \int_{\mathbb{R}^N\backslash\mathcal{K}} |x|^{-2\beta\alpha} \left(|\nabla\psi_R|^2 + |\psi_R||\Delta\psi_R|\right)^{2\beta+1} dx \\
&\leq C \int_{R_0\leq|x|\leq 2R_0} |x|^{-2\beta\alpha} \left(|\nabla\eta_0|^2 + \eta_0|\Delta\eta_0|\right)^{2\beta+1} dx \\
&\quad + C \int_{R\leq|x|\leq 2R} |x|^{-2\beta\alpha} \left(|\nabla\psi_R|^2 + |\psi_R||\Delta\psi_R|\right)^{2\beta+1} dx \\
&\leq A + BR^{N-2(2\beta+1)-2\beta\alpha}.
\end{aligned}$$

As the constants A, B depend only on η_0, R_0, ϕ and β, we get the claim (2.7). Using Proposition 2.2 with the test function $\phi_r(x - y)$, we obtain easily the estimate (2.8).

Consider $\Gamma(\beta) = N - 4\beta - 2 - 2\beta\alpha^-$. As $3 \leq N < 10 + 4\alpha^-$, $\Gamma(0) > 0$ and $\Gamma(2) < 0$, so there exist $\beta_1 \in (0,2)$ and $\varepsilon_0 \in (0,2)$ such that

$$2\beta_1 + 1 \geq 2\beta_1 + 1 + 2\beta_1\alpha^- > \theta := \frac{N}{2-\varepsilon_0} > \frac{N}{2}.$$

Let $|y| > 4R_0$ and $R = \frac{|y|}{4}$, so $B(y,2R) \subset \mathbb{R}^N \setminus \overline{B(0,R_0)}$. By Hölder's inequality and (2.8),

$$\begin{aligned}
\int_{B(y,R)} (|x|^\alpha e^u)^\theta \, dx &\leq \left(\int_{B(y,R)} |x|^\alpha e^{(2\beta_1+1)u} dx\right)^{\frac{\theta}{2\beta_1+1}} \left(\int_{B(y,R)} |x|^{\frac{2\beta_1\alpha\theta}{2\beta_1+1-\theta}} dx\right)^{\frac{2\beta_1+1-\theta}{2\beta_1+1}} \\
&\leq C\left(R^{N-2(2\beta_1+1)-2\beta_1\alpha}\right)^{\frac{\theta}{2\beta_1+1}} \left(R^{N+\frac{2\beta_1\alpha\theta}{2\beta_1+1-\theta}}\right)^{\frac{2\beta_1+1-\theta}{2\beta_1+1}} \\
&= CR^{N-2\theta},
\end{aligned}$$

that is,

$$\int_{B(y,R)} (|x|^\alpha e^u)^\theta \, dx \leq CR^{N-2\theta}, \quad \forall\, |y| > 4R_0 \text{ and } R = \frac{|y|}{4}. \tag{2.9}$$

We claim now

$$\lim_{|x|\to+\infty} |x|^{2+\alpha} e^{u(x)} = 0. \tag{2.10}$$

To prove that, we need a well-known result of Serrin in [18] (see also Theorem 7.1.1 in [17]):

Lemma 2.4 *Let $\theta = \frac{N}{2-\varepsilon_0}$, $\varepsilon_0 \in (0,2)$, $q \in (1,\infty]$ and $\delta > 0$. For any weak solution of $-\Delta\eta = a(x)\eta$ in $B(y,2R) \subset \mathbb{R}^N$, if $R^{\varepsilon_0}\|a(x)\|_{L^\theta(B(y,2R))} \leq \delta$, there holds*

$$\|\eta\|_{L^\infty(B(y,R))} \leq CR^{-\frac{N}{q}} \|\eta\|_{L^q(B(y,2R))} \tag{2.11}$$

where C is a constant depending only on N, q, θ and δ.

Moreover, the estimate (2.11) holds also for any weak nonnegative function verifying $-\Delta\eta \leq a(x)\eta$ in $B(y,2R) \subset \mathbb{R}^N$ assuming that $R^{\varepsilon_0}\|a(x)\|_{L^\theta(B(y,2R))} \leq \delta$.

Set

$$\beta_2 = \frac{N-2}{2(2+\alpha)}, \quad \lambda = \frac{2\beta_2+1}{2} = \frac{N+\alpha}{2(2+\alpha)} > 0 \quad \text{and} \quad w = e^{\lambda u}.$$

Then $\beta_2 \in (0,2)$ since $3 \leq N < 10 + 4\alpha$ and $N - 2(2\beta_2+1) - 2\beta_2\alpha = 0$. Take $\beta = \beta_2$ in (2.7) and tending R to ∞, we obtain

$$\int_{|x|\geq 2R_0} |x|^\alpha w^2 dx < \infty. \tag{2.12}$$

We have also

$$-\Delta w - \lambda|x|^\alpha e^u w = -\lambda^2 w|\nabla u|^2 \leq 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N).$$

Let $|y| > 8R_0$ and $R = \frac{|y|}{8}$. Using the estimate (2.9), as $\theta = \frac{N}{2-\varepsilon_0}$,

$$R^{\varepsilon_0}\|\lambda|x|^\alpha e^u\|_{L^\theta(B(y,2R))} \leq \lambda R^{\varepsilon_0}\left(CR^{N-2\theta}\right)^{\frac{1}{\theta}} = C'R^{\varepsilon_0+\frac{N}{\theta}-2} = C'. \tag{2.13}$$

Applying Serrin's result with $\eta = w$, $a(x) = \lambda|x|^\alpha e^u$, $q = 2$ and $\delta = C'$ of (2.13), by (2.12),

$$w(y) \leq CR^{-\frac{N}{2}}\|w\|_{L^2(B(y,2R))} \leq CR^{-\frac{N}{2}}R^{-\frac{\alpha}{2}}\||x|^{\frac{\alpha}{2}}w\|_{L^2(B(y,2R))} = o\left(R^{-\frac{N+\alpha}{2}}\right) \quad \text{as } |y| \to \infty.$$

Consequently

$$e^{u(y)} = w(y)^{\frac{1}{\lambda}} = o\left(R^{-\frac{N+\alpha}{2\lambda}}\right) = o\left(R^{-2-\alpha}\right) \quad \text{as } |y| \to \infty,$$

hence the claim (2.10) holds true.

To finish the proof, consider v, the average of u over spheres. Fix $M > 0$ large satisfying $\alpha + 2 - \frac{2}{(N-2)M} > 0$ (recall that $\alpha > -2$). By (2.10), there exists $R_M > 0$ such that

$$-\Delta v(r) = \overline{r^\alpha e^u} \le \frac{1}{Mr^2}, \quad \forall\, r \ge R_M.$$

Integrating from R_M to r, we deduce then

$$v'(r) \ge -\frac{C}{r^{N-1}} - \frac{1}{(N-2)Mr}, \quad \forall\, r \ge R_M.$$

As $N \ge 3$, there exists $R' > R_M$ such that

$$v'(r) \ge -\frac{2}{(N-2)Mr}, \quad \forall\, r \ge R'.$$

Integrating on $[R', r]$, we get

$$r^{2+\alpha} e^{v(r)} \ge C r^{\alpha+2-\frac{2}{(N-2)M}}, \quad \forall\, r \ge R',$$

which yields

$$\sup_{|x|=r} \left(|x|^{2+\alpha} e^{u(x)}\right) = r^{2+\alpha} \sup_{|x|=r} e^{u(x)} \ge r^{2+\alpha} e^{v(r)} \ge C r^{\alpha+2-\frac{2}{(N-2)M}} \to \infty,$$

which contradicts (2.10), the proof is completed. $\square$

3 Hénon equation with power growth

Here we consider the equation (1.2) in $\mathbb{R}^N$ or $\mathbb{R}^N_+$. As for Theorem 1.4, the basic argument is always the *a priori* estimates resulting from the stability condition (1.4).

Proposition 3.1 *Let Ω be a domain (bounded or not) in $\mathbb{R}^N$, $N \ge 2$. Let u be a weak stable solution of (1.2) with $p > 1$ and $\alpha > -2$. Then for any $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1)$ and any integer $m \ge \max\left(\frac{p+\gamma}{p-1}, 2\right)$, there holds*

$$\int_\Omega \left(|\nabla(|u|^{\frac{\gamma-1}{2}} u)|^2 + |x|^\alpha |u|^{\gamma+p}\right) |\psi|^{2m} dx \le C \int_\Omega |x|^{-\frac{(\gamma+1)\alpha}{p-1}} \left(|\nabla \psi|^2 + |\psi||\Delta\psi|\right)^{\frac{p+\gamma}{p-1}} dx \tag{3.1}$$

for all test functions $\psi \in C_c^\infty(\Omega)$ verifying $\|\psi\|_\infty \le 1$, where the constant C depends on p, m, γ and α.

Similarly, if we suppose that the weak solution of (1.2) u belongs to $H^1_{loc}(\Omega)$ such that $u = 0$ on $\partial\Omega$ and u is stable outside a compact set $\mathcal{K} \subset \Omega$, then the estimate (3.1) holds for all test functions $\psi \in C_c^\infty(\mathbb{R}^N \backslash \mathcal{K})$ verifying $\|\psi\|_\infty \le 1$.

The proof follows the main lines of the demonstration of Proposition 1.7 in [4] or Proposition 6 in [11], with small modifications. As for Proposition 2.2, we need to consider truncations of the weak solution u, but the calculation is easier here (see also [6]).

Set just $\zeta_k(t) = \max(-k, \min(t, k))$, $k \in \mathbb{N}$ and use the test function $|\zeta_k(u)|^{\frac{\gamma-1}{2}} u\varphi \in H_0^1(\Omega)$ in (1.4), with $\varphi \in C_c^\infty(\Omega)$ and $\varphi \in C_c^\infty(\mathbb{R}^N \backslash \mathcal{K})$ respectively. The rest of proof can be proceeded as for Proposition 1.7 in [4], and by taking k tends to ∞. We omit the details.

Assume for example that u is a weak solution of (1.2), stable outside a compact set of $\mathbb{R}^N$, i.e. there exists $R_0 > 0$ such that (3.1) holds true with $\Omega = \mathbb{R}^N \backslash \overline{B(0, R_0)}$. Let $\alpha > -2$ and $p > 1$, using (3.1), we can prove

- Let $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1)$ and $r > 2R_0$, then

$$\int_{2R_0<|x|<r} \left[|\nabla(u^{\frac{\gamma+1}{2}})|^2 + |x|^\alpha |u|^{\gamma+p}\right] dx \leq A + Br^{N-\frac{(2+\alpha)\gamma+2p+\alpha}{p-1}}, \tag{3.2}$$

where A and B are constants independent of $r > 2R_0$.

- Let $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1)$ and $B(y, 2r) \subset \mathbb{R}^N \setminus \overline{B(0, R_0)}$, then

$$\int_{B(y,r)} \left[|\nabla(u^{\frac{\gamma+1}{2}})|^2 + |x|^\alpha |u|^{\gamma+p}\right] dx \leq Cr^{N-\frac{(2+\alpha)\gamma+2p+\alpha}{p-1}}, \tag{3.3}$$

where $C > 0$ is independent of y and r.

The proof is very similar to that for (2.7) and (2.8) (see Step 1 of the proof for Theorem 3.3 in [4]), we leave the details for interested readers.

3.1 Slow decay estimate and proof of Theorem 1.9

Before proving the fast decay (1.9), we prove slow decay estimates for solutions on punctured domains or at infinity for $1 < p < p(N, 0)$, and a special regularity result when $p < p(N, \alpha^-)$.

Theorem 3.2 *Let $N \geq 2$, $1 < p < p(N, 0)$ and u be a weak solution to (1.2) with finite Morse index. If the domain Ω contains $B(0, r) \setminus \{0\}$ (resp. $\mathbb{R}^N \setminus B(0, r)$), there hold $u \in C^{2,\beta}_{loc}(\Omega \setminus \{0\})$ for some $\beta \in (0, 1)$ and*

$$u(x) = O\left(|x|^{-\frac{2+\alpha}{p-1}}\right), \quad \nabla u(x) = O\left(|x|^{-1-\frac{2+\alpha}{p-1}}\right), \quad \text{for } |x| \to 0 \text{ (resp. } |x| \to \infty). \tag{3.4}$$

Moreover, when $1 < p < p(N, \alpha^-)$, there exists $\beta \in (0, 1)$ such that $u \in C^{0,\beta}_{loc}(\Omega)$.

Proof. First, since each single point is of zero capacity in $\mathbb{R}^N$ when $N \geq 2$, all arguments for [6, Proposition 2.1] (see also [9]) are still valid although the equation is different, that is

Lemma 3.3 *Let $N \geq 2$ and u be a weak solution to (1.2) with finite Morse index. For every $x_0 \in \Omega$, there exists $r_0 > 0$ such that u is stable in $B(x_0, 4r_0)$.*

Similar to (3.3), by using standard cut-off function $\psi \in C^2_0(B(x_0, 4r_0))$ with (3.1), we obtain $|x|^\alpha |u|^{\gamma+p} \in L^1(B(x_0, 3r_0))$ for any $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1)$. If $p < p(N, 0)$ and $x_0 \neq 0$, as in the proof of [4, Theorem 2.1], we can claim that

$$\int_{B(x_0,2r_0)} \left(|x|^\alpha |u|^{p-1}\right)^{\frac{N}{2-\varepsilon_0}} dx < C \quad \text{for some } \varepsilon_0 \in (0, 2) \text{ and } r_0 < \frac{|x_0|}{4},$$

where C is a constant depending on α, r_0, p and ε_0. Applying now Lemma 2.4, since $-\Delta u = |x|^\alpha |u|^{p-1} u$ and $u \in L^2_{loc}(\Omega)$, we get $u \in L^\infty(B(x_0, r_0))$. As x_0 can be any point in $\Omega \setminus \{0\}$, it means that $u \in L^\infty_{loc}(\Omega \setminus \{0\})$. Hence $|x|^\alpha |u|^{p-1} u \in L^q_{loc}(\Omega \setminus \{0\})$ for some $q > \frac{N}{2}$ because $\alpha > -2$. We get $u \in C^{2,\beta}_{loc}(\Omega \setminus \{0\})$ by classical regularity theory.

When $x_0 = 0$, we can still use the above idea. But the estimate $|x|^\alpha |u|^{p-1} \in L^{\frac{N}{2-\varepsilon_0}}(B(0, 2r_0))$ needs also the condition $N + \frac{\alpha(\theta\xi-1)}{\xi-1} > 0$ (see line 5 of page 3291 in [4]), which requires $\Delta(p, \gamma, \alpha) < 0$ so $p < p(N, \alpha)$. When $p < p(N, \alpha^-)$, Lemma 2.4 yields again $u \in L^\infty(B(0, r_0))$, therefore u is Hölder continuous at 0 by equation.

The key argument to prove (3.4) is a uniform estimate inspired by the interesting work of Phan & Souplet [15].

Lemma 3.4 *Let $1 < p < p(N,0)$, $B_1 := B(0,1)$ and $\beta > 0$. Assume that $c \in C^{0,\beta}(\overline{B}_1)$ verifies $\|c\|_{C^{0,\beta}(\overline{B}_1)} \le C_1$ and $c(x) \ge C_2 > 0$ in B_1. Then there exists a constant $C > 0$ depending only on β, C_1, C_2, p and N, such that any stable solution u to $-\Delta u = c(x)|u|^{p-1}u$ in B_1 satisfies*

$$|u(x)|^{\frac{p-1}{2}} + |\nabla u(x)|^{\frac{p-1}{p+1}} \le \frac{C}{1-|x|}, \quad \forall\, x \in B_1.$$

Indeed, arguing by contradiction, we can repeat exactly the proof of Lemma 2.1 in [15] and arrive at a nontrivial *stable* solution verifying $-\Delta v = C_0|v|^{p-1}v$ in $\mathbb{R}^N$, with a constant $C_0 > 0$. However, this is impossible by Farina's classification result [11, Theorem 1] since $p < p(N,0)$.

Using the scaling argument, the proof of (3.4) is the same as for [15, Theorem 1.2]. Look at the situation near the origin. Applying Lemma 3.3, u is stable in $B(x_0, R)$ when $|x_0| = 2R > 0$ is small. Define

$$U(y) = R^{\frac{2+\alpha}{p-1}} u(x_0 + Ry), \quad \text{where } y \in B_1 = B(0,1). \tag{3.5}$$

Therefore U is a stable solution of

$$-\Delta U = c(y)|U|^{p-1}U \;\text{ in } B_1 \quad \text{with} \quad c(y) = \left|\frac{x_0}{R} + y\right|^{\alpha}.$$

We can check that $1 \le c(y) \le 3$ in B_1 and $\|c\|_{C^{1,\alpha}(\overline{B}_1)} \le C_\alpha$, hence $|U(0)| + |\nabla U(0)| \le C$ by Lemma 3.4, which is equivalent to say

$$|u(x_0)| + R|\nabla u(x_0)| \le CR^{-\frac{2+\alpha}{p-1}}, \quad \text{for } |x_0| = 2R \text{ small enough.}$$

So we are done. $\square$

Remark 3.5 *The estimate* (3.4) *was proved in Theorems 5.1 and 5.2 of [4] with different method.*

Proof of Theorem 1.9. Thanks to Theorem 3.2, we need only to consider (1.9), which is also a direct consequence of scaling argument. Let $|x_0| > 0$ be small such that $B(0, 2|x_0|) \subset \Omega$. Define $V(y) = u(x_0 + Ry)$ in B_1, where $2R = |x_0|$. As $u \in C(\Omega)$, we have $\|\Delta V\|_\infty \le CR^{2+\alpha}$, so $|\nabla V(0)| \le CR^{2+\alpha}$ by standard elliptic theory. Hence $|\nabla u(x_0)| \le C|x_0|^{1+\alpha}$, we get easily (1.9) since $\alpha > -2$. $\square$

3.2 Proof of Theorem 1.7 for subcritical p

Consider the subcritical cases, that is

$$1 < p < \frac{N+2+2\alpha}{N-2} \quad \text{and} \quad p < p(N,0). \tag{3.6}$$

Let $\gamma = 1$, so

$$N - \frac{(2+\alpha)\gamma + 2p + \alpha}{p-1} = N - \frac{2p+2+2\alpha}{p-1} < 0.$$

Consequently, since $u \in C(\mathbb{R}^N)$ by Theorem 3.2 and taking $r \to \infty$ in (3.2), we have $\nabla u \in L^2(\mathbb{R}^N)$ and $|x|^{\frac{\alpha}{p+1}}u \in L^{p+1}(\mathbb{R}^N)$.

Let $\phi_R(x) = \phi\left(R^{-1}|x|\right)$, where $\phi \in C_c^\infty(-2,2)$ is a cut-off function such that $0 \le \phi \le 1$ in $\mathbb{R}$ and $\phi(t) = 1$ for $|t| \le 1$. As $u\phi_R \in H_0^1 \cap L^\infty$ is compactly supported, by density argument, we can use it as test function in (1.2), to obtain

$$\int_{\mathbb{R}^N} |\nabla u|^2 \phi_R dx - \int_{\mathbb{R}^N} |x|^\alpha |u|^{p+1}\phi_R dx = \frac{1}{2}\int_{\mathbb{R}^N} |u|^2 \Delta\phi_R dx. \tag{3.7}$$

By Hölder's inequality and the estimate (3.3), we get

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^N} |u|^2 \Delta\phi_R dx\right| \\
\le\ &\left[\int_{R<|x|<2R} \left(|x|^{\frac{\alpha}{p+1}}|u|\right)^{p+1} dx\right]^{\frac{2}{p+1}} \left[\int_{R<|x|<2R} \left(|x|^{-\frac{2\alpha}{p+1}}|\Delta\phi_R|\right)^{\frac{p+1}{p-1}} dx\right]^{\frac{p-1}{p+1}} \\
\le\ &CR^{N-\frac{2p+2+2\alpha}{p-1}}.
\end{aligned}
\tag{3.8}
$$

As $u \in C^2(\mathbb{R}^N \setminus \{0\})$ by Theorem 3.2, we can apply the classical Pohozaev identity to u in $\Omega_{\varepsilon,R} := B(0,R) \setminus \overline{B(0,\varepsilon)}$ with $R > \varepsilon > 0$, then

$$
\begin{aligned}
&\frac{N+\alpha}{p+1}\int_{\Omega_{\varepsilon,R}} |x|^\alpha |u|^{p+1} dx - \frac{N-2}{2}\int_{\Omega_{\varepsilon,R}} |\nabla u|^2 dx \\
=\ &\int_{\partial\Omega_{\varepsilon,R}} |x|^\alpha \langle x,\nu\rangle |u|^{p+1} d\sigma + \int_{\partial\Omega_{\varepsilon,R}} \frac{\partial u}{\partial \nu}\langle x,\nabla u\rangle d\sigma - \int_{\partial\Omega_{\varepsilon,R}} \frac{|\nabla u|^2}{2}\langle x,\nu\rangle d\sigma.
\end{aligned}
\tag{3.9}
$$

Using (3.4) and $p < \frac{N+2+2\alpha}{N-2}$,

$$
\int_{\partial B(0,R)} \left(|x||\nabla u|^2 + |x|^{\alpha+1}|u|^{p+1}\right) d\sigma \le CR^{N-2-\frac{2(2+\alpha)}{p-1}} \to 0, \quad \text{if } R \to \infty. \tag{3.10}
$$

On the other hand, we have $-\Delta u = O(|x|^\alpha)$ near the origin and $u \in C(\mathbb{R}^N)$. As $\alpha > -2$, by regularity theory and Sobolev embedding we can claim that $\nabla u \in L^q_{loc}(\mathbb{R}^N)$ for some $q > 2$. Applying again Hölder's inequality,

$$
\frac{1}{\varepsilon}\int_0^\varepsilon \left(\int_{\partial B(0,s)} |x||\nabla u|^2 d\sigma\right) ds \le \int_{B(0,\varepsilon)} |\nabla u|^2 dx \le \varepsilon^\sigma \|\nabla u\|_{L^q(B(0,\varepsilon))}, \quad \text{where } \sigma > 0.
$$

Thus there exists a sequence $\varepsilon_j \to 0$ such that

$$
\lim_{j\to\infty} \int_{\partial B(0,\varepsilon_j)} |x||\nabla u|^2 d\sigma = 0. \tag{3.11}
$$

Take $\varepsilon = \varepsilon_j$ in (3.9) then tend R and j to ∞. It follows from (3.10) and (3.11) that

$$
\frac{N+\alpha}{p+1}\int_{\mathbb{R}^N} |x|^\alpha u^{p+1} dx - \frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx = 0.
$$

Combining with (3.7) and (3.8), we have

$$
\left(\frac{N-2}{2} - \frac{N+\alpha}{p+1}\right)\int_{\mathbb{R}^N} |x|^\alpha |u|^{p+1} dx = 0.
$$

As $\frac{N-2}{2} - \frac{N+\alpha}{p+1} < 0$ for subcritical p, we conclude that $u \equiv 0$ under the assumption (3.6). □

3.3 Fast decay behavior at infinity

We show here Theorem 1.10. Consider first

$$
p \ge \frac{N+2+2\alpha}{N-2} \quad \text{and} \quad p < p(N,\alpha^-). \tag{3.12}
$$

Let $\gamma(p) = 2p + 2\sqrt{p(p-1)} - 1$, $p > 1$ and $\alpha > -2$. Clearly

$$\frac{(2+\alpha)\gamma(p) + 2p + \alpha}{p-1} = 2 + (4+2\alpha)\left[\frac{p}{p-1} + \sqrt{\frac{p}{p-1}}\right]$$

is decreasing in $(1, \infty)$. If $N \leq 10 + 4\alpha$, $(2+\alpha)\gamma(p) + 2p + \alpha > N(p-1)$ for any $p > 1$. We can check also that if $N > 10 + 4\alpha$, $(2+\alpha)\gamma(p) + 2p + \alpha = N(p-1)$ with $p = p(N, \alpha)$ given in Theorem 1.1. Therefore under the assumption (3.12), as $p(N, \alpha)$ is increasing with respect to α, there exists $\gamma_1 \in [1, 2p + 2\sqrt{p(p-1)} - 1)$ such that $N(p-1) - (2+\alpha)\gamma_1 - 2p - \alpha = 0$.

Set now

$$\beta_0 = \frac{\gamma_1 + p}{2} > 1 \quad \text{and} \quad \omega(x) = |u(x)|^{\beta_0} \geq 0,$$

Using (3.2) with $\gamma = \gamma_1$ and letting $r \to \infty$, we have

$$\int_{|x|>2R_0} |x|^\alpha \omega^2 dx < \infty. \tag{3.13}$$

and

$$-\Delta\omega - \beta_0 |x|^\alpha |u|^{p-1}\omega = -\beta_0(\beta_0 - 1)|u|^{\beta_0 - 2}|\nabla u|^2 \leq 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^N).$$

Moreover, as $p < p(N, \alpha^-) \leq p(N, 0)$, it follows from (3.3) that there exists $\varepsilon_0(p, N) \in (0, 2)$ such that

$$\int_{B(y,r)} \left(|x|^\alpha |u|^{p-1}\right)^{\frac{N}{2-\varepsilon_0}} dx \leq C r^{N - \frac{2N}{2-\varepsilon_0}}, \quad \forall\, B(y, 2r) \subset \mathbb{R}^N \setminus \overline{B(0, R_0)}$$

where the constant C is independent of y and r. Let $|y| > 8R_0$ and $R = \frac{|y|}{8}$, so $B(y, 4R) \subset \mathbb{R}^N \setminus \overline{B(0, R_0)}$. Therefore

$$R^{\varepsilon_0} \|\beta_0 |x|^\alpha |u|^{p-1}\|_{L^{\frac{N}{2-\varepsilon_0}}(B(y,2R))} \leq \beta_0 R^{\varepsilon_0} \left(C R^{N - \frac{2N}{2-\varepsilon_0}}\right)^{\frac{2-\varepsilon_0}{N}} = C',$$

Applying (3.13) and Lemma 2.4,

$$\omega(y) \leq C R^{-\frac{N}{2}} \|\omega\|_{L^2(B(y,2R))} \leq C R^{-\frac{N}{2}} R^{-\frac{\alpha}{2}} \||x|^{\frac{\alpha}{2}}\omega\|_{L^2(B(y,2R))} = o\left(R^{-\frac{N+\alpha}{2}}\right) \quad \text{as} \quad |y| \to \infty.$$

Hence

$$|u(y)| = \omega(y)^{\frac{1}{\beta_0}} = o\left(R^{-\frac{N+\alpha}{2\beta_0}}\right) = o\left(R^{-\frac{2+\alpha}{p-1}}\right) \quad \text{as} \quad |y| \to \infty,$$

which shows the fast decay of u. To prove the second claim of (1.10), we observe that

$$-\Delta u(y) = |y|^\alpha |u|^{p-1} u = o\left(R^{-\frac{2+\alpha}{p-1} - 2}\right) \quad \text{as} \quad |y| \to \infty.$$

The scaling argument and standard elliptic theory imply then

$$|\nabla u(y)| = o\left(R^{-\frac{2+\alpha}{p-1} - 1}\right) \quad \text{as} \quad |y| \to \infty.$$

Now we consider the remaining case

$$\underline{p}(N, \alpha) < p < \frac{N + 2 + 2\alpha}{N - 2} \quad \text{and} \quad p < p(N, \alpha^-). \tag{3.14}$$

Here we will use Kelvin's transformation as in [4]. Let

$$v(x) = |x|^{2-N} u\left(\frac{x}{|x|^2}\right), \quad \text{for } |x| > 0 \text{ small},$$

then v verifies $-\Delta v = |x|^\beta |v|^{p-1} v$ with $\beta = (N-2)(p-1) - (4+\alpha)$. We can verify that $\beta > -2$ since $p > \frac{N+\alpha}{N-2}$. Moreover, we have the following properties (see Proposition 3.1 and the proof of Theorem 3.4 in [4]):

- The solution v is stable over $B(0,r)\setminus\{0\}$ for $r>0$ small since u is stable outside $B(0,r^{-1})$.
- The assumption (3.14) implies $\frac{N+2+2\beta}{N-2} < p < p(N,\beta^-)$. Indeed, $\underline{p}(N,\alpha) < p < p(N,\alpha)$ yields that $\beta < p(N,\beta)$.

Using estimate (3.4) for v, we can prove that v is a weak stable solution of $-\Delta v = |x|^\beta |v|^{p-1}v$ in $B(0,r)$. For example, there holds

$$\int_{B(0,r)} |\nabla v|^2 dx \le C\int_0^r s^{N-3-\frac{2(2+\beta)}{p-1}} ds < \infty, \quad \text{since } N-2-\frac{2(2+\beta)}{p-1} > 0.$$

Moreover v is continuous in $B(0,r)$ by Theorem 3.2 as $\beta < p(N,\beta^-)$, so we get

$$u(x) = |x|^{2-N} v\left(\frac{x}{|x|^2}\right) = O\left(|x|^{2-N}\right) = o\left(|x|^{-\frac{2+\alpha}{p-1}}\right) \quad \text{as } |x|\to\infty.$$

As $p < p(N,0)$, combining with (1.9) for v, we have

$$\nabla u(x) = O\left(|x|^{1-N}\right) + o\left(|x|^{\frac{2+\beta}{p-1}+1-N}\right) = o\left(|x|^{-1-\frac{2+\alpha}{p-1}}\right) \quad \text{as } |x|\to\infty.$$

The proof is completed. □

3.4 Proof of Theorem 1.7 for supercritical p

Now we are in position to prove Theorem 1.7 for p verifying

$$p > \frac{N+2+2\alpha}{N-2} \quad \text{and} \quad p < p(N,\alpha^-).$$

In fact, combining Theorems 1.9 and 1.10, the regularity result in Theorem 3.2, the supercritical exponent situation for Theorem 1.7 is a direct consequence of the following nonexistence result.

Proposition 3.6 *Let $\alpha > -2$ and $p > 1$ and $p \ne \frac{N+2+2\alpha}{2+\alpha}$. Then there does not exist any nontrivial weak solution of* (1.2) *in $\mathbb{R}^N$ satisfying $u \in C^2(\mathbb{R}^N\setminus\{0\})$, the estimates* (1.10) *and*

$$\lim_{|x|\to 0} |x|^{\frac{2+\alpha}{p-1}}|u(x)| = \lim_{|x|\to 0} |x|^{1+\frac{2+\alpha}{p-1}}|\nabla u(x)| = 0. \tag{3.15}$$

As in [11, 20], we use the Emden-Fowler change of variable

$$u(r,\sigma) = r^{-\frac{2+\alpha}{p-1}} w(t,\sigma), \quad t = \ln r.$$

Therefore w satisfies

$$w_{tt} + A_1 w_t + \Delta_{S^{N-1}} w + A_2 w + |w|^{p-1}w = 0 \quad \text{in } \mathbb{R}\times S^{N-1},$$

where

$$A_1 = N-2-\frac{2+\alpha}{p-1} \ne 0, \quad A_2 = -\frac{2+\alpha}{p-1}\left(N-2-\frac{2+\alpha}{p-1}\right)$$

and $\Delta_{S^{N-1}}$ denotes the Laplace-Beltrami operator on the unit sphere $S^{N-1}\subset\mathbb{R}^N$. Set

$$E(w)(t) = \int_{S^{N-1}} \left(\frac{1}{2}|\nabla_{S^{N-1}} w|^2 - \frac{A_2}{2}w^2 - \frac{1}{p+1}|w|^{p+1} - \frac{1}{2}w_t^2\right) d\sigma.$$

Clearly,

$$\frac{d}{dt}E(w)(t) = A_1 \int_{S^{N-1}} w_t^2 d\sigma.$$

The estimates (1.10) and (3.15) yield

$$\lim_{t\to\pm\infty} w(t,\sigma) = \lim_{t\to\pm\infty} |w_t(t,\sigma)| = \lim_{t\to\pm\infty} |\nabla_{S^{N-1}} w(t,\sigma)| = 0,$$

hence $\lim_{t\to\pm\infty} E(w)(t) = 0$. On the other hand, integrating the equation of w, we get

$$0 = A_1 \int_{\mathbb{R}} \int_{S^{N-1}} w_t^2 d\sigma dt = A_1 \int_{\mathbb{R}^N} w_t^2 dx,$$

which means that $w_t = 0$ in $\mathbb{R}^N$. Therefore $w \equiv 0$ since $\lim_{t\to\pm\infty} w(t,\sigma) = 0$, so $u \equiv 0$. □

Remark 3.7 *We can see that the above proof works also for*

$$\underline{p}(N,\alpha) < p < \min\left(\frac{N+2+2\alpha}{N-2},\ p(N,\alpha^-)\right).$$

3.5 Proof of Theorem 1.8

The proof is a simple adaptation of ideas for Theorem 9(b) in [11]. The main point is that we can consider v, the odd extension of u. Then v is a weak solution of (1.2) in $\mathbb{R}^N$. The crucial argument is that we can use test function as $\zeta_k(u)\psi$ with $\psi \in C_c^\infty(\mathbb{R}^N)$ as u verifies the Dirichlet boundary condition on $\partial\mathbb{R}^N_+$, we get all the corresponding estimates since all the boundary terms are zero when doing the integration by parts (see more details in [11]). For example, u is stable outside a compact set $\mathcal{K} \subset \mathbb{R}^N_+$, so we can obtain the estimate (3.1) for $\psi \in C_c^\infty(\mathbb{R}^N \setminus \mathcal{K})$. By symmetry, the same estimate holds with v in $\mathbb{R}^N_-$ with $\psi \in C_c^\infty(\mathbb{R}^N \setminus \mathcal{K}')$ where $\mathcal{K}'$ is the mirror symmetry of $\mathcal{K}$. By taking the sum, we can claim that for any $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1)$ and $m \in \mathbb{N}$ large enough, there holds

$$\int_\Omega \left(|\nabla(|v|^{\frac{\gamma-1}{2}} v)|^2 + |x|^\alpha |v|^{\gamma+p}\right) |\psi|^{2m} dx \le C \int_\Omega |x|^{-\frac{(\gamma+1)\alpha}{p-1}} \left(|\nabla\psi|^2 + |\psi||\Delta\psi|\right)^{\frac{p+\gamma}{p-1}} dx,$$

for any $\psi \in C_c^\infty(\mathbb{R}^N \setminus (\mathcal{K} \cup \mathcal{K}'))$ verifying $\|\psi\|_\infty \le 1$.

In other words, we refind all the regularity results and the corresponding estimates for v as for weak solution in $\mathbb{R}^N$ with finite Morse index. For example, we can prove the corresponding result of Lemma 3.4 to solution of $-\Delta u = c(x)|u|^{p-1}u$ in $B_1 \cap \mathbb{R}^N_+$ verifying $u = 0$ on $B_1 \cap \partial\mathbb{R}^N_+$, where the contradiction comes from the classification result on half space, Theorem 9(b) in [11] instead of [11, Theorem 1].

Proceeding as for Theorem 1.7, we refine then $v \equiv 0$, so is u. □

4 Further remarks and open questions

In general, the Hénon equation $-\Delta u = |x|^\alpha g(u)$ is more delicate to handle than the corresponding autonomous equation, i.e. when $\alpha = 0$. Many properties for the autonomous equation are no longer true or less understood for the nonautonomous situation.

For example, let $\alpha > 0$; we don't know if a *positive* and *continuous* solution to $-\Delta u = |x|^\alpha u^p$ in $\mathbb{R}^N$ could exist with a subcritical exponent

$$\frac{N + \max(2,\alpha)}{N-2} < p < p_\alpha := \frac{N+2+2\alpha}{N-2}.$$

Only very recently, Phan and Souplet proved the nonexistence of *bounded positive* solution in $\mathbb{R}^3$ for any $1 < p < p_\alpha$, see [15] and the references therein.

For the critical exponent case $p = p_\alpha$ with $\alpha > -2$, the existence of radial solutions in $\mathbb{R}^N$ is known via (1.8), and they are stable outside a compact set. When $\alpha = 0$ and $p = \frac{N+2}{N-2}$, Farina proved in [11, Theorem 2] a very interesting equivalence between the following assumptions:

(i) u is a finite Morse index solution of (1.2) in $\mathbb{R}^N$;

(ii) u is a solution of (1.2) in $\mathbb{R}^N$ and u is stable outside a compact set;

(iii) u is a solution of (1.2) in $\mathbb{R}^N$ and $\nabla u \in L^2(\mathbb{R}^N)$.

Combining with [8], we get infinitely many (conformally non-equivalent) sign-changing solutions to $-\Delta u = |u|^{\frac{4}{n-2}}u$ in $\mathbb{R}^N$ with finite Morse index.

For weak solution to (1.2) with general $p \leq p_\alpha$ and $\alpha > -2$, the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) hold true. For (ii) $\Rightarrow$ (iii), we can take just $\gamma = 1$ in the estimate (3.2). However it is difficult to give a general positive or negative answer for implication (ii) $\Rightarrow$ (i) or (iii) $\Rightarrow$ (ii), which seem to be completely open, even for $\alpha = 0$, except Farina's result.

An interesting work linked to that is [13] (see also the references therein), where the authors proved some relationships between the symmetry and low Morse index solutions for $-\Delta u = f(|x|, u)$. For example, using [13, Theorem 1.5], we can claim that for $\alpha > 1$, any stable solution to (1.2) in $\mathbb{R}^N$ verifying $\nabla u \in L^2(\mathbb{R}^N)$ must be radial.

When $\alpha \neq 0$ and $p = p_\alpha$, the existence of *non-radial* or *sign-changing* entire solution to (1.2) with finite Morse index seems also unknown.

When $p \geq p(N, \alpha)$ and $\alpha > -2$, we have the following question:

Does there exist classical or weak solution u to (1.2) in $\mathbb{R}^N$ with $p \geq p(N, \alpha)$ and $0 < \mathrm{ind}(u) < \infty$?

As far as we are aware, the problem is open, even for $\alpha = 0$. All known solutions for $\alpha = 0$ and $p \geq p(N, 0)$, as radial solutions or lower dimensional solutions are either stable or have infinite Morse index in $\mathbb{R}^N$. Of course, we can ask the similar question for (1.1) with $N \geq 10 + 4\alpha$.

The understanding of finite Morse index but unstable solutions to (1.2) in $\mathbb{R}^N$ for $p \geq p(N, \alpha)$ is far away from evident, even they have necessarily finite energy, i.e. $\nabla u \in L^2(\mathbb{R}^N)$. For example, when $\alpha = 0$, if such a solution u exists and if $u \to 0$ as $|x| \to \infty$, it must be *non-radial* and *sign-changing*, since any positive solution is radial and all radial solutions are stable as $p \geq p(N, 0)$. Theorem 1.6 in [13] yields then $\mathrm{ind}(u) \geq N + 1 \geq 12$, which means that u must have a large Morse index.

The assumption $\lim_{|x|\to\infty} u(x) = 0$ seems to be *reasonable* for any finite Morse index solution of (1.2), but we don't know if it holds true in general for $p \geq p(N, 0)$.

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