On stable solutions of the biharmonic problem with polynomial growth
Hatem Hajlaoui, Abdellaziz Harrabi, Dong Ye

To cite this version:

HAL Id: hal-01095026
https://hal.archives-ouvertes.fr/hal-01095026
Submitted on 14 Dec 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers. L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
ON STABLE SOLUTIONS OF BIHARMONIC PROBLEM WITH POLYNOMIAL GROWTH

HATEM HAJLAOUI, ABDELLAZIZ HARRABI, AND DONG YE

Abstract. We prove the nonexistence of smooth stable solution to the biharmonic problem
\[ \Delta^2 u = u^p, \quad u > 0 \text{ in } \mathbb{R}^N \]
for \( 1 < p < \infty \) and \( N < 2(1 + x_0) \), where \( x_0 \) is the largest root of the equation:
\[ x^4 - \frac{32p(p+1)}{(p-1)^2} x^2 + \frac{32p(p+1)(p+3)}{(p-1)^3} x - \frac{64p(p+1)^2}{(p-1)^4} = 0. \]
In particular, as \( x_0 > 5 \) when \( p > 1 \), we obtain the nonexistence of smooth stable solution for any \( N \leq 12 \) and \( p > 1 \). Moreover, we consider also the corresponding problem in the half space \( \mathbb{R}^N_+ \), or the elliptic problem \( \Delta^2 u = \lambda(u+1)^p \) on a bounded smooth domain \( \Omega \) with the Navier boundary conditions. We will prove the regularity of the extremal solution in lower dimensions. Our results improve the previous works in [20, 19, 2, 4].

1. Introduction

Consider the biharmonic equation
\[ \Delta^2 u = u^p, \quad u > 0 \text{ in } \mathbb{R}^N \]
where \( N \geq 5 \) and \( p > 1 \). Let
\[ \Lambda(\phi) := \int_{\mathbb{R}^N} |\Delta \phi|^2 dx - p \int_{\mathbb{R}^N} u^{p-1} \phi^2 dx, \quad \forall \phi \in H^2(\mathbb{R}^N). \]
A solution \( u \) is said stable if \( \Lambda(\phi) \geq 0 \) for any test function \( \phi \in H^2(\mathbb{R}^N) \).

In this note, we prove the following classification result.

Theorem 1.1. Let \( N \geq 5 \) and \( p > 1 \). The equation (1.1) has no classical stable solution, if \( N < 2 + 2x_0 \) where \( x_0 \) is the largest root of the polynomial
\[ H(x) = x^4 - \frac{32p(p+1)}{(p-1)^2} x^2 + \frac{32p(p+1)(p+3)}{(p-1)^3} x - \frac{64p(p+1)^2}{(p-1)^4}. \]
Moreover, we have \( x_0 > 5 \) for any \( p > 1 \). Consequently, if \( N \leq 12 \), (1.1) has no classical stable solution for all \( p > 1 \).

For the corresponding second order problem:
\[ \Delta u + |u|^{p-1} u = 0 \text{ in } \mathbb{R}^N, \quad p > 1. \]
Farina has obtained the optimal Liouville type result for all finite Morse index solutions. He proved in [7] that a smooth finite Morse index solution to (1.4) exists, if and only if \( p \geq p_{JL} \) and \( N \geq 11 \), or \( p = \frac{N+2}{N-2} \) and \( N \geq 3 \). Here \( p_{JL} \) is the so-called Joseph-Lundgren exponent, see (1.11) in [12].

1991 Mathematics Subject Classification. Primary 35J91; Secondary 35J30, 35J40.
Key words and phrases. stable solutions, biharmonic equations, polynomial growths.
The nonexistence of positive solutions to (1.1) are showed if \( p < \frac{N+4}{N-4} \), and all entire solutions are classified if \( p = \frac{N+4}{N-4} \). On the other hand, the radially symmetric solutions to (1.1) are studied in [8, 9, 13, 14]. In particular, Karageorgis proved that the radial entire solution to (1.1) is stable if and only if \( p \geq p_{JL4} \) and \( N \geq 13 \). Here \( p_{JL4} \) stands for the corresponding Joseph-Lundgren exponent to \( \Delta^2 \), see [14].

The general fourth order case (1.1) is more delicate, since the integration by parts argument used by Farina cannot be adapted easily. The first nonexistence result for general stable solution was proved by Wei & Ye [20], they proposed to consider (1.1) as a system

\[
-\Delta u = v, \quad -\Delta v = u^p \quad \text{in} \quad \mathbb{R}^N,
\]

and introduced the idea to use different test functions with \( u \) but also \( v \). Using estimates in [17] they showed that for \( N \leq 8 \), (1.1) has no smooth stable solutions. For \( N \geq 9 \), using a blow-up argument, they proved that the classification holds still for \( p < \frac{N}{N-8} + \epsilon N \) with \( \epsilon N > 0 \), but without any explicit value of \( \epsilon N \). This result was improved by Wei, Xu & Yang in [19] for \( N \geq 20 \) with a more explicit bound.

Using the stability for system (1.5) and an interesting iteration argument, Cowan proved that, see Theorem 2 in [2], there is no smooth stable solution to (1.1), if

\[
N < 2 + \frac{4(p+1)}{p-1} t_0, \quad \text{where} \quad t_0 = \sqrt{\frac{2p}{p+1}} + \sqrt{\frac{2p}{p+1}} - \sqrt{\frac{2p}{p+1}}, \quad \forall \, p > 1.
\]

In particular, if \( N \leq 10 \), (1.1) has no stable solution for any \( p > 1 \).

However, the study for radial solutions in [14] suggests the following conjecture:

A smooth stable solution to (1.1) exists if and only if \( p \geq p_{JL4} \) and \( N \geq 13 \).

Consequently, the Liouville type result for stable solutions of (1.1) should hold true for \( N \leq 12 \) with any \( p > 1 \), that’s what we prove here. More precisely, by Theorem 1 in [14], the radial entire solutions to (1.1) are unstable if and only if

\[
\frac{N^2(N-4)^2}{16} < pQ_4 \left( -\frac{4}{p-1} \right), \quad \text{where} \quad Q_4(m) = m(m-2)(m+N-2)(m+N-4).
\]

The l.h.s. comes from the best constant of the Hardy-Rellich inequality (see [16]): Let \( N \geq 5 \),

\[
\int_{\mathbb{R}^N} |\Delta \varphi|^2 dx \geq \frac{N^2(N-4)^2}{16} \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^4} dx, \quad \forall \, \varphi \in H^2(\mathbb{R}^N).
\]

The r.h.s. of (1.7) comes from the weak radial solution \( w(x) = |x|^{-\frac{4}{p-1}} \). When \( p > \frac{N+4}{N-4} \), we can check that \( w \in H^2_{loc}(\mathbb{R}^N) \) and

\[
\Delta^2 w = Q_4 \left( -\frac{4}{p-1} \right) w^p \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^N).
\]

Since \( w^{p-1}(x) = |x|^{-4} \), using the Hardy-Rellich inequality, the condition (1.7) means just that \( w \) is not a stable solution in \( \mathbb{R}^N \), i.e.

\[
\exists \, \varphi \in H^2(\mathbb{R}^N) \text{ such that } \Lambda_w(\varphi) := \int_{\mathbb{R}^N} |\Delta \varphi|^2 dx - p \int_{\mathbb{R}^N} Q_4 \left( -\frac{4}{p-1} \right) w^{p-1} \varphi^2 dx < 0.
\]
If we denote $N = 2 + 2x$, a direct calculation shows that (1.7) is equivalent to $H_{JL4}(x) < 0$, where

$$H_{JL4}(x) := (x^2 - 1)^2 - \frac{32p(p + 1)}{(p - 1)^2} x^2 + \frac{32p(p + 1)(p + 3)}{(p - 1)^3} x - \frac{64p(p + 1)^2}{(p - 1)^4}.$$

By [9], (1.7) is equivalent to $N < 2 + 2x_1$ if $x_1$ denotes the largest root of $H_{JL4}$. We can remark the nearness between the polynomial $H$ in Theorem 1.1 and $H_{JL4}$, since $H(x) - H_{JL4}(x) = 2x^2 - 1$.

Furthermore, Theorem 1.1 improves the bound given in [2] for all $p > 1$. Indeed, there holds $x_0 > \frac{2(p+1)}{p-1} t_0$, see Lemmas 2.2 and 2.4 below.

Recall that to handle the equation (1.1), we prove in general that $v = -\Delta u > 0$ in $\mathbb{R}^N$ using average functions on the sphere, see [18]. Applying the blow up argument as in [17, 20], we can assume then $u$ and $v$ are uniformly bounded in $\mathbb{R}^N$. Therefore the following Souplet’s estimate in [17] holds true in $\mathbb{R}^N$, which was established for any bounded solution $u$ of (1.1):

$$v \geq \sqrt{\frac{2}{p+1}} \frac{u^{p+1}}{u}. \hspace{1cm} (1.8)$$

Here we propose a new approach. Without assuming the boundedness of $u$ or showing immediately the positivity of $v$, we prove first some integral estimates for stable solutions of (1.1), which will enable us the estimate (1.8). This idea permits us to handle more general biharmonic equations: Let $N \geq 5$ and $p > 1$, and consider

$$\Delta^2 u = u^p, \quad u > 0 \text{ in } \Sigma \subset \mathbb{R}^N, \quad u = \Delta u = 0 \text{ on } \partial \Sigma. \hspace{1cm} (1.9)$$

Let $E = H^2(\Sigma) \cap H^1_0(\Sigma)$ and

$$\Lambda_0(\phi) := \int_{\Sigma} |\Delta \phi|^2 dx - p \int_{\Sigma} u^{p-1} \phi^2 dx, \quad \forall \phi \in E. \hspace{1cm} (1.10)$$

A solution $u$ of (1.9) is said to be stable if $\Lambda_0(\phi) \geq 0$ for any $\phi \in E$.

**Proposition 1.2.** Let $u$ be a classical stable solution of (1.9) with $\Sigma = \mathbb{R}^N$, or the half space $\Sigma = \mathbb{R}^N_+, \text{ or the exterior domain } \Sigma = \mathbb{R}^N \setminus \overline{\Omega}, \mathbb{R}^N_+ \setminus \overline{\Omega} \text{ where } \Omega \text{ is a bounded smooth domain of } \mathbb{R}^N$. Then the inequality (1.8) holds in $\Sigma$, consequently $v > 0$ in $\Sigma$.

Using this, we obtain a Liouville type result for (1.9) in the half space situation, which improves the result in [20] for wider range of $N$, and without assuming the boundedness of $u$ or $v = -\Delta u$.

**Theorem 1.3.** Let $x_0$ be defined as in Theorem 1.1. If $N < 2 + 2x_0$, there exists no classical stable solution of (1.9) if $\Sigma = \mathbb{R}^N_+$.

Our proof combines also many ideas coming from [20, 4, 2]. Briefly, for (1.1), we apply different test functions to both equations of the system (1.5) and make use of the following inequality in [4] (see also [2, 5]): If $u$ is a stable solution of (1.1), then

$$\int_{\mathbb{R}^N} \sqrt{pu^{p-1}} \varphi^2 dx \leq \int_{\mathbb{R}^N} |\nabla \varphi|^2 dx, \quad \forall \varphi \in C^1_0(\mathbb{R}^N). \hspace{1cm} (1.11)$$
This will enable us to make two estimates. By these two estimates, we prove that for any stable solution \( u \) to (1.1), \( \phi \in C^2_0(\mathbb{R}^N) \) and \( s \geq 1 \), there holds

\[
L(s) < 0 \Rightarrow \int_{\mathbb{R}^N} u^p v^{s-1} \phi^2 dx \leq C \int_{\mathbb{R}^N} v^s (|\Delta(\phi^2)| + |\nabla \phi|^2) dx
\]

Here \( L \) is a polynomial of degree 4, see (2.9) below, and the constant \( C \) depends only on \( p \) and \( s \). Applying then the iteration argument of Cowan in [2], we show that \( u \equiv 0 \) if \( N < 2 + 2x_0 \), which is a contradiction, since \( u \) is positive.

Using similar ideas, we consider the elliptic equation on bounded domains:

\[
(P_\lambda) \quad \left\{ \begin{array}{ll}
\Delta^2 u = \lambda(u + 1)^p & \text{in a bounded smooth domain } \Omega \subset \mathbb{R}^N, \quad N \geq 1 \\
u = \Delta u = 0 & \text{on } \partial \Omega.
\end{array} \right.
\]

It is well known (see [1, 10]) that there exists a critical value \( \lambda^* > 0 \) depending on \( p > 1 \) and \( \Omega \) such that

- If \( \lambda \in (0, \lambda^*) \), \((P_\lambda)\) has a minimal and classical solution \( u_\lambda \) which is stable;
- If \( \lambda = \lambda^* \), \( u^* = \lim_{\lambda \to \lambda^*} u_\lambda \) is a weak solution to \((P_{\lambda^*})\), \( u^* \) is called the extremal solution.
- No solution of \((P_\lambda)\) exists whenever \( \lambda > \lambda^* \).

In [3, 20], it was proved that if \( 1 < p < \left( \frac{N-8}{8N} \right)^- \) or equivalently when \( N < \frac{8p}{p-1} \), the extremal solution \( u^* \) is smooth. Recently, Cowan & Ghoussoub improved the above result by showing that \( u^* \) is smooth if \( N < 2 + \frac{4(p+1)}{p-1} t_0 \) with \( t_0 \) in (1.6), so \( u^* \) is smooth for any \( p > 1 \) when \( N \leq 10 \). Our result is

**Theorem 1.4.** The extremal solution \( u^* \) is smooth if \( N < 2 + 2x_0 \) with \( x_0 \) given by Theorem 1.1. In particular, \( u^* \) is smooth for any \( p > 1 \) if \( N \leq 12 \).

We remark that our proof does not use the a priori estimate of \( v = -\Delta u \) as in [3, 4].

The paper is organized as follows. We prove some preliminary results and Proposition 1.2 in section 2. The proofs of Theorems 1.1, 1.3 and 1.4 are given respectively in section 3 and 4.

2. Preliminaries

We show first how to obtain the estimate (1.8) for stable solutions of (1.9). Our idea is to use the stability condition (1.10) to get some decay estimate for stable solutions of (1.9). In the following, we denote by \( B_r \) the ball of center 0 and radius \( r > 0 \).

**Lemma 2.1.** Let \( u \) be a stable solution to (1.9) and set \( v = -\Delta u \), there holds

\[
\int_{\Sigma \cap B_R} (v^2 + u^{p+1}) \ dx \leq CR^{N-4-\frac{8}{p-1}}, \quad \forall \ R > 0.
\]

**Proof.** We proceed similarly as in Step 1 of the proof for Theorem 1.1 in [20], but we do not assume here that \( v > 0 \) or \( u \) is bounded in \( \Sigma \). For any \( \xi \in C^4(\Sigma) \) verifying \( \xi = \Delta \xi = 0 \) on \( \partial \Sigma \) and \( \eta \in C_0^\infty(\mathbb{R}^N) \), we have

\[
\int_{\Sigma} (\Delta^2 \xi) \xi \eta^2 \ dx = \int_{\Sigma} [\Delta(\xi \eta)]^2 \ dx + \int_{\Sigma} [-4(\nabla \xi \cdot \nabla \eta)^2 + 2\xi \Delta \xi |\nabla \eta|^2] \ dx
\]

\[
+ \int_{\Sigma} \xi^2 \left[ 2\nabla(\Delta \eta) \cdot \nabla \eta + (\Delta \eta)^2 \right] \ dx.
\]

4
The proof is direct as for Lemma 2.3 in [20], noticing just that in the integrations by parts, all boundary integration terms on \( \partial \Sigma \) vanish under the Navier conditions for \( \xi \).

Let \( u \) be a solution of (1.9). Take \( \xi = u \) in (2.2), there holds
\[
\int_{\Sigma} [\Delta(u\eta)]^2 dx - \int_{\Sigma} u^{p+1}\eta^2 dx = 4 \int_{\Sigma} (\nabla u \nabla \eta)^2 dx + 2 \int_{\Sigma} uv|\nabla \eta|^2 dx - \int_{\Sigma} u^2 \left[ 2\nabla(\Delta \eta) \cdot \nabla \eta + (\Delta \eta)^2 \right] dx
\]
where \( v = -\Delta u \). Using \( \phi = u\eta \) in (1.10), we obtain easily
\[
\int_{\Sigma} \left[ (\Delta(u \eta))^2 + u^{p+1}\eta^2 \right] dx \leq C_1 \int_{\Sigma} \left[ |\nabla u|^2 |\nabla \eta|^2 + u^2 |\nabla(\Delta \eta) \cdot \nabla \eta| + u^2 (\Delta \eta)^2 \right] dx + C_2 \int_{\Sigma} uv|\nabla \eta|^2 dx.
\]
Here and below, \( C \) or \( C_i \) denotes generic positive constants independent of \( u \), which could be changed from one line to another. As \( \Delta(u \eta) = 2\nabla u \cdot \nabla \eta + u \Delta \eta - v\eta \), we get from (2.3),
\[
\int_{\Sigma} \left[ v^2 \eta^2 + u^{p+1}\eta^2 \right] dx \leq C_1 \int_{\Sigma} \left[ |\nabla u|^2 |\nabla \eta|^2 + u^2 |\nabla(\Delta \eta) \cdot \nabla \eta| + u^2 (\Delta \eta)^2 \right] dx + C_2 \int_{\Sigma} uv|\nabla \eta|^2 dx.
\]
On the other hand, as \( u = 0 \) on \( \partial \Sigma \),
\[
2 \int_{\Sigma} |\nabla u|^2 |\nabla \eta|^2 dx = \int_{\Sigma} \Delta(u^2) |\nabla \eta|^2 dx + 2 \int_{\Sigma} uv |\nabla \eta|^2 dx = \int_{\Sigma} u^2 \Delta(|\nabla \eta|^2) dx + 2 \int_{\Sigma} uv|\nabla \eta|^2 dx.
\]
Inputting the above inequality into (2.5) with \( \varphi = \varphi_0(R^{-1}x) \) for \( R > 0 \), \( \eta = \varphi^m \) and \( m = \frac{2p+2}{p-1} > 2 \), we

arrive at
\[
\int_{\Sigma} (u^2 + u^{p+1}) \varphi^{2m} dx \leq \frac{C}{R^4} \int_{\Sigma} u^2 \varphi^{2m-4} dx
\]
(2.6)
\[
\leq \frac{C}{R^4} \left( \int_{\Sigma} u^{p+1} \varphi^{(p+1)(m-2)} dx \right)^{\frac{2}{p+1}} R^{-\frac{N(p-1)}{p+1}}
\]
\[
= \frac{C}{R^4} \left( \int_{\Sigma} u^{p+1} \varphi^{2m} dx \right)^{\frac{2}{p+1}} R^{-\frac{N(p-1)}{p+1}}.
\]
Hence
\[
\int_{\Sigma} u^{p+1} \varphi^{2m} dx \leq C R^{-4 - \frac{8}{p+1} - \frac{1}{p+1}}.
\]
Combining with (2.6), as \( \varphi^{2m} = 1 \) for \( x \in B_R := \{ x \in \mathbb{R}^N, |x| \leq R \} \), (2.1) is proved. \( \square \)

**Proof of Proposition 1.2.** Let
\[
\zeta = \beta u^\frac{p+1}{2} - v, \quad \text{where} \quad \beta = \sqrt{\frac{2}{p+1}}.
\]
Then a direct computation shows that \( \Delta \zeta \geq \beta^{-1} u^\frac{p+1}{2} \zeta \) in \( \Sigma \). Consider \( \zeta_+ := \max(\zeta, 0) \), there holds, for any \( R > 0 \)
\[
\int_{\Sigma \cap B_R} |\nabla \zeta_+|^2 dx = -\int_{\Sigma \cap B_R} \zeta_+ \Delta \zeta dx + \int_{\partial(\Sigma \cap B_R)} \zeta_+ \frac{\partial \zeta}{\partial \nu} d\sigma \leq \int_{\Sigma \cap B_R} \zeta_+ \frac{\partial \zeta}{\partial \nu} d\sigma.
\]
(2.7)
Here we used \( \zeta_+ \Delta \zeta \geq 0 \) in \( \Sigma \) and \( \zeta = 0 \) on \( \partial \Sigma \). Denote now \( S^{N-1} \) the unit sphere in \( \mathbb{R}^N \) and
\[
e(r) = \int_{S^{N-1}(r^{-1} \Sigma)} \zeta_+^2 (r \sigma) d\sigma \quad \text{for} \quad r > 0.
\]
We remark that \( \exists R_0 > 0 \) verifying
\[
\int_{\Sigma \cap \partial B_r} \zeta_+ \frac{\partial \zeta}{\partial \nu} d\sigma = \frac{r^{N-1}}{2} e'(r), \quad \forall \; r \geq R_0.
\]
(2.8)
Moreover, for \( R \geq R_0 \), we deduce from (2.1) that
\[
\int_{R_0}^{R} r^{N-1} e(r) dr \leq \int_{B_R \cap \Sigma} \zeta_+^2 dx \leq C \int_{B_R \cap \Sigma} (v^2 + u^{p+1}) dx \leq C R^{-4 - \frac{8}{p+1}} = o \left( R^N \right).
\]
This means that the function \( e \) cannot be nondecreasing at infinity, so that there exists \( R_j \to \infty \) satisfying \( e'(R_j) \leq 0 \). Combining (2.7) and (2.8) with \( R = R_j \to \infty \), there holds
\[
\int_{\Sigma} |\nabla \zeta_+|^2 dx = 0.
\]
Using \( \zeta = 0 \) on \( \partial \Sigma \), we have \( \zeta_+ \equiv 0 \) in \( \Sigma \), or equivalently (1.8) holds true in \( \Sigma \). Clearly \( v > 0 \) in \( \Sigma \) by (1.8). \( \square \)

In the following, we show some properties of the polynomials \( L \) and \( H \), useful for our proofs. Let
\[
L(s) = s^4 - 32 \frac{p}{p+1} s^2 + 32 \frac{p(p+3)}{(p+1)^2} s - 64 \frac{p}{(p+1)^2}, \quad s \in \mathbb{R}.
\]
(2.9)
**Lemma 2.2.** \( L(2t_0) < 0 \) and \( L \) has a unique root \( s_0 \) in the interval \((2t_0, \infty)\).
Proof. Obviously
\[ L(2t_0) = 16t_0^4 - 128 \frac{p}{p+1} t_0^2 + 64 \frac{p(p+3)}{(p+1)^2} t_0 - 64 \frac{p}{(p+1)^2} \]

By \( \frac{t_0^2}{2t_0-1} = \sqrt{\frac{2p}{p+1}} \) (see [2]), there holds \( t_0^4 = \frac{2p}{p+1} (2t_0 - 1)^2 \). A direct computation yields
\[
\frac{(p+1)^2 L(2t_0)}{32p} = (p+1)(2t_0 - 1)^2 - 4(p+1)t_0^2 + 2(p+3)t_0 - 2
\]
\[ = (p-1)(1-2t_0). \]

As \( t_0 > 1 \) for any \( p > 1 \), we have \( L(2t_0) < 0 \). Furthermore, \( \forall \, p > 1, \, s \geq 2t_0, \) we have
\[
(p+1)L''(s) = 12(p+1)s^2 - 64p \geq 48(p+1)t_0^2 - 64p
\]
\[ \geq 48(p+1) \frac{2p}{p+1} - 64p
\]
\[ = 32p > 0 \]
in \( [2t_0, \infty) \), where we used \( t_0^2 \geq \frac{2p}{p+1} \) which holds by (1.6). Therefore \( L \) is convex in \( [2t_0, \infty) \).
Since \( \lim_{s \to \infty} L(s) = \infty \) and \( L(2t_0) < 0 \), it’s clear that \( L \) admits a unique root in \( (2t_0, \infty) \). □

Remark 2.3. After the change of variable \( x = \frac{p+1}{p-1}s \), a direct calculation gives
\[ H(x) = \left( \frac{p+1}{p-1} \right)^4 L(s), \text{ hence } H(x) < 0 \text{ if and only if } L(s) < 0. \]
Using the above Lemma, \( x_0 = \frac{p+1}{p-1}s_0 \) is the largest root of the polynomial \( H \), and \( x_0 \) is the unique root of \( H \) for \( x \geq \frac{2(p+1)}{p-1} t_0 \).

Lemma 2.4. Let \( x_0 = \frac{p+1}{p-1}s_0 \) be the largest root of \( H \). Then \( x_0 > 5 \) for any \( p > 1 \).

Proof. As \( x_0 \) is the largest root of \( H \), to have \( x_0 > 5 \), it suffices to show \( H(5) < 0 \). Let \( J(p) = (p-1)^4 H(5) \), then \( J(p) = -15p^4 - 1284p^3 + 4262p^2 - 3844p + 625 \). Therefore,
\[ J'(p) = -60p^3 - 3852p^2 + 8524p - 8844, \quad J''(p) = -180p^2 - 7704p + 8524. \]
We see that \( J'' < 0 \) in \( [2, \infty) \). Consequently \( J'(p) < 0 \) and \( J(p) < 0 \) for \( p \geq 2 \). Hence \( x_0 > 5 \) if \( p \geq 2 \). For \( p \in (1, 2) \), there holds \( x_0 > \frac{2(p+1)}{p-1} t_0 \geq 6t_0 \) which exceeds 5 as \( t_0 > 1 \). □

3. Proof of Theorems 1.1 and 1.3

We will prove only Theorem 1.1, since the proof of Theorem 1.3 is completely similar, where we just change \( B_r \) by \( B_r \cap \mathbb{R}_+^N \).

The following result generalizes Lemma 4 in [2], which is a crucial argument for our proof. As above, the constant \( C \) always denotes a positive number which may change term by term, but does not depend on the solution \( u \). For \( k \in \mathbb{N} \), let \( R_k := 2^k R \) with \( R > 0 \).

Lemma 3.1. Assume that \( u \) is a classical stable solution of (1.1). Then for all \( 2 \leq s \leq s_0 \), there is \( C < \infty \) such that
\[
(3.1) \quad \int_{B_{R_k}} u^p v^{s-1} dx \leq \frac{C}{R^2} \int_{B_{R_{k+1}}} v^s dx, \quad \forall \, R > 0.
\]
Proof. Let \( u \) be a classical stable solution of (1.1). Let \( \phi \in C^2_0(\mathbb{R}^N) \) and \( \varphi = u^{q+1} \) with \( q \geq 1 \). Take \( \varphi \) into the stability inequality (1.11), we obtain

\[
\sqrt{p} \int_{\mathbb{R}^N} u^{p-1} u^{q+1} \phi^2 dx \leq \int_{\mathbb{R}^N} u^{q+1} |\nabla \phi|^2 + \int_{\mathbb{R}^N} |\nabla u^{q+1/2} \phi^2| + (q + 1) \int_{\mathbb{R}^N} u^{q} \phi \nabla u \nabla \phi
\]

Integrating by parts, we get

\[
\int_{\mathbb{R}^N} |\nabla u^{q+1/2} \phi^2| dx = \frac{1}{2} \int_{\mathbb{R}^N} u^{q-1} |\nabla u| \phi^2 dx
\]

(3.3)

\[
= \frac{(q + 1)^2}{4q} \int_{\mathbb{R}^N} \phi^2 \nabla(u^q) \nabla u dx
\]

and

\[
(q + 1) \int_{\mathbb{R}^N} u^{q} \phi \nabla u \nabla \phi dx = \frac{1}{2} \int_{\mathbb{R}^N} \nabla(u^{q+1}) \nabla(\phi^2) dx = -\frac{1}{2} \int_{\mathbb{R}^N} u^{q+1} \Delta(\phi^2) dx.
\]

Combining (3.2)-(3.4), we conclude that

\[
a_1 \int_{\mathbb{R}^N} u^{p-1} u^{q+1} \phi^2 dx \leq \int_{\mathbb{R}^N} u^{q+1} \phi^2 dx + C \int_{\mathbb{R}^N} \nabla(u^{q+1}) (|\Delta(\phi^2)| + |\nabla \phi|^2) dx
\]

where \( a_1 = \frac{4q}{(q+1)^2} \). Choose now \( \phi(x) = h(R_k^{-1} x) \) where \( h \in C^\infty_0(B_2) \) such that \( h \equiv 1 \) in \( B_1 \), there holds then

\[
\int_{\mathbb{R}^N} u^{p-1} u^{q+1} \phi^2 dx \leq \frac{1}{a_1} \int_{\mathbb{R}^N} u^{q+1} \phi^2 dx + \frac{C}{R^2} \int_{B_{R_k+1}} u^{q+1} dx
\]

(3.6)

Now, apply the stability inequality (1.11) with \( \varphi = v^{r+1} \phi \), \( r \geq 1 \), to obtain

\[
\sqrt{p} \int_{\mathbb{R}^N} u^{p-1} v^{r+1} \phi^2 \leq \int_{\mathbb{R}^N} v^{r+1} |\nabla \phi|^2 + \int_{\mathbb{R}^N} |\nabla v^{r+1} \phi^2| + (r + 1) \int_{\mathbb{R}^N} v^{r} \phi \nabla v \nabla \phi
\]

By a very similar computation (recalling that \( -\Delta v = u^p \)), we have

\[
\int_{\mathbb{R}^N} u^{p-1} v^{r+1} \phi^2 dx \leq \frac{1}{a_2} \int_{\mathbb{R}^N} u^{p} v^r \phi^2 dx + \frac{C}{R^2} \int_{B_{R_k+1}} v^{r+1} dx
\]

where \( a_2 = \frac{4r\sqrt{p}}{(r+1)^2} \).

Using (3.6) and (3.7), there holds

\[
I_1 + a_2^{r+1} I_2 := \int_{\mathbb{R}^N} u^{p-1} u^{q+1} \phi^2 dx + a_2^{r+1} \int_{\mathbb{R}^N} u^{p-1} v^{r+1} \phi^2 dx
\]

(3.8)

\[
\leq \frac{1}{a_1} \int_{\mathbb{R}^N} u^{q+1} \phi^2 dx + a_2 \int_{\mathbb{R}^N} u^{p} v^r \phi^2 dx + \frac{C}{R^2} \int_{B_{R_k+1}} (u^{q+1} + v^{r+1}) dx.
\]

Fix now

\[
2q = (p + 1)r + p - 1, \quad \text{or equivalently} \quad q + 1 = \frac{(p + 1)(r + 1)}{2}.
\]
By Young’s inequality, we get
\[
\frac{1}{a_1} \int_{\mathbb{R}^N} u^q v \phi^2 dx = \frac{1}{a_1} \int_{\mathbb{R}^N} u^{\frac{p-1}{2}} u^{\frac{p+1}{2}} r v \phi^2 dx
\]
\[
= \frac{1}{a_1} \int_{\mathbb{R}^N} u^{\frac{p-1}{2}} u^{\frac{r}{r+1}} (q+1) v \phi^2 dx
\]
\[
\leq \frac{r}{r+1} \int_{\mathbb{R}^N} u^{\frac{p-1}{2}} u^{q+1} \phi^2 dx + \frac{1}{a_1^{r+1}(r+1)} \int_{\mathbb{R}^N} u^{\frac{p}{2}} v^{r+1} \phi^2 dx
\]
\[
= \frac{r}{r+1} I_1 + \frac{1}{a_1^{r+1}(r+1)} I_2
\]
and similarly
\[
a_r^2 \int_{\mathbb{R}^N} u^p v r \phi^2 dx \leq \frac{1}{r+1} I_1 + \frac{a_r^{r+1}}{r+1} I_2
\]
Combining the above two inequalities and (3.8), we deduce then
\[
a_r^{r+1} I_2 \leq \left[ \frac{a_r^{r+1} r}{r+1} + \frac{1}{a_1^{r+1}(r+1)} \right] I_2 + \frac{C}{R^2} \int_{B_{R^{k+1}}} (u^{q+1} + v^{r+1}) dx,
\]
and hence
\[
\frac{(a_1 a_2)^{r+1}}{r+1} I_2 \leq \frac{C a_r^{r+1}}{R^2} \int_{B_{R^{k+1}}} (u^{q+1} + v^{r+1}) dx.
\]
Thus, if \(a_1 a_2 > 1\), by the choice of \(\phi\),
\[
\int_{B_{R^{k}}} u^{\frac{p-1}{2}} v^{r+1} dx \leq I_2 \leq \frac{C}{R^2} \int_{B_{R^{k+1}}} (u^{q+1} + v^{r+1}) dx.
\]
From (1.8) and (3.9), we get \(u^{q+1} \leq C v^{r+1}\). Denote \(s = r+1\), we can conclude that if \(a_1 a_2 > 1\),
\[
\int_{B_{R^{k}}} u^{p} v^{s-1} dx \leq C_1 \int_{B_{R^{k}}} u^{\frac{p-1}{2}} v^{s} dx \leq \frac{C_2}{R^2} \int_{B_{R^{k+1}}} (u^{q+1} + v^{r+1}) dx \leq \frac{C_3}{R^2} \int_{B_{R^{k+1}}} v^{s} dx.
\]
On the other hand, a simple verification shows that
\[
a_1 a_2 > 1\) is equivalent to \(L(s) < 0\).
\]
By Lemma 2.2, for \(s \in [2t_0, s_0)\), there holds \(L(s) < 0\). So the inequality (3.10), i.e. (3.1) holds true for any \(2t_0 \leq s < s_0\). On the other hand, by Lemma 4 of [2], the estimate (3.1) is valid for \(2 \leq s < 2t_0\), hence for \(2 \leq s < s_0\).
\[
\Box
\]

We can follow then the iteration process in [2] (see Proposition 1 or Corollary 2 there) to obtain this consequence:

**Corollary 3.2.** Suppose \(u\) is a classical stable solution of (1.1). For all \(2 \leq \beta < \frac{N}{N-2} s_0\), there are \(\ell \in \mathbb{N}\) and \(C < \infty\) such that
\[
\left( \int_{B_R} v^{\beta} dx \right)^{\frac{1}{\beta}} \leq C R^\frac{N}{2} \left( \int_{B_{R^{\ell}}} v^{2} dx \right)^{\frac{1}{2}}, \quad \forall R > 0.
\]
Now we are in position to complete the proof of Theorem 1.1. Let \( u \) be a smooth stable solution to (1.1), applying Corollary 3.2 and (2.1): For any \( 2 \leq \beta < \frac{N}{N-2}s_0 \), there exists \( C > 0 \) such that
\[
\left( \int_{B_R} v^\beta \, dx \right)^{\frac{1}{\beta}} \leq CR^{\frac{N}{2}(\frac{\beta}{2}-1)+\frac{N}{2}}\frac{1}{2-p}, \quad \forall \, R > 0.
\]
Note that
\[
\frac{N}{2} \left( \frac{2}{\beta} - 1 \right) + \frac{N}{2} - 2 - \frac{4}{p-1} < 0 \iff N < \frac{2(p+1)}{p-1}\beta.
\]
Considering the allowable range of \( \beta \) given in Corollary 3.2, if \( N < 2 + 2\frac{p+1}{p-1}s_0 \), after sending \( R \to \infty \) we get then \( \|v\|_{L^\beta(\mathbb{R}^N)} = 0 \), which is impossible since \( v \) is positive. To conclude, the equation (1.1) has no classical stable solution if \( N < 2 + 2s_0 \) where \( s_0 = \frac{p+1}{p-1} \).

Moreover, by Lemma 2.4, \( x_0 > 5 \) for any \( p > 1 \), which means that if \( N \leq 12 \), (1.1) has no classical stable solution for all \( p > 1 \). \( \square \)

4. Proof of Theorem 1.4

In this section, we consider the elliptic problem \((P_\lambda)\). Let \( u_\lambda \) be the minimal solution of \((P_\lambda)\), it is well known that \( u_\lambda \) is stable. To simplify the presentation, we erase the index \( \lambda \). By [4, 5], there holds
\[
\sqrt{\lambda^p} \int_\Omega (u + 1)^{\frac{p-1}{2}} \varphi^2 \, dx \leq \int_\Omega |\nabla \varphi|^2 \, dx, \quad \forall \, \varphi \in H_0^1(\Omega)
\]
Using \( \varphi = u^{\frac{q+1}{2}} \) as test function in (3.2), by similar computation as for (3.5) in section 3, we obtain
\[
a_1 \sqrt{\lambda} \int_\Omega (u + 1)^{\frac{p-1}{2}} u^{q+1} \, dx \leq \int_\Omega u^q v \, dx, \quad \text{where} \quad a_1 = \frac{4q \sqrt{p}}{(q+1)^2}.
\]
Here we do not need a cut-off function \( \phi \), because all boundary terms appearing in the integrations by parts vanish under the Navier boundary conditions, hence the calculations are even easier. We can use the Young’s inequality as for Theorem 1.1, but we show here a proof inspired by [6].

Similarly as for (3.7), using \( \varphi = v^{r+1} \) in (4.1), we have
\[
a_2 \sqrt{\lambda} \int_\Omega (u + 1)^{\frac{p-1}{2}} v^{r+1} \, dx \leq \int_\Omega \lambda(u + 1)^p v^r \, dx, \quad \text{where} \quad a_2 = \frac{4r \sqrt{p}}{(r+1)^2}.
\]
Take always \( 2q = (p+1)r + p - 1 \). Applying Hölder’s inequality, there hold
\[
\int_\Omega u^q v \, dx \leq \left( \int_\Omega u^{\frac{p-1}{2}} v^{r+1} \, dx \right)^{\frac{r}{r+1}} \left( \int_\Omega u^{\frac{p-1}{2}+q+1} \, dx \right)^{\frac{1}{r+1}}
\]
\[
\leq \left[ \int_\Omega (u + 1)^{\frac{p-1}{2}} v^{r+1} \, dx \right]^{\frac{1}{r+1}} \left[ \int_\Omega (u + 1)^{\frac{p-1}{2}+q+1} \, dx \right]^{\frac{1}{r+1}}
\]
and
\[
\int_\Omega (u + 1)^p v^r \, dx \leq \left[ \int_\Omega (u + 1)^{\frac{p-1}{2}} v^{r+1} \, dx \right]^{\frac{r}{r+1}} \left[ \int_\Omega (u + 1)^{\frac{p-1}{2}+q+1} \, dx \right]^{\frac{1}{r+1}}.
\]
Multiplying (4.2) with (4.3), using (4.4) and (4.5), we get immediately

\[(4.6) \quad \left[ \int_{\Omega} (u + 1)^{p - 1 - q} dx \right]^{\frac{1}{p+1}} \leq \frac{1}{a_1 a_2} \left[ \int_{\Omega} (u + 1)^{p - 1 + q} dx \right]^{\frac{1}{p+1}}. \]

On the other hand, for any \( \varepsilon > 0 \) there exists \( C_{\varepsilon} > 0 \) such that

\[ (u + 1)^{p - 1 + q} \leq (1 + \varepsilon)(u + 1)^{p - 1} + C_{\varepsilon} \quad \text{in} \quad \mathbb{R}_+. \]

If \( a_1 a_2 > 1 \), there exists \( \varepsilon_0 > 0 \) satisfying \( 1 + \varepsilon_0 < (a_1 a_2)^{r+1} \). We deduce from (4.6) that

\[ \left[ 1 - \frac{1 + \varepsilon_0}{(a_1 a_2)^{r+1}} \right] \int_{\Omega} (u + 1)^{p - 1 - q} dx \leq C. \]

Therefore, when \( L(s) < 0 \), i.e. when \( a_1 a_2 > 1 \), there is \( C > 0 \) such that

\[ \int_{\Omega} u^{p - 1 + q} dx \leq \int_{\Omega} (u + 1)^{p - 1} u^{q+1} dx \leq C. \]

As \( u^* = \lim_{\lambda \to 0^+} u_\lambda \), we conclude, using Lemma 2.2,

\[(4.7) \quad u^* \in L^{\frac{p-1}{2} + q+1}(\Omega), \quad \text{for all} \quad q \text{ satisfying} \quad \frac{2(q + 1)}{p+1} = r + 1 = s < s_0. \]

Furthermore, by [10], we know that \( u^* \in H^2(\Omega) \). As \( u^* \geq 0 \) verifies \( \Delta^2 u^* = \lambda^*(u^* + 1)^p \leq C(u^*)^{p-1} u^* + C \) with \( u^* = \Delta u^* = 0 \) on \( \partial \Omega \), by standard elliptic estimate, we know that \( u^* \) is smooth if

\[ \frac{N}{4} < \left( \frac{p - 1}{2} + q + 1 \right) - 1 = \frac{1}{2} \left( 1 + \frac{p + 1}{p - 1} s \right). \]

Therefore, \( u^* \) is smooth if \( N < 2 + 2x_0 \). By Lemma 2.4, \( u^* \) is smooth for any \( p > 1 \) if \( N \leq 12 \). \( \square \)

Acknowledgments D.Y. is partially supported by the French ANR project referenced ANR-08-BLAN-0335-01. This work was partially realized during a visit of A.H. at the University of Lorraine, he would like to thank the Laboratoire de Mathématiques et Applications de Metz for the kind hospitality.

References


**Institut de mathématiques appliquées et d’informatiques, Kairouan, Tunisia**

*E-mail address: hajlaouihatem@gmail.com*

**Institut de mathématiques appliquées et d’informatiques, Kairouan, Tunisia**

*E-mail address: abdellaziz.harrabi@yahoo.fr*

**IECL, UMR 7502, Université de Lorraine, 57045 Metz, France**

*E-mail address: dong.ye@univ-lorraine.fr*