Integral Equations for Acoustic Scattering by Partially Impenetrable Composite Objects

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Abstract. We study direct first-kind boundary integral equations arising from transmission problems for the Helmholtz equation with piecewise constant coefficients and Dirichlet boundary conditions imposed on a closed surface. We identify necessary and sufficient conditions for the occurrence of so-called spurious resonances, that is, the failure of the boundary integral equations to possess unique solutions.

Following [A. Buffa and R. Hiptmair, *Regularized combined field integral equations*, Numer. Math., 100 (2005), pp. 1–19] we propose a modified version of the boundary integral equations that is immune to spurious resonances. Via a gap construction it will serve as the basis for a universally well-posed stabilized global multi-trace formulation that generalizes the method of [X. Claeys and R. Hiptmair, *Multi-trace boundary integral formulation for acoustic scattering by composite structures*, Communications on Pure and Applied Mathematics, 66 (2013), pp. 1163–1201] to situations with Dirichlet boundary conditions.

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1 Introduction

We are concerned with boundary integral equations (BIE) describing the propagation of acoustic waves in so-called composite media composed of parts with linear and spatially homogenous material properties. Such media are rather common in mathematical models in engineering and well-posed BIE are important as foundation for boundary element methods (BEM), a well established and widely used technique for computational acoustics.

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The bulk of mathematical investigations on BIE has addressed the case of only two different homogeneous media, with one occupying a bounded volume in space, see, for instance, [16], [26, Ch. 9], [32, Sect. 3.9], and the monographs [27, 15]. Apparently, the first profound mathematical derivation and analysis of particular direct BIEs for acoustics with composite media was given in [37]. Of course, boundary element methods for composite scattering had been devised before in computational engineering, notably the so-called Poggio-Miller-Chew-Harrington-Wu-Tsai (PMCHWT) integral equations [31, 7, 39, 18] for electromagnetic scattering.

The BIEs proposed in [37] may be dubbed a *single trace formulation* (STF), because they involve a single pair of Cauchy data on each interface as unknowns. They can legitimately be regarded as *the* standard direct BIEs for transmission problems, because they immediately arise from fundamental Calderón identities and the transmission conditions are imposed strongly through the trial trace spaces. If all participating media are penetrable, the BIEs of STF are well-posed in natural trace spaces, see [12, Sect. 3.2], [11, Prop. A.1]. However, if impenetrable media are admitted, the standard STF may be affected by the notorious *spurious resonance phenomenon*, that is, for particular combinations of wave numbers the BIE may fail to possess unique solutions. This has not been properly addressed in [37] and in Section 4 we provide a detailed analysis of when the STF becomes vulnerable to spurious resonances. In short, spurious resonances can occur, if an impenetrable part is completely surrounded by another homogeneous medium, see Theorem 4.8.

To restore unconditional well-posedness of the STF, we adapt the classical idea of combined field integral equations (CFIE), both in its indirect and direct version, *cf.* [1, 24, 28] for the former, and [6] for the latter. Sloppily speaking, CFIEs exploit the capacity of (approximate) absorbing boundary conditions to ensure unique solvability of time-harmonic wave propagation problems even on bounded domains, whereas a discrete set of resonant frequencies will always haunt pure Dirichlet or Neumann boundary conditions. The simplest choice of approximate absorbing boundary conditions is plain impedance or Robin boundary conditions with non-zero purely imaginary impedance, see [32, Sect. 3.4.9]. Yet, in this work, we rely on regularized or modified versions of CFIEs from [3, 36], which are compatible with variational formulations in natural trace spaces. The corresponding extensions of the single trace boundary integral equations are studied in Section 5.

Another drawback of the classical STF-BIEs, when used as the foundation for low-order Galerkin boundary element discretization, is their failure to be amenable to the powerful and popular Calderón preconditioning techniques [19, 34, 8]. For lucid explanations refer to [12, Sect. 4]. Lately, this shortcoming of the STF has prompted the development of so-called *multi-trace formulations* (MTF) for scattering at composite objects. They feature four unknown traces at (some) material interfaces and come in two flavors: global MTFs as introduced in [9, 11, 12, 10] and [12, Sect. 5], and local MTFs presented in [20, 21]. They all have in common that they enforce the transmission conditions only weakly, in contrast to the STF. Thus, trial and test spaces can neatly be split into contributions of different sub-domains, and, in the spirit of *domain decomposition*, this paves the way for local preconditioning.

Thus far, all mathematical analyses of MTFs eschew non-penetrable media, except for [13], which is confined to pure diffusion problems. Only in computational engineering some recent variants of local MTF for computational electromagnetics [29, 30] include CFIE ideas in order to treat impenetrable, that is, perfectly electrically conducting, bodies. In this article, in Section 5, we propose a CFIE-type extension of the global MTF introduced in [11]. It naturally emerges from single trace CFIEs appealing to the "gap idea" described in [11, Sect. 5] and [12, Sect. 5.2]. The new global multi-trace CFIEs inherit unconditional stability and turn out to be a compact perturbation of the previously known global MTF. Thus, the customary Calderón preconditioning technique [12, Sect. 4] can be applied to them.

Discretization, for instance, by Galerkin boundary element methods, will not be addressed in this article. However, coercivity of variational formulations in spaces of Cauchy traces together with uniqueness of solutions, immediately allows to conclude quasi-optimality of conforming Galerkin BEM, see [17], [38], and [32, Sect 4.2.3]. Hence, our theory paves the way for predicting the convergence of all varieties of Galerkin BEM for both single- and multi-trace CFIE provided that the smoothness of Cauchy traces of the exact field solution is known.

List of notations

Ω_i	material sub-domains $\subset \mathbb{R}^d$, Ω_0 unbounded, see Fig. 1
n	number of (bounded) sub-domains with penetrable medium
$\Sigma := \partial\Omega_\Sigma$	Boundary where homogeneous Dirichlet boundary conditions are imposed
Γ	union of interfaces (skeleton), see (2.1)
γ_{D}^j, γ_{N}^j	Dirichlet and Neumann trace operators on $\partial\Omega_j$, see (2.4)
γ^j	Cauchy trace operator defined in (2.5)
$\mathbb{H}(\partial\Omega_j)$	Cauchy trace space associated with $\partial\Omega_j$, see (3.1)
$\mathbb{H}(\Gamma)$	Multi-trace space as defined in (3.1)
$\langle\cdot,\cdot\rangle_j$	Duality pairing between Dirichlet and Neumann traces on $\partial\Omega_j$
$[\cdot,\cdot]$	self-duality pairing on $\mathbb{H}(\Gamma)$
$\mathbb{X}^{\pm\frac{1}{2}}(\Gamma)$, $\mathbb{X}(\Gamma)$	single trace Dirichlet/Neumann/Cauchy spaces, see (3.5), (3.6)
T_{D}, T_{N}, T	restriction of single trace functions onto Σ, see Propositions 3.1, 3.2
SL_κ^j	single layer potential defined on $\partial\Omega_j$
DL_κ^j	double layer potential defined on $\partial\Omega_j$
G_κ^j	total potential defined on $\partial\Omega_j$
$\mathcal{C}_\kappa(\partial\Omega_j)$	space of Cauchy data on $\partial\Omega_j$

$\mathsf{A}^{j}_{\kappa_j}$	boundary integral operator on $\partial\Omega_j$
$\mathsf{B}_{i,j}$	non-local "remote" coupling boundary integral operators
$\mathbb{X}_0(\Gamma)$	single trace space with vanishing Dirichlet data on Σ, see (4.1)

2 Setting of the problem

In the present article, we consider a partition $\mathbb{R}^d = \cup_{j=0}^n \overline{\Omega}_j \cup \overline{\Omega}_\Sigma$ where Ω_Σ and the Ω_j for $j \neq 0$ are open, bounded, and mutually disjoint, and each Ω_Σ, Ω_j is a Lipschitz domain [26, Def. 3.28].

In addition, we assume that Ω_Σ, $\mathbb{R}^d\backslash\overline{\Omega}_\Sigma$, and each Ω_j are connected. An important consequence of these assumptions is that Ω_Σ does not contain any hole which rules out the presence of an internal resonant cavity. We set

$$\Gamma := \cup_{j=0}^n \partial\Omega_j \quad \text{(the "skeleton")} \quad \text{and} \quad \Sigma := \partial\Omega_\Sigma\ . \tag{2.1}$$

As in Figure 1 there may exist points where three or more sub-domains abut, which is precisely the situation that we wish to tackle. We consider the following transmission problem for the Helmholtz equation: Find $U \in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^d\backslash\Omega_\Sigma)$ [1]such that

$$\begin{cases} -\Delta U - \kappa_j^2 U = 0 \quad \text{in } \Omega_j \\ U - U_{\mathrm{inc}} \text{ is } \kappa_0\text{-outgoing in } \Omega_0 \end{cases} \tag{2.2a}$$

$$\begin{cases} U|_{\partial\Omega_j} - U|_{\partial\Omega_k} &= 0 \\ \partial_{n_j} U|_{\partial\Omega_j} - \partial_{n_k} U|_{\partial\Omega_k} &= 0 \end{cases} \quad \text{on } \partial\Omega_j \cap \partial\Omega_k \tag{2.2b}$$

$$\{ \quad U|_\Sigma = 0\ . \tag{2.2c}$$

In equation (2.2a), the outgoing condition refers to Sommerfeld's radiation condition, i.e. if $\omega \subset \mathbb{R}^d$ is any bounded open subset, we shall say that $V \in \mathrm{H}^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d\backslash\omega)$ is κ-outgoing if

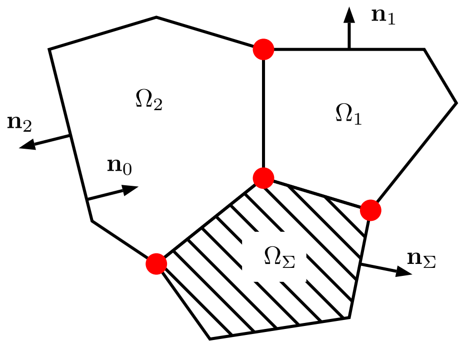
$$\lim_{\rho\to\infty} \int_{\partial\mathrm{B}_\rho} |\partial_r V - i\kappa V|^2 d\sigma_\rho = 0$$

where B_ρ is the ball of center 0 and radius ρ, $d\sigma_\rho$ is the surface measure on $\partial\mathrm{B}_\rho$, and ∂_r refers to the radial derivative. Sommerfeld's radiation is presented in detail for example in [27, §.2.6.5] or [25, §.4.4]. For the sake of simplicity and clarity, we asume that all wave numbers are positive

$$\kappa_j > 0\ , \quad j = 0, \ldots, n\ . \tag{2.3}$$

Then Problem (2.2) admits a unique solution U, as proved in [37, Sect. 2].

[1]We follow the usual notations; given some open subset $\omega \subset \mathbb{R}^d$, we define $\mathrm{H}^1(\omega) := \{v \in \mathrm{L}^2(\omega) \mid \nabla v \in \mathrm{L}^2(\omega)\}$ with $\|v\|^2_{\mathrm{L}^2(\omega)} := \|v\|^2_{\mathrm{L}^2(\omega)} + \|\nabla v\|^2_{\mathrm{L}^2(\omega)}$, and $\mathrm{H}^1(\Delta,\omega) := \{v \in \mathrm{H}^1(\omega) \mid \Delta v \in \mathrm{L}^2(\omega)\}$. If $\mathrm{H}(\omega)$ is any one of these spaces, $\mathrm{H}_{\mathrm{loc}}(\overline{\omega}) := \{v \mid \varphi v \in \mathrm{H}(\omega)\ \ \forall\varphi \in \mathscr{C}^\infty_{\mathrm{K}}(\mathbb{R}^d)\}$, where $\mathscr{C}^\infty_{\mathrm{K}}(\mathbb{R}^d)$ refers to the space of C^∞ function with compact support.



Ω_0 = exterior domain
For each j the vector $\mathbf{n}_j$ refers to the normal vector on $\partial\Omega_j$ directed toward the *exterior* of Ω_j, and $\mathbf{n}_\Sigma$ denotes the vector normal to Σ directed toward the exterior of Ω_Σ. The existence of such vector fields is guaranteed by Rademacher's theorem [32,Thm. 2.7.1].

FIGURE 1. Geometric setting for the Helmholtz transmission problem for composite media with impenetrable Ω_Σ.

As it involves transmission conditions, and since we will be interested in the derivation of boundary integral equations adapted to this problem, we need to introduce suitable trace operators. According to [32, Thm. 2.6.8 and Thm. 2.7.7], for every subdomain $\Omega_j, j = 0 \dots n$, there exist continuous *trace operators* $\gamma_{\mathrm{D}}^j : \mathrm{H}^1_{\mathrm{loc}}(\overline{\Omega}_j) \to \mathrm{H}^{1/2}(\partial\Omega_j)$ and $\gamma_{\mathrm{N}}^j : \mathrm{H}^1_{\mathrm{loc}}(\Delta, \overline{\Omega}_j) \to \mathrm{H}^{-1/2}(\partial\Omega_j)$ (so-called Dirichlet and Neumann traces) by density defined through

$$\gamma_{\mathrm{D}}^j(\varphi) := \varphi|_{\partial\Omega_j} \quad \text{and} \quad \gamma_{\mathrm{N}}^j(\varphi) := \mathbf{n}_j \cdot \nabla\varphi|_{\partial\Omega_j} \qquad \forall\varphi \in \mathscr{C}^\infty(\overline{\Omega}_j)\ . \tag{2.4}$$

We use similar notations for traces on Σ with $\mathbf{n}_\Sigma$ fixing the orientation of the Neumann trace, see Figure 1. Both traces can be merged into the *interior Cauchy trace operators*

$$\gamma^j(v) := \begin{bmatrix} \gamma_{\mathrm{D}}^j(v) \\ \gamma_{\mathrm{N}}^j(v) \end{bmatrix} \qquad \forall v \in \mathrm{H}^1_{\mathrm{loc}}(\Delta, \overline{\Omega}_j)\ . \tag{2.5}$$

Traces from the exterior of Ω_j spawn the exterior Cauchy trace operators $\gamma_c^j : \mathrm{H}^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d \backslash \overline{\Omega}_j) \to \mathrm{H}^{1/2}(\partial\Omega_j) \times \mathrm{H}^{-1/2}(\partial\Omega_j)$, whose Neumann trace is still based on the normal $\mathbf{n}_j$.

Remark 2.1. Forgoing generality in favor of clarity and brevity, we focus on the rather simple problem (2.2) as typical specimen of transmission problem describing acoustic scattering. Straightforward extensions of the approach in this article can cope with the following situations:

- several impenetrable subdomains (not just one),
- Neumann (instead of Dirichlet) boundary conditions imposed on Σ,
- wave-numbers κ_j with non-vanishing imaginary part,
- piecewise constant coefficients in the second-order part of the differential operator as in [12],
- more general source terms (for example, general inhomogeneous transmission and boundary conditions).

These points would entail only minor adjustments in our analysis. We refer the reader to [11, 12] for more details on how to deal with more complex situations. In [12] electromagnetic scattering problems are treated alongside their acoustic counterparts in a unified setting. Following this policy and the CFIE ideas of [2], the considerations of this article could also be generalized to electromagnetic wave propagation.

3 Trace spaces

We want to recast Problem (2.2) into variational boundary integral equations, so that these are immune to spurious resonances. We aim for BIE set in natural trace spaces. The most fundamental trace space we can introduce consist is the *multi-trace space* [11, Sect. 2.1], the Cartesian product of local traces:

$$\begin{aligned} \mathbb{H}(\Gamma) \;&:=\; \mathbb{H}(\partial\Omega_0)\times\cdots\times\mathbb{H}(\partial\Omega_n) \\ \text{where}\quad \mathbb{H}(\partial\Omega_j) &:= \mathrm{H}^{+\frac{1}{2}}(\partial\Omega_j)\times\mathrm{H}^{-\frac{1}{2}}(\partial\Omega_j)\;. \end{aligned} \tag{3.1}$$

We endow each $\mathbb{H}(\partial\Omega_j)$ with the norm given by $\|(v,q)\|_{\mathbb{H}(\partial\Omega_j)} := (\|v\|^2_{\mathrm{H}^{1/2}(\partial\Omega_j)}+\|q\|^2_{\mathrm{H}^{-1/2}(\partial\Omega_j)})^{1/2}$, and equip $\mathbb{H}(\Gamma)$ with the norm naturally associated with the cartesian product

$$\|\mathfrak{u}\|_{\mathbb{H}(\Gamma)} := \Big(\;\|\mathfrak{u}_0\|^2_{\mathbb{H}(\partial\Omega_0)}+\cdots+\|\mathfrak{u}_n\|^2_{\mathbb{H}(\partial\Omega_n)}\;\Big)^{1/2}$$

for $\mathfrak{u}=(\mathfrak{u}_0,\dots,\mathfrak{u}_n)\in\mathbb{H}(\Gamma)$[2]. We write $\langle\cdot,\cdot\rangle_j$ for the duality pairing between $\mathrm{H}^{+\frac{1}{2}}(\partial\Omega_j)$ and $\mathrm{H}^{-\frac{1}{2}}(\partial\Omega_j)$. We also need a bilinear duality pairing for $\mathbb{H}(\partial\Omega_j)$ and $\mathbb{H}(\Gamma)$; we opt for the skew-symmetric version

$$\begin{aligned} [\mathfrak{u},\mathfrak{v}] &:= \sum_{j=0}^{n}[\mathfrak{u}_j,\mathfrak{v}_j]_j \\ \text{where}\quad \left[\begin{pmatrix}u_j\\p_j\end{pmatrix},\begin{pmatrix}v_j\\q_j\end{pmatrix}\right]_j \;&:=\; \langle u_j,q_j\rangle_j-\langle v_j,p_j\rangle_j\;. \end{aligned} \tag{3.2}$$

This particular choice of a duality pairing is well adapted to the forthcoming analysis. Note that under the duality pairing [,], the space $\mathbb{H}(\Gamma)$ is its own topological dual, and it is easy to show, using the duality between $\mathrm{H}^{1/2}(\partial\Omega_j)$ and $\mathrm{H}^{-1/2}(\partial\Omega_j)$, that the pairing [,] induces an isometric isomorphism between $\mathbb{H}(\Gamma)$ and its dual $\mathbb{H}(\Gamma)'$, equivalent to the inf-sup condition

$$\inf_{\mathfrak{v}\in\mathbb{H}(\Gamma)}\sup_{\mathfrak{u}\in\mathbb{H}(\Gamma)}\frac{|\,[\mathfrak{u},\mathfrak{v}]\,|}{\|\mathfrak{u}\|_{\mathbb{H}(\Gamma)}\|\mathfrak{v}\|_{\mathbb{H}(\Gamma)}}=1. \tag{3.3}$$

[2]Functions in Dirichlet trace spaces like $\mathrm{H}^{+\frac{1}{2}}(\partial\Omega_j)$ will be denoted by u,v,w, whereas we use p,q,r for Neumann traces. Small fraktur font symbols $\mathfrak{u}$, $\mathfrak{v}$, $\mathfrak{w}$ are reseved for Cauchy traces, with an integer subscript indicating restriction to a subdomain boundary. Capital letters will be used to designate scalar functions on domains, whereas small bold letters will be used for vector fields.

We also write $\mathbb{H}(\Sigma) := \mathrm{H}^{1/2}(\Sigma) \times \mathrm{H}^{-1/2}(\Sigma)$ and equip this space of Cauchy traces with the norm $\|(v,q)\|^2_{\mathbb{H}(\Sigma)} := \|v\|^2_{\mathrm{H}^{1/2}(\Sigma)} + \|q\|^2_{\mathrm{H}^{-1/2}(\Sigma)}$. Analogous to (3.2), on this space we shall consider the following skew-symmetric duality pairing

$$\left[\begin{pmatrix} u \\ p \end{pmatrix}\begin{pmatrix} v \\ q \end{pmatrix}\right]_\Sigma := \langle u, q\rangle_\Sigma - \langle v, p\rangle_\Sigma \ . \tag{3.4}$$

3.1 Single-trace spaces

Next, as in [11, Sect. 2.2], [12, Sect. 3.1], we introduce subspaces of traces that respect the transmission conditions (2.2b) across interfaces. We first focus on traces of Dirichlet/Neumann type introducing

$$\begin{aligned} \mathbb{X}^{+\frac{1}{2}}(\Gamma) &:= \Big\{ (u_j)_{j=0}^n \in \Pi_{j=0}^n \mathrm{H}^{\frac{1}{2}}(\partial\Omega_j) \ \Big| \\ &\qquad \exists V \in \mathrm{H}^1(\mathbb{R}^d) \ \text{ s.t. } \ V|_{\partial\Omega_j} = u_j \ \forall j \Big\} , \\ \mathbb{X}^{-\frac{1}{2}}(\Gamma) &:= \Big\{ (p_j)_{j=0}^n \in \Pi_{j=0}^n \mathrm{H}^{-\frac{1}{2}}(\partial\Omega_j) \ \Big| \\ &\qquad \exists \mathbf{q} \in \mathrm{H}(\mathrm{div}, \mathbb{R}^d) \ \text{ s.t. } \ \mathbf{n}_j \cdot \mathbf{q}|_{\partial\Omega_j} = p_j \ \forall j \Big\} . \end{aligned} \tag{3.5}$$

The Cartesian product (up to some permutation of indices) $\mathbb{X}^{1/2}(\Gamma) \times \mathbb{X}^{-1/2}(\Gamma)$ yields the *single-trace space* $\mathbb{X}(\Gamma) \subset \mathbb{H}(\Gamma)$ defined by

$$\begin{aligned} \mathbb{X}(\Gamma) &:= \left\{ \mathfrak{u} = \begin{pmatrix} u_j \\ p_j \end{pmatrix}_{j=0}^n \ \right| \\ &\qquad (u_j)_{j=0}^n \in \mathbb{X}^{+\frac{1}{2}}(\Gamma), \ (p_j)_{j=0}^n \in \mathbb{X}^{-\frac{1}{2}}(\Gamma) \Big\} . \end{aligned} \tag{3.6}$$

Observe that a function $U \in \mathrm{H}^1(\Delta, \Omega_0) \times \cdots \times \mathrm{H}^1(\Delta, \Omega_n)$ satisfies the transmission conditions (2.2b), if and only if $(\gamma^j(U))_{j=0}^n \in \mathbb{X}(\Gamma)$. In particular, if $U \in \mathrm{H}^1(\Delta, \mathbb{R}^d \backslash \Omega_\Sigma)$ then $(\gamma^j(U))_{j=0}^n \in \mathbb{X}(\Gamma)$. Indeed, from an intuitive point of view, the space $\mathbb{X}(\Gamma)$ can be viewed as the space of traces of functions that satisfy the transmission conditions (2.2b). Thus, in the sequel, we will use this space to enforce transmission conditions.

Since every $\mathbf{x} \in \Sigma$ also belongs to some $\partial\Omega_j$, $j = 0, \ldots, n$, functions in $\mathbb{X}^{\pm 1/2}(\Gamma)$ can be expected to induce traces in $\mathrm{H}^{\pm 1/2}(\Sigma)$. This is made precise in the following proposition.

Proposition 3.1. *For every element* $(u_j)_{j=0}^n \in \mathbb{X}^{+1/2}(\Gamma)$*, there exists a unique* $u_\Sigma \in \mathrm{H}^{1/2}(\Sigma)$ *such that* $V|_\Sigma = u_\Sigma$ *for any* $V \in \mathrm{H}^1(\mathbb{R}^d)$ *that satisfies* $V|_{\partial\Omega_j} = u_j$, $j = 0 \ldots n$. *Moreover the linear operator* $\mathsf{T}_{\mathrm{D}} : \mathbb{X}(\Gamma) \to \mathrm{H}^{1/2}(\Sigma)$ *defined by* $\mathsf{T}_{\mathrm{D}}(\,(u_j, p_j)_{j=0}^n\,) := u_\Sigma$ *is continuous and surjective.*

Similarly, for every element $(p_j)_{j=0}^n \in \mathbb{X}^{-1/2}(\Gamma)$*, there exists a unique* $p_\Sigma \in \mathrm{H}^{-1/2}(\Sigma)$ *such that* $\mathbf{n}_\Sigma \cdot \mathbf{p}|_\Sigma = p_\Sigma$ *for any* $\mathbf{p} \in \mathrm{H}(\mathrm{div}, \mathbb{R}^d)$ *that satisfies* $\mathbf{n}_j \cdot \mathbf{p}|_{\partial\Omega_j} = p_j$, $j = 0 \ldots n$. *Moreover the linear mapping* $\mathsf{T}_{\mathrm{N}} : \mathbb{X}(\Gamma) \to \mathrm{H}^{-1/2}(\Sigma)$ *defined by* $\mathsf{T}_{\mathrm{N}}(\,(u_j, p_j)_{j=0}^n\,) := p_\Sigma$ *is continuous and surjective.*

Proof: We prove only the first part of the proposition, as the proof of the second part follows along the same lines. Assume that $u_\Sigma \in \mathrm{H}^{-1/2}(\Sigma)$ satisfies $V|_\Sigma = u_\Sigma$ for one particular $V \in \mathrm{H}^1(\mathbb{R}^d)$ such that $V|_{\partial\Omega_j} = u_j$, $\forall j = 0\dots n$. If $V' \in \mathrm{H}^1(\mathbb{R}^d)$ also satisfies $V|_{\partial\Omega_j} = u_j$, $j = 0\dots n$, then V and V' coincide on Σ since $\Sigma \subset \cup_{j=0}^n \partial\Omega_j$. Hence $u_\Sigma = V'|_\Sigma$. This proves the uniqueness of u_Σ.

Let us construct the map T_{D} explicitely. First, for every subdomain Ω_j we consider a continuous lifting operator $\mathsf{E}_j : \mathrm{H}^{1/2}(\partial\Omega_j) \to \mathrm{H}^1(\Omega_j)$ satisfying $\gamma_{\mathrm{D}}^j \cdot \mathsf{E}_j(v_j) = v_j$. Then define $\mathsf{E} : \mathbb{X}^{1/2}(\Gamma) \to \mathrm{L}^2(\mathbb{R}^d\backslash\overline{\Omega}_\Sigma)$ by combining the E_j according to $\mathsf{E}(\,(u_j)_{j=0}^n)|_{\Omega_j} := \mathsf{E}_j(u_j)$, $j = 0\dots n$.

Actually $\mathsf{E}(\mathbb{X}^{1/2}(\Gamma)) \subset \mathrm{H}^1(\mathbb{R}^d\backslash\overline{\Omega}_\Sigma)$. Indeed, note that $\gamma_{\mathrm{D}}^k \cdot \mathsf{E}(\,(u_j)_{j=0}^n) = u_k$ for all $k = 0\dots n$ and for any choice of the u_j's. Choose $u := (u_j)_{j=0}^n$ arbitrarily in $\mathbb{X}^{1/2}(\Gamma)$. There exists $V \in \mathrm{H}^1(\mathbb{R}^d)$ such that $\gamma_{\mathrm{D}}^j(V) = u_j = \gamma_{\mathrm{D}}^j(\mathsf{E}(u))$, which implies $\gamma_{\mathrm{D}}^j(V - \mathsf{E}(u)) = 0$. From this we conclude $\mathsf{E}(u) - V \in \mathrm{H}^1(\mathbb{R}^d\backslash\overline{\Omega}_\Sigma)$ and finally $\mathsf{E}(u) \in V + \mathrm{H}^1(\mathbb{R}^d\backslash\overline{\Omega}_\Sigma) = \mathrm{H}^1(\mathbb{R}^d\backslash\overline{\Omega}_\Sigma)$.

Now consider any continuous extension operator $\widetilde{\mathsf{E}} : \mathrm{H}^1(\mathbb{R}^d\backslash\overline{\Omega}_\Sigma) \to \mathrm{H}^1(\mathbb{R}^d)$ such that $\widetilde{\mathsf{E}}(V)|_{\mathbb{R}^d\backslash\overline{\Omega}_\Sigma} = V$. Whenever $\mathfrak{u} = (u_j, p_j)_{j=0}^n$ belongs to $\mathbb{X}(\Gamma)$, we have in particular $(u_j)_{j=0}^n \in \mathbb{X}^{1/2}(\Gamma)$, so we can define

$$\mathsf{T}_{\mathrm{D}}(U) := \big(\gamma_{\mathrm{D}}^\Sigma \circ \widetilde{\mathsf{E}} \circ \mathsf{E}\big)\big(\,(u_j)_{j=0}^n\big) \qquad \text{for any } \mathfrak{u} = (u_j, p_j)_{j=0}^n \in \mathbb{X}(\Gamma)\ .$$

With this definition, T_{D} is clearly continuous. In addition, it fulfills the other requirements: setting $V = \widetilde{\mathsf{E}} \circ \mathsf{E}\,\big(\,(u_j)_{j=0}^n\big)$ we have $V \in \mathrm{H}^1(\mathbb{R}^d)$ and $V|_{\partial\Omega_j} = u_j$, $j = 0\dots n$, by construction. In particular, this implies that $u_\Sigma = V|_\Sigma = \mathsf{T}_{\mathrm{D}}(U)$. □

The following elementary result generalizes [11, Eq. (2.2)] and [12, Theorem 3.1] and it will be crucial for many manipulations.

Proposition 3.2. *Define the continuous operator* $\mathsf{T} : \mathbb{X}(\Gamma) \to \mathbb{H}(\Sigma)$ *by the formula* $\mathsf{T}(\mathfrak{u}) = (\mathsf{T}_{\mathrm{D}}(\mathfrak{u}), \mathsf{T}_{\mathrm{N}}(\mathfrak{u}))$. *Then we have*

$$[\mathfrak{u}, \mathfrak{v}] = -\,[\mathsf{T}(\mathfrak{u}), \mathsf{T}(\mathfrak{v})]_\Sigma \qquad \forall \mathfrak{u}, \mathfrak{v} \in \mathbb{X}(\Gamma)\ .$$

Proof: According to the explicit expression of $[\ ,\ ]$ and $[\ ,\ ]_\Sigma$ given by (3.2) and (3.4), it suffices to show that, whenever $\mathfrak{u} = (u_j, p_j)_{j=0}^n \in \mathbb{X}(\Gamma)$ and $\mathfrak{v} = (v_j, q_j)_{j=0}^n \in \mathbb{X}(\Gamma)$, we have

$$\sum_{j=0}^n \langle u_j, q_j\rangle_j = -\,\langle \mathsf{T}_{\mathrm{D}}(\mathfrak{u}), \mathsf{T}_{\mathrm{N}}(\mathfrak{v})\rangle_\Sigma \quad \text{and} \quad \sum_{j=0}^n \langle v_j, p_j\rangle_j = -\,\langle \mathsf{T}_{\mathrm{N}}(\mathfrak{u}), \mathsf{T}_{\mathrm{D}}(\mathfrak{v})\rangle_\Sigma\ .$$

We will prove only the first identity above, as the second can be shown in exactly the same manner, exchanging the roles of $\mathfrak{u}$ and $\mathfrak{v}$. First of all note that $(u_j)_{j=0}^n \in \mathbb{X}^{1/2}(\Gamma)$ since $\mathfrak{u} \in \mathbb{X}(\Gamma)$, and $(q_j)_{j=0}^n \in \mathbb{X}^{-1/2}(\Gamma)$ since $\mathfrak{v} \in \mathbb{X}(\Gamma)$. In addition, according to Proposition 3.1, there exist $G \in \mathrm{H}^1(\mathbb{R}^d)$ and $\mathbf{h} \in \mathrm{H}(\mathrm{div}, \mathbb{R}^d)$ such that

$$G|_{\partial\Omega_j} = u_j\ ,\ \ G|_\Sigma = \mathsf{T}_{\mathrm{D}}(\mathfrak{u}) \quad \text{and} \quad \mathbf{n}_j \cdot \mathbf{h}|_{\partial\Omega_j} = q_j\ ,\ \ \mathbf{n}_\Sigma \cdot \mathbf{h}|_\Sigma = \mathsf{T}_{\mathrm{N}}(\mathfrak{v})\ .$$

As a consequence, applying Green's formula in each Ω_j, Ω_Σ and then in $\mathbb{R}^d$, we obtain

$$\begin{aligned}\langle \mathsf{T}_{\mathrm{D}}(\mathfrak{u}), \mathsf{T}_{\mathrm{N}}(\mathfrak{v})\rangle_\Sigma + \sum_{j=0}^{n} \langle u_j, q_j\rangle_j \\ = \int_{\Omega_\Sigma} \mathrm{div}(\mathbf{h})\, G + \mathbf{h}\cdot\nabla G\, d\mathbf{x} + \sum_{j=0}^{n}\int_{\Omega_j} \mathrm{div}(\mathbf{h})\, G + \mathbf{h}\cdot\nabla G\, d\mathbf{x} \\ = \int_{\mathbb{R}^d} \mathrm{div}(\mathbf{h})\, G + \mathbf{h}\cdot\nabla G\, d\mathbf{x} = 0\,.\end{aligned}$$

□

3.2 Review of potential operators

In this paragraph we recapitulate well-known results concerning the integral representation of solutions of the homogeneous Helmholtz equation in Lipschitz domains. Detailed proofs can be found, for example, in [32, Chap.3].

Let the function $\mathscr{G}_\kappa(\mathbf{x})$ designate the κ-outgoing fundamental solution for the Helmholtz operator $-\Delta - \kappa^2$. For each subdomain Ω_j, for any $\mathfrak{u} = (u, p) \in \mathbb{H}(\partial\Omega_j)$ and any $\mathbf{x} \in \mathbb{R}^d\backslash\partial\Omega_j$, define the single/double layer potential operators by[3]

$$\begin{aligned}\mathsf{SL}_\kappa^j(p)(\mathbf{x}) &:= \int_{\partial\Omega_j} p(\mathbf{y})\, \mathscr{G}_\kappa(\mathbf{x}-\mathbf{y})\, d\sigma(\mathbf{y})\,,\\ \mathsf{DL}_\kappa^j(u)(\mathbf{x}) &:= -\int_{\partial\Omega_j} u(\mathbf{y})\, \mathbf{n}_j(\mathbf{y})\cdot\nabla_{\mathbf{y}}\big(\mathscr{G}_\kappa(\mathbf{x}-\mathbf{y})\big)\, d\sigma(\mathbf{y})\,, \qquad (3.7)\\ \mathsf{G}_\kappa^j(\mathfrak{u})(\mathbf{x}) &:= \mathsf{DL}_\kappa^j(u)(\mathbf{x}) + \mathsf{SL}_\kappa^j(p)(\mathbf{x})\,, \qquad \mathbf{x}\notin\partial\Omega_j\,.\end{aligned}$$

The operator G_κ^j defined above maps continuously $\mathbb{H}(\partial\Omega_j)$ into $\mathrm{H}^1_{\mathrm{loc}}(\Delta, \overline{\Omega}_j) \times \mathrm{H}^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d\backslash\Omega_j)$, see [32, Thm 3.1.16]. In particular G_κ^j can be applied to a pair of traces, *i.e.* Cauchy traces, of the form $\mathfrak{u} = \gamma^j(V)$. This potential operator can be used to write a representation formula for solutions of the homogeneous Helmholtz equation, see [32, Thm 3.1.6].

Proposition 3.3. *Let $U \in \mathrm{H}^1_{\mathrm{loc}}(\overline{\Omega}_j)$ satisfy $\Delta U + \kappa_j^2 U = 0$ in Ω_j. In addition, assume that U is κ_j-outgoing, if $j = 0$. Then we have the representation formula*

$$\mathsf{G}_{\kappa_j}^j(\gamma^j(U))(\mathbf{x}) = \begin{cases} U(\mathbf{x}) & \text{for } \mathbf{x}\in\Omega_j\,,\\ 0 & \text{for } \mathbf{x}\in\mathbb{R}^d\backslash\overline{\Omega}_j\,.\end{cases}$$

Similarly, if $V \in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^d\backslash\Omega_j)$ satisfies $\Delta V + \kappa_j^2 V = 0$ in $\mathbb{R}^d\backslash\overline{\Omega}_j$, as well as a radiation condition in the case $j \neq 0$, then we have $\mathsf{G}_{\kappa_j}^j(\gamma^j(V))(\mathbf{x}) = -V(\mathbf{x})$ for $\mathbf{x}\in\mathbb{R}^d\backslash\overline{\Omega}_j$, and $\mathsf{G}_{\kappa_j}^j(\gamma^j(V))(\mathbf{x}) = 0$ for $\mathbf{x}\in\Omega_j$.

---

[3]We point out that in order to maintain symmetry of formulas our choice of signs differs from what is commonly adopted in the literature.

The potential operator G^j_κ also satisfies a remarkable identity, known as jump relations, describing the relationship of traces of $\mathsf{G}^j_{\kappa_j}(\mathfrak{u})$ from both sides of $\partial\Omega_j$. Using the jump operator for Cauchy traces $[\gamma^j] := \gamma^j - \gamma^j_c$, they can concisely be expressed as

$$[\gamma^j]\cdot\mathsf{G}^j_{\kappa_j}(\mathfrak{u}_j) = \mathfrak{u}_j \qquad \forall\mathfrak{u}_j\in\mathbb{H}(\partial\Omega_j)\;,\quad j=0,\dots,n\;. \tag{3.8}$$

We refer the reader to [32, Thm.3.3.1] (the jump formulas are often given in the form of four equations in literature). Proposition 3.3 shows that, if U is solution to a homogeneous Helmholtz equation in Ω_j (and is κ_j-outgoing, if $j=0$) then $\big(\gamma^j\circ\mathsf{G}^j_{\kappa_j}\big)(\gamma^j(U)) = \gamma^j(U)$. This actually provides a caracterization of solutions of the homogeneous Helmholtz equation, *cf.* [11, Prop. 3.2], [27, Thm. 3.1.3], [32, Sect. 3.6].

Proposition 3.4. *Define the space of* Cauchy data

$$\begin{aligned}\mathcal{C}_\kappa(\partial\Omega_j) := \{\gamma^j(U)\mid U\in \mathrm{H}^1_{\mathrm{loc}}(\overline{\Omega}_j)\ \textit{and}\ \Delta U+\kappa^2U=0\ \textit{ in }\ \Omega_j,\\ U\ \kappa_j\textit{-outgoing, if } j=0\,\}\;.\end{aligned}$$

Then $\gamma^j\circ\mathsf{G}^j_\kappa:\mathbb{H}(\partial\Omega_j)\to\mathbb{H}(\partial\Omega_j)$ *is a continuous projector, called the interior* Calderón projector *of* Ω_j*, whose range coincides with* $\mathcal{C}_\kappa(\partial\Omega_j)$*, i.e. for any* $\mathfrak{u}_j\in\mathbb{H}(\partial\Omega_j)$

$$\gamma^j\cdot\mathsf{G}^j_\kappa(\mathfrak{u}_j) = \mathfrak{u}_j \quad\iff\quad \mathfrak{u}_j\in\mathcal{C}_\kappa(\partial\Omega_j)\;.$$

For a detailed proof of this proposition, see [32, Prop. 3.6.2]. This characterization of Cauchy traces of (outgoing) Helmholtz solutions is instrumental for deriving direct boundary integral equations for the subdomains Ω_j. The next lemma gives another caracterization of the space of Cauchy data, which was established in [11, Lemma 6.2].

Lemma 3.5. *Consider any* $j=0,\dots n$*, and any* $\kappa>0$*. Then for any* $\mathfrak{u}_j\in\mathbb{H}(\partial\Omega_j)$ *we have*

$$\mathfrak{u}_j\in\mathcal{C}_\kappa(\partial\Omega_j) \quad\iff\quad [\mathfrak{u}_j,\mathfrak{v}_j]_j = 0 \quad \forall\mathfrak{v}_j\in\mathcal{C}_\kappa(\partial\Omega_j)\;. \tag{3.9}$$

Applying traces to potentials yields boundary integral operators. In our compact notation, the crucial local boundary integral operators are

$$\begin{aligned}\mathsf{A}^j_{\kappa_j} &:= \{\gamma^j\}\circ\mathsf{G}^j_{\kappa_j}\\ &:= \tfrac{1}{2}(\gamma^j+\gamma^j_c)\circ\mathsf{G}^j_{\kappa_j} = \begin{pmatrix}-\mathsf{K}_j & \mathsf{V}_j\\ \mathsf{W}_j & \mathsf{K}'_j\end{pmatrix}\;,\quad j=0,\dots,n\;.\end{aligned} \tag{3.10}$$

We adopted the notations of [32, Sect. 3.1] for the atomic boundary integral operators, the double layer operators K_j, the single layer operators V_j, the adjoint double layer operators K'_j, and the hypersingular boundary integral operators W_j.

The operators $\mathsf{A}^j_{\kappa_j}$ satisfy an intriguing symmetry property, which seems to be well known in literature, see for example [4, Thm 3.9] (that concerns the Maxwell case, though). Since, apparently, the proof for acoustic waves is not published, we give it for the sake of completeness.

Lemma 3.6. *For any $j = 0, \dots, n$, and any wave number κ_j we have,*

$$\left[\mathsf{A}^j_{\kappa_j}(\mathfrak{u}_j), \mathfrak{v}_j\right]_j = \left[\mathsf{A}^j_{\kappa_j}(\mathfrak{v}_j), \mathfrak{u}_j\right]_j \qquad \forall \mathfrak{u}_j, \mathfrak{v}_j \in \mathbb{H}(\partial\Omega_j) \; .$$

Proof: This result is just a consequence of the jump formulas (3.8), as well as of Lemma 3.5 applied repeatedly in Ω_j and $\mathbb{R}^d \backslash \overline{\Omega}_j$:

$$\begin{aligned}
\left[\mathsf{A}^j_{\kappa_j}\,\mathfrak{u}_j, \mathfrak{v}_j\right]_j &\overset{(3.10)}{=} \left[\{\gamma^j\}\,\mathsf{G}^j_{\kappa_j}\,\mathfrak{u}_j, \mathfrak{v}_j\right]_j \overset{(3.8)}{=} \left[\{\gamma^j\}\,\mathsf{G}^j_{\kappa_j}\,\mathfrak{u}_j, [\gamma^j]\,\mathsf{G}^j_{\kappa_j}\,\mathfrak{v}_j\right]_j \\
&\overset{(3.9)}{=} -\left[\gamma^j\,\mathsf{G}^j_{\kappa_j}\,\mathfrak{u}_j, \gamma^j_c\,\mathsf{G}^j_{\kappa_j}\,\mathfrak{v}_j\right]_j + \left[\gamma^j_c\,\mathsf{G}^j_{\kappa_j}\,\mathfrak{u}_j, \gamma^j\,\mathsf{G}^j_{\kappa_j}\,\mathfrak{v}_j\right]_j \\
&\overset{(3.9)}{=} -\left[[\gamma^j]\,\mathsf{G}^j_{\kappa_j}\,\mathfrak{u}_j, \{\gamma^j\}\,\mathsf{G}^j_{\kappa_j}\,\mathfrak{v}_j\right]_j \overset{(3.8)}{=} \left[\{\gamma^j\}\,\mathsf{G}^j_{\kappa_j}\,\mathfrak{v}_j, \mathfrak{u}_j\right]_j \\
&\overset{(3.10)}{=} \left[\mathsf{A}^j_{\kappa_j}\,\mathfrak{v}_j, \mathfrak{u}_j\right]_j \; .
\end{aligned}$$

□

Another symmetry of potentials and their traces applies to the coupling between different subdomains:

Lemma 3.7. *Take two arbitrary subdomains Ω_j, Ω_k with $j \neq k$, any wave number κ_0. We have*

$$\left[\gamma^j\,\mathsf{G}^k_{\kappa_0}(\mathfrak{v}_k), \mathfrak{v}_j\right]_j = \left[\gamma^k\,\mathsf{G}^j_{\kappa_0}(\mathfrak{v}_j), \mathfrak{v}_k\right]_k \quad \forall \mathfrak{v}_j \in \mathbb{H}(\partial\Omega_j),\; \forall \mathfrak{v}_k \in \mathbb{H}(\partial\Omega_k) \; .$$

Proof: First of all, applying Lemma in 3.5 in Ω_j yields

$$\left[\gamma^j\,\mathsf{G}^k_{\kappa_0}(\mathfrak{v}_k), \mathfrak{v}_j\right]_j = \left[\gamma^j\,\mathsf{G}^k_{\kappa_0}(\mathfrak{v}_k), [\gamma^j]\,\mathsf{G}^j_{\kappa_0}(\mathfrak{v}_j)\right]_j = -\left[\gamma^j\,\mathsf{G}^k_{\kappa_0}(V_k), \gamma^j_c\,\mathsf{G}^j_{\kappa_0}(V_j)\right]_j \; .$$

Consider two Cauchy traces $\mathfrak{w}^j = (\mathfrak{w}^j_q)^n_{q=0}$, $\mathfrak{w}^k = (\mathfrak{w}^k_q)^n_{q=0}$, defined by the following formulas (with $\alpha = j, k$)

$$\mathfrak{w}^\alpha_q := \gamma^q\,\mathsf{G}^\alpha_{\kappa_0}(\mathfrak{v}_\alpha) \quad \text{for} \;\; q \neq \alpha, \qquad \mathfrak{w}^\alpha_\alpha := \gamma^\alpha_c\,\mathsf{G}^\alpha_{\kappa_0}(\mathfrak{v}_\alpha) \; .$$

With these notations

$$\left[\gamma^j\,\mathsf{G}^k_{\kappa_0}(\mathfrak{u}_k), \gamma^j_c\,\mathsf{G}^j_{\kappa_0}(\mathfrak{v}_j)\right]_j = \left[\mathfrak{w}^k_j, \mathfrak{w}^j_j\right]_j \; .$$

Observe that $\mathfrak{w}^j, \mathfrak{w}^k \in \mathbb{X}(\Gamma)$. As a consequence, we can apply Proposition 3.2 and obtain

$$\left[\mathfrak{w}^k_j, \mathfrak{w}^j_j\right]_j = -\left[\mathsf{T}(\mathfrak{w}^k), \mathsf{T}(\mathfrak{w}^j)\right]_\Sigma - \sum_{\substack{q=0\ldots n \\ q\neq j}} \left[\mathfrak{w}^k_q, \mathfrak{w}^q_j\right]_q \; .$$

In addition, note that $\mathfrak{w}^j_q, \mathfrak{w}^k_q \in \mathcal{C}_{\kappa_0}(\partial\Omega_q)$ for $q \neq j, k$, and similarly $\mathsf{T}(\mathfrak{w}^j)$,$\mathsf{T}(\mathfrak{w}^k)$ $\in \mathcal{C}_{\kappa_0}(\partial\Omega_\Sigma)$. Now we apply Lemma 3.5 on $\partial\Omega_q$ for $q \neq j, k$ and on $\partial\Omega_\Sigma$, which shows that all the terms vanish on the right hand side of (3.2), except the one associated to $q = k$. This yields $\left[\mathfrak{w}_j^k, \mathfrak{w}_j^j\right]_j = \left[\mathfrak{w}_k^k, \mathfrak{w}_k^j\right]_k$. Finally we conclude the proof by applying Lemma 3.5 once more in Ω_k to obtain

$$\begin{aligned}\left[\mathfrak{w}_k^k, \mathfrak{w}_k^j\right]_k &= \left[\gamma_c^k\, \mathsf{G}_{\kappa_0}^k(\mathfrak{v}_k), \gamma^k\, \mathsf{G}_{\kappa_0}^j(\mathfrak{v}_j)\right]_k \\ &= -\left[[\gamma^k]\, \mathsf{G}_{\kappa_0}^k(\mathfrak{v}_k), \gamma^k\, \mathsf{G}_{\kappa_0}^j(\mathfrak{v}_j)\right]_k = \left[\gamma^k\, \mathsf{G}_{\kappa_0}^j(\mathfrak{v}_j), \mathfrak{v}_k\right]_k . \qquad \square\end{aligned}$$

Since we will also use potential operators $\mathsf{SL}_\kappa^\Sigma$, $\mathsf{DL}_\kappa^\Sigma$ and G_κ^Σ that are defined by (3.7) with Ω_j replaced by Ω_Σ, we would like to mention that all the above results also hold for the subdomain Ω_Σ.

4 Classical single-trace formulation of the first kind

Now we present a first direct boundary integral formulation for Problem (2.2). This first formulation was already introduced and analysed in [37]. Since it is pivotal for our later developments, we recall its derivation and main properties.

4.1 Boundary and transmission conditions

The classical single-trace formulation takes into account the homogenous Dirichlet boundary conditions (2.2c) on Σ by incorporating them into the variational space. Set $\mathfrak{u} := (\gamma^j(U))_{j=0}^n$ where U is the unique solution to Problem (2.2). To arrive at an integral equation formulation, one first enforces the transmission conditions across the interfaces, and the Dirichlet boundary conditions on Σ by demanding that $\mathfrak{u} \in \mathbb{X}_0(\Gamma)$ where

$$\mathbb{X}_0(\Gamma) \;:=\; \{\, \mathfrak{u} \in \mathbb{X}(\Gamma) \mid \mathsf{T}_{\mathrm{D}}(\mathfrak{u}) = 0 \,\} . \tag{4.1}$$

Note that in the case $n = 0$ where $\mathbb{R}^d = \overline{\Omega}_0 \cup \overline{\Omega}_\Sigma$ and $\Gamma = \Sigma$, this space is simply given by $\mathbb{X}_0(\Gamma) = \{0\} \times \mathrm{H}^{-1/2}(\Sigma)$. Thanks to the continuity of $\mathsf{T}_{\mathrm{D}} : \mathbb{X}(\Gamma) \to \mathrm{H}^{1/2}(\Sigma)$, the space $\mathbb{X}_0(\Gamma)$ is a closed subspace of $\mathbb{X}(\Gamma)$. In addition, the function $U \in \mathrm{H}^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d\backslash\Omega_\Sigma)$ satisfies the boundary and transmission conditions in (2.2), if and only if $(\gamma^j(U))_{j=0}^n \in \mathbb{X}_0(\Gamma)$. In order to impose these conditions in a variational manner, one may rely on the following elementary characterization of $\mathbb{X}_0(\Gamma)$.

Lemma 4.1. *For any* $\mathfrak{u} \in \mathbb{H}(\Gamma)$*, we have,*

$$\mathfrak{u} \in \mathbb{X}_0(\Gamma) \iff [\mathfrak{u}, \mathfrak{v}] = 0 \quad \forall \mathfrak{v} \in \mathbb{X}_0(\Gamma).$$

Proof: Let $\mathfrak{u} \in \mathbb{X}_0(\Gamma)$. Take any element $\mathfrak{v} \in \mathbb{X}_0(\Gamma)$. Denote by $u, v \in \mathrm{H}^{1/2}(\Sigma)$ and $p, q \in \mathrm{H}^{-1/2}(\Sigma)$ the traces such that $\mathsf{T}(\mathfrak{u}) = (u, p)$ and $\mathsf{T}(\mathfrak{v}) = (v, q)$. According to the definition of $\mathbb{X}_0(\Gamma)$ we must have $u = v = 0$. Applying Proposition 3.2, we obtain

$$[\mathfrak{u}, \mathfrak{v}] = -\,[\mathsf{T}(\mathfrak{u}), \mathsf{T}(\mathfrak{v})]_\Sigma = \langle 0, q\rangle_\Sigma - \langle 0, p\rangle_\Sigma = 0 .$$

Now assume that $\mathfrak{u} \in \mathbb{H}(\Gamma)$ satisfies $[\mathfrak{u}, \mathfrak{v}] = 0$, for all $\mathfrak{v} \in \mathbb{X}_0(\Gamma)$. It is a direct consequence of Proposition 7.1 in [11] that actually $\mathfrak{u} \in \mathbb{X}(\Gamma)$ (note that notations are different in [11]). Let $u \in \mathrm{H}^{1/2}(\Sigma)$ and $p \in \mathrm{H}^{-1/2}(\Sigma)$ satisfy $\mathsf{T}(\mathfrak{u}) = (u, p)$. Take any trace $q \in \mathrm{H}^{-1/2}(\Sigma)$ and consider $\mathbf{q} \in \mathrm{H}(\mathrm{div}, \mathbb{R}^d)$ such that $\mathbf{n}_\Sigma \cdot \mathbf{q}|_\Sigma = q$. Finally denote $q_j := \mathbf{n}_j \cdot \mathbf{q}|_{\partial\Omega_j}$ and set $\mathfrak{v} = (0, q_j)_{j=0}^n$. Clearly $\mathfrak{v} \in \mathbb{X}(\Gamma)$ since $\mathbf{q} \in \mathrm{H}(\mathrm{div}, \mathbb{R}^d)$, and $\mathsf{T}_{\mathrm{D}}(\mathfrak{v}) = 0, \mathsf{T}_{\mathrm{N}}(\mathfrak{v}) = q$ by construction, so $\mathfrak{v} \in \mathbb{X}_0(\Gamma)$. Finally we obtain

$$0 = [\mathfrak{u}, \mathfrak{v}] = -\left[\mathsf{T}(\mathfrak{u}), \mathsf{T}(\mathfrak{v})\right]_\Sigma = -\langle u, q \rangle_\Sigma \ .$$

Since this holds for every $q \in \mathrm{H}^{-1/2}(\Sigma)$, we finally conclude that $u = \mathsf{T}_{\mathrm{D}}(\mathfrak{u}) = 0$, which implies $\mathfrak{u} \in \mathbb{X}_0(\Gamma)$. □

4.2 Integral formulation

Define $\mathfrak{u}^{\mathrm{inc}} := (\gamma^0(U_{\mathrm{inc}}), 0, \ldots, 0)$. According to the characterization of Cauchy data given by Proposition 3.4, the trace $\mathfrak{u} := (\gamma^0 U, \ldots, \gamma^n U)$ of a solution U of the boundary transmission problem (2.2) satisfies

$$(-\mathrm{Id}/2 + \mathsf{A})(\mathfrak{u} - \mathfrak{u}^{\mathrm{inc}}) = 0 \ ,$$

where the operator $\mathsf{A} : \mathbb{H}(\Gamma) \to \mathbb{H}(\Gamma)$ is defined subdomain-wise by

$$\mathsf{A}(\mathfrak{u}) := (\ \mathsf{A}^j_{\kappa_j}(\mathfrak{u}_j)\ )_{j=0}^n = (\ \{\gamma^j\} \cdot \mathsf{G}^j_{\kappa_j}(\mathfrak{u}_j)\ )_{j=0}^n$$

$$= \begin{bmatrix} \mathsf{A}^0_{\kappa_0} & 0 & \cdots & 0 \\ 0 & \mathsf{A}^1_{\kappa_1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mathsf{A}^n_{\kappa_n} \end{bmatrix} \cdot \begin{bmatrix} \mathfrak{u}_0 \\ \vdots \\ \vdots \\ \mathfrak{u}_n \end{bmatrix}, \tag{4.2}$$

for $\mathfrak{u} = (\mathfrak{u}_0, \ldots, \mathfrak{u}_n) \in \mathbb{H}(\Gamma)$. Summing up, Problem (2.2) spawns the *boundary integral equations*

$$\mathfrak{u} \in \mathbb{X}_0(\Gamma) \quad \text{such that} \quad (-\mathrm{Id}/2 + \mathsf{A})(\mathfrak{u} - \mathfrak{u}^{\mathrm{inc}}) = 0 \ . \tag{4.3}$$

To cast Equation (4.3) into a variational form, one must first test it with suitable traces. Choosing test traces $\mathfrak{v} \in \mathbb{X}_0(\Gamma)$, and taking into account Lemma 4.1, we see that if $\mathfrak{u}$ satisfies (4.3), then it also solves the *STF variational formulation* [12, Eq. (3.19)]

$$\boxed{\begin{cases} \text{find} \quad \mathfrak{u} \in \mathbb{X}_0(\Gamma) \quad \text{such that} \\ [\mathsf{A}(\mathfrak{u}), \mathfrak{v}] = -\left[\mathfrak{u}^{\mathrm{inc}}, \mathfrak{v}\right] \qquad \forall \mathfrak{v} \in \mathbb{X}_0(\Gamma)\ . \end{cases}} \tag{4.4}$$

It was established, in [37, §4.1], that the bilinear form $(\mathfrak{u}, \mathfrak{v}) \mapsto [\mathsf{A}(\mathfrak{u}), \mathfrak{v}]$ satisfies a generalized Gårding inequality, see also [13, Thm. 3.4], [11, Thm. 10.4], [12, Thm. 3.3].

Proposition 4.2. *Let the isometric isomorphism* $\Theta : \mathbb{H}(\Gamma) \to \mathbb{H}(\Gamma)$ *be defined by*[4] $\Theta(\mathfrak{v}) := (-\overline{v}_j, \overline{q}_j)_{j=0}^n$ *for* $\mathfrak{v} = (v_j, q_j)_{j=0}^n \in \mathbb{H}(\Gamma)$. *There exists a compact operator* $\mathsf{K} : \mathbb{H}(\Gamma) \to \mathbb{H}(\Gamma)$, *and a constant* $\beta > 0$ *such that*

$$|\,[(\mathsf{A}+\mathsf{K})\mathfrak{v}, \Theta(\mathfrak{v})]\,| \geqslant \beta \|\mathfrak{v}\|^2_{\mathbb{H}(\Gamma)} \quad \forall \mathfrak{v} \in \mathbb{H}(\Gamma)\ .$$

A direct consequence of this proposition is that the operator $\mathsf{A} : \mathbb{X}_0(\Gamma) \to \mathbb{X}_0(\Gamma)$ is of Fredholm type with index 0. As a consequence, $\dim(\ker(\mathsf{A}))$ is finite and will depend on the wave numbers $\kappa_0, \kappa_1, \dots, \kappa_n$. Fredholm alternative arguments [32, Sect. 2.1.4] bear out that injectivity of A already ensures stability of the variational problem (4.4).

Corollary 4.3. *If* $\ker(\mathsf{A}) = \{0\}$ *then there is* $\alpha > 0$ *such that*

$$\begin{aligned} &\inf_{\mathfrak{u}\in\mathbb{X}_0(\Gamma)} \sup_{\mathfrak{v}\in\mathbb{X}_0(\Gamma)} \frac{|\,[\mathsf{A}(\mathfrak{u}), \mathfrak{v}]\,|}{\|\mathfrak{u}\|_{\mathbb{H}(\Gamma)}\|\mathfrak{v}\|_{\mathbb{H}(\Gamma)}} > \alpha \quad \textit{and} \\ &\inf_{\mathfrak{v}\in\mathbb{X}_0(\Gamma)} \sup_{\mathfrak{u}\in\mathbb{X}_0(\Gamma)} \frac{|\,[\mathsf{A}(\mathfrak{u}), \mathfrak{v}]\,|}{\|\mathfrak{u}\|_{\mathbb{H}(\Gamma)}\|\mathfrak{v}\|_{\mathbb{H}(\Gamma)}} > 0\ . \end{aligned} \tag{4.5}$$

The link between the STF variational formulation (4.4) and the transmission boundary value problem (2.2) has been established in [37, §4.1]:

Proposition 4.4. *Provided that* $\ker(\mathsf{A}) = \{0\}$, *the traces* $\mathfrak{u} = (\gamma^j(U))_{j=0}^N$ *solve (*4.4*), if and only if* $U \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d \backslash \Omega_\Sigma)$ *is solution to (*2.2*), where* $U(\mathbf{x})$ *is defined by*

$$\begin{aligned} U(\mathbf{x}) &:= U_{\mathrm{inc}}(\mathbf{x}) + \mathsf{G}^0_{\kappa_0}(\mathfrak{u}_0)(\mathbf{x}) \quad \textit{for} \quad \mathbf{x} \in \Omega_0\ , \\ U(\mathbf{x}) &:= \mathsf{G}^j_{\kappa_j}(\mathfrak{u}_j)(\mathbf{x}) \quad \textit{for} \quad \mathbf{x} \in \Omega_j\ , \quad j = 1, \dots, n\ . \end{aligned} \tag{4.6}$$

4.3 Spurious resonances

As mentioned in the introduction, an important drawback of Formulation (4.4), is the possibility that $\ker(\mathsf{A}) \neq \{0\}$, which is commonly referred to as “spurious resonance phenomenon” in literature. Of course, this is highly undesirable, because, in case $\ker(\mathsf{A}) \neq \{0\}$, then (4.4) is not well posed, whereas Problem (2.2) always has a unique solution. In this section, we examine in what situations spurious resonances can occur. First of all, we need to establish an auxiliary result.

Lemma 4.5. *Let* $\mathfrak{u} = (\mathfrak{u}_0, \dots, \mathfrak{u}_n) \in \mathbb{X}_0(\Gamma)$ *satisfy* $[\mathsf{A}(\mathfrak{u}), \mathfrak{v}] = 0$ *for all* $\mathfrak{v} \in \mathbb{X}_0(\Gamma)$, *and set* $W_j(\mathbf{x}) = \mathsf{G}^j_{\kappa_j}(\mathfrak{u}_j)(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^d \backslash \Omega_j$ *Then, for each* $j = 0 \dots n$, *we have* $W_j \equiv 0$ *on any* connected component *of* $\mathbb{R}^d \backslash \overline{\Omega}_j$ *that does* not coincide *with* Ω_Σ.

Proof: The proof takes the cue from [37, Sect. 2] and combines elements of the proofs of [12, Lemma 3.4], [11, Prop. A.1]. We split it into three steps.

[4]We use overbars to designate complex conjugation.

❶ Let $\mathfrak{u}$ satisfy the assumptions of the lemma and define W_j as above. By the definition of A and Lemma 4.1 (⇒) we conclude

$$[\mathsf{A}(\mathfrak{u}), \mathfrak{v}] = \sum_{j=0}^{n} \left[\gamma_c^j(W_j), \mathfrak{v}_j\right]_j = 0 \quad \forall \mathfrak{v} \in \mathbb{X}_0(\Gamma) .$$

Appealing to Lemma 4.1 (⇐), this implies that

$$\mathfrak{w} := (\gamma_c^j(W_j))_{j=0}^n \in \mathbb{X}_0(\Gamma) . \tag{4.7}$$

❷ Next we establish that $W_j \equiv 0$ in any *unbounded* connected component of $\mathbb{R}^d \backslash \Omega_j$. To see this, note that for any $j = 0 \dots n$, we have $\Delta W_j + \kappa_j^2 W_j = 0$ in $\mathbb{R}^d \backslash \overline{\Omega}_j$ and W_j is κ_j-outgoing (radiating). Take $\rho > 0$ large enough to ensure that $\mathbb{R}^d \backslash \Omega_0 \subset \mathrm{B}_\rho$, where $\mathrm{B}_\rho \subset \mathbb{R}^d$ denotes the ball centered at 0 with radius ρ. Applying Green's formula in $\mathrm{B}_\rho \backslash \overline{\Omega}_j$ for $j = 0 \dots n$ yields

$$\begin{aligned} \int_{\partial \mathrm{B}_\rho} W_j \partial_r \overline{W}_j \, d\sigma &= \int_{\mathrm{B}_\rho \backslash \overline{\Omega}_j} |\nabla W_j|^2 - \kappa_j^2 |W_j|^2 \, d\mathbf{x} + \int_{\partial \Omega_j} \gamma_{\mathrm{D},c}^j(W_j) \gamma_{\mathrm{N},c}^j(\overline{W}_j) \, d\sigma \\ 0 &= \int_{\mathrm{B}_\rho \backslash \overline{\Omega}_0} |\nabla W_0|^2 - \kappa_0^2 |W_0|^2 \, d\mathbf{x} + \int_{\partial \Omega_0} \gamma_{\mathrm{D},c}^0(W_0) \gamma_{\mathrm{N},c}^0(\overline{W}_0) \, d\sigma \end{aligned}$$

In the equations above, ∂_r refers to the radial derivative. Take the imaginary part of the identity above, and sum over $j = 0 \dots n$, taking into account that $\mathfrak{w} := (\gamma_c^j(W_j))_{0 \leqslant j \leqslant n} \in \mathbb{X}(\Gamma)$. This yields

$$\begin{aligned} \sum_{j=1}^{n} \mathrm{Im}\Big\{ \int_{\partial \mathrm{B}_\rho} W_j \partial_r \overline{W}_j d\sigma \Big\} &= \mathrm{Im}\Big\{ \sum_{j=0}^{n} \int_{\partial \Omega_j} \gamma_{\mathrm{D},c}^j(W_j) \gamma_{\mathrm{N},c}^j(\overline{W}_j) \, d\sigma \Big\} \\ &= \frac{1}{2} \mathrm{Im}\{ \, [\mathfrak{w}, \overline{\mathfrak{w}}] \, \} = 0. \end{aligned}$$

In the last equality above we used Lemma 4.1. By construction, the functions W_j are κ_j-outgoing radiating, so that $0 = \lim_{r \to \infty} \int_{\partial \mathrm{B}_\rho} |\partial_r W_j - i\kappa_j W_j|^2 d\sigma$. As a consequence we obtain

$$\begin{aligned} &\sum_{j=1}^{n} \frac{1}{\kappa_j} \int_{\partial \mathrm{B}_\rho} |\partial_r W_j|^2 + \kappa_j^2 |W_j|^2 \, d\sigma \\ &\quad = \sum_{j=1}^{n} \frac{1}{\kappa_j} \int_{\partial \mathrm{B}_\rho} |\partial_r W_j - i\kappa_j W_j|^2 d\sigma - 2 \sum_{j=1}^{n} \mathrm{Im} \Big\{ \int_{\partial \mathrm{B}_\rho} W_j \partial_r \overline{W}_j \, d\sigma \Big\} \\ &\qquad = \sum_{j=1}^{n} \frac{1}{\kappa_j} \int_{\partial \mathrm{B}_\rho} |\partial_r W_j - i\kappa_j W_j|^2 d\sigma \quad \underset{\rho \to +\infty}{\longrightarrow} \quad 0 . \end{aligned}$$

This shows in particular that $\lim_{\rho \to \infty} \int_{\partial \mathrm{B}_\rho} |W_j|^2 d\sigma = 0$ for all $j = 1 \dots n$. As a consequence, we can apply Rellich Lemma, see Lemma 2.11 in [14], which implies that $W_j = 0$ in the unbounded component of $\mathbb{R}^d \backslash \overline{\Omega}_j, j = 1 \dots n$.

❸ Consider an arbitrary $j \in \{0, \dots n\}$, and let $\mathcal{O}_j$ be a *bounded* connected component of $\mathbb{R}^d \backslash \overline{\Omega}_j$ with $\mathcal{O}_j \neq \Omega_\Sigma$. Since (i) $\Omega_\Sigma, \Omega_0, \dots, \Omega_n$ form a partition of $\mathbb{R}^d$, (ii) all these domains are connected, and (iii) $\mathbb{R}^d \backslash \Omega_\Sigma$ is connected, we find that

- $\Sigma \neq \partial\mathcal{O}_j$,
- there is a $\ell \in \{1,\dots,n\}$, $\ell \neq j$, such that $\Omega_l \subset \mathcal{O}_j$ and $|\partial\Omega_\ell \cap \partial\Omega_j| > 0$.

A typical situation is depicted in Figure 2. Hence, there exists $\mathbf{x}_j \in \partial\mathcal{O}_j \cap \partial\Omega_\ell$ and an open ball $\mathrm{D} = \mathrm{B}(\mathbf{x}_j, \rho_j)$, $\rho_j > 0$, such that

$$\mathrm{D} \cap \partial\mathcal{O}_j = \mathrm{D} \cap \partial\Omega_\ell \neq \varnothing\ .$$

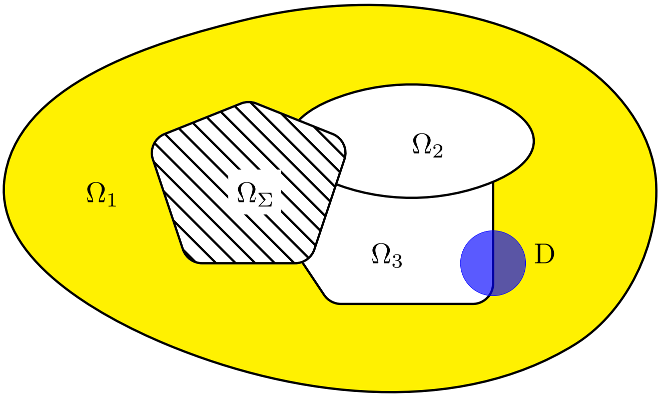


FIGURE 2. Geometrical situation for part ❸ of the proof of Lemma 4.5. Here $j = 1$, $\overline{\mathcal{O}}_1 = \overline{\Omega}_\Sigma \cup \overline{\Omega}_2 \cup \overline{\Omega}_3$ and $\ell = 3$.

Since both $\mathcal{O}_j$ and Ω_j are connected and bounded, the set $\mathbb{R}^d\backslash\overline{\mathcal{O}}_j$ is unbounded and connected. Thus, it is entirely contained in the unbounded connected component of $\mathbb{R}^d\backslash\overline{\Omega}_\ell$ that we denote by $\mathcal{U}_\ell$. From part ❷ of the proof we know that $W_\ell = 0$ in $\mathcal{U}_\ell$.

Obviously, $\partial\mathcal{U}_\ell \subset \partial\Omega_\ell$ as well as $\partial\mathcal{O}_j \subset \partial\Omega_j$. Moreover we know that $\mathrm{D} \cap \partial\mathcal{O}_j = \mathrm{D} \cap \partial\mathcal{U}_\ell$ has positive measure. Since $\mathfrak{w} = (\gamma_c^k(W_k))_{k=0}^n \in \mathbb{X}_0(\Gamma)$ according to (4.7) from Part ❶ of the proof, we deduce that on $D \cap \partial\mathcal{O}_j \cap \partial\mathcal{U}_\ell \subset \partial\Omega_j \cap \partial\Omega_\ell$ holds

$$\gamma^j_{\mathrm{D},c}(W_j) = \gamma^\ell_{\mathrm{D},c}(W_\ell) = 0$$
$$\gamma^j_{\mathrm{N},c}(W_j) = -\gamma^\ell_{\mathrm{N},c}(W_\ell) = 0 \quad \text{on } \mathrm{D} \cap \partial\mathcal{O}_j \cap \partial\mathcal{U}_\ell\ .$$

This means that $\gamma_c^j(W_j) = 0$ on $\partial\mathcal{O}_j \cap D$. As $\Delta W_j + \kappa_j^2 W_j = 0$ in $\mathcal{O}_j$, by analytic continuation this implies $W_j = 0$ in $\mathcal{O}_j$ according to Lemma 2.2 in [37]. □

Our final goal is to find sufficient and necessary conditions, under which the assumptions of Lemma 4.5 imply $\mathfrak{u} = 0$. The next result teaches that we need to examine the functions W_j outside Ω_j.

Lemma 4.6. *Let $\mathfrak{u} \in \mathbb{X}_0(\Gamma)$ satisfy $[\mathsf{A}\,\mathfrak{u}, \mathfrak{v}] = 0$ for all $\mathfrak{v} \in \mathbb{X}_0(\Gamma)$. Set $W_j(\mathbf{x}) := \mathsf{G}^j_{\kappa_j}(\mathfrak{u}_j)(\mathbf{x})$, $\mathbf{x} \notin \partial\Omega_j$, and assume that $\gamma_c^j(W_j) = 0$ for all $j = 0\dots n$. Then $\mathfrak{u} = 0$.*

Proof: We have $\gamma^j(W_j) = [\gamma^j(W_j)] = [\gamma^j]\cdot\mathsf{G}^j_{\kappa_j}(\mathfrak{u}_j) = \mathfrak{u}_j$ so $(\gamma^j(W_j))_{j=0}^n = \mathfrak{u} \in \mathbb{X}_0(\Gamma)$. Moreover, by construction $\Delta W_j + \kappa_j^2 W_j = 0$ in Ω_j for all $j = 0, \ldots, n$. We conclude that $V \in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^d \backslash \Omega_\Sigma)$ defined by $V|_{\Omega_j} = W_j|_{\Omega_j}$ satisfies all the equations of Problem (2.2) without incident field, $U_{\mathrm{inc}} = 0$. Since this transmission problem is well-posed V must vanish. Hence $\gamma^j(W_j) = 0$ for all $j = 0 \ldots n$, and finally $\mathfrak{u}_j = [\gamma^j(W_j)] = 0$ for all $j = 0 \ldots n$, i.e. $\mathfrak{u} = 0$. □

The previous lemma together with Lemma 4.5 sends the message that $\ker(\mathsf{A}) \neq \{0\}$ can occur only if Ω_Σ agrees with a connected component of the complement of some subdomain. Now we describe a simple setting in which this is the case.

Example ([33, Sect. 3.1]). Consider the case where $n = 0$, so that the scatterer reduces to a single impenetrable part $\mathbb{R}^d = \overline{\Omega}_0 \cup \overline{\Omega}_\Sigma$, and $\Gamma = \partial\Omega_0 = \Sigma$, see Figure 3. In this geometrical setting we have $\mathbb{X}_0(\Gamma) = \{0\} \times \mathrm{H}^{-1/2}(\Sigma)$.

Choose $\kappa_0 \in \mathbb{R}_+$ such that there exists $V \in \mathrm{H}^1(\Delta, \Omega_\Sigma)\backslash\{0\}$ that satisfies $\Delta V + \kappa_0^2 V = 0$ in Ω_Σ, and $V = 0$ on $\partial\Omega_\Sigma$. The existence of such non-trivial functions V is a classical consequence of spectral theory. Formulation (4.4) then reduces to the well-known single-layer integral formulation of the first kind: seek $p \in \mathrm{H}^{-\frac{1}{2}}(\Gamma)$ such that

$$\left\langle q, (\{\gamma^0_{\mathrm{D}}\} \circ \mathsf{SL}^0_{\kappa_0})(p)\right\rangle_0 \;=\; -\left\langle q, \gamma^0_{\mathrm{D}}(U_{\mathrm{inc}})\right\rangle_0 \quad \forall q \in \mathrm{H}^{-\frac{1}{2}}(\Gamma)\,. \tag{4.8}$$

Note that $[\gamma^0_{\mathrm{D}}] \cdot \mathsf{SL}^0_{\kappa_0} = 0$ according to (3.8), so we have $\{\gamma^0_{\mathrm{D}}\} \cdot \mathsf{SL}^0_{\kappa_0} = \gamma^0_{\mathrm{D},c} \cdot \mathsf{SL}_{\kappa_0}$. Coming back to the function V considered above, we have $\gamma^0_{\mathrm{D},c}(V) = 0$ and $\gamma^0_{\mathrm{N},c}(V) \neq 0$. In addition, a direct application of Proposition 3.3 yields $V(\mathbf{x}) = -\mathsf{SL}^0_{\kappa_0}(\gamma^0_{\mathrm{N},c}(V))(\mathbf{x})$ for $\mathbf{x} \in \Omega_\Sigma$, so $\{\gamma^0_{\mathrm{D}}\} \cdot \mathsf{SL}^0_{\kappa_0}(\gamma^0_{\mathrm{N},c}(V)) = \gamma^0_{\mathrm{D},c} \cdot \mathsf{SL}^0_{\kappa_0}(\gamma^0_{\mathrm{N},c}(V)) = 0$, which means that $p := \gamma^0_{\mathrm{N},c}(V) \neq 0$ solves (4.8), although $U_{\mathrm{inc}} = 0$.

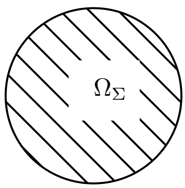


FIGURE 3. Homogeneous impenetrable scatterer giving rise to an exterior Dirichlet problem for the Helmholtz equation.

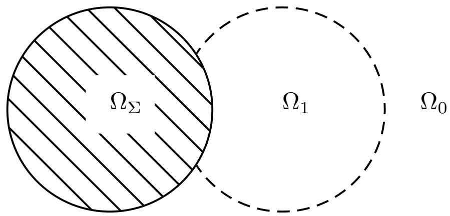


FIGURE 4. Situation without spurious resonances, *cf.* Corollary 4.7

We have assumed that $\mathbb{R}^d \backslash \Omega_\Sigma$ is connected. Therefore it is evident, that, if Ω_Σ coincides with a bounded component of $\mathbb{R}^d \backslash \Omega_j$, the boundary Σ of Ω_Σ must be contained in $\partial\Omega_j$.

Corollary 4.7. *Assume that* $\Sigma \not\subset \partial\Omega_j$ *for all* $j = 0 \dots n$. *Then, for any choice of wave numbers satisfying (2.3), we have* $\ker(\mathsf{A}) = \{0\}$

The insights we have gained so far are not exactly intuitive as demonstrated by the following example.

Example. Consider Problem (2.2) where $n = 1$, so that $\mathbb{R}^d = \overline{\Omega}_0 \cup \overline{\Omega}_1 \cup \overline{\Omega}_\Sigma$. Assume that $\kappa_0 = \kappa_1$ so that the interface $\partial\Omega_0 \cap \partial\Omega_1$ is "artificial". In fact, we face the very same scattering problem as in Example 4.3 above. Suppose that $\mathrm{mes}(\Sigma \cap \partial\Omega_0) > 0$ and $\mathrm{mes}(\Sigma \cap \partial\Omega_1) > 0$ like in Figure 4. Then, no matter what the value of κ_0 (even if $\kappa_0 \in \mathfrak{S}(\Delta, \Omega_\Sigma)$), there is no spurious resonance!

The following lemma generalizes the observation made in Example 4.3. In the interest of a concise statement we introduce the (discrete) set of interior Dirichlet eigenvalues of $-\Delta$ on Ω_Σ:

$$\mathfrak{S}(\Delta, \Omega_\Sigma) := \left\{ \kappa > 0 \mid \exists V \in \mathrm{H}^1(\Delta, \Omega_\Sigma)\backslash\{0\} : \begin{array}{rcll} -\Delta V & = & \kappa^2 V & \text{in } \Omega_\Sigma\,, \\ V & = & 0 & \text{on } \partial\Omega_\Sigma \end{array} \right\}. \tag{4.9}$$

Theorem 4.8. *For Problem (2.2), for any choice of wave numbers satisfying (2.3), we have the equivalence*

$$\ker(\mathsf{A}) \neq \{0\} \quad \Longleftrightarrow \quad \left\{ \begin{array}{c} \Sigma \subset \partial\Omega_j \quad \textit{for a } j \in \{0, \dots, n\} \\ \textit{and} \\ \kappa_j \in \mathfrak{S}(\Delta, \Omega_\Sigma) \end{array} \right\}.$$

Proof: Without loss of generality assume that $\Sigma \subset \partial\Omega_0$ (the proof below can easily be adapted to the case $\Sigma \subset \partial\Omega_j$ for $j \neq 0$).

❶ There exists a connected component $\mathcal{O}_0$ of $\mathbb{R}^d\backslash\overline{\Omega}_0$ such that $\Sigma \subset \partial\mathcal{O}_0$. We necessarily have $\Sigma = \partial\mathcal{O}_0$, otherwise Σ would admit a boundary as a Lipschitz manifold of dimension $d-1$, and this is not possible since $\Sigma = \partial\Omega_\Sigma$. The set $\mathbb{R}^d\backslash\overline{\mathcal{O}}_0$ is connected, it is contained in $\mathbb{R}^d\backslash\overline{\Omega}_\Sigma$, and it is maximal as a connected subset of $\mathbb{R}^d\backslash\overline{\Omega}_\Sigma$. As a consequence $\mathbb{R}^d\backslash\overline{\mathcal{O}}_0 = \mathbb{R}^d\backslash\overline{\Omega}_\Sigma$ since $\mathbb{R}^d\backslash\overline{\Omega}_\Sigma$ is assumed to be connected. In conclusion, Ω_Σ is exactly one bounded connected component of $\mathbb{R}^d\backslash\overline{\Omega}_0$. In particular, Ω_Σ is separated from the other subdomains Ω_j, $j = 1, \dots, n$:

$$\overline{\Omega}_\Sigma \cap \cup_{j=1}^n \overline{\Omega}_j = \varnothing. \tag{4.10}$$

❷ Assume first that $\kappa_0 \in \mathfrak{S}(\Delta, \Omega_\Sigma)$. As in Example 4.3, consider a function $V \in \mathrm{H}^1(\Omega_\Sigma)\backslash\{0\}$ such that $\Delta V + \kappa_0^2 V = 0$ in Ω_Σ, and $V = 0$ on Σ. Consider $\mathfrak{u}_0 = (u_0, p_0) \in \mathbb{H}(\partial\Omega_0)$ with $u_0 = 0$, $p_0 = 0$ on $\partial\Omega_0\backslash\Sigma$, and $p_0 = \gamma_{\mathrm{N}}^\Sigma(V) \neq 0$ on Σ. Applying Proposition 3.3 to V, we see that $\mathsf{G}_{\kappa_0}^0(\mathfrak{u}_0)(\mathbf{x}) = \mathsf{SL}_{\kappa_0}^\Sigma(p_0)(\mathbf{x}) = 0$ for $\mathbf{x} \in \Omega_0 \subset \mathbb{R}^d\backslash\overline{\Omega}_\Sigma$, so that $\gamma_{\mathrm{D}}^0\,\mathsf{SL}_{\kappa_0}^\Sigma(p_0) = 0$. Now set $\mathfrak{u} = (\mathfrak{u}_0, 0, \dots, 0) \in \mathbb{X}_0(\Gamma)\backslash\{0\}$. For any $\mathfrak{v} = (\mathfrak{v}_0, \dots, \mathfrak{v}_n) \in \mathbb{X}_0(\Gamma)$ we have

$$[\mathsf{A}(\mathfrak{u}), \mathfrak{v}] = \left[\gamma^0\,\mathsf{SL}_{\kappa_0}^0(p_0), \mathfrak{v}_0\right]_0 = \left\langle \gamma_{\mathrm{D}}^0\,\mathsf{SL}_{\kappa_0}^0(p_0), q_0 \right\rangle_0 = 0\,,$$

where $\mathfrak{v}_0 = (0, q_0)$ on Σ. Hence, $\mathfrak{u} \in \mathrm{Ker}(\mathsf{A})\backslash\{0\}$.

❸ Now assume that $\kappa_0 \notin \mathfrak{S}(\Delta, \Omega_\Sigma)$. We have to confirm that necessarily $\mathfrak{u} = 0$. Thanks to Lemma 4.5 $W_j = 0$ in $\mathbb{R}^d\backslash\overline{\Omega}_j$ for $j = 1 \dots n$, and $W_0 = 0$ in $\mathbb{R}^d\backslash(\overline{\Omega}_0 \cup \overline{\Omega}_\Sigma)$, which implies

$$\gamma_c^j(W_j) = 0 \quad \text{for } j = 1 \dots n, \quad \text{and} \quad \gamma_c^0(W_0)|_{\partial\Omega_0\backslash\Sigma} = 0 \,.$$

Now let us show that $\gamma_c^0(W_0) = 0$ on Σ as well, i.e. $\gamma^\Sigma(W_0) = 0$. We already know that, with $\mathfrak{w}$ from (4.7), $\mathsf{T}_{\mathrm{D}}(\mathfrak{w}) = \gamma_{\mathrm{D}}^\Sigma(W_0) = 0$ since $\mathfrak{w} \in \mathbb{X}_0(\Gamma)$. According to Proposition 3.3, we have

$$W_0(\mathbf{x}) = -\,\mathsf{G}^0_{\kappa_0}(\gamma_c^0(W_0))(\mathbf{x}) = \mathsf{G}^\Sigma_{\kappa_0}(\mathsf{T}(\mathfrak{w}))(\mathbf{x}) = \mathsf{SL}^\Sigma_{\kappa_0}(\,\mathsf{T}_{\mathrm{N}}(\mathfrak{w})\,)(\mathbf{x})$$

for all $\mathbf{x} \in \Omega_\Sigma \subset \mathbb{R}^d\backslash\overline{\Omega}_0$. So we conclude that $0 = \gamma_{\mathrm{D}}^\Sigma(W_0) = \gamma_{\mathrm{D}}^\Sigma \cdot \mathsf{SL}^\Sigma_{\kappa_0}(\,\mathsf{T}_{\mathrm{N}}(\mathfrak{w})\,)$. It is well known, see for example [32, Thm. 3.9.1], that $\mathrm{Ker}(\gamma_{\mathrm{D}}^\Sigma\,\mathsf{SL}_{\kappa_0}) = \{0\}$, if $\kappa_0 \notin \mathfrak{S}(\Delta, \Omega_\Sigma)$, hence we finally conclude that $\mathsf{T}_{\mathrm{N}}(\mathfrak{w}) = \gamma^0_{\mathrm{N},c} W_0 = 0$, which means $\gamma_c^0(W_0) = 0$. To finish the proof we apply Lemma 4.6. □

5 Single-trace combined field integral equation

We have discovered that the STF (4.4) is free of spurious resonnance except for the situation $\Sigma \subset \partial\Omega_j$. As a remedy we are going to devise an augmented STF taking the cue from the CFIE approach already mentioned in the Introduction. We will not restrict ourselves to geometries that allow spurious resonances because, if Σ is largely contained in $\partial\Omega_j$ with the exception of a small section, discretizations of the STF may already suffer from poor conditioning. Thus, even when spurious resonances cannot occur, the CFIE augmentation may enhance numerical stability!

The classical CFIEs resort to simple complex combinations of Dirichlet and Neumann traces, ignoring the fact that they belong to different function spaces. This compounds the difficulties of a rigorous analysis of the resulting boundary integral equations. We refer to the discussion in [3, Sect. 3.1]. This problem can be overcome by using regularized CFIE that rely on compact operators which map between Dirichlet and Neumann traces. This was first employed for theoretical investigations [28] and, more recently, used for the design of stable Galerkin boundary element methods [5, 3, 22, 23, 35, 2]. Our approach is inspired by [3].

5.1 Transformed traces

The principle of regularized CFIE boils down to enforcing generalized impedance (Robin type) boundary conditions for potentials on Σ. As in [3, Sect. 3.2], these impedance boundary conditions rely on a regularizing linear operator $\mathsf{M} : \mathrm{H}^{-1/2}(\Sigma) \to \mathrm{H}^{+1/2}(\Sigma)$ that satisfies

$$\textbf{(i)} \qquad \mathsf{M} \text{ is } \textit{compact}, \tag{5.1a}$$

$$\textbf{(ii)} \qquad \mathrm{Im}\{\langle \varphi, \mathsf{M}\,\overline{\varphi}\rangle_\Sigma\} > 0 \quad \forall \varphi \in \mathrm{H}^{-1/2}(\Sigma)\backslash\{0\}\,. \tag{5.1b}$$

There exist many operators satisfying (i)-(ii). Indeed if $\tilde{\mathsf{M}}$ is any second order strongly coercive real symmetric surface differential operator on Σ, then $\mathsf{M} = -\imath\,\tilde{\mathsf{M}}$ matches these conditions. The particular choice $\mathsf{M} = -\imath\,(\Delta_\Sigma + \mathrm{Id})^{-1}$ will be further commented in §5.4. Based on M we define the space of traces complying with generalized impedance boundary conditions

$$\mathbb{X}_{\mathsf{M}}(\Gamma) := \{\, \mathfrak{u} \in \mathbb{X}(\Gamma) \mid \mathsf{T}_{\mathrm{D}}(\mathfrak{u}) = \mathsf{M}\,\mathsf{T}_{\mathrm{N}}(\mathfrak{u}) \,\} \,. \tag{5.2}$$

Appealing to the duality of the trace spaces $\mathrm{H}^{-1/2}(\Sigma)$ and $\mathrm{H}^{+1/2}(\Sigma)$ we can define the adjoint regularizing operator $\mathsf{M}^* : \mathrm{H}^{-1/2}(\Sigma) \to \mathrm{H}^{+1/2}(\Sigma)$ by the formula

$$\langle q, \mathsf{M}^*\, p\rangle_\Sigma \;:=\; \langle p, \mathsf{M}\, q\rangle_\Sigma \quad \forall p, q \in \mathrm{H}^{-1/2}(\Sigma) \,. \tag{5.3}$$

It is immediate that M^* satisfies (5.1), if and only if M does. As a consequence, we can define the space $\mathbb{X}_{\mathsf{M}^*}(\Gamma)$ analogously to (5.2). It can be used to obtain a weak characterization of $\mathbb{X}_{\mathsf{M}}(\Gamma)$:

Lemma 5.1. *For any* $\mathfrak{u} \in \mathbb{H}(\Gamma)$, *we have* $\mathfrak{u} \in \mathbb{X}_{\mathsf{M}}(\Gamma) \iff [\mathfrak{u}, \mathfrak{v}] = 0 \;\; \forall \mathfrak{v} \in \mathbb{X}_{\mathsf{M}^*}(\Gamma)$.

Proof: ❶($\Rightarrow$) From Proposition 3.2 and (3.4) we obtain the identity

$$[\mathfrak{u}, \mathfrak{v}] = \langle \mathsf{T}_{\mathrm{N}}(\mathfrak{u}), \mathsf{T}_{\mathrm{D}}(\mathfrak{v})\rangle_\Sigma - \langle \mathsf{T}_{\mathrm{D}}(\mathfrak{u}), \mathsf{T}_{\mathrm{N}}(\mathfrak{v})\rangle_\Sigma \;, \quad \mathfrak{u}, \mathfrak{v} \in \mathbb{X}(\Gamma) \,. \tag{5.4}$$

For $\mathfrak{u} \in \mathbb{X}_{\mathsf{M}}(\Gamma)$ we infer

$$\begin{aligned} [\mathfrak{u}, \mathfrak{v}] &= \langle \mathsf{T}_{\mathrm{N}}(\mathfrak{u}), \mathsf{T}_{\mathrm{D}}(\mathfrak{v})\rangle_\Sigma - \langle \mathsf{M}\,\mathsf{T}_{\mathrm{N}}(\mathfrak{u}), \mathsf{T}_{\mathrm{N}}(\mathfrak{v})\rangle_\Sigma \\ &= \Big\langle \mathsf{T}_{\mathrm{N}}(\mathfrak{u}), \underbrace{(\mathsf{T}_{\mathrm{D}}(\mathfrak{v}) - \mathsf{M}^*\,\mathsf{T}_{\mathrm{N}}(\mathfrak{v}))}_{=0} \Big\rangle_\Sigma = 0 \quad \forall \mathfrak{v} \in \mathbb{X}_{\mathsf{M}^*}(\Gamma) \,. \end{aligned}$$

❷($\Leftarrow$) To begin with, as in the proof of [12, Thm. 3.1], we conclude with (5.4) that $\mathfrak{u} \in \mathbb{X}(\Gamma)$. Then, for $\mathfrak{v} \in \mathbb{X}_{\mathsf{M}^*}(\Gamma)$, (5.4) becomes

$$[\mathfrak{u}, \mathfrak{v}] = \langle \mathsf{T}_{\mathrm{N}}(\mathfrak{u}), \mathsf{M}^*\,\mathsf{T}_{\mathrm{N}}(\mathfrak{v})\rangle_\Sigma - \langle \mathsf{T}_{\mathrm{D}}(\mathfrak{u}), \mathsf{T}_{\mathrm{N}}(\mathfrak{v})\rangle_\Sigma = \langle (\mathsf{T}_{\mathrm{D}}(\mathfrak{u}) - \mathsf{M}\,\mathsf{T}_{\mathrm{N}}(\mathfrak{u})), \mathsf{T}_{\mathrm{N}}(\mathfrak{v})\rangle_\Sigma \,.$$

As T_{N} is surjective, the second assertion of the lemma follows. □

The regularizing operator will enter the definition of a *trace transformation* $\mathsf{R} : \mathbb{X}(\Gamma) \to \mathbb{X}(\Gamma)$ that realizes an isomorphism of the form "identity + compact". Its definition involves a continuous extension operator $\mathsf{E}_\Sigma : \mathrm{H}^{+\frac{1}{2}}(\Sigma) \to \mathrm{H}^1(\mathbb{R}^d)$ that furnishes a right inverse of the trace $\gamma_{\mathrm{D}}^{\Sigma}$. Then we define

$$\mathsf{R} = \mathrm{Id} + \mathsf{C} \quad , \quad \mathsf{C} := \big((\gamma_{\mathrm{D}}^{j} \circ \mathsf{E}_\Sigma \circ \mathsf{M} \circ \mathsf{T}_{\mathrm{N}}, 0)\big)_{j=0}^{n} \;, \tag{5.5}$$

where $\mathsf{C} : \mathbb{X}(\Gamma) \to \mathbb{X}(\Gamma)$ inherits compactness from M.

Lemma 5.2. R *is an isomorphism and we have* $\mathsf{R}(\,\mathbb{X}_0(\Gamma)\,) = \mathbb{X}_{\mathsf{M}}(\Gamma)$.

Proof: Observe that $\mathsf{C}^2 = 0$, so that $\mathsf{R}^{-1} = \mathrm{Id} - \mathsf{C}$, which proves the first statement. Now let $\gamma_{\mathrm{D}} : \mathrm{H}^1(\mathbb{R}^d) \to \mathbb{X}(\Gamma)$ refer to the global trace operator defined by $\gamma_{\mathrm{D}}(u) = (\gamma_{\mathrm{D}}^{j}(u))_{j=0}^{n}$. Since $\mathsf{T}_{\mathrm{D}} \circ \gamma_{\mathrm{D}} \circ \mathsf{E}_{\Sigma} = \mathrm{Id}$ and $\mathsf{T}_{\mathrm{N}}(\mathsf{C}\,\mathfrak{u}) = 0$, we easily see that for $\mathfrak{u} \in \mathbb{X}_0(\Gamma)$

$$\mathsf{T}_{\mathrm{D}}(\mathsf{R}\,\mathfrak{u}) - \mathsf{M}\,\mathsf{T}_{\mathrm{N}}(\mathsf{R}\,\mathfrak{u}) = \underbrace{\mathsf{T}_{\mathrm{D}}(\mathfrak{u})}_{=0} + \mathsf{M}\,\mathsf{T}_{\mathrm{N}}(\mathfrak{u}) - \mathsf{M}\,\mathsf{T}_{\mathrm{N}}(\mathfrak{u}) = 0\,.$$

this shows that $\mathsf{R}(\,\mathbb{X}_0(\Gamma)\,) \subset \mathbb{X}_{\mathsf{M}}(\Gamma)$. We show in the same manner that $(\mathrm{Id} - \mathsf{C})(\,\mathbb{X}_{\mathsf{M}}(\Gamma)\,) \subset \mathbb{X}_0(\Gamma)$, which finishes the proof. □

Remark 5.3. If $\Sigma \subset \partial\Omega_j$ for some $j \in \{0, \ldots, n\}$, we can pick an extension E_{Σ} that is local in the sense that

$$\mathrm{supp}(\mathsf{E}\,u) \subset \Omega_{\Sigma} \cup \Sigma \cup \Omega_j\,, \quad u \in H^{+1/2}(\Sigma)\,. \tag{5.6}$$

5.2 Direct single trace CFIE

The STF (4.4) is a direct BIE in the sense that its unknowns are Cauchy traces of the solution of the transmission problem (2.2). This property is preserved by the CFIE augmentation proposed in this section.

As in Section 4.2 let $\mathfrak{u} = (\mathfrak{u}_j)_{j=0}^{n} \in \mathbb{X}_0(\Gamma)$ denote the Cauchy traces of the solution U of Problem (2.2) i.e. $\mathfrak{u}_j = \gamma^j(u)$, $j = 0, \ldots, n$. We have seen that it satisfies the integral equation (4.3). The derivation of a direct combined field integral equation starts from this identity and, as before, casts it into a weak form similar to (4.4). However, this time we employ test functions $\widetilde{\mathfrak{v}} \in \mathbb{X}_{\mathrm{M}}(\Gamma)$ instead of taking $\mathfrak{v} \in \mathbb{X}_0(\Gamma)$! We end up with: seek $\mathfrak{u} \in \mathbb{X}_0(\Gamma)$ such that

$$\begin{aligned} [(-\mathrm{Id}/2 + \mathsf{A})\mathfrak{u}, \widetilde{\mathfrak{v}}] &= \sum_{j=0}^{n} \left[\gamma_c^j\,\mathsf{G}_{\kappa_j}^{j}(\mathfrak{u}_j), \widetilde{\mathfrak{v}}_j\right]_j \\ &= -\left[\mathfrak{u}^{\mathrm{inc}}, \widetilde{\mathfrak{v}}\right] \quad \forall \widetilde{\mathfrak{v}} \in \mathbb{X}_{\mathsf{M}}(\Gamma)\,. \end{aligned} \tag{5.7}$$

Thanks to Lemmas 5.2 and 4.1, an equivalent reformulation of (5.7) is

$$\begin{aligned} [(-\mathrm{Id}/2 + \mathsf{A})\mathfrak{u}, (\mathrm{Id} + \mathsf{C})\mathfrak{v}] &= [\mathsf{A}\,\mathfrak{u}, \mathfrak{v}] + \mathsf{c}(\mathfrak{u}, \mathfrak{v}) \\ &= -\left[\mathfrak{u}^{\mathrm{inc}}, (\mathrm{Id} + \mathsf{C})\mathfrak{v}\right] \quad \forall \mathfrak{v} \in \mathbb{X}_0(\Gamma)\,, \end{aligned} \tag{5.8}$$

where we define the *compact* bilinear form $\mathsf{c} : \mathbb{X}(\Gamma) \times \mathbb{X}(\Gamma) \to \mathbb{C}$ according to

$$\mathsf{c}(\mathfrak{w}, \mathfrak{v}) := [(-\mathrm{Id}/2 + \mathsf{A})\mathfrak{w}, \mathsf{C}\,\mathfrak{v}]\,, \quad \mathfrak{w}, \mathfrak{v} \in \mathbb{X}(\Gamma)\,. \tag{5.9}$$

Compactness of c is an immediate consequence of the compactness of $\mathsf{C} : \mathbb{X}(\Gamma) \to \mathbb{X}(\Gamma)$. We may also introduce the unique element $\widetilde{\mathfrak{u}}_{\mathrm{inc}} \in \mathbb{H}(\Gamma)$ such that $[\widetilde{\mathfrak{u}}_{\mathrm{inc}}, \mathfrak{v}] = -\left[\mathfrak{u}^{\mathrm{inc}}, (\mathrm{Id} + \mathsf{C})\mathfrak{v}\right]$. This makes it possible to write the *direct single trace CFIE* in variational form:

$$\boxed{\begin{cases} \text{seek } \mathfrak{u} \in \mathbb{X}_0(\Gamma) \text{ such that} \\ [\mathsf{A}\,\mathfrak{u}, \mathfrak{v}] + \mathsf{c}(\mathfrak{u}, \mathfrak{v}) = -\,[\widetilde{\mathfrak{u}}_{\mathrm{inc}}, \mathfrak{v}] \quad \forall \mathfrak{v} \in \mathbb{X}_0(\Gamma)\,. \end{cases}} \tag{5.10}$$

Below we write a_{M} for the bilinear form from (5.10). Obviously, (5.10) amounts to a compact perturbation of (4.4) so that it preserves many key properties. In particular, it satisfies a generalized Gårding inequality analogous to Proposition 4.2.

Corollary 5.4. *Recall the isomorphism* $\Theta : \mathbb{H}(\Gamma) \to \mathbb{H}(\Gamma)$ *from Proposition* 4.2*, defined by* $\Theta(\mathfrak{v}) = (-\overline{v}_j, \overline{q}_j)_{j=0}^n$ *for* $\mathfrak{v} = (v_j, q_j)_{j=0}^n \in \mathbb{H}(\Gamma)$*. The bilinear form* a_M *on the left side of* (5.10) *satisfies*

$$|\,\mathsf{a}_\mathsf{M}(\mathfrak{v}, \Theta(\mathfrak{v})) + \mathsf{k}(\mathfrak{v}, \Theta(\mathfrak{v}))| \geqslant \beta\, \|\mathfrak{v}\|^2_{\mathbb{H}(\Gamma)} \quad \forall \mathfrak{v} \in \mathbb{X}_0(\Gamma)\,,$$

with a compact *sesqui-linear form* $\mathsf{k} : \mathbb{X}(\Gamma) \times \mathbb{X}(\Gamma) \to \mathbb{C}$.

Denote $\mathscr{A}_\mathrm{M} : \mathbb{X}_0(\Gamma) \to \mathbb{X}_0(\Gamma)'$ the operator induced by a_M. The previous proposition shows that $\mathscr{A}_\mathrm{M}$ is of Fredholm type with index 0. Thanks to Fredholm alternative arguments injectivity of $\mathscr{A}_\mathrm{M}$ is sufficient for stability of the variational problem (5.10) (in the sense of an inf-sup condition like (4.5)).

Lemma 5.5. *For any choice of the wave numbers* $\kappa_0, \dots \kappa_n$ *satisfying (*2.3*),* $\mathrm{Ker}(\mathscr{A}_\mathrm{M})$ *is trivial.*

Proof: By and large, the proof runs parallel to that of Lemma 4.5 and Theorem 4.8. Thus, some parts will only be sketched and for details the reader may refer to Section 4.3.

❶ Pick $\mathfrak{u} = (\mathfrak{u}_0, \mathfrak{u}_1, \dots, \mathfrak{u}_n) \in \mathbb{X}_0(\Gamma)$ such that it solves (5.7)/(5.8) with $\mathfrak{u}_{\mathrm{inc}} = 0$. As in Section 4.3 we set $W_j(\mathbf{x}) = \mathsf{G}^j_{\kappa_j}(\mathfrak{u}_j)(\mathbf{x})$ and $\mathfrak{w} := \big(\gamma^j_c W_j\big)_{j=0}^n \in \mathbb{H}(\Gamma)$, *cf.* (4.7). Since (5.7) with $\mathfrak{u}_{\mathrm{inc}} = 0$ implies $\big[\mathfrak{w}, \tilde{\mathfrak{v}}\big] = 0$ for all $\tilde{\mathfrak{v}} \in \mathbb{X}_\mathsf{M}(\Gamma)$, Lemma 5.1 confirms $\mathfrak{w} \in \mathbb{X}_{\mathsf{M}^*}(\Gamma)$.

❷ We exploit (5.1b) and exactly as in Step ❷ of the proof of Lemma 4.5 we show that $W_j \equiv 0$ in any unbounded connected component of $\mathbb{R}^d \backslash \overline{\Omega}_j$.

❸ The arguments employed in Step ❸ of the proof of Lemma 4.5 completely carry over to the present situation and confirm that $W_j \equiv 0$ in any connected component of $\mathbb{R}^d \backslash \overline{\Omega}_j$ that does not coincide with Ω_Σ. This is the counterpart of the statement of Lemma 4.5 for (5.10).

❹ If $\Sigma \not\subset \partial\Omega_j$ for every $j = 0, \dots, n$, we find $\mathfrak{w} = 0$ as explained when justifying Corollary 4.7. Then apply Lemma 4.6 and the proof is finished.

❺ Assume $\Sigma \subset \partial\Omega_j$ for some $j = 0, \dots, n$. By above arguments all W_k, $k \neq j$, vanish on $\mathbb{R}^d \backslash \overline{\Omega}_k$. However, W_j may not vanish on Ω_Σ, recall Step ❷ of the proof of Theorem 4.8. However, from $\mathfrak{w} \in \mathbb{X}_{\mathsf{M}^*}(\Gamma)$ we conclude

$$\gamma^\Sigma_{\mathrm{D}}(W_j) = \mathsf{M}^*\, \gamma^\Sigma_{\mathrm{N}}(W_j)\,.$$

Thus, In Ω_Σ the function W_j satisfies $\Delta W_j + \kappa_j^2 W_j = 0$ in Ω_Σ and $\gamma^\Sigma_{\mathrm{D}}(W_j) = \mathsf{M}^*\, \gamma^\Sigma_{\mathrm{N}}(W_j)$. By Green's formula, we obtain as in [3]

$$0 = \mathrm{Im}\Big\{ \int_{\Omega_\Sigma} |\nabla W_j|^2 - \kappa_j^2 |W_j|^2 d\mathbf{x} \Big\} = \mathrm{Im}\Big\{ \int_\Sigma \gamma^\Sigma_{\mathrm{N}}(\overline{W}_j) \cdot \mathsf{M}^* \cdot \gamma^\Sigma_{\mathrm{N}}(W_j)\, d\sigma \Big\}\,.$$

According to property (5.1b) of M^*, this implies $\gamma^\Sigma_{\mathrm{N}}(W_j) = 0$, hence $\gamma^\Sigma_{\mathrm{D}}(W_j) = \mathsf{M}^*\, \gamma^\Sigma_{\mathrm{N}}(W_j) = 0$. Finally this yields $\gamma^j_c(W_j) = 0$ and $W_j \equiv 0$ in Ω_Σ, so that we know $\mathfrak{w} = 0$. Appealing to Lemma 4.6 finishes the proof. □

As in Section 4.2, via Fredholm theory, from this lemma we conclude that (5.10) always possesses a unique solution.

Remark 5.6. In the case $n = 0$ of a single impenetrable scatterer the spaces and operators reduce to

$$\mathbb{X}_0(\Gamma) = \{0\} \times H^{+\frac{1}{2}}(\Sigma)\ , \quad \mathsf{A} \overset{(4.2)}{=} \mathsf{A}^0_{\kappa_0}\ , \quad \mathsf{C} \overset{(5.5)}{=} \begin{pmatrix} \mathsf{M} \circ \mathsf{T}_{\mathrm{N}} \\ 0 \end{pmatrix}\ . \tag{5.11}$$

As a consequence, with (3.10) the variational equation (5.8) becomes: seek $\mathfrak{u} = (0, p_0) \in \mathbb{X}_0(\Gamma)$

$$\left[\left(-\tfrac{\mathrm{Id}}{2} + \begin{pmatrix} -\mathsf{K}_0 & \mathsf{V}_0 \\ \mathsf{W}_0 & \mathsf{K}_0' \end{pmatrix}\right)\begin{pmatrix} 0 \\ p_0 \end{pmatrix}, \left(\mathrm{Id} - \begin{pmatrix} 0 & \mathsf{M} \\ 0 & 0 \end{pmatrix}\right)\begin{pmatrix} 0 \\ q_0 \end{pmatrix}\right]$$
$$= -\left[\begin{pmatrix} u_{\mathrm{inc}} \\ p_{\mathrm{inc}} \end{pmatrix}, \left(\mathrm{Id} - \begin{pmatrix} 0 & \mathsf{M} \\ 0 & 0 \end{pmatrix}\right)\begin{pmatrix} 0 \\ q_0 \end{pmatrix}\right]$$

for all $q_0 \in H^{-\frac{1}{2}}(\Sigma)$. Owing to (3.2) and with $\mathfrak{u}_{\mathrm{inc}} = (u_{\mathrm{inc}}, p_{\mathrm{inc}})$ this is equivalent to finding $p_0 \in H^{-\frac{1}{2}}(\Sigma)$ such that

$$\langle \mathsf{V}_0\, p_0, q_0 \rangle + \left\langle (-\mathrm{Id}/2 + \mathsf{K}_0')p_0, \mathsf{M}\, q_0 \right\rangle = \langle u_{\mathrm{inc}}, q_0 \rangle + \langle p_{\mathrm{inc}}, \mathsf{M}\, q_0 \rangle$$
$$\Updownarrow$$
$$\left\langle \left(\mathsf{V}_0 + \mathsf{M}^*(-\mathrm{Id}/2 + \mathsf{K}_0')\right) p_0, q_0 \right\rangle = \left\langle u_{\mathrm{inc}} + \mathsf{M}^*\, p_{\mathrm{inc}}, q_0 \right\rangle\ ,$$

for all $q_0 \in H^{-\frac{1}{2}}(\Sigma)$. This agrees with the regularized CFIE from [3, Sect. 4].

5.3 Indirect CFIE

Both the STF (4.4) and the regularized CFIE (5.10) are *direct* BIE, since their unique solutions provide true Cauchy traces of the solution U of (2.2). If the solution of a BIE does not agree with traces of the solution of the related boundary value problem, it is classified as *indirect*. In [3, Sect. 3] a regularized indirect CFIE was devised for the simple situation $n = 0$. In this section we adapt this approach to the STF. We obtain a variational equation that is dual to the direct CFIE introduced in the previous section.

The indirect CFIE stems from a representation of the solution to Problem (2.2) in the following form

$$\begin{aligned} U(\mathbf{x}) &= G^0_{\kappa_0}(\tilde{\mathfrak{u}}_0)(\mathbf{x}) + U_{\mathrm{inc}}(\mathbf{x}) && \text{for } \mathbf{x} \in \Omega_0, \\ U(\mathbf{x}) &= G^j_{\kappa_j}(\tilde{\mathfrak{u}}_j)(\mathbf{x}) && \text{for } \mathbf{x} \in \Omega_j, \quad j = 1 \ldots n, \\ &\text{where } \tilde{\mathfrak{u}} = (\tilde{\mathfrak{u}}_j)_{j=0}^n \in \mathbb{X}_{\mathsf{M}}(\Gamma)\ . \end{aligned} \tag{5.12}$$

Admittedly, existence of such a representation of U is not obvious at first glance. Assume for a moment that such a representation can be found. Then the boundary and transmission conditions of Problem (2.2) can be expressed as $(\gamma^j(U))_{j=0}^n \in \mathbb{X}_0(\Gamma)$. Using Lemma 4.1 and representation (5.12) yields

$$\left[\gamma^0(U_{\mathrm{inc}}), \mathfrak{v}_0\right]_0 + \sum_{j=0}^{n} \left[\gamma^j\, \mathsf{G}^j_{\kappa_j}(\tilde{\mathfrak{u}}_j), \mathfrak{v}_j\right]_j = 0 \qquad \forall \mathfrak{v} = (\mathfrak{v}_j)_{j=0}^n \in \mathbb{X}_0(\Gamma)\ . \tag{5.13}$$

Definition (3.10) together with the jump relations (3.8) give the equivalent statement

$$\left[(\frac{1}{2}\mathrm{Id} + \mathsf{A})\tilde{\mathfrak{u}}, \mathfrak{v}\right] = -\left[\gamma^0(U_{\mathrm{inc}}), \mathfrak{v}_0\right]_0 \qquad \forall \mathfrak{v} = (\mathfrak{v}_j)_{j=0}^n \in \mathbb{X}_0(\Gamma)\ . \tag{5.14}$$

Thanks to Lemma 5.2 there exists $\mathfrak{u} = (\mathfrak{u}_0, \ldots, \mathfrak{u}_n) \in \mathbb{X}_0(\Gamma)$ such that $\tilde{\mathfrak{u}} = \mathsf{R}\,\mathfrak{u} = (\mathrm{Id}+\mathsf{C})\mathfrak{u}$. Plugging this into (5.14), and taking account of the definition of $\mathfrak{u}_{\mathrm{inc}}$ and Lemma 4.1, we obtain

$$[\mathsf{A}(\mathfrak{u}), \mathfrak{v}] + \left[(\mathsf{A} + \frac{1}{2}\mathrm{Id})\,\mathsf{C}\,\mathfrak{u}, \mathfrak{v}\right] = -\left[\mathfrak{u}^{\mathrm{inc}}, \mathfrak{v}\right] \quad \forall \mathfrak{v} \in \mathbb{X}_0(\Gamma)\ . \tag{5.15}$$

Clearly, this equation is a perturbed version of Formulation (4.4). Introduce the bilinear form

$$\mathsf{c}'(\mathfrak{w}, \mathfrak{v}) := \left[(\mathsf{A} + \frac{1}{2}\mathrm{Id})\,\mathsf{C}\,\mathfrak{w}, \mathfrak{v}\right]\ , \tag{5.16}$$

the variational problem of the *indirect single trace CFIE* can be stated as:

$$\boxed{\begin{cases} \text{seek } \mathfrak{u} \in \mathbb{X}_0(\Gamma) \text{ such that} \\ [\mathsf{A}\,\mathfrak{u}, \mathfrak{v}] + \mathsf{c}'(\mathfrak{u}, \mathfrak{v}) = -\left[\mathfrak{u}_{\mathrm{inc}}, \mathfrak{v}\right] \quad \forall \mathfrak{v} \in \mathbb{X}_0(\Gamma)\ . \end{cases}} \tag{5.17}$$

Lemma 5.7. *We have* $\mathsf{c}'(\mathfrak{w}, \mathfrak{v}) = \mathsf{c}(\mathfrak{v}, \mathfrak{w})$ *for all* $\mathfrak{v}, \mathfrak{w} \in \mathbb{X}_0(\Gamma)$.

Proof: This is an immediate consequence of the definitions (5.9), (5.16), of Lemma 3.6, and of the skew-symmetry of the pairing $[\cdot,\cdot]$. □

Corollary 5.8. *For any choice of the wave numbers* $\kappa_0, \ldots \kappa_n$ *satisfying (*2.3*), the indirect single trace CFIE* (5.17) *has a unique solution.*

Proof: Lemma 5.7 tells us that the bilinear forms of (5.17) and (5.10) are adjoint to each other. As a consequence, Corollary 5.4 and Lemma 5.5 carry over to (5.17) verbatim. A Fredholm alternative argument clinches the case. □

The previous proposition makes clear that Formulation (5.17) is always well posed. Now, assume that $\mathfrak{u}$ is defined as the solution to Formulation (5.17). Undo the substitution made above by setting $\tilde{\mathfrak{u}} = \mathsf{R}^{-1}\,\mathfrak{u} = (\mathrm{Id} - \mathsf{C})\mathfrak{u}$. Then, by construction, the function U defined by (5.12) solves Problem (2.2) and coincides with its unique solution. Ultimately, this proves that a representation according to (5.12) can always be found for a solution of Problem (2.2). In addition, by means of (5.12) the field can be recovered.

Remark 5.9. In the case $n = 0$ already discussed in Remark 5.6 the variational problem (5.17) boils down to the indirect CFIE derived in [3, Sect. 3].

5.4 Mixed variational formulations

A convenient concrete choice for an operator M satisfying (5.1a) and (5.1b) was proposed in [3, Sect. 4], namely $\mathsf{M} = (\Delta_\Sigma + \mathrm{Id})^{-1} : H^{-1}(\Sigma) \to H^1(\Sigma)$, where Δ_Σ stands for the Laplace-Beltrami operator on the closed surface Σ. The variational definition of this operator reads:

$$\mathsf{M}\,\varphi \in H^1(\Sigma) : \quad \mathsf{d}_\Sigma(\mathsf{M}\,\varphi, v_\Sigma) = -\imath\,\langle \varphi, v_\Sigma\rangle_\Sigma \quad \forall v_\Sigma \in H^1(\Sigma),\ \varphi \in H^{-1}(\Sigma)\ , \tag{5.18}$$

with sesqui-linear form ($\mathbf{grad}_\Sigma$ is the surface gradient on Σ)

$$\mathsf{d}_\Sigma(z, v) := \int_\Sigma \mathbf{grad}_\Sigma\, z \cdot \mathbf{grad}_\Sigma\, v + z\, v\, \mathrm{d}S\ , \quad z, v \in H^1(\Sigma)\ . \tag{5.19}$$

Compactness of $\mathsf{M} : H^{-1/2}(\Sigma) \to H^{+1/2}(\Sigma)$ is immediate from the continuity $\mathsf{M} : H^{-1}(\Sigma) \to H^1(\Sigma)$ and the compact embeddings $H^{-1/2}(\Sigma) \subset H^{-1}(\Sigma)$ and $H^1(\Sigma) \subset H^{+1/2}(\Sigma)$. This operator is also symmetric in the sense that

$$\langle \mathsf{M}\,\varphi, \psi\rangle_\Sigma = \langle \mathsf{M}\,\psi, \varphi\rangle_\Sigma\ , \quad \varphi, \psi \in H^{-1}(\Sigma)\ . \tag{5.20}$$

The bilinear forms of the variational formulations (5.10) and (5.15) of single-trace CFIEs involve evaluations of M. With Galerkin boundary element discretization in mind, it is desirable to avoid these and rely on the variational definition of M instead. As in [3, Sect. 4.2 & Sect. 3.2], this can be achieved by introducing auxiliary variables. In light of Lemma 5.7 we will restrict the discussion to the direct formulation (5.8). Using (4.2), the bilinear form c from (5.9) can be rewritten as ($\mathfrak{w}, \mathfrak{v} \in \mathbb{X}_0(\Gamma)$)

$$\begin{aligned}
\mathsf{c}(\mathfrak{w}, \mathfrak{v}) &= \sum_{j=0}^{n} \Big[\big(-\mathrm{Id}/2 + \mathsf{A}^j_{\kappa_j}\big)\mathfrak{w}_j, \big(\mathsf{C}\,\mathfrak{v}\big)_j \Big]_j = -\tfrac{1}{2}\,[\mathfrak{w}, \mathsf{C}\,\mathfrak{v}] + \sum_{j=0}^{n} \Big[\mathsf{A}^j_{\kappa_j}\,\mathfrak{w}_j, \big(\mathsf{C}\,\mathfrak{v}\big)_j \Big]_j \\
&\overset{①}{=} \tfrac{1}{2}\,[\mathsf{T}\,\mathfrak{w}, \mathsf{T}(\mathsf{C}\,\mathfrak{v})]_\Sigma - \sum_{j=0}^{n} \Big\langle \big(\mathsf{A}^j_{\kappa_j}\,\mathfrak{w}_j\big)_{\mathrm{N}}, \gamma^j_{\mathrm{D}}\,\mathsf{E}_\Sigma\,\mathsf{M}(\mathsf{T}_{\mathrm{N}}\,\mathfrak{v}) \Big\rangle_j \\
&\overset{②}{=} -\frac{1}{2}\,\langle \mathsf{T}_{\mathrm{N}}(\mathfrak{w}), \mathsf{M}(\mathsf{T}_{\mathrm{N}}\,\mathfrak{v})\rangle_\Sigma - \sum_{j=0}^{n} \Big\langle (\gamma^j_{\mathrm{D}}\,\mathsf{E}_\Sigma)'\big(\mathsf{A}^j_{\kappa_j}\,\mathfrak{w}_j\big)_{\mathrm{N}}, \mathsf{M}(\mathsf{T}_{\mathrm{N}}\,\mathfrak{v}) \Big\rangle_\Sigma \\
&\overset{③}{=} \Big\langle \mathsf{M}\Big(-\tfrac{1}{2}\,\mathsf{T}_{\mathrm{N}}(\mathfrak{w}) - \sum_{j=0}^{n} (\gamma^j_{\mathrm{D}}\,\mathsf{E}_\Sigma)'\big(\mathsf{A}^j_{\kappa_j}\,\mathfrak{w}_j\big)_{\mathrm{N}}\Big), \mathsf{T}_{\mathrm{N}}(\mathfrak{v}) \Big\rangle_\Sigma
\end{aligned}$$

In step ① we appeal to Proposition 3.2 for the first term and use the notation $(\cdot)_{\mathrm{N}}$ to extract the Neumann component of Cauchy traces. We also exploit that $\mathsf{C}\,\mathfrak{v}$ has vanishing Neumann component and the definition (5.5) of C. The step ② uses that $\mathsf{T}_{\mathrm{D}} \circ \mathsf{E}_\Sigma = \mathrm{Id}$ and the adjoint operator $(\gamma^j_{\mathrm{D}}\,\mathsf{E}_\Sigma)' : H^{-1/2}(\partial\Omega_j) \to H^{-1/2}(\Sigma)$. In ③ we apply (5.20). These manipulations suggest that we introduce the new unknown

$$z_\Sigma := \mathsf{M}\Big(-\tfrac{1}{2}\,\mathsf{T}_{\mathrm{N}}(\mathfrak{w}) - \sum_{j=0}^{n} (\gamma^j_{\mathrm{D}}\,\mathsf{E}_\Sigma)'\big(\mathsf{A}^j_{\kappa_j}\,\mathfrak{w}_j\big)_{\mathrm{N}}\Big) \in H^1(\Sigma)\ , \tag{5.21}$$

which satisfies

$$\begin{aligned}\mathsf{d}_\Sigma(z_\Sigma, v_\Sigma) &= \left\langle -\tfrac{1}{2}\,\mathsf{T}_{\mathrm{N}}\,\mathfrak{w} - \sum_{j=0}^{n} (\gamma_{\mathrm{D}}^{j}\,\mathsf{E}_\Sigma)'\big(\mathsf{A}_{\kappa_j}^{j}\,\mathfrak{w}_j\big)_{\mathrm{N}}, v_\Sigma \right\rangle_\Sigma \\ &= -\tfrac{1}{2}\,\langle \mathsf{T}_{\mathrm{N}}\,\mathfrak{w}, v_\Sigma\rangle_\Sigma - \sum_{j=0}^{n} \left\langle \big(\mathsf{A}_{\kappa_j}^{j}\,\mathfrak{w}_j\big)_{\mathrm{N}}, \gamma_{\mathrm{D}}^{j}\,\mathsf{E}_\Sigma(v_\Sigma) \right\rangle_j \quad \forall v_\Sigma \in \mathrm{H}^1(\Sigma)\ . \end{aligned} \tag{5.22}$$

By means of z_Σ we can express $\mathsf{c}(\mathfrak{w}, \mathfrak{v}) = \langle z_\Sigma, \mathsf{T}_{\mathrm{N}}\,\mathfrak{v}\rangle_\Sigma$, which converts the variational problem (5.10) of the direct single-trace CFIE into mixed form: seek $\mathfrak{u} \in \mathbb{X}_0(\Gamma)$, $z_\Sigma \in \mathrm{H}^1(\Gamma)$ such that, $\forall \mathfrak{v} \in \mathbb{X}_0(\Gamma)$, $\forall v_\Sigma \in \mathrm{H}^1(\Sigma)$,

$$\begin{aligned} [\mathsf{A}\,\mathfrak{u}, \mathfrak{v}] + \langle z_\Sigma, \mathsf{T}_{\mathrm{N}}\,\mathfrak{v}\rangle_\Sigma &= -\,[\widetilde{\mathfrak{u}}_{\mathrm{inc}}, \mathfrak{v}]\ , \\ \Big\langle \tfrac{1}{2}\,\mathsf{T}_{\mathrm{N}}(\mathfrak{u}) + \sum_{j=0}^{n} (\mathsf{A}_{\kappa_j}^{j}\,\mathfrak{u}_j)_{\mathrm{N}}, \gamma_{\mathrm{D}}^{j}\,\mathsf{E}_\Sigma(v_\Sigma)\Big\rangle_\Sigma + \mathsf{d}_\Sigma(z_\Sigma, v_\Sigma) &= 0\ . \end{aligned} \tag{5.23}$$

This variational problem inherits coercivity from (5.10), because the compact embedding $\mathrm{H}^1(\Sigma) \subset \mathrm{H}^{+1/2}(\Sigma)$ renders the off-diagonal operators of (5.23) compact. Uniqueness also carries over from (5.10). Moreover, (5.23) is amenable to Galerkin discretization by means of standard boundary elements, for instance, piecewise linear continuous functions on a triangular surface mesh of Σ for the approximation of z_Σ.

6 Multi-trace Combined Field Integral Equations

As pointed out in the Introduction, a shortcoming of the classical single-trace formulation (4.4) and also of its stabilized versions(5.10) and (5.17) is the tight coupling between subdomains implicit in the use of the single trace variational space $\mathbb{X}_0(\Gamma)$, which contains the transmission conditions "in strong form". This limits flexibility in using Galerkin trial spaces locally on the subdomains. More severely, it turned out to be a big obstacle to the use of operator preconditioning techniques. We skip a detailed explanation here and recommend that the reader study [12, Sect. 4]. We only quote the conclusion drawn in [12] and [13] that switching to variational formulations posed on decoupled local trace spaces will pave the way for effective operator preconditioning.

This has been the main motivation behind the development of so-called multi-trace formulations (MTFs). Here the expression "multi-trace" refers to a family of BIE where the unknowns are doubled on each interface that separates two (bounded) subdomains. In [11] and [12, Sect. 5] a global MTF was devised based on the classical STF (4.4). In this section we are going to derive and study its CFIE counterpart related to the formulations that we have established in Sections 4 and 5.

6.1 The gap idea

The global MTF was discovered through a heuristic geometric limit process, which is elucidated and justified in [11, Sect. 5], [12, Sect. 5.2], and [13, Sect. 4.2]. Tersely speaking, we imagine an (infinitely) narrow gap between bounded subdomains Ω_j, $j = 1, \ldots, N$, including Ω_Σ. This gap is filled with the same ambient medium as Ω_0, see Figure 5 for an illustration. For this arrangement where all bounded subdomains are isolated from each other we consider variational single trace formulations. Sloppily speaking, the corresponding global MTFs then boil down to STFs applied to gap configurations with vanishing gap.

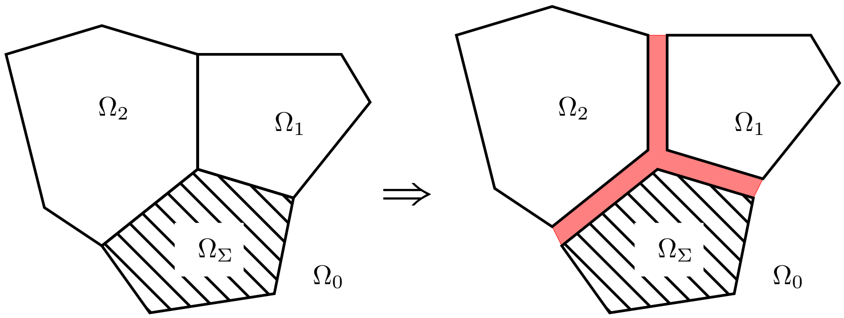


FIGURE 5. Illustration of the gap idea (gap highlighted)

Recalling Theorem 4.8, the alert reader will have noticed that *the gap configuration as in Figure* 5 *(right) is exactly the situation, in which spurious resonances may afflict the classical STF* (4.4), because $\Sigma \subset \partial\Omega_0$. More precisely, uniqueness of solutions will be lost, if $\kappa_0 \in \mathfrak{S}(\Delta, \Omega_\Sigma)$, where the latter set comprises the interior Dirichlet eigenvalues for $-\Delta$ on Ω_Σ, see (4.9). Thus,

(E1). we expect that the standard global MTF will suffer from spurious resonances whenever $\kappa_0 \in \mathfrak{S}(\Delta, \Omega_\Sigma)$.

On the other hand,

(E2). we also expect that the MTFs we obtain from pursuing the gap construction for CFIE extensions of the STF, will be stable for all combinations of wave numbers.

This hope relies on Lemmas 5.5 and Corollary 5.8. In the sequel we give rigorous justifications of our conjectures. We are not going to employ vanishing gap arguments, which entail difficult geometric limit processes, but directly scrutinize the variational formulations as in [11].

In the gap setting (Figure 5, right) we face a partitioning $\partial\Omega_0 = \cup_{j=1}^n \partial\Omega_j \cup \Sigma$ so that, in this special case, the variational space $\mathbb{X}_0(\Gamma)$ from (4.1) for the STF variational formulations is clearly isomorphic to a product of Cauchy trace spaces on the subdomain boundary and Neumann traces on Σ:

$$\widehat{\mathbb{H}}(\Gamma) := \mathbb{H}(\partial\Omega_1) \times \cdots \times \mathbb{H}(\partial\Omega_n) \times \mathrm{H}^{-\frac{1}{2}}(\Sigma) \,. \tag{6.1}$$

This space will supply the functional framework for the global MTF, including for general configurations (such as in Figure 5, left). The main difference between $\widehat{\mathbb{H}}(\Gamma)$ and the space $\mathbb{H}(\Gamma)$ introduced in (3.1) is that the former does not contain any contribution from $\partial\Omega_0$. Instead, it comprises contributions from Σ, via a trace chosen in $\mathrm{H}^{-1/2}(\Sigma)$. We equip the new space $\widehat{\mathbb{H}}(\Gamma)$ with a norm defined by

$$\|\widehat{\mathfrak{u}}\|^2_{\widehat{\mathbb{H}}(\Gamma)} := \|\widehat{\mathfrak{u}}_1\|^2_{\mathbb{H}(\partial\Omega_1)} + \cdots + \|\widehat{\mathfrak{u}}_n\|^2_{\mathbb{H}(\partial\Omega_n)} + \|p_\Sigma\|^2_{\mathrm{H}^{-1/2}(\Sigma)}$$

for all $\widehat{\mathfrak{u}} = (\widehat{\mathfrak{u}}_1, \dots, \widehat{\mathfrak{u}}_n, p_\Sigma)$. Clearly, the dual space of $\widehat{\mathbb{H}}(\Gamma)$ with respect to local L^2-type duality pairings is $\check{\mathbb{H}}(\Gamma) := \mathbb{H}(\partial\Omega_1) \times \cdots \times \mathbb{H}(\partial\Omega_n) \times \mathrm{H}^{1/2}(\Sigma)$. In concrete terms, for $\widehat{\mathfrak{u}} = (\widehat{\mathfrak{u}}_1, \dots, \widehat{\mathfrak{u}}_n, u_\Sigma) \in \check{\mathbb{H}}(\Gamma)$, and $\widehat{\mathfrak{v}} = (\widehat{\mathfrak{v}}_1, \dots, \widehat{\mathfrak{v}}_n, q_\Sigma) \in \widehat{\mathbb{H}}(\Gamma)$ the underlying duality pairing between $\widehat{\mathbb{H}}(\Gamma)$ and $\check{\mathbb{H}}(\Gamma)$ is defined by the bilinear form

$$[\![\widehat{\mathfrak{u}}, \widehat{\mathfrak{v}}]\!] := \sum_{j=1}^{n} \left[\widehat{\mathfrak{u}}_j, \widehat{\mathfrak{v}}_j\right]_j + \langle u_\Sigma, q_\Sigma\rangle_\Sigma \ .$$

Routine verifications show that this bilinear form is non-degenerate and satisfies inf-sup conditions. We will use it to derive variational formulations.

6.2 Multi-trace formulations (MTFs)

Guided by the gap idea, and the STF (4.4) in gap settings, we can embark on the lengthy manipulations elaborated in [11, Sect. 8] and [12, Eq. (5.8)]. Since no new complications arise in the presence of essential boundary conditions, we omit the details. In the end we arrive at a *multi-trace formulation* for the transmission boundary value problem with Dirichlet boundary conditions on Σ:

$$\boxed{\left\{\begin{array}{l} \text{find } \widehat{\mathfrak{u}} \in \widehat{\mathbb{H}}(\Gamma) \quad \text{such that} \\[2ex] \left[\!\!\left[\widehat{\mathsf{A}}(\widehat{\mathfrak{u}}), \widehat{\mathfrak{v}}\right]\!\!\right] = \left[\!\!\left[\widehat{\mathfrak{f}}, \widehat{\mathfrak{v}}\right]\!\!\right] \quad \forall \widehat{\mathfrak{v}} \in \widehat{\mathbb{H}}(\Gamma)\ , \end{array}\right.} \tag{6.2}$$

where $\widehat{\mathfrak{f}} = (\widehat{\mathfrak{f}}_1, \dots, \widehat{\mathfrak{f}}_n, f_\Sigma) \in \check{\mathbb{H}}(\Gamma)$ is defined by $\widehat{\mathfrak{f}}_j = \gamma^j(U_{\mathrm{inc}})$ and $f_\Sigma = \gamma^\Sigma_{\mathrm{D}}(U_{\mathrm{inc}})$, and $\widehat{\mathsf{A}} : \widehat{\mathbb{H}}(\Gamma) \to \check{\mathbb{H}}(\Gamma)$ is a continuous linear operator defined by

$$\widehat{\mathsf{A}} := \begin{bmatrix} \mathsf{A}^1_{\kappa_1} + \mathsf{A}^1_{\kappa_0} & \gamma^1\,\mathsf{G}^2_{\kappa_0} & \cdots & \gamma^1\,\mathsf{G}^n_{\kappa_0} & \gamma^1\,\mathsf{SL}^\Sigma_{\kappa_0} \\ \gamma^2\,\mathsf{G}^1_{\kappa_0} & \mathsf{A}^2_{\kappa_2} + \mathsf{A}^2_{\kappa_0} & \cdots & \gamma^2\,\mathsf{G}^n_{\kappa_0} & \gamma^2\,\mathsf{SL}^\Sigma_{\kappa_0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \gamma^n\,\mathsf{G}^1_{\kappa_0} & \gamma^n\,\mathsf{G}^2_{\kappa_0} & \cdots & \mathsf{A}^n_{\kappa_n} + \mathsf{A}^n_{\kappa_0} & \gamma^n\,\mathsf{SL}^\Sigma_{\kappa_0} \\ \gamma^\Sigma_{\mathrm{D}}\,\mathsf{G}^1_{\kappa_0} & \gamma^\Sigma_{\mathrm{D}}\,\mathsf{G}^2_{\kappa_0} & \cdots & \gamma^\Sigma_{\mathrm{D}}\,\mathsf{G}^n_{\kappa_0} & \mathsf{V}^\Sigma_{\kappa_0} \end{bmatrix} \tag{6.3}$$

Definitions of the potentials $\mathsf{SL}^\Sigma_\kappa$ and $\mathsf{G}^j_{\kappa_0}$ can be found in (3.7), and $\mathsf{V}^\Sigma_{\kappa_0} := \gamma^\Sigma_{\mathrm{D}}\,\mathsf{SL}^\Sigma_{\kappa_0}$ is a single layer boundary integral operator on Σ. Hence, with $\widehat{\mathfrak{u}} =$

$(\widehat{\mathfrak{u}}_1,\ldots,\widehat{\mathfrak{u}}_n,p_\Sigma)$, $\widehat{\mathfrak{v}}=(\widehat{\mathfrak{v}}_1,\ldots,\widehat{\mathfrak{v}}_n,q_\Sigma)$, the bilinear form of (6.2) boils down to

$$\begin{aligned}\left[\!\left[\widehat{\mathsf{A}}(\widehat{\mathfrak{u}}),\widehat{\mathfrak{v}}\right]\!\right] = & \sum_{j=1}^{n}\left[(\mathsf{A}^j_{\kappa_j}+\mathsf{A}^j_{\kappa_0})(\widehat{\mathfrak{u}}_j),\widehat{\mathfrak{v}}_j\right]_j + \sum_{j=1}^{n}\sum_{\substack{i=1\\ i\neq j}}\left[\gamma^j\,\mathsf{G}^i(\widehat{\mathfrak{u}}_i),\widehat{\mathfrak{v}}_j\right]_j + \\ & \sum_{j=1}^{n}\left[\gamma^j\,\mathsf{SL}^\Sigma_{\kappa_0}(p_\Sigma),\widehat{\mathfrak{v}}_j\right]_j + \sum_{j=1}^{n}\left\langle\gamma^\Sigma_{\mathrm{D}}\,\mathsf{G}^j_{\kappa_0}(\widehat{\mathfrak{u}}_j),q_\Sigma\right\rangle_\Sigma \\ & + \left\langle\gamma^\Sigma_{\mathrm{D}}\,\mathsf{SL}^\Sigma_{\kappa_0}(p_\Sigma),q_\Sigma\right\rangle_\Sigma \end{aligned} \tag{6.4}$$

Remark 6.1. The key observation is that all building blocks of $\widehat{\mathsf{A}}$ and the terms in (6.4) remain well defined, even if we dispense with a gap between the subdomains Ω_j, $j\geqslant 1$ and Ω_Σ. Thus, $\widehat{\mathsf{A}}$ and the multi-trace variational problem (6.2) remain meaningful in the generic setting with junction points depicted in Figure 5, left, and introduced in Section 2. The gap idea instills confidence that (6.2) will inherit all properties of the single-trace problem (4.4) on isolated subdomains. In the next section, we are going to provide a rigorous foundation for this intuition.

6.3 Analysis of standard MTF

We consider the *standard global MTF* variational problem (6.2)/(6.4) in the general "non-gap" setting with possible junction points (Figure 5, left). Obviously, the bilinear form $(\widehat{\mathfrak{u}},\widehat{\mathfrak{v}})\to[\![\widehat{\mathsf{A}}(\widehat{\mathfrak{u}}),\widehat{\mathfrak{v}}]\!]$ is continuous on $\widehat{\mathbb{H}}(\Gamma)$. Also let us point out a symmetry property of this bilinear form that will be useful later. Due to the definition of $\widehat{\mathsf{A}}$ from (6.3), the next result is a direct consequence of Lemma 3.6 and Lemma 3.7:

$$\left[\!\left[\widehat{\mathsf{A}}(\widehat{\mathfrak{u}}),\widehat{\mathfrak{v}}\right]\!\right] = \left[\!\left[\widehat{\mathsf{A}}(\widehat{\mathfrak{v}}),\widehat{\mathfrak{u}}\right]\!\right] \quad \forall\widehat{\mathfrak{u}},\widehat{\mathfrak{v}}\in\widehat{\mathbb{H}}(\Gamma)\ . \tag{6.5}$$

Now, extending Proposition 4.4 to the standard global MTF, the following proposition exhibits the precise relationship between Formulation (6.2) and Problem (2.2). Corresponding results for the pure transmission problem can be found in [11, Sect. 9].

Proposition 6.2. *If* $\widehat{\mathfrak{u}}=(\widehat{\mathfrak{u}}_1,\ldots,\widehat{\mathfrak{u}}_n,p_\Sigma)\in\widehat{\mathbb{H}}(\Gamma)$ *solves (*6.2*) then* $U\in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d\backslash\Omega_\Sigma)$ *defined by*

$$\begin{aligned} U(\mathbf{x}) &= \mathsf{G}^j_{\kappa_j}(\widehat{\mathfrak{u}}_j)(\mathbf{x}) && \textit{for } \mathbf{x}\in\Omega_j,\ j=1\ldots n \\ U(\mathbf{x}) &= U_{\mathrm{inc}}(\mathbf{x})-\mathsf{SL}^\Sigma_{\kappa_0}(p_\Sigma)(\mathbf{x})-\textstyle\sum_{j=1}^n\mathsf{G}^j_{\kappa_0}(\widehat{\mathfrak{u}}_j)(\mathbf{x}) && \textit{for } \mathbf{x}\in\Omega_0 \end{aligned} \tag{6.6}$$

*is the unique solution of Problem (*2.2*).*

Proof: By construction, the function U defined by (6.6) satisfies $\Delta U+\kappa_j^2U=0$ in Ω_j for $j=0\ldots n$, and the radiation conditions at ∞ (with respect to κ_0). The only property we have to verify is the transmission conditions (2.2b), that is, $(\gamma^j(U))_{j=0}^n\in\mathbb{X}_0(\Gamma)$. Owing to Lemma 4.1 this is equivalent to showing that for all $\mathfrak{v} = (\mathfrak{v}_j)_{j=0}^n \in \mathbb{X}_0(\Gamma)$ we have $\sum_{j=0}^n \left[\gamma^j(U), \mathfrak{v}_j\right]_j = 0$ which, see (6.6), is equivalent to

$$\left[\gamma^0 U_{\mathrm{inc}} - \gamma^0\,\mathsf{SL}_{\kappa_0}^{\Sigma}(p_\Sigma) - \sum_{j=1}^n \gamma^0\,\mathsf{G}_{\kappa_0}^j(\hat{\mathfrak{u}}_j), \hat{\mathfrak{v}}_0\right]_0 + \sum_{j=1}^n \left[\gamma^j\,\mathsf{G}_{\kappa_0}^j(\hat{\mathfrak{u}}_j), \mathfrak{v}_j\right]_j = 0\,. \tag{6.7}$$

We fix some $\mathfrak{v} \in \mathbb{X}_0(\Gamma)$, and denote $\mathfrak{v}_\star := (\mathfrak{v}_1, \ldots, \mathfrak{v}_n, \mathsf{T}_{\mathrm{N}}(\mathfrak{v})) \in \widehat{\mathbb{H}}(\Gamma)$. For the remainder of the proof it is important to remember that $\mathsf{T}_{\mathrm{D}}(\mathfrak{v}) = 0$. From the jump relations (3.8) and (3.10) we can conclude $\mathsf{A}_{\kappa_j}^j + \mathsf{A}_{\kappa_0}^j = \gamma^j\,\mathsf{G}_{\kappa_j}^j + \gamma_c^j\,\mathsf{G}_{\kappa_0}^j$. We use this identity and infer from (6.2) and (6.4) with $\hat{\mathfrak{v}} = \mathfrak{v}_\star$

$$\begin{aligned}
0 = \left[\!\left[\hat{\mathsf{A}}\hat{\mathfrak{u}}, \mathfrak{v}_\star\right]\!\right] - \left[\!\left[\hat{\mathfrak{f}}, \mathfrak{v}_\star\right]\!\right] = \sum_{j=1}^n \Big(&\left[\gamma^j\,\mathsf{G}_{\kappa_j}^j(\hat{\mathfrak{u}}_j), \mathfrak{v}_j\right]_j + \left[\gamma_c^j\,\mathsf{G}_{\kappa_0}^j(\hat{\mathfrak{u}}_j), \mathfrak{v}_j\right]_j \\
&+ \sum_{\substack{i=1\\ i\neq j}}^n \left[\gamma^j\,\mathsf{G}_{\kappa_0}^i(\hat{\mathfrak{u}}_i), \mathfrak{v}_j\right]_j + \left\langle \gamma_D^\Sigma\,\mathsf{G}_{\kappa_0}^j(\hat{\mathfrak{u}}_j), \mathsf{T}_{\mathrm{N}}(\mathfrak{v})\right\rangle_\Sigma \Big) && (6.8a)\\
&+ \sum_{j=1}^n \left[\gamma^j\,\mathsf{SL}_{\kappa_0}^\Sigma(p_\Sigma), \mathfrak{v}_j\right]_j + \left\langle \gamma_{\mathrm{D}}^\Sigma\,\mathsf{SL}_{\kappa_0}^\Sigma(p_\Sigma), \mathsf{T}_{\mathrm{N}}(\mathfrak{v})\right\rangle_\Sigma && (6.8b)\\
&- \sum_{j=1}^n \left[\gamma^j U_{\mathrm{inc}}, \mathfrak{v}_j\right]_j - \left\langle \gamma_{\mathrm{D}}^\Sigma U_{\mathrm{inc}}, \mathsf{T}_{\mathrm{N}}(\mathfrak{v})\right\rangle_\Sigma\,. && (6.8c)
\end{aligned}$$

For $j = 1, \ldots, n$, evidently $\mathsf{G}_{\kappa_0}^j(\hat{\mathfrak{u}}_j) \in \mathrm{H}_{\mathrm{loc}}^1(\Delta, \mathbb{R}^d \backslash \Omega_j)$. As a consequence, when we take the trace on $\partial\Omega_j$ from *outside*, we have $\mathfrak{z} := \left(\gamma^0\,\mathsf{G}_{\kappa_0}^j(\hat{\mathfrak{u}}_j), \ldots, \gamma_c^j\,\mathsf{G}_{\kappa_0}^j(\hat{\mathfrak{u}}_j), \ldots, \gamma^n\,\mathsf{G}_{\kappa_0}^j(\hat{\mathfrak{u}}_j)\right) \in \mathbb{X}(\Gamma)$. Thus, we can invoke Proposition 3.2, and find $[\mathfrak{z}, \mathfrak{v}] = -\left[\mathsf{T}(\mathfrak{z}), \mathsf{T}(\mathfrak{v})\right]_\Sigma$, which means

$$\begin{aligned}
\left[\gamma_c^j\,\mathsf{G}_{\kappa_0}^j(\hat{\mathfrak{u}}_j), \mathfrak{v}_j\right]_j + \sum_{\substack{i=1\\ i\neq j}}^n \left[\gamma^i\,\mathsf{G}_{\kappa_0}^j\,\hat{\mathfrak{u}}_j, \mathfrak{v}_i\right]_i + \left\langle \gamma_{\mathrm{D}}^\Sigma\,\mathsf{G}_{\kappa_0}^j(\hat{\mathfrak{u}}_j), \mathsf{T}_{\mathrm{N}}(\mathfrak{v})\right\rangle_\Sigma \\
= -\left[\gamma^0\,\mathsf{G}_{\kappa_0}^j\,\hat{\mathfrak{u}}_j, \mathfrak{v}_0\right]_0\,.
\end{aligned} \tag{6.9}$$

In the same vein, we can set $\mathfrak{y} := \left(\gamma^0\,\mathsf{SL}_{\kappa_0}^\Sigma(p_\Sigma), \ldots, \gamma^n\,\mathsf{SL}_{\kappa_0}^\Sigma(p_\Sigma)\right) \in \mathbb{X}(\Gamma)$, which, again by Proposition 3.2, satisfies $[\mathfrak{y}, \mathfrak{v}] = -\left[\mathsf{T}(\mathfrak{y}), \mathsf{T}(\mathfrak{v})\right]_\Sigma$, equivalent to

$$\begin{aligned}
\sum_{j=1}^n \left[\gamma^j\,\mathsf{SL}_{\kappa_0}^\Sigma(p_\Sigma), \mathfrak{v}_j\right]_j + \left\langle \gamma_{\mathrm{D}}^\Sigma\,\mathsf{SL}_{\kappa_0}^\Sigma(p_\Sigma), \mathsf{T}_{\mathrm{N}}(\mathfrak{v})\right\rangle_\Sigma \\
= -\left[\gamma^0\,\mathsf{SL}_{\kappa_0}^\Sigma(p_\Sigma), \mathfrak{v}_0\right]_0\,.
\end{aligned} \tag{6.10}$$

Similarly, since $\Delta U_{\mathrm{inc}} + \kappa_0^2 U_{\mathrm{inc}} = 0$ everywhere, Proposition 3.2 yields $\sum_{j=0}^n \left[\gamma^j U_{\mathrm{inc}}, \mathfrak{v}_j\right]_j = -\left[\gamma^\Sigma U_{\mathrm{inc}}, \mathsf{T}\,\mathfrak{v}\right]_\Sigma = -\left\langle \gamma_{\mathrm{D}}^\Sigma U_{\mathrm{inc}}, \mathsf{T}_{\mathrm{N}}\,\mathfrak{v}\right\rangle_\Sigma$. Obviously, we aim to use this last identity to tackle (6.8c), (6.9) (summed over $j = 1, \ldots, n$) to simplify (6.8a), and (6.10) to replace (6.8b). Thus we arrive at

$$0 = \sum_{j=1}^{n} \left[\gamma^j \,\mathsf{G}^j_{\kappa_0}(\hat{\mathfrak{u}}_j), \mathfrak{v}_j\right]_j \quad - \sum_{j=1}^{n} \left[\gamma^0 \,\mathsf{G}^j_{\kappa_0}(\hat{\mathfrak{u}}_j), \mathfrak{v}_0\right]_0 \\ - \left[\gamma^0 \,\mathsf{SL}^{\Sigma}_{\kappa_0}(p_\Sigma), \mathfrak{v}_0\right]_0 + \left[\gamma^0 U_{\mathrm{inc}}, \mathfrak{v}_0\right]_\Sigma \ , \tag{6.11}$$

which agrees the equation (6.7)! Since $\mathfrak{v}$ was chosen arbitrarily in $\mathbb{X}_0(\Gamma)$, this finishes the proof. □

The gap construction hints that the operators $\hat{\mathsf{A}}$ defined in (6.3) will enjoy coercivity analogous to the assertions of Theorem 4.2. This is confirmed by the next result, which generalizes [11, Thm. 10.4].

Proposition 6.3. *Define the operators* $\theta_j : \mathbb{H}(\partial\Omega_j) \to \mathbb{H}(\partial\Omega_j)$ *by* $\theta_j(v, q_\Sigma) = (-\overline{v}, \overline{q}_\Sigma)$*, and let* $\Phi : \hat{\mathbb{H}}(\Gamma) \to \hat{\mathbb{H}}(\Gamma)$ *denote the operator* $\Phi(\hat{\mathfrak{v}}) = (\theta_1(\hat{\mathfrak{v}}_1), \ldots, \theta_n(\hat{\mathfrak{v}}_n), \overline{q})$ *for* $\hat{\mathfrak{v}} = (\hat{\mathfrak{v}}_1, \ldots, \hat{\mathfrak{v}}_n, q)$*. There exists a compact operator* $\mathsf{K} : \hat{\mathbb{H}}(\Gamma) \to \check{\mathbb{H}}(\Gamma)$*, and a constant* $\beta > 0$ *such that*

$$\left|\left[\!\left[(\hat{\mathsf{A}} + \mathsf{K})\hat{\mathfrak{v}}, \Phi(\hat{\mathfrak{v}})\right]\!\right]\right| \geqslant \beta \|\hat{\mathfrak{v}}\|^2_{\hat{\mathbb{H}}(\Gamma)} \quad \textit{for all} \quad \hat{\mathfrak{v}} \in \hat{\mathbb{H}}(\Gamma) \ .$$

Proof: Since a change of the wave numbers $\kappa_0, \kappa_1, \ldots, \kappa_n$ only induces a compact perturbation of $\hat{\mathsf{A}}$ [32, Lemma 3.9.8], it suffices to prove the result for the case where $\kappa_0 = \cdots = \kappa_n = \imath$ where $\imath = \sqrt{-1}$. Take any $\hat{\mathfrak{v}} = (\hat{\mathfrak{v}}_1, \ldots, \hat{\mathfrak{v}}_n, q_\Sigma) \in \hat{\mathbb{H}}(\Gamma)$. Denote $W_j(\mathbf{x}) := \mathsf{G}^j_{\kappa_0}(\hat{\mathfrak{v}}_j)(\mathbf{x})$ for $j = 1 \ldots n$, and $W_{n+1}(\mathbf{x}) := \mathsf{G}^{\Sigma}_{\kappa_0}(\hat{\mathfrak{v}}_{n+1})$ where $\hat{\mathfrak{v}}_{n+1} := (0, q_\Sigma) \in \mathbb{H}(\Sigma)$.

For the sake of concise notations, in the remainder of this proof, we will write $[\cdot,\cdot]_{n+1} := [\cdot,\cdot]_\Sigma$, $\mathsf{G}^{n+1}_{\kappa_0} := \mathsf{G}^{\Sigma}_{\kappa_0}$, $\mathsf{A}^{n+1}_{\kappa_0} := \{\gamma^\Sigma\}\,\mathsf{G}^{\Sigma}_{\kappa_0}$ and $\Omega_{n+1} := \Omega_\Sigma$. Then we have

$$\begin{aligned}\mathrm{Re}\left[\!\left[\hat{\mathsf{A}}(\hat{\mathfrak{v}}), \Phi(\hat{\mathfrak{v}})\right]\!\right] \;=\; & \mathrm{Re}\left[\mathsf{A}^{n+1}_{\kappa_0}(\hat{\mathfrak{v}}_{n+1}), \theta_{n+1}(\hat{\mathfrak{v}}_{n+1})\right]_{n+1} \\ & + \sum_{j=1}^{n} 2\ \mathrm{Re}\left[\mathsf{A}^j_{\kappa_0}(\hat{\mathfrak{v}}_j), \theta_j(\hat{\mathfrak{v}}_j)\right]_j \\ & + \sum_{j=1}^{n+1} \sum_{\substack{q=1\\ q\neq j}}^{n+1} \mathrm{Re}\left[\gamma^q \,\mathsf{G}^j_{\kappa_0}(\hat{\mathfrak{v}}_j), \theta_q(\hat{\mathfrak{v}}_q)\right]_q \ .\end{aligned}$$

Proceeding exactly as in the proof of Proposition 10.3 in [11], and in particular applying Proposition 10.1 and 10.2 of [11], we have

$$\begin{aligned}\mathrm{Re}\left[\!\left[\hat{\mathsf{A}}(\hat{\mathfrak{v}}), \Phi(\hat{\mathfrak{v}})\right]\!\right] \;=\; & \sum_{q=0}^{n+1} \sum_{j=1}^{n} \int_{\Omega_q} |\nabla W_j|^2 + |W_j|^2 d\mathbf{x} \\ & + \sum_{q=0}^{n+1} \int_{\Omega_q} \left|\nabla\Big(\sum_{j=1}^{n+1} W_j\Big)\right|^2 + \left|\sum_{j=1}^{n+1} W_j\right|^2 d\mathbf{x} \\ \geqslant\; & \sum_{q=0}^{n+1} \sum_{j=1}^{n} \|W_j\|^2_{\mathrm{H}^1(\Omega_q)} \ .\end{aligned} \tag{6.12}$$

Note that $(a_1 + \cdots + a_k)^2 \leqslant k\,(a_1^2 + \ldots a_k^2)$ for any $a_1,\ldots a_k \in \mathbb{R}$. Applying this elementary identity to (6.12) allows to conclude that

$$\begin{aligned} 2(n+1)\,\mathrm{Re}\,\left[\!\left[\widehat{\mathsf{A}}(\widehat{\mathfrak{v}}),\Phi(\widehat{\mathfrak{v}})\right]\!\right] & \\ &\geqslant 2n\sum_{q=0}^{n+1}\sum_{j=1}^{n}\|W_j\|^2_{\mathrm{H}^1(\Omega_q)} + 2\sum_{q=0}^{n+1}\Big\|\sum_{j=1}^{n+1}W_j\Big\|^2_{\mathrm{H}^1(\Omega_q)} \\ &\geqslant \sum_{q=0}^{n+1}\|W_{n+1}\|^2_{\mathrm{H}^1(\Omega_q)}\,. \end{aligned} \tag{6.13}$$

Now, since $-\Delta W_j + W_j = 0$ in Ω_q for any j,q, and since, by the jump relations (3.8), $\widehat{\mathfrak{v}}_j = [\gamma^j(W_j)]$, the continuity of trace operations yields $\|\widehat{\mathfrak{v}}_j\|_{\mathbb{H}(\partial\Omega_j)} \leqslant C\sum_{q=0}^{n+1}\|W_j\|_{\mathrm{H}^1(\Omega_q)}$. Combining this with (6.12) and (6.13) concludes the proof. $\square$

A direct consequence of the previous proposition is that the operator $\widehat{\mathsf{A}}$ is Fredholm with index 0. Hence it is an isomorphism if it is injective, which can fail only in case of spurious resonnance, since Problem (2.2) is well posed. Recalling the gap idea and the characterization of the kernel of A from Theorem 4.8, the following result about spurious resonances of the global MTF is not surprising, *cf.* Section 6.1.

Proposition 6.4. $\mathrm{Ker}(\widehat{\mathsf{A}}) = \{\ (0,\ldots,0,p)\mid p\in\mathrm{Ker}(\gamma_{\mathrm{D}}^{\Sigma}\,\mathsf{SL}_{\kappa_0}^{\Sigma})\ \}$. *As a consequence, for any choice of wave numbers κ_j, the operator $\widehat{\mathsf{A}}$ is a bijection if and only if $\kappa_0 \notin \mathfrak{S}(\Delta,\Omega_\Sigma)$.*

Proof: Since $\widehat{\mathsf{A}}$ is Fredholm with index 0, it is a bijection, if and only if it is injective. Assume that $\widehat{\mathfrak{u}} = (\widehat{\mathfrak{u}}_1,\ldots,\widehat{\mathfrak{u}}_n,p_\Sigma) \in \widehat{\mathbb{H}}(\Gamma)$ satisfies $\widehat{\mathsf{A}}(\widehat{\mathfrak{u}}) = 0$. In this case Proposition 6.2 applies with $U_{\mathrm{inc}} = 0$. Since Problem (2.2) is well posed this shows that, in Formula (6.6), $U = 0$ as well, so we conclude that $\mathsf{G}^j_{\kappa_j}(\widehat{\mathfrak{u}}_j)(\mathbf{x}) = 0$ for $\mathbf{x}\in\Omega_j$, and finally

$$\gamma^j\,\mathsf{G}^j_{\kappa_j}(\widehat{\mathfrak{u}}_j) = 0 \quad \forall j = 1,\ldots,n\,. \tag{6.14}$$

Now pick an arbitrary $l = 1\ldots n$, and an arbitrary $\mathfrak{v}_l \in \mathcal{C}_{\kappa_l}(\partial\Omega_l)$ from the space of Cauchy data defined in (3.4). We have $\widehat{\mathfrak{v}} := (0,\ldots,0,\mathfrak{v}_l,0\ldots,0) \in \widehat{\mathbb{H}}(\Gamma)$. We can apply (6.2) in the form $\left[\!\left[\widehat{\mathsf{A}}(\widehat{\mathfrak{u}}),\widehat{\mathfrak{v}}\right]\!\right] = 0$, take into account the definition of $\widehat{\mathsf{A}}$, see (6.4), use (6.14), which yields

$$0 = \left[\gamma^l\,\mathsf{SL}^{\Sigma}_{\kappa_0}(p),\mathfrak{v}_l\right]_l + \sum_{j=1}^{n}\left[\gamma^l\,\mathsf{G}^j_{\kappa_0}(\widehat{\mathfrak{u}}_j),\mathfrak{v}_l\right]_l + \left[\gamma^l_c\,\mathsf{G}^l_{\kappa_l}(\widehat{\mathfrak{u}}_l),\mathfrak{v}_l\right]_l\,. \tag{6.15}$$

In the computations above, we used the identity $\mathsf{A}^l_{\kappa_l} + \mathsf{A}^l_{\kappa_0} = \gamma^l_c\,\mathsf{G}^l_{\kappa_l} + \gamma^l\,\mathsf{G}^l_{\kappa_0}$. Next, as $\mathfrak{v}_l\in\mathcal{C}_{\kappa_l}(\partial\Omega_l)$, Lemma 3.5 show that the following terms vanish

$$\left[\gamma^l\,\mathsf{G}^l_{\kappa_0}(\widehat{\mathfrak{u}}_l),\mathfrak{v}_l\right]_l = 0 \quad,\quad \left[\gamma^l\,\mathsf{SL}^{\Sigma}_{\kappa_0}(p),\mathfrak{v}_l\right]_l = 0\,. \tag{6.16}$$

In addition, we have $\mathbb{H}(\partial\Omega_l) = \mathrm{Im}(\gamma_c^l\,\mathsf{G}_{\kappa_l}^l)\oplus\mathcal{C}_{\kappa_0}(\partial\Omega_l)$ according to [11, Lemma A.2]. Combining (6.15) and (6.16) we obtain that

$$\left[\gamma_c^l\,\mathsf{G}_{\kappa_l}^l(\hat{\mathfrak{u}}_l), \mathfrak{v}_l\right]_l = 0 \quad \text{for all} \quad \mathfrak{v}_l \in \mathbb{H}(\partial\Omega_l)\ . \tag{6.17}$$

Finally, we conclude that $\gamma_c^l\,\mathsf{G}_{\kappa_l}^l(\hat{\mathfrak{u}}_l) = 0$ for all $l = 1\ldots n$. As a consequence, we obtain from the jump relations

$$\hat{\mathfrak{u}}_j = [\gamma^j]\,\mathsf{G}_{\kappa_j}^j(\hat{\mathfrak{u}}_j) = \gamma^j\,\mathsf{G}_{\kappa_j}^j(\hat{\mathfrak{u}}_j) - \gamma_c^j\,\mathsf{G}_{\kappa_j}^j(\hat{\mathfrak{u}}_j) = 0.$$

Since $\widehat{\mathsf{A}}(\hat{\mathfrak{u}}) = 0$, from the bottom row of 6.3 we finally obtain that $\gamma_{\mathrm{D}}^{\Sigma}\,\mathsf{SL}_{\kappa_0}^{\Sigma}(p_\Sigma) = 0$. Hence $p_\Sigma \in \mathrm{Ker}(\gamma_{\mathrm{D}}^{\Sigma}\,\mathsf{SL}_{\kappa_0}^{\Sigma})$. Recall that the single layer operator $\gamma_{\mathrm{D}}^{\Sigma}\,\mathsf{SL}_{\kappa_0}^{\Sigma}$ is a Fredholm operator with index 0, and it is an ismorphism (i.e. admits a trivial kernel) if and only if $\kappa_0 \notin \mathfrak{S}(\Delta, \Omega_\Sigma)$, see [32, Thm 3.9.1]. From this we conclude that, if $\kappa_0 \notin \mathfrak{S}(\Delta, \Omega_\Sigma)$, then $p_\Sigma = 0$, and $\mathrm{Ker}(\widehat{\mathsf{A}}) = \{0\}$. In case $\kappa_0 \in \mathfrak{S}(\Delta, \Omega_\Sigma)$, then $\mathsf{SL}_{\kappa_0}^{\Sigma}(p_\Sigma)(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbb{R}^d\backslash\Omega_\Sigma$, so that $\gamma^l\,\mathsf{SL}_{\kappa_0}^{\Sigma}(p_\Sigma) = 0\ \forall l = 1\ldots n$, hence $(0,\ldots,0,p_\Sigma) \in \mathrm{Ker}(\widehat{\mathsf{A}})$. □

Comparing Proposition 4.8, Proposition 4.7 and Proposition 6.4, we see that if Formulation (4.4) suffers spurious resonnances, then so does Formulation (6.2). On the other hand, we point out that for *any* geometric arrangement with $\Omega_\Sigma \neq \emptyset$, there are certain κ_0 where Formulation (6.2) breaks down, while Formulation (4.4) remains well posed.

6.4 Direct multi-trace CFIE

Since we expect spurious resonances for (6.2), recall (E1), we also study multi-trace counterparts of CFIE formulations. The focus will be first on the direct single-trace CFIE proposed in Section 5.2 and its variational formulation on $\widehat{\mathbb{H}}(\Gamma)$. By the structure of (5.10), we need only elaborate how to adapt the compact bilinear form c from (5.9).

Again we take inspiration from geometrical configurations involving a gap between the different scatterers (Figure 5, left). In gap configurations there exists a natural isomorphism $\widehat{\mathbb{H}}(\Gamma) \cong \mathbb{X}_0(\Gamma)$, we look for $\hat{\mathsf{c}} : \widehat{\mathbb{H}}(\Gamma) \times \widehat{\mathbb{H}}(\Gamma) \to \mathbb{C}$ such that $\hat{\mathsf{c}}(\hat{\mathfrak{u}}, \hat{\mathfrak{v}}) = \mathsf{c}(\mathfrak{u}, \mathfrak{v})$, where we have the correspondences $\hat{\mathfrak{u}} \leftrightarrow \mathfrak{u}$ and $\hat{\mathfrak{v}} \leftrightarrow \mathfrak{v}$ in the isomorphism mapping $\widehat{\mathbb{H}}(\Gamma)$ onto $\mathbb{X}_0(\Gamma)$. Observe that c defined by (5.9) can be re-written as

$$\mathsf{c}(\mathfrak{u}, \mathfrak{v}) = \sum_{j=0}^{n} \left[\gamma_c^j\,\mathsf{G}_{\kappa_j}^j(\mathfrak{u}_j), \mathsf{C}(\mathfrak{v}_j)\right]_j\ , \quad \mathfrak{u}, \mathfrak{v} \in \mathbb{X}_0(\Gamma)\ . \tag{6.18}$$

In the gap situation (i.e. the situation of disjoint subdomains), the extension operator E_Σ can be chosen to map into functions, whose support is inside Ω_0, which means that $\gamma_{\mathrm{D}}^j \circ \mathsf{E}_\Sigma = 0$ for $j \neq 0$, and that, essentially, C maps into $\mathrm{H}^{1/2}(\Sigma)$. This brings about a substantial simplification of the operator C and leads to

$$\mathsf{c}(\mathfrak{u},\mathfrak{v}) = \left[\gamma_c^0\,\mathsf{G}^0_{\kappa_0}(\mathfrak{u}_0),(\mathsf{C}\,\mathfrak{v})_0\right]_0 = \left\langle \gamma_{\mathrm{N}}^{\Sigma}\,\mathsf{G}^0_{\kappa_0}(\mathfrak{u}_0),\mathsf{M}\,\mathrm{T}_{\mathrm{N}}(\mathfrak{v})\right\rangle_{\Sigma}\ ,\quad \mathfrak{u},\mathfrak{v}\in\mathbb{X}_0(\Gamma)\ . \tag{6.19}$$

For any $(v,q)\in\mathrm{H}^{1/2}(\partial\Omega_j)\times\mathrm{H}^{-1/2}(\partial\Omega_j)$, denote $\theta_j(v,q) := (v,-q)$. Since $(\mathfrak{u}_j)_{j=0}^n\in\mathbb{X}_0(\Gamma)$ and $\partial\Omega_0=\Sigma\cup\partial\Omega_1\cup\cdots\cup\partial\Omega_n$, the trace $\mathfrak{u}_0$ is equal to $\theta_j(\mathfrak{u}_j)$ on each $\partial\Omega_j, j=1\ldots n$, and equal to $(0,-p_\Sigma)$ on Σ. This yields $\mathsf{G}^0_{\kappa_0}(\mathfrak{u}_0) = -\,\mathsf{SL}^{\Sigma}_{\kappa_0}(p_\Sigma)-\sum_{j=0}^n\mathsf{G}^j_{\kappa_0}(\mathfrak{u}_j)$. Hence

$$\hat{\mathsf{c}}(\hat{\mathfrak{u}},\hat{\mathfrak{v}}) = -\left\langle \mathsf{M}^*\left(\gamma_{\mathrm{N}}^{\Sigma}\,\mathsf{SL}^{\Sigma}_{\kappa_0}(p_\Sigma)\right),q_\Sigma\right\rangle_{\Sigma} - \sum_{j=1}^{n}\left\langle \mathsf{M}^*\left(\gamma_{\mathrm{N}}^{\Sigma}\,\mathsf{G}^j_{\kappa_0}(\mathfrak{u}_j)\right),q_\Sigma\right\rangle_{\Sigma}\ , \tag{6.20}$$

for $\hat{\mathfrak{u}}=(\mathfrak{u}_1,\ldots,\mathfrak{u}_n,p_\Sigma)\in\widehat{\mathbb{H}}(\Gamma)$, $\hat{\mathfrak{v}}=(\mathfrak{v}_1,\ldots,\mathfrak{v}_n,q_\Sigma)\in\widehat{\mathbb{H}}(\Gamma)$. From (5.10), 6.3, and (6.20) we deduce the operator $\widehat{\mathsf{A}}_{\mathsf{M}}:\widehat{\mathbb{H}}(\Gamma)\to\check{\mathbb{H}}(\Gamma)$ defined as

$$\widehat{\mathsf{A}}_{\mathsf{M}} := \begin{bmatrix} \mathsf{A}^1_{\kappa_1}+\mathsf{A}^1_{\kappa_0} & \cdots & \gamma^1\,\mathsf{G}^n_{\kappa_0} & \gamma^1\,\mathsf{SL}^{\Sigma}_{\kappa_0} \\ \gamma^2\,\mathsf{G}^1_{\kappa_0} & \cdots & \gamma^2\,\mathsf{G}^n_{\kappa_0} & \gamma^2\,\mathsf{SL}^{\Sigma}_{\kappa_0} \\ \vdots & \ddots & \vdots & \vdots \\ \gamma^n\,\mathsf{G}^1_{\kappa_0} & \cdots & \mathsf{A}^n_{\kappa_n}+\mathsf{A}^n_{\kappa_0} & \gamma^n\,\mathsf{SL}^{\Sigma}_{\kappa_0} \\ \left(\gamma_{\mathrm{D}}^{\Sigma}-\mathsf{M}^*\,\gamma_{\mathrm{N}}^{\Sigma}\right)\mathsf{G}^1_{\kappa_0} & \cdots & \left(\gamma_{\mathrm{D}}^{\Sigma}-\mathsf{M}^*\,\gamma_{\mathrm{N}}^{\Sigma}\right)\mathsf{G}^n_{\kappa_0} & \left(\gamma_{\mathrm{D}}^{\Sigma}-\mathsf{M}^*\,\gamma_{\mathrm{N}}^{\Sigma}\right)\mathsf{SL}^{\Sigma}_{\kappa_0} \end{bmatrix} \tag{6.21}$$

Similar considerations yield an expression in $\check{\mathbb{H}}(\Gamma)$ for the right hand side of the direct single trace CFIE in the gap setting; we find

$$\hat{\mathfrak{f}}_{\mathsf{M}} := \left(\gamma^1 U_{\mathrm{inc}},\ldots,\gamma^n U_{\mathrm{inc}},\gamma_{\mathrm{D}}^{\Sigma}U_{\mathrm{inc}}-\mathsf{M}^*(\gamma_{\mathrm{N}}^{\Sigma}U_{\mathrm{inc}})\right)\in\check{\mathbb{H}}(\Gamma)\ . \tag{6.22}$$

Then the direct multi-trace CFIE in variational form and in the gap setting reads:

$$\boxed{\begin{cases} \text{find } \hat{\mathfrak{u}}\in\widehat{\mathbb{H}}(\Gamma) \quad \text{such that} \\ \left[\!\left[\widehat{\mathsf{A}}_{\mathsf{M}}(\hat{\mathfrak{u}}),\hat{\mathfrak{v}}\right]\!\right] = \left[\!\left[\hat{\mathfrak{f}}_{\mathsf{M}},\hat{\mathfrak{v}}\right]\!\right] \quad \forall\hat{\mathfrak{v}}\in\widehat{\mathbb{H}}(\Gamma)\ , \end{cases}} \tag{6.23}$$

Although we have derived Formulation (6.23) in a gap setting where all scatterers were distant from each other, this formulation still makes sense in a general geometric configuration (such as in Figure 5, left). We justifies in the next paragraph the validity of (6.23) for a general setting. In addition, we give rigorous arguments for conjecture (E2) on Page 27, where we claimed that the *direct global multi-trace CFIE* (6.23) is immune to spurious resonances for any choice of wave numbers κ_j.

Obviously, thanks to the compactness of M, see (5.1), the operator $\widehat{\mathsf{A}}_{\mathsf{M}}$ from (6.21) is a compact perturbation of $\widehat{\mathsf{A}}$ from 6.3, and the bilinear form of (6.2) is a compact perturbation of that of (6.23). The next result exhibits the precise relationship between the solution to (6.23) and the solution to (6.2).

Proposition 6.5. *A solution of the global multi-trace CFIE (*6.23*) is also a solution of the standard global MTF (*6.2*).*

Proof: Take a solution $\widehat{\mathfrak{u}} = (\widehat{\mathfrak{u}}_1, \dots, \widehat{\mathfrak{u}}_n, p_\Sigma) \in \widehat{\mathbb{H}}(\Gamma)$ of (6.23). Consider the function $W(\mathbf{x}) := U_{\mathrm{inc}}(\mathbf{x}) - \mathsf{SL}^{\Sigma}_{\kappa_0}(p_\Sigma)(\mathbf{x}) - \sum_{j=1}^{n} \mathsf{G}^{j}_{\kappa_0}(\widehat{\mathfrak{u}}_j)(\mathbf{x})$. Take test traces $\widehat{\mathfrak{v}} \in \widehat{\mathbb{H}}(\Gamma)$ of the form $\widehat{\mathfrak{v}} = (0, \dots, 0, q_\Sigma)$, where $q_\Sigma \in \mathrm{H}^{-1/2}(\Sigma)$ is arbitrary. Formulation (6.23) yields

$$\int_{\Sigma} q \left(\gamma^{\Sigma}_{\mathrm{D}}(W) - \mathsf{M}^{*}\, \gamma^{\Sigma}_{\mathrm{N}}(W) \right) d\sigma = 0 \quad \forall q \in \mathrm{H}^{-\frac{1}{2}}(\Gamma) \;,$$

which implies $\gamma^{\Sigma}_{\mathrm{D}}(W) = \mathsf{M}^{*}\, \gamma^{\Sigma}_{\mathrm{N}}(W)$. Since we have $\Delta W + \kappa_0^2 W = 0$ in Ω_Σ, applying Green's formula provides

$$0 = \mathrm{Im}\{ \int_{\Omega_\Sigma} |\nabla W|^2 - \kappa_0^2 |W|^2 d\mathbf{x} \} = 2\ \mathrm{Im}\{ \int_{\Sigma} \gamma^{\Sigma}_{\mathrm{N}}(\overline{W})\, \mathsf{M}\, \gamma^{\Sigma}_{\mathrm{N}}(W) d\sigma \} \;,$$

hence $\gamma^{\Sigma}_{\mathrm{N}}(\psi) = 0$. We conclude that $\gamma^{\Sigma}_{\mathrm{D}}(\psi) = \mathsf{M}^{*}\, \gamma^{\Sigma}_{\mathrm{N}}(\psi) = 0$. This corresponds to the equation of (6.2) associated with the last line of (6.3). Since the only difference between (6.23) and (6.2) is this equation, we are done with the proof. □

A corollary of the previous result is that, if U solves (6.23), then the unique solution to Problem (2.2) is given by (6.6). This justifies considering (6.23) for general geometric configurations. Now, since $\widehat{\mathfrak{c}}$ is compact, Proposition 6.3 implies that the bilinear form of (6.23) also satisfies a generalized Garding inequality.

Corollary 6.6. *The assertion of Proposition* 6.3 *holds with* $\widehat{\mathsf{A}}$ *replaced with* $\widehat{\mathsf{A}}_{\mathsf{M}}$*.*

A consequence of the above proposition is that the operator $\widehat{\mathsf{A}}_{\mathsf{M}}$ is of Fredholm type with index 0. One advantage of Formulation (6.23) over Formulation (6.2) is the absence of spurious resonnances, which is proved by the following result.

Proposition 6.7. *For any choice of wave numbers $\kappa_j > 0$, the global multi-trace CFIE (*6.23*) possesses a unique solution.*

Proof: Pick an element $\widehat{\mathfrak{u}} \in \mathrm{Ker}(\widehat{\mathsf{A}}_{\mathsf{M}})$. This means that $\widehat{\mathfrak{u}}$ is a solution of (6.23) where $\widehat{\mathfrak{f}}_{\mathsf{M}} = 0$. As a consequence of Proposition 6.5, we have $\widehat{\mathfrak{u}} \in \mathrm{Ker}(\widehat{\mathsf{A}})$, so that, by Proposition 6.4, $\widehat{\mathfrak{u}} = (0, \dots, 0, p_\Sigma)$ for some $p_\Sigma \in \mathrm{H}^{-1/2}(\Sigma)$. Coming back to (6.23), and choosing $\widehat{\mathfrak{v}} \in \widehat{\mathbb{H}}(\Gamma)$ of the form $\widehat{\mathfrak{v}} = (0, \dots, 0, q_\Sigma)$ with some $q_\Sigma \in \mathrm{H}^{-1/2}(\Sigma)$, we obtain

$$\int_{\Sigma} q_\Sigma \left(\gamma^{\Sigma}_{\mathrm{D}}\, \mathsf{SL}^{\Sigma}_{\kappa_0}(p_\Sigma) - \mathsf{M}^{*} \left(\gamma^{\Sigma}_{\mathrm{N}}\, \mathsf{SL}^{\Sigma}_{\kappa_0}(p_\Sigma) \right) \right) d\sigma = 0 \;.$$

It was established in [3, Lemma 4.1] that the operator $\gamma_{\mathrm{D}}^{\Sigma}\,\mathsf{SL}_{\kappa_0}^{\Sigma} - \mathsf{M}^*\,\gamma_{\mathrm{N}}^{\Sigma}\,\mathsf{SL}_{\kappa_0}^{\Sigma}$ is injective for all $\kappa_0 > 0$. So we conclude that $p_\Sigma = 0$ which finishes the proof. □

Corollary 6.8. *For any choice of the wave numbers $\kappa_0, \ldots \kappa_n$ satisfying (2.3), Formulation (6.23) is well posed i.e. $\widehat{\mathsf{A}}_{\mathsf{M}} : \widehat{\mathbb{H}}(\Gamma) \to \check{\mathbb{H}}(\Gamma)$ is an isomorphism.*

Proof: Since $\widehat{\mathsf{A}}_{\mathsf{M}}$ is a Fredholm operator with index 0, this holds true if and only if it is injective, which is the statement of Proposition 6.7. □

6.5 Indirect multi-trace CFIE

Of course, there is a multi-trace version also of the indirect CFIE presented in Section 5.3. Since developments are largely parallel to that for the direct CFIE, we do not give details. As is clear from (5.17), which serves as the starting point, the operator of the indirect multi-trace CFIE will be a perturbed version of $\widehat{\mathsf{A}}$. More precisely, the potential operator $\mathsf{SL}_{\kappa_0}^{\Sigma}$ is replaced with $\mathsf{SL}_{\kappa_0}^{\Sigma} + \mathsf{DL}_{\kappa_0}^{\Sigma} \cdot \mathsf{M}$. As in Section 6.2 the perturbation is encoded in a bilinear form $\widehat{\mathsf{c}}^* : \widehat{\mathbb{H}}(\Gamma) \times \widehat{\mathbb{H}}(\Gamma) \to \mathbb{C}$, defined by

$$\widehat{\mathsf{c}}^*(\widehat{\mathfrak{u}}, \widehat{\mathfrak{v}}) := \sum_{j=1}^{n} \left[\gamma^j\,\mathsf{DL}_{\kappa_0}^{\Sigma}(\mathsf{M}\,p_\Sigma), \mathfrak{v}_j\right]_j + \left\langle \gamma_{\mathrm{D}}^{\Sigma}\,\mathsf{DL}_{\kappa_0}^{\Sigma}(\mathsf{M}\,p_\Sigma), q_\Sigma \right\rangle_\Sigma , \tag{6.24}$$

for $\widehat{\mathfrak{u}} = (\mathfrak{u}_1, \ldots, \mathfrak{u}_n, p_\Sigma) \in \widehat{\mathbb{H}}(\Gamma)$ and $\widehat{\mathfrak{v}} = (\mathfrak{v}_1, \ldots, \mathfrak{v}_n, q_\Sigma) \in \widehat{\mathbb{H}}(\Gamma)$. This bilinear form inherits compactness from M is. It can be used to state the *indirect global multi-trace CFIE* in variational form

$$\left\{ \begin{array}{l} \text{Find } \widehat{\mathfrak{u}} \in \widehat{\mathbb{H}}(\Gamma) \quad \text{such that} \\[2ex] \left[\!\left[\widehat{\mathsf{A}}(\widehat{\mathfrak{u}}), \widehat{\mathfrak{v}} \right]\!\right] + \widehat{\mathsf{c}}^*(\widehat{\mathfrak{u}}, \widehat{\mathfrak{v}}) = \left[\!\left[\widehat{\mathfrak{f}}, \widehat{\mathfrak{v}} \right]\!\right] \quad \forall \widehat{\mathfrak{v}} \in \widehat{\mathbb{H}}(\Gamma) . \end{array} \right. \tag{6.25}$$

Compared to Formulation (6.2), this variational problem features an additional compact term. The next proposition gives a precise description of the relation between the solutions of (6.25) and the solutions to (2.2).

Proposition 6.9. *If $\widehat{\mathfrak{u}} = (\widehat{\mathfrak{u}}_1, \ldots, \widehat{\mathfrak{u}}_n, p_\Sigma) \in \widehat{\mathbb{H}}(\Gamma)$ is a solution of (6.25), then $U \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d \backslash \Omega_\Sigma)$ defined by (for $j = 1, \ldots, n$)*

$$\begin{aligned} U(\mathbf{x}) &= \mathsf{G}_{\kappa_j}^j(\widehat{\mathfrak{u}}_j)(\mathbf{x}) , \quad \mathbf{x} \in \Omega_j \\ U(\mathbf{x}) &= U_{\mathrm{inc}}(\mathbf{x}) - \mathsf{SL}_{\kappa_0}^{\Sigma}(p_\Sigma)(\mathbf{x}) - \mathsf{DL}_{\kappa_0}^{\Sigma}(\mathsf{M}^*\,p_\Sigma)(\mathbf{x}) \\ &\qquad - \sum_{j=1}^{n} \mathsf{G}_{\kappa_0}^j(\widehat{\mathfrak{u}}_j)(\mathbf{x}), \quad \mathbf{x} \in \Omega_0 , \end{aligned} \tag{6.26}$$

is the unique solution of the transmission boundary value problem (2.2).

We do not give the proof of this result as it is identical to the proof of Proposition 6.2. The only difference is that $\mathsf{SL}_{\kappa_0}^{\Sigma}(p_\Sigma)$ has to be replaced by $\mathsf{SL}_{\kappa_0}^{\Sigma}(p_\Sigma) + \mathsf{DL}_{\kappa_0}^{\Sigma}(\mathsf{M}\,p_\Sigma)$. Now let us underline the close relationship between

(6.25) and (6.23), that are dual to each other in the sense of the following lemma.

Lemma 6.10. *The bilinear forms of the direct global multi-trace CFIE* (6.23) *and its indirect counterpart* (6.25) *are adjoint to each other:*

$$\left[\!\left[\widehat{\mathsf{A}}(\widehat{\mathfrak{u}}),\widehat{\mathfrak{v}}\right]\!\right] + \widehat{\mathsf{c}}(\widehat{\mathfrak{u}},\widehat{\mathfrak{v}}) = \left[\!\left[\widehat{\mathsf{A}}(\widehat{\mathfrak{v}}),\widehat{\mathfrak{u}}\right]\!\right] + \widehat{\mathsf{c}}^*(\widehat{\mathfrak{v}},\widehat{\mathfrak{u}}) \quad \forall \widehat{\mathfrak{u}},\widehat{\mathfrak{v}} \in \widehat{\mathbb{H}}(\Gamma)\;.$$

Proof: We already know that $[\![\widehat{\mathsf{A}}(\widehat{\mathfrak{u}}),\widehat{\mathfrak{v}}]\!] = [\![\widehat{\mathsf{A}}(\widehat{\mathfrak{v}}),\widehat{\mathfrak{u}}]\!]$, according to (6.5), so we have to show that $\widehat{\mathsf{c}}(\widehat{\mathfrak{u}},\widehat{\mathfrak{v}}) = \widehat{\mathsf{c}}^*(\widehat{\mathfrak{v}},\widehat{\mathfrak{u}})$. Take two elements $\widehat{\mathfrak{u}} = (\mathfrak{u}_1,\dots,\mathfrak{u}_n,p_\Sigma)$ and $\widehat{\mathfrak{v}} = (\mathfrak{v}_1,\dots,\mathfrak{v}_n,q_\Sigma)$ in $\widehat{\mathbb{H}}(\Gamma)$. We have

$$\widehat{\mathsf{c}}(\widehat{\mathfrak{u}},\widehat{\mathfrak{v}}) = -\left\langle \mathsf{M}^*\,\gamma_{\mathrm{N}}^{\Sigma}\,\mathsf{SL}_{\kappa_0}^{\Sigma}(p_\Sigma),q_\Sigma\right\rangle_\Sigma - \sum_{j=1}^{n}\left\langle \mathsf{M}^*\,\gamma_{\mathrm{N}}^{\Sigma}\,\mathsf{G}_{\kappa_0}^{j}(\mathfrak{u}_j),q_\Sigma\right\rangle_\Sigma \tag{6.27}$$

We examine successively each term in the sum above. $\mathfrak{v}_\Sigma = (\mathrm{M}\,q,0) \in \mathbb{H}(\Sigma)$ and $\mathfrak{u}_\Sigma = (0,p) \in \mathbb{H}(\Sigma)$. Applying symmetry property given by Lemma 3.6 in Ω_Σ yields

$$-\langle \gamma_{\mathrm{N}}^{\Sigma}\,\mathsf{SL}_{\kappa_0}^{\Sigma}(p),\mathsf{M}\,q\rangle_\Sigma = \langle \gamma_{\mathrm{D},c}^{\Sigma}\,\mathsf{DL}_{\kappa_0}^{\Sigma}(\mathrm{M}q),p\rangle_\Sigma\;. \tag{6.28}$$

Similarly we have $-\langle \mathsf{M}^*\,\gamma_{\mathrm{N}}^{\Sigma}\,\mathsf{G}_{\kappa_0}^{j}(\mathfrak{u}_j),q\rangle_\Sigma = [\gamma^\Sigma\,\mathsf{G}_{\kappa_0}^{j}(\mathfrak{u}_j),\mathfrak{v}_\Sigma]_\Sigma$. We can apply Lemma 3.7 (taking Ω_Σ as one of the subdomains) to obtain $[\gamma^\Sigma\,\mathsf{G}_{\kappa_0}^{j}(\mathfrak{u}_j),\mathfrak{v}_\Sigma]_\Sigma = [\gamma^j\,\mathsf{G}_{\kappa_0}^{\Sigma}(\mathfrak{v}_\Sigma),\mathfrak{u}_j]_j$ which can be written in the present case

$$-\langle \mathsf{M}^*\,\gamma_{\mathrm{N}}^{\Sigma}\,\mathsf{G}_{\kappa_0}^{j}(\mathfrak{u}_j),q\rangle_\Sigma = [\gamma^j\,\mathsf{DL}_{\kappa_0}^{\Sigma}(\mathsf{M}\,q),\mathfrak{u}_j]_j \tag{6.29}$$

according to the explicit expression of $\mathfrak{v}_\Sigma$. Plugging (6.28) and (6.29) into the explicit expression of $\widehat{\mathsf{c}}$ given by (6.27), and comparing with the definition of $\widehat{\mathsf{c}}^*$, this concludes the proof. □

Let $\widehat{\mathsf{A}}'_{\mathsf{M}} : \widehat{\mathbb{H}}(\Gamma) \to \check{\mathbb{H}}(\Gamma)$ refer to the continuous operator associated to the bilinear form in the left-hand side of (6.25). The previous lemma, combined with the inf-sup conditions satisfied by $\widehat{\mathsf{A}}$, shows that $\widehat{\mathsf{A}}'_{\mathsf{M}}$ is bijective if and only if $\widehat{\mathsf{A}}_{\mathsf{M}}$ is bijective, which is systematically true according to Proposition 6.6. In addition, since $\widehat{\mathsf{A}}$ and $\widehat{\mathsf{A}}'_{\mathsf{M}}$ only differ by a compact contribution, Proposition 6.3 implies that the bilinear form associated to Formulation (6.25) satisfies a generalized Garding inequality. We sum up all these results in the next proposition.

Proposition 6.11. *The assertion of Proposition* 6.3 *holds with* $\widehat{\mathsf{A}}$ *replaced by* $\widehat{\mathsf{A}}'_{\mathsf{M}}$. *In addition, for any choice of the wave numbers* $\kappa_0,\dots\kappa_n$ *satisfying* (2.3)*, Formulation* (6.25) *is well posed i.e. it admits a unique solution and* $\widehat{\mathsf{A}}'_{\mathsf{M}} : \widehat{\mathbb{H}}(\Gamma) \to \check{\mathbb{H}}(\Gamma)$ *is an isomorphism.*

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