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HAL Id: hal-01093998
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Submitted on 11 Dec 2014

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SIGN CHANGES IN SHORT INTERVALS OF COEFFICIENTS OF SPINOR ZETA FUNCTION OF A SIEGEL CUSP FORM OF GENUS 2

EMMANUEL ROYER, JYOTI SENGUPTA, AND JIE WU

Abstract. In this paper, we establish a Voronoi formula for the spinor zeta function of a Siegel cusp form of genus 2. We deduce from this formula quantitative results on the number of its positive (resp. negative) coefficients in some short intervals.

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1. Introduction

Let $S_k$ be the space of Siegel cusp forms of integral weight $k$ on the group $\text{Sp}_4(\mathbb{Z}) \subset \text{GL}_4(\mathbb{Q})$ and let $F \in S_k$ be an eigenfunction of all the Hecke operators. Let

$$Z_F(s) := \prod_{p \in \mathcal{P}} Z_{F,p}(p^{-s}) \quad (\text{Res} > 1)$$

be the spinor zeta function of $F$. Here $\mathcal{P}$ is the set of prime numbers and if $\alpha_{0,p},\alpha_{1,p},\alpha_{2,p}$ are the Satake $p$-parameters attached to $F$ then

$$Z_{F,p}(t)^{-1} := (1 - \alpha_{0,p} t)(1 - \alpha_{0,p} \alpha_{1,p} t)(1 - \alpha_{0,p} \alpha_{2,p} t)(1 - \alpha_{0,p} \alpha_{1,p} \alpha_{2,p} t).$$

They satisfy

$$\alpha_{0,p}^2 \alpha_{1,p} \alpha_{2,p} = 1$$

for all $p$. A Siegel form is in the Maass subspace $S^M_k$ of $S_k$ if it is a linear combination of Siegel forms $F$ that are eigenvectors of all the Hecke operators and for which there
exists a primitive modular form, \( f \), of weight \( 2k - 2 \) such that

\[
Z_F(s) = \zeta\left(s - \frac{1}{2}\right)\zeta\left(s + \frac{1}{2}\right) L(f, s).
\]

Here \( L(f, s) \) is the \( L \)-function of \( f \). This happens only if \( k \) is even. The bijective linear application between \( S^M_k \) and the space of modular forms of weight \( 2k - 2 \) is called the Saito-Kurokawa lifting [16]. The Ramanujan-Petersson conjecture says that

\[ |\alpha_{j,p}| = 1 \text{ for } j = 0, 1, 2 \text{ and all primes } p. \]

It is not true for Siegel Hecke-eigenforms in \( S^M_k \). But, if \( k \) is odd or, if \( k \) is even and in the orthogonal complement of \( S^M_k \), then it has been established by Weissauer [15]. We denote by \( H^*_k \) the set of Siegel cuspidal Hecke-eigenforms of weight \( k \) and genus 2 that, if \( k \) is even, are in the orthogonal complement of \( S^M_k \). The forms we consider in this paper all belong to \( H^*_k \). According to Breulmann [2], a Siegel Hecke-eigenform is in \( S^M_k \) if and only if all its Hecke eigenvalues are positive.

According to [1, 6], the function

\[
\Lambda_F(s) := (2\pi)^{-s} \Gamma\left(s + k - \frac{3}{2}\right) \Gamma\left(s + \frac{1}{2}\right) Z_F(s)
\]

has an entire continuation to \( \mathbb{C} \) since \( F \in H^*_k \). Further it satisfies the functional equation

\[ \Lambda_F(s) = (-1)^k \Lambda_F(1-s) \]

on \( \mathbb{C} \). The spinor zeta function of \( F \) has the Dirichlet expansion:

\[
Z_F(s) = \sum_{n \geq 1} a_F(n) n^{-s}
\]

for \( \text{Res} > 1 \). By using (1), one sees that

\[ |a_F(n)| \leq d_4(n) \]

for all \( n \geq 1 \), where \( d_4(n) \) is the number of solutions in positive integers \( a, b, c, d \) of \( n = abcd \).

In this paper, we investigate the problem of sign changes for the sequence \((a_F(n))_{n \geq 1}\) in short intervals. Define

\[
\mathcal{N}^*_F(x) := \sum_{n \leq x \atop a_F(n) \leq 0} 1.
\]

We apply a method due to Lau & Tsang [11] to establish the following Theorem. Convergence issues however appear and we have to deal with them.

**Theorem**— Let \( F \) be in \( H^*_k \) and \( \epsilon > 0 \). There are constants \( c > 0 \) absolute and \( x_0(F) \) depending only on \( F \) such that for all \( x \geq x_0(F) \), we have

\[
\mathcal{N}^*_F(x + cx^{3/4}) - \mathcal{N}^*_F(x) \gg x^{3/8 - \epsilon},
\]

where the implied constant in \( \gg \) depends only on \( \epsilon \).
Remark—An ingredient of our proof is the inequality
\[
\sum_{n \leq x} a_F(n) \ll_{F, \epsilon} x^{3/5+\epsilon} \quad (x \geq 2).
\]
(see Lemma 1). We also prove, and use an Omega-result:
\[
\sum_{n \leq x} a_F(n) = \Omega_{\pm}(x^{3/8})
\]
(see Lemma 2).

Two related problems have already been studied. Denote by \(\lambda_F(n)\) the \(n\)-th normalised Hecke eigenvalue of \(F\). Then we have
\[
\sum_{n=1}^{\infty} \frac{\lambda_F(n)}{n^s} = \frac{Z_F(s)}{\zeta(2s+1)} \quad (\text{Res} > 1).
\]

In [8], Kohnen proved that
\[
\#\{n \leq x: \lambda_F(n) \geq 0\} \to \infty \quad (x \to \infty).
\]

Then, Das [5] proved that, as \(x\) tends to \(+\infty\), the quantities
\[
\frac{1}{\#\{p \in \mathcal{P}: p \leq x\}} \#\{p \in \mathcal{P} \cap [1,x]: \lambda_F(p) \geq 0\}
\]
are bounded from below (and naturally also bounded from above). In [9], Kohnen & Sengupta proved that under the same assumption there is an integer \(n \ll k^2(\log k)^{20}\) such that \(\lambda_F(n) < 0\). Their result has been generalised to higher levels by Brown [3].

Remark—Das’s result is on the counting function of the Hecke eigenvalues. It implies however the result on the coefficients of the spinor zeta function since (5) implies
\[
a_F(n) = \sum_{(d,m)\in \mathbb{N}^2 \atop d^2m=n} \frac{\lambda_F(m)}{d}.
\]

Moreover, the proof of Kohnen & Sengupta can be adapted to prove that there is an integer \(n \ll k^2(\log k)^{20}\) such that \(a_F(n) < 0\).

2. Truncated Voronoi Formula

The aim of this section is to establish the following truncated Voronoi formula, which will be needed in the proof of the Theorem.

**Lemma 1**—Let \(F\) be in \(H_k^*\). Then for any \(A > 0\) and \(\epsilon > 0\), we have
\[
\sum_{n \leq x} a_F(n) = \frac{x^{3/8}}{(2\pi)^{3/4}} \sum_{n \leq M} \frac{a_F(n)}{n^{3/8}} \cos \left(4\sqrt{2\pi(nx)^{1/4}} + \frac{\pi}{4}\right)
+ O_{A,\epsilon}\left((x^3M^{-1})^{1/4+\epsilon} + (xM)^{1/4+\epsilon}\right)
\]
uniformly for \( x \geq 2 \) and \( 1 \leq M \leq x^A \), where the implied constant depends on \( A, F \) and \( \varepsilon \) only. In particular

\[
\sum_{n \leq x} a_F(n) \ll_F, x^{3/5+\varepsilon} \quad (x \geq 2).
\]

**Proof.** Without loss of generality, we assume that \( M \in \mathbb{N} \). Let \( \kappa := 1 + \varepsilon \) and

\[
T^4 = 4\pi^2 (M + \frac{1}{2})\kappa.
\]

By the Perron formula (see [14, Corollary II.2.4]) we have

\[
\sum_{n \leq x} a_F(n) = \frac{1}{2\pi i} \int_{\kappa-iT}^{\kappa+iT} Z_F(s) \frac{x^s}{s} \, ds + O_{F,\varepsilon} \left( x^{3/4 + \varepsilon} M^{-1/4} + x^\varepsilon \right).
\]

We shift the line of integration horizontally to \( \text{Res} = -\varepsilon \), the main term gives

\[
\frac{1}{2\pi i} \int_{\kappa-iT}^{\kappa+iT} Z_F(s) \frac{x^s}{s} \, ds = Z_F(0) + \frac{1}{2\pi i} \int_{\mathcal{L}} Z_F(s) \frac{x^s}{s} \, ds,
\]

where \( \mathcal{L} \) is the contour joining the points \( \kappa \pm iT \) and \(-\varepsilon \pm iT\). Using the convexity bound [12, §1.3]

\[
Z_F(\sigma + it) \ll_{F,\varepsilon} (|t| + 1)^{\max\{2(1-\sigma),0\}+\varepsilon} \quad (-\varepsilon \leq \sigma \leq \kappa),
\]

the integrals over the horizontal segments and the term \( Z_F(0) \) can be absorbed in

\[
O_{F,\varepsilon} \left( (Tx)^{\varepsilon} (T + T^{-1}x) \right) = O_{F,\varepsilon} \left( x^{1/4+\varepsilon} M^{1/4} + x^{3/4+\varepsilon} M^{-1/4} \right).
\]

To handle the integral over the vertical segment \( \mathcal{L}_v := [-\varepsilon - iT, -\varepsilon + iT] \), we invoke the functional equation (2). We deduce that

\[
\frac{1}{2\pi i} \int_{\mathcal{L}_v} Z_F(s) \frac{x^s}{s} \, ds = (-1)^k \sum_{n \geq 1} \frac{a_F(n)}{n} I_{\mathcal{L}_v}(nx),
\]

where

\[
I_{\mathcal{L}_v}(y) := \frac{1}{2\pi i} \int_{\mathcal{L}_v} (2\pi i)^{2s-1} \Gamma(k - \frac{1}{2} - s) \Gamma(s - \frac{1}{2}) y^s \Gamma(\frac{3}{2} - s) \Gamma(s + k - \frac{3}{2}) \, ds.
\]

By using the Stirling formula

\[
\Gamma(\sigma + it) = \sqrt{2\pi} |t|^{\sigma-1/2} e^{-\pi |t|/2} \left( 1 + \mathcal{O} \left( t^{-1} \right) \right)
\]

uniformly for \( \sigma_1 \leq \sigma \leq \sigma_2 \) and \( |t| \geq 1 \), the quotient of the four gamma factors is

\[
|t|^{2-4\varepsilon} e^{-\frac{4}{3}(1+\varepsilon)\log|t| + i\varepsilon \log(1-k)} \left( 1 + \mathcal{O} \left( t^{-1} \right) \right)
\]

for bounded \( \sigma \) and any \( |t| \geq 1 \), where the implied constant depends on \( \sigma \) and \( k \). Together with the second mean value theorem for integrals [14, Theorem I.0.3], we obtain

\[
I_{\mathcal{L}_v}(nx) \ll (nx)^{-1} \left( \int_T^{T + 4\varepsilon} |t^{1+4\varepsilon} e^{-ig(t)} \, dt | + T^{1+4\varepsilon} \right)
\]

\[
\ll T \left( \frac{T^4}{nx} \right)^{1/\varepsilon} \left( \int_T e^{-i\varepsilon g(t)} \, dt \right) + 1
\]

\[
\ll T \left( \frac{T^4}{nx} \right)^{1/\varepsilon} \left( \int_T e^{-i\varepsilon g(t)} \, dt \right) + 1
\]
for some $1 \leq a \leq T$, where $g(t) := t \log \left( \frac{t^4}{(4\pi^2 nx)} \right) - 4t$. In view of (8), we have
\[ g'(t) = -\log(4\pi^2 nx/t^4) < 0 \quad \text{and} \quad |g'(t)| \geq |\log(n/(M + \frac{1}{2}))|
\]
for $n \geq M + 1$ and $1 \leq t \leq T$. Using (3) and [14, Theorem I.6.2], we infer that
\[
\sum_{n>M} \frac{a_T(n)}{n} I_{\mathcal{L}_v}(nx) \ll T \left( \frac{T^4}{x} \right) \sum_{n>M} \frac{d_4(n)}{n^{1+\varepsilon}} \left( \left| \log \frac{n}{M + \frac{1}{2}} \right|^{-1} + 1 \right) 
\]
\[
\ll T \left( \frac{T^4}{x} \right) \left\{ \sum_{M<n<2M} \frac{d_4(n)(M + \frac{1}{2})}{n^{1+\varepsilon}|n-M-\frac{1}{2}|} + \frac{1}{M^{\varepsilon/2}} \right\} 
\]
\[
\ll T \left( \frac{T^4}{\sqrt{M}x} \right) 
\]
\[
\ll T x^\varepsilon. 
\]

For $n \leq M$, we extend the segment of integration $\mathcal{L}_v$ to an infinite line $\mathcal{L}_v^*$ in order to apply Lemma 1 in [4]. Write
\[
\mathcal{L}_v^* := [\frac{1}{2} + \varepsilon \pm iT, \frac{1}{2} + \varepsilon \pm i\infty), \quad \mathcal{L}_h^* := [-\varepsilon \pm iT, \frac{1}{2} + \varepsilon \pm iT]
\]
and define $\mathcal{L}_h^*$ to be the positively oriented contour consisting of $\mathcal{L}_v$, $\mathcal{L}_v^*$ and $\mathcal{L}_h^*$. In view of (11), the contribution over the horizontal segments $\mathcal{L}_h^*$ is
\[
I_{\mathcal{L}_h^*}(nx) \ll \int_{-\varepsilon}^{1/2-\varepsilon} (2\pi)^{2\sigma-1} T^{2-4\sigma} \frac{(nx)^{\sigma}}{T^{\sigma}} \, d\sigma 
\]
\[
\ll T \int_{-\varepsilon}^{1/2-\varepsilon} \left( \frac{nx}{T^{\sigma}} \right)^\sigma \, d\sigma 
\]
\[
\ll T x^\varepsilon. 
\]

As in (12), for $n \leq M$ we get that
\[
I_{\mathcal{L}_v^*}(nx) \ll (nx)^{1/2+\varepsilon} \left( \int_T^\infty t^{-1-4\varepsilon} e^{-i g(t)} \, dt + \frac{1}{T^{1+4\varepsilon}} \right) 
\]
\[
\ll T \left( \frac{nx}{T^{\sigma}} \right)^{1/2+\varepsilon} \left( \left| \log \frac{M + \frac{1}{2}}{n} \right|^{-1} + 1 \right) 
\]
\[
\ll T \left( \left| \log \frac{M + \frac{1}{2}}{n} \right|^{-1} + 1 \right). 
\]

So
\[
\sum_{n<M} \frac{a_T(n)}{n} (I_{\mathcal{L}_v}(nx) + I_{\mathcal{L}_h^*}(nx)) \ll T x^{\varepsilon/2} \sum_{n<M} \frac{d_4(n)}{n} \left( \left| \log \frac{M + \frac{1}{2}}{n} \right|^{-1} + 1 \right) 
\]
\[
\ll T x^{\varepsilon/2} \sum_{n<M} \frac{d_4(n)(M + \frac{1}{2})}{n|n-M-\frac{1}{2}|} + T x^\varepsilon 
\]
\[
\ll T x^\varepsilon. 
\]
Define
\[ I_{s'}(y) = \frac{1}{4\pi^2i} \int_{s'} \frac{\Gamma(k - \frac{1}{2} - s)\Gamma\left(\frac{3}{2} - s\right)\Gamma(s)}{\Gamma(s + k - \frac{3}{2})\Gamma(s + \frac{1}{2})\Gamma(1 + s)} (4\pi^2 y)^k ds. \]

After a change of variable \( s \) into \( 1 - s \), we see that
\[ I_{s'}(y) = I_0(4\pi^2 y), \]
with
\[ I_0(t) := \frac{1}{2\pi i} \int_{s'} \frac{\Gamma(s + k - \frac{3}{2})\Gamma(s + \frac{1}{2})\Gamma(1 - s)}{\Gamma(k - \frac{1}{2} - s)\Gamma\left(\frac{3}{2} - s\right)\Gamma(2 - s)} t^{1-s} ds. \]

Here \( s' \) consists of the line \( s = \frac{1}{2} - \varepsilon + iT \) with \( |t| \geq T \), together with three sides of the rectangle whose vertices are \( \frac{1}{2} - \varepsilon - iT, 1 + \varepsilon - iT, 1 + \varepsilon + iT \) and \( \frac{1}{2} - \varepsilon + iT \). Note that all the poles of the integrand in \( I_0(t) \) lie on the left of the line \( s' \).

Using a result due to Chandrasekharan and Narasimhan [4, Lemma 1] generalised by Lau & Tsang [11, Lemma 2.2] we obtain (note that a factor \( \sqrt{2} \) is missing for the definition of \( c_0 \) in both references)
\[ I_0(t) = (-1)^k \left( \frac{1}{2\pi} \right)^{k/3} t^{3/8} \cos \left( 4t^{1/4} + \frac{\pi}{4} \right) + O(t^{1/8}). \]

It hence follows that
\[ I_{s'}(nx) = (-1)^k \left( \frac{nx}{21} \right)^{3/8} \cos \left( 4\sqrt{2\pi} (nx)^{1/4} + \frac{\pi}{4} \right) + O((nx)^{1/8}). \]

We conclude
\[ \sum_{n \in \mathbb{M}} a_F(n) I_{s'}(nx) = (-1)^k \left( \frac{nx}{21} \right)^{3/8} \sum_{n \in \mathbb{M}} \frac{a_F(n)}{n^{3/8}} \cos \left( 4\sqrt{2\pi} (nx)^{1/4} + \frac{\pi}{4} \right) + O(x^{1/4+c} \mathbb{M}^{1/4}) \]
from (14) and (15). Finally the asymptotic formula (6) by (9)-(10), (13) and (16).

Since
\[ x^{3/8} \sum_{n \in \mathbb{M}} \frac{a_f(n)}{n^{3/8}} \cos \left( 4\sqrt{2\pi} (nx)^{1/4} + \frac{\pi}{4} \right) \ll (x\mathbb{M})^{3/8+c}, \]
the choice of \( \mathbb{M} = x^{3/5} \) in (6) gives (7).

### 3. Proof of the Theorem

We establish a lemma that has a similar statement as a one due to Lau & Wu [10, Lemma 3.2]. However, due to convergence issue, the proof is more delicate.

**Lemma 2**—Let \( F \) be in \( H_v^* \). Define
\[ S_F(x) := \sum_{n \leq x} a_F(n). \]

There exist positive absolute constants \( C, c_1, c_2 \) and \( X_0(F) \) depending only on \( F \) such that for all \( X \geq X_0(F) \), we can find \( x_1, x_2 \in [X, X + CX^{3/4}] \) for which
\[ S_F(x_1) > c_1 X^{3/8} \quad \text{and} \quad S_F(x_2) < -c_2 X^{3/8}. \]
Proof. We begin the proof with Theorem C of Hafner [7]. In order to use this result, it is more convenient to introduce the notion of \((C, \ell)\)-summability and to present related simple facts (see [13] for more details). Let \(\{g_n(t)\}_{n \geq 0}\) be a sequence of functions. We write

\[
  s(g; n) := \sum_{0 \leq v \leq n} g_v(t), \quad \sigma(g; n) := \frac{1}{C_n^{(\ell+1)}} \sum_{v=0}^{n} C_n^{(\ell)} g_{n-v}(g; v),
\]

where \(C_n^{(\ell)} := \binom{\ell+n-1}{n}\). We say that the series of general term \(g_n(t)\) is uniformly \((C, \ell)\)-summable to the sum \(G(t)\) if \(\sigma(g; n)\) converges uniformly to \(G(t)\) as \(n \to \infty\). We have \(C_0^{(\ell)} + \cdots + C_n^{(\ell)} = C_n^{(\ell+1)}\) and if the series \(\sum_n g_n(t) dt\) converges then the series of general term \(\int g_n(t) dt\) is also \((C, \ell)\)-summable and their limits are the same.

As in [7, page 151], for \(\rho > -1\) and \(x \notin 2\pi \mathbb{N}\), define

\[
  A_\rho(x) := \frac{1}{\Gamma(\rho+1)} \sum_{2\pi n \leq x} a_F(n) (x - 2\pi n)^\rho.
\]

Now let \(\mathcal{C}\) be the rectangle with vertices \(c \pm iR\) and \(1 - b \pm iR\) (taken in the counterclockwise direction), where \(b > c > \max \{|1, |k - \frac{3}{2}|\}\) and \(R > \left|k - \frac{3}{2}\right|\) are real numbers. Let

\[
  Q_\rho(x) := \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\Gamma(s)(2\pi)^{-s}Z_F(s)}{\Gamma(s + \rho + 1)} x^{\rho + s} ds.
\]

Denote by \(\mathcal{C}_{0,b}\) the oriented polygonal path with vertices \(-i\infty, -iR, b - iR, b + iR, iR\) and \(+i\infty\). Let

\[
  f_\rho(x) := \frac{1}{2\pi i} \int_{\mathcal{C}_{0,b}} \frac{\Gamma(1-s)\Delta(s)}{\Gamma(2+s-\rho-s)\Delta(1-s)} x^{1+\rho-s} ds
\]

where

\[
  \Delta(s) = \Gamma(s+k-\frac{3}{2})\Gamma(s+\frac{1}{2}).
\]

By [7, Theorem C], the series of general term \((-1)^k(2\pi n)^{-1-\rho}a_F(n) f_\rho(2\pi nx)\) is uniformly \((C, \ell)\)-summable for \(\ell > \max\{\frac{1}{2} - \rho, 0\}\) on any finite closed interval in \((0, \infty)\) only under the condition \(\rho > -1\) and the sum is \(A_\rho(x) - Q_\rho(x)\). In particular, we can fix \(\ell = 1\) and \(\rho = 0\). We shall say \(C\)-summable for \((C, 1)\)-summable.

The only pole of the integrand of \(Q_\rho(x)\) is 0, it is encircled by \(\mathcal{C}\) hence

\[
  Q_\rho(x) \ll_F 1 \quad (x \geq 1).
\]

To estimate \(f_\rho(x)\), we use again the result by Lau & Tsang [11, Lemma 2.2] already used to establish Voronoi formula. We get

\[
  f_0(y) = \frac{(-1)^k}{\sqrt{2\pi}} y^{3/8} \cos\left(4y^{1/4} + \frac{\pi}{4}\right) + (-1)^k e_1 y^{1/8} \cos\left(4y^{1/4} + \frac{3\pi}{4}\right) + O\left(\frac{1}{y^{1/8}}\right),
\]

where \(e_1\) is a absolute constant.
Let
\[
\Phi(v) := (2\pi)^{3/4} A_0(2\pi v^4)^{1/2},
\]
\[
g_n(v) := \frac{\alpha_F(n)}{n^{3/4}} \cos\left(4\sqrt{2\pi n^{1/4}} v + \frac{\pi}{4}\right),
\]
\[
g_n^*(v) := \frac{\epsilon_1 \alpha_F(n)}{v n^{7/8}} \sin\left(4\sqrt{2\pi n^{1/4}} v + \frac{\pi}{4}\right).
\]
Then the series of general term \(g_n(v) - g_n^*(v)\) is uniformly \(C\)-summable on any finite closed interval in \((0, \infty)\) and the sum is \(\Phi(v) + O(v^{-3/2})\) (here the term \(O(v^{-3/2})\) comes from \(Q_0(2\pi v^4)\) and the \(O\)-term of \((17)\)). In view of \((4)\), a simple partial integration shows that the series of general term \(g_n^*(v)\) converges to the sum \(\sum_n g_n^*(v)\) uniformly on any finite closed interval in \((0, \infty)\). Thus the series of general term \(g_n(v)\) is uniformly \(C\)-summable on any finite closed interval in \((0, \infty)\) and the sum is \(\Phi(v) + \sum_n g_n^*(v) + O(v^{-3/2})\).

Let \(t\) be any large natural number, \(\kappa > 1\) a large parameter that will be fixed later. Write
\[
K_{\tau}(u) = (1 - |u|)(1 + \tau \cos(4\sqrt{2\pi \kappa u}))
\]
with \(\tau = \pm 1\). We consider the integral
\[
J_{\tau} = \int_{-1}^{1} \Phi(t + \kappa u) K_{\tau}(u) du.
\]
We have
\[
\int_{-1}^{1} g_n(t + \kappa u) K_{\tau}(u) du = r_{\beta} \frac{\alpha_F(n)}{n^{3/8}},
\]
\[
\int_{-1}^{1} g_n^*(t + \kappa u) K_{\tau}(u) du = s_{\beta} \frac{\epsilon_1 \alpha_F(n)}{n^{7/8}},
\]
where
\[
r_{\beta} := \int_{-1}^{1} K_{\tau}(u) \cos\left(4\sqrt{2\pi \beta(t + \kappa u)} + \frac{\pi}{4}\right) du,
\]
\[
s_{\beta} := \int_{-1}^{1} K_{\tau}(u) \sin\left(4\sqrt{2\pi \beta(t + \kappa u)} + \frac{\pi}{4}\right) du.
\]
As in \([10, (3.13)]\), we have
\[
r_{\beta} = \delta_{\beta = 1} \frac{\tau}{2} + O\left(\frac{1}{\kappa^2 \beta^2} + \delta_{\beta \neq 1} \frac{1}{\kappa^2 (\beta - 1)^2}\right)
\]
and
\[
s_{\beta} \ll (t \beta \kappa)^{-1}.
\]
It follows that
\[
\int_{-1}^{1} g_1(t + \kappa u) K_\tau(u) \, du = \frac{\tau}{2} + O\left(\frac{1}{\kappa^2}\right),
\]
\[
\int_{-1}^{1} g_n(t + \kappa u) K_\tau(u) \, du \ll \frac{d_4(n)}{\kappa^2 n^{9/8}} \quad (n \geq 2),
\]
\[
\int_{-1}^{1} g_0^*(t + \kappa u) K_\tau(u) \, du \ll \frac{d_4(n)}{\kappa t n^{9/8}},
\]
where all the implied constants are absolute. These estimates show that
\[
\sum_{n \geq 1} \int_{-1}^{1} g_n(t + \kappa u) K_\tau(u) \, du = \frac{\tau}{2} + O\left(\frac{1}{\kappa^2}\right),
\]
\[
\sum_{n \geq 1} \int_{-1}^{1} g_0^*(t + \kappa u) K_\tau(u) \, du \ll \frac{1}{\kappa t}.
\]

In view of the remark about C-summability, we obtain
\[
J_\tau = \frac{\tau}{2} + O\left(\frac{1}{\kappa t} + \frac{1}{t^{3/2}}\right).
\]

We fix \( \kappa \) large enough. When \( X \geq \kappa^4 \), we take \( t = \left\lfloor \frac{X}{\kappa^4} \right\rfloor \). So \( t > 2\kappa \) and the O-term in \( J_\tau \) is \( \ll \kappa^{-2} \), so the main term dominates if \( \kappa \) has been chosen sufficiently large. Therefore
\[
J_{-1} < -\frac{1}{4} \quad \text{and} \quad J_1 > \frac{1}{4}.
\]

Since \( S_F(x) = A_0(2\pi x) \), we rewrite this as
\[
\int_{-1}^{1} \frac{S_F(t + \kappa u)}{(t + \kappa u)^{3/2}} K_{-1}(u) \, du < -\frac{1}{4(2\pi)^{3/4}} \quad \text{and} \quad \int_{-1}^{1} \frac{S_F(t + \kappa u)}{(t + \kappa u)^{3/2}} K_1(u) \, du > \frac{1}{4(2\pi)^{3/4}}.
\]

The kernel function \( K_\tau(u) \) is nonnegative and satisfies
\[
1 - (3\pi \kappa)^{-2} \leq \int_{-1}^{1} K_\tau(u) \, du \leq 2 \quad (\tau = \pm 1).
\]

As a consequence, we have
\[
\frac{S_F((t + \kappa \eta_+)^4)}{(t + \kappa \eta_+)^{3/2}} \geq \frac{1}{2(2\pi)^{3/4}}
\]
and
\[
\frac{S_F((t + \kappa \eta_-)^4)}{(t + \kappa \eta_-)^{3/2}} \leq -\frac{1}{4(1 - (3\pi \kappa)^{-2})(2\pi)^{3/4}}
\]
for some \( \eta_\pm \in [-1, 1] \). These two points deviate from \( X \) by a distance \( \ll X^{3/4} \), since the difference between \( (t \pm \kappa)^4 \) is \( \ll \kappa^3 \times X^{3/4} \).

This implies the result of Lemma 2. \( \square \)
Now we are ready to prove the Theorem.

By Lemma 2, for any $x \geq X_0(F)$ we can pick three points $x < x_1 < x_2 < x_3 < x + 3C x^{3/4}$ such that $S_F(x_i) < -c x^{3/8}$ ($i = 1, 3$) and $S_F(x_2) > c x^{3/8}$ for some absolute constant $c > 0$. (Note that $y + C y^{3/4} \leq x + 3C x^{3/4}$ for $y = x + C x^{3/4}$.) Hence we deduce that

$$\sum_{x_1 < n < x_2 \atop a_F(n) > 0} a_F(n) \geq S_F(x_2) - S_F(x_1) > 2c x^{3/8}$$

and

$$\sum_{x_2 < n < x_3 \atop a_F(n) < 0} (-a_F(n)) \geq - (S_F(x_3) - S_F(x_2)) > 2c x^{3/8}.$$

Thus, the Theorem follows as each term in the two sums are positive and $\ll \varepsilon n^{\varepsilon}$.

References


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