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PARTIAL CLONES CONTAINING ALL BOOLEAN MONOTONE SELF-DUAL PARTIAL FUNCTIONS

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ABSTRACT. The study of partial clones on $\mathbf{2} := \{0, 1\}$ was initiated by R. V. Freivald. In his fundamental paper published in 1966, Freivald showed, among other things, that the set of all monotone partial functions and the set of all self-dual partial functions are both maximal partial clones on $\mathbf{2}$.

Several papers dealing with intersections of maximal partial clones on $\mathbf{2}$ have appeared after Freivald work. It is known that there are infinitely many partial clones that contain the set of all monotone self-dual partial functions on $\mathbf{2}$, and the problem of describing them all was posed by some authors.

In this paper we show that the set of partial clones that contain all monotone self-dual partial functions is of continuum cardinality on $\mathbf{2}$.

1. PRELIMINARIES

Let A be a finite non-singleton set. Without loss of generality we assume that $A = \mathbf{k} := \{0, \ldots, k-1\}$. For a positive integer n, an n-ary partial function on $\mathbf{k}$ is a map $f : \operatorname{dom}(f) \to \mathbf{k}$ where $\operatorname{dom}(f)$ is a subset of $\mathbf{k}^n$ called the *domain* of f. Let $\operatorname{Par}^{(n)}(\mathbf{k})$ denote the set of all n-ary partial functions on $\mathbf{k}$ and let

$$\operatorname{Par}(\mathbf{k}) := \bigcup_{n \geq 1} \operatorname{Par}^{(n)}(\mathbf{k}).$$

For $n, m \geq 1$, $f \in \mathrm{Par}^{(n)}(\mathbf{k})$ and $g_1, \ldots, g_n \in \mathrm{Par}^{(m)}(\mathbf{k})$, the *composition* of f and $g_1, \ldots, g_n$, denoted by $f[g_1, \ldots, g_n] \in \mathrm{Par}^{(m)}(\mathbf{k})$, is defined by

$$\mathrm{dom}\,(f[g_1, \ldots, g_n]) \quad := \quad \{\vec{a} \in \mathbf{k}^m \mid \vec{a} \in \bigcap_{i=1}^{m} \mathrm{dom}\,(g_i)$$
$$\text{and } (g_1(\vec{a}), \ldots, g_m(\vec{a})) \in \mathrm{dom}\,(f)\}$$

and

$$f[g_1, \ldots, g_n](\vec{a}) := f(g_1(\vec{a}), \ldots, g_n(\vec{a})),$$

for all $\vec{a} \in \mathrm{dom}\,(f[g_1, \ldots, g_n])$.

For every positive integer n and each $1 \leq i \leq n$, let e_i^n denote the *n-ary i-th projection function* defined by

$$e_i^n(a_1, \ldots, a_n) = a_i$$

for all $(a_1, \ldots, a_n) \in \mathbf{k}^n$. Furthermore, let

$$J_{\mathbf{k}} := \{e_i^n : 1 \leq i \leq n\}$$

be the set of all (total) projections.

Definition 1. A *partial clone* on $\mathbf{k}$ is a composition closed subset of $\mathrm{Par}(\mathbf{k})$ containing $J_{\mathbf{k}}$.

Remark 1. *There are two other equivalent definitions for partial clones. One definition uses Mal'tsev's formalism and the other uses the concept of one point extension. These definitions can be found in chapter 20 of [7].*

The partial clones on $\mathbf{k}$, ordered by inclusion, form a lattice $\mathcal{L}_{P_{\mathbf{k}}}$ in which the infimum is the set-theoretical intersection. That means that the intersection of an arbitrary family of partial clones on $\mathbf{k}$ is also a partial clone on $\mathbf{k}$. A *maximal partial clone* on $\mathbf{k}$ is a coatom of the lattice $\mathcal{L}_{P_{\mathbf{k}}}$. Therefore a partial clone M is maximal if there is no partial clone C over $\mathbf{k}$ such that $M \subset C \subset \mathrm{Par}(\mathbf{k})$.

Example 1. *The set of partial functions*

$$\Omega_k := \bigcup_{n \geq 1} \{f \in \mathrm{Par}^{(n)}(\mathbf{k}) \mid \mathrm{dom}\,(f) \neq \emptyset \Longrightarrow \mathrm{dom}\,(f) = \mathbf{k}^n\}$$

is a maximal partial clone on $\mathbf{k}$.

Definition 2. *For $h \geq 1$, let ρ be an h-ary relation on $\mathbf{k}$ and f be an n-ary partial function on $\mathbf{k}$. We say that f preserves ρ if for every $h \times n$ matrix $M = [M_{ij}]$ whose columns $M_{*j} \in \rho$, $(j = 1, \ldots n)$ and whose rows $M_{i*} \in \mathrm{dom}\,(f)$ $(i = 1, \ldots, h)$, the h-tuple $(f(M_{1*}), \ldots, f(M_{h*})) \in \rho$. Define*

$$\mathrm{pPol}\,\rho := \{f \in \mathrm{Par}(\mathbf{k}) \mid f \text{ preserves } \rho\}.$$

It is well known that $\mathrm{pPol}\,\rho$ is a partial clone called the partial clone determined by the relation ρ.

Notice that if there is no $h \times n$ matrix $M = [M_{ij}]$ whose columns $M_{*j} \in \rho$ and whose rows $M_{i*} \in \mathrm{dom}\,(f)$, then $f \in \mathrm{pPol}\,\rho$.

Example 2. *Let $\mathbf{2} := \{0, 1\}$ and let $\{(0, 0), (0, 1), (1, 1)\}$ be the natural order on $\mathbf{2}$. Consider the binary relation $\{(0, 1), (1, 0)\}$ on $\mathbf{2}$. Then*

$$\mathrm{pPol}\,\{(0, 0), (0, 1), (1, 1)\}$$

is the set of all monotone partial functions and

$$\mathrm{pPol}\,\{(0, 1), (1, 0)\}$$

is the set of all self-dual partial functions on $\mathbf{2}$.

For simplicity we will write $\mathrm{pPol}\,(\leq)$ and $\mathrm{pPol}\,(\neq)$ for

$$\mathrm{pPol}\,(\{(0, 0), (0, 1), (1, 1)\}) \quad and \quad \mathrm{pPol}\,(\{(0, 1), (1, 0)\}),$$

respectively. It is not difficult to see that

$$\mathrm{pPol}\,(\leq) \quad := \quad \{f \in \mathrm{Par}(\mathbf{2}) \mid [\mathbf{a}, \mathbf{b} \in \mathrm{dom}\,(f), \mathbf{a} \leq \mathbf{b}] \Longrightarrow f(\mathbf{a}) \leq f(\mathbf{b})\}, \quad and$$

$$\mathrm{pPol}\,(\neq) \quad := \quad \{f \in \mathrm{Par}(\mathbf{2}) \mid [\mathbf{a}, \mathbf{a} + 1 \in \mathrm{dom}\,(f)] \Longrightarrow f(\mathbf{a} + 1) = f(\mathbf{a}) + 1\}$$

where the above sums are taken mod 2.

As mentioned earlier, Freivald showed that there are exactly eight maximal partial clones on $\mathbf{2}$. The following two relations are needed to state Freivald's result. Set

$$R_1 \quad := \quad \{(x, x, y, y) \mid x, y \in \mathbf{2}\} \cup \{(x, y, y, x) \mid x, y \in \mathbf{2}\} \quad \text{and}$$

$$R_2 \quad := \quad R_1 \cup \{(x, y, x, y) \mid x, y \in \mathbf{2}\}.$$

Theorem 2 ([2]). *There are exactly 8 maximal partial clones on $\mathbf{2}$, namely, $\mathrm{pPol}\,\{0\}$, $\mathrm{pPol}\,\{1\}$, $\mathrm{pPol}\,\{(0, 1)\})$, $\mathrm{pPol}\,(\leq)$, $\mathrm{pPol}\,(\neq)$, $\mathrm{pPol}\,(R_1)$, $\mathrm{pPol}\,(R_2)$, and Ω_2.*

Notice that the total functions in $\mathrm{pPol}\,R_2$ (i.e., the functions with full domain) form the maximal clone of all (total) linear functions over $\mathbf{2}$ (see, e.g., chapter 3 of [7]).

An interesting and somehow difficult problem in clone theory is to study intersections of maximal partial clones. It is shown in [1] that the set of all partial clones on $\mathbf{2}$ that contain the maximal clone consisting of all *total* linear functions on $\mathbf{2}$ is of continuum cardinality (for details see [1, 4] and Theorem 20.7.13 of [7]). A consequence of this is that the interval of partial clones $[\mathrm{pPol}\,(R_2) \cap \Omega_2, \mathrm{Par}(\mathbf{2})]$ is of continuum cardinality on $\mathbf{2}$.

A similar result, (but slightly easier to prove) is established in [3] where it is shown that the interval of partial clones $[\mathrm{pPol}\,(R_1) \cap \Omega_2, \mathrm{Par}(\mathbf{2})]$ is also of continuum cardinality. Notice that the three maximal partial clones $\mathrm{pPol}\,R_1$, $\mathrm{pPol}\,R_2$ and Ω_2 contain all unary functions (i.e., maps) on $\mathbf{2}$. Such partial clones are called *Słupecki type* partial clones in [4, 10]. These are the only three maximal partial clones of Słupecki type on $\mathbf{2}$.

For a complete study of the pairwise intersections of all maximal partial clones of Słupecki type on a finite non-singleton set $\mathbf{k}$, see [4].

The papers [5, 6, 8, 12, 13] focus on the case $k = 2$ where various interesting, and sometimes hard to obtain, results are established.

For instance, the intervals

$$[\text{pPol}\,\{0\} \cap \text{pPol}\,\{1\} \cap \text{pPol}\,\{(0,1)\} \cap \text{pPol}\,(\leq),\ \text{Par}(\mathbf{2})] \quad \text{and}$$

$$[\text{pPol}\,\{0\} \cap \text{pPol}\,\{1\} \cap \text{pPol}\,\{(0,1)\} \cap \text{pPol}\,(\neq),\ \text{Par}(\mathbf{2})]$$

are shown to be finite and are completely described in [5]. Some of the results in [5] are included in [12, 13] where partial clones on $\mathbf{2}$ are handled via the one point extension approach (see Section 20.2 in [7]).

In view of results from [1, 3, 5, 12, 13], it was thought that if $2 \leq i \leq 5$ and $M_1, \ldots, M_i$ are non-Słupecki maximal partial clones on $\mathbf{2}$, then the interval

$$[M_1 \cap \cdots \cap M_i, \text{Par}(\mathbf{2})]$$

is either finite or countably infinite.

Now it was shown in [6] that the interval of partial clones $[\text{pPol}\,(\leq) \cap \text{pPol}\,(\neq), \text{Par}(\mathbf{2})]$ is infinite. This result is mentioned in Theorem 20.8 of [7] (with an independent proof given in [8]) and in chapter 8 of the PhD thesis [11]. However, it remained an open problem to determine whether $[\text{pPol}\,(\leq) \cap \text{pPol}\,(\neq), \text{Par}(\mathbf{2})]$ is countably or uncountably infinite.

In this paper we settle this question by proving that the interval of partial clones

$$[\text{pPol}\,(\leq) \cap \text{pPol}\,(\neq), \text{Par}(\mathbf{2})]$$

is of continuum cardinality on $\mathbf{2}$.

2. The construction

For $n \geq 5$ and $n > k > 1$ we denote by $\sigma_k^n \subseteq \mathbf{2}^{2n}$ the $(2n)$-ary relation defined by

$$\sigma_k^n := \{(x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbf{2}^{2n} \mid \forall\, i = 1, \ldots, n,\ x_i \neq y_i, \quad \text{and}$$
$$\forall\, i = 1, \ldots, n,\ y_{i+1} \leq x_i \text{ and } y_{i+2} \leq x_i \ldots \text{ and } y_{i+k} \leq x_i\},$$

where the subscripts $i + j$ in the above definition are taken modulo n. It is not difficult to see that

$$\sigma_k^n := \{(x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbf{2}^{2n} \mid \forall\, i = 1, \ldots, n,\ x_i \neq y_i, \quad \text{and}$$
$$\forall\, i = 1, \ldots, n,\ x_i = 0 \Longrightarrow [x_{i+1} = x_{i+2} = \cdots = x_{i+k} = 1]\}.$$

By the *Definability Lemma* established by B. Romov in [9] (see also Lemma 20.3.4 in [7] and [4, 5, 6] for details), we have that

$$\text{pPol}\,(\leq) \cap \text{pPol}\,(\neq) \subseteq \text{pPol}\,(\sigma_k^n)$$

for all $n \geq 5$ and all $k \geq 1$.

For $n \geq 5$ and $n > k \geq 1$, we denote by $\rho_k^n \subseteq \mathbf{2}^{4n}$ the $(4n)$-ary relation defined by

$$\rho_k^n := \{(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n}, y_1, \ldots, y_n, y_{n+1}, \ldots, y_{2n}) \in \mathbf{2}^{4n} \mid$$
$$(x_1, \ldots, x_n, y_1, \ldots, y_n) \in \sigma_1^n, \quad \text{and} \quad (x_{n+1}, \ldots, x_{2n}, y_{n+1}, \ldots, y_{2n}) \in \sigma_k^n\}.$$

Again by the Definability Lemma, we have that

$$\mathrm{pPol}\,(\sigma_1^n) \cap \mathrm{pPol}\,(\sigma_k^n) \subseteq \mathrm{pPol}\,(\rho_k^n),$$

and thus $\mathrm{pPol}\,(\le) \cap \mathrm{pPol}\,(\neq) \subseteq \mathrm{pPol}\,(\rho_k^n)$ for all $n \ge 5$ and all $k \ge 1$.

Our goal is to construct an infinite set of odd integers X and an infinite family of partial functions $\{g_t,\ t \in X\}$ so that for every $t, t' \in X$, we have $g_t \in \mathrm{pPol}\,\rho_{t'}^{n(t')}$ if and only if $t \neq t'$.

Remark 3. *Since every tuple in σ_k^n (resp. ρ_k^n) is completely determined by its first n entries (resp. $2n$ entries), we will omit the second half of such tuples. We therefore denote by S_k^n and R_k^n the relations obtained from σ_k^n and ρ_k^n, respectively, by deleting the second half of every tuple in σ_k^n and ρ_k^n, i.e.,*

$$S_k^n := \{(x_1, \ldots, x_n) \in \mathbf{2}^n \mid (x_1, \ldots, x_n, 1+x_1, \ldots, 1+x_n) \in \sigma_k^n\}$$

and

$$R_k^n := \{(x_1, \ldots, x_{2n}) \in \mathbf{2}^{2n} \mid (x_1, \ldots, x_{2n}, 1+x_1, \ldots, 1+x_{2n}) \in \rho_k^n\}$$

where the above sums are taken mod 2.

Note that S_k^n is the n-ary relation on $\mathbf{2}$ whose members are tuples in which any two 0's are separated by at least k symbols 1 (in particular, if the first position is 0, then the last k positions must be 1). Furthermore, R_k^n is the cartesian product $S_1^n \times S_k^n$.

As mentioned earlier we will use the relations S_1^n, S_k^n and R_k^n with the understanding that we are omitting the second parts of the relations σ_1^n, σ_k^n and ρ_k^n in order to simplify the notation.

Notations. In the sequel $k \ge 4$ stands for an **even** integer. Set $n(k) := k(k+1) + 1$. We will write $\rho^{n(k)}$ for $\rho_k^{n(k)}$ and $R^{n(k)}$ for $R_k^{n(k)}$. Let $M_\uparrow^k$ be the $n(k) \times n(k)$ matrix with columns in $S_1^{n(k)}$, the first being $c_1 = [0110101\ldots0101]^T$ and the remaining columns are obtained by applying cyclic shifts to c_1, i.e.,

$$c_2 = [10110101\cdots010]^T,$$
$$c_3 = [010110101\cdots01]^T,$$
$$\cdots$$
$$c_{n(k)} = [110101\cdots010]^T.$$

Remark 4. *Let r_i and r_j be two rows of $M_\uparrow^k$. If $|i - j| \ge 2 (\mathrm{mod}\, n(k))$, then r_i and r_j have a 0 in the same position.*

Lemma 5. *If $k' < k$, then there is no $n(k') \times n(k)$ matrix N whose columns are in $S_1^{n(k')}$ and whose rows are rows of $M_\uparrow^k$.*

Proof. Suppose that $k' < k$ and that N is an $n(k') \times n(k)$ matrix whose columns are in $S_1^{n(k')}$. Suppose, by way of contradiction, that the rows of N

are rows of $M_\uparrow^k$. By Remark 4, the only possible adjacent rows of a row r in N are exactly the predecessor and successor rows of r in $M_\uparrow^k$. But then $n(k')$ would be even, thus yielding the desired contradiction. ∎

Let $M_\downarrow^k$ be the $n(k) \times n(k)$ matrix with columns in $S_k^{n(k)}$, and such that the first is $c_1' = [0\underbrace{1\cdots1}_{k+1}0\underbrace{1\cdots1}_{k}\cdots0\underbrace{1\cdots1}_{k}]^T$ and the remaining columns are obtained by applying cyclic shifts to c_1' as before.

Remark 6. *Since $k \geq 4$ is even, if r_i is a row of $M_\uparrow^k$, and r_j' is a row of $M_\downarrow^k$, then r_i and r_j' have a 0 in the same position.*

Lemma 7. *If $k' > k$, then there is no $n(k') \times n(k)$ matrix N whose columns are in $S_k^{n(k')}$ and whose rows are rows of $M_\downarrow^k$.*

Proof. Suppose that $k' > k$ and that N is an $n(k') \times n(k)$ matrix whose columns are in $S_k^{n(k')}$. Assume, by way of contradiction, that the rows of N are rows of $M_\downarrow^k$. Since each row of $M_\downarrow^k$ has exactly k 0's, we have that N has $k \times n(k')$ 0's. Hence the matrix N has a column with at least $\frac{k \times n(k')}{n(k)}$ symbols 0. It is easy to verify that since $k' > k \geq 4$, we have that $\frac{k \times n(k')}{n(k)} > k'$. But this yields the desired contradiction, since all columns of N are members of $S_k^{n(k')}$, and each has at most k' 0's. ∎

Define M_k as the $2n(k) \times n(k)$ matrix given by

$$M_k = \begin{pmatrix} M_\uparrow^k \\ M_\downarrow^k \end{pmatrix}.$$

Notice that each column of M_k is a tuple of $R^{n(k)}$.

Lemma 8. *Let N be a $2n(k') \times n(k)$ matrix whose columns are in $R^{n(k')}$ and whose rows are rows of M^k. Then, either all rows of N are rows of $M_\downarrow^k$, or the first $n(k')$ are rows of $M_\uparrow^k$ and the remaining $n(k')$ are rows of $M_\downarrow^k$.*

Proof. By Remark 4 and the fact that $R^{n(k')} := S_1^{k'} \times S_k^{k'}$, there cannot be more than 2 rows of $M_\uparrow^k$ among the last $n(k')$. In fact, by Remark 6 there can only be rows from $M_\downarrow^k$ among the last $n(k')$ rows of N. Furthermore, from Remark 6 and the fact that $R^{n(k')} := S_1^{k'} \times S_k^{k'}$, it follows that either all of the first $n(k')$ rows of N are rows of $M_\uparrow^k$ or all of the first $n(k')$ rows of N are rows of $M_\downarrow^k$. ∎

Let f_k be the $n(k)$-ary partial function whose domain is the set of rows of M_k, and such that f_k is constant 0 on the rows of $M_\uparrow^k$ and constant 1 on the rows of $M_\downarrow^k$.

Theorem 9. *Let $k, k' \geq 4$ be even integers. Then $f_k \in \mathrm{pPol}\, R^{n(k')}$ if and only if $k \neq k'$.*

Proof. Since $[0 \cdots 01 \cdots 1]^T$ does not belong to $R^{n(k)}$, we see that $f_k \notin$ pPol $R^{n(k)}$.

So suppose that $k \neq k'$. If $k < k'$, then it follows from Definition 2 and Lemmas 7 and 8 that $f_k \in$ pPol $R^{n(k')}$.

Suppose now that $k > k'$. If N is an $2n(k') \times n(k)$ matrix whose columns are in $R^{n(k')}$ and whose rows are rows of M^k (otherwise we are done for the domain of f_k is exactly the set of rows of M^k), then by Lemmas 5 and 8 it follows that all rows of N are rows of $M_\downarrow^k$. Since f_k is constant 1 on the rows of $M_\downarrow^k$, and since the constant 1 $2n(k')$ tuple belongs to $R^{n(k')}$, we conclude that $f_k \in$ pPol $R^{n(k')}$. ∎

Let $\overline{M_k}$ be the $2n(k) \times n(k)$ matrix obtained by replacing every row of the matrix M_k by its dual tuple (obtained by interchanging 1's and 0's) and define L_k as the $4n(k) \times n(k)$ matrix given by

$$L_k = \left(\begin{array}{c} M_k \\ \hline \overline{M_k} \end{array} \right).$$

Moreover, let g_k be the $n(k)$-ary partial function whose domain is the set of rows of L_k, and such that $g_k(\vec{u}) = f_k(\vec{u})$ if $\vec{u}$ is a row of M_k and $g_k(\vec{u}) = 1 + f_k(\vec{u})$ (mod 2) if $\vec{u}$ is a row of $\overline{M_k}$. Then, Theorem 9 can be restated as follows:

Main Theorem. *Let $k, k' \geq 4$ be even integers. Then $g_k \in$ pPol $\rho^{n(k')}$ if and only if $k \neq k'$.*

Let $\mathbf{E}_{\geq 4} := \{4, 6, 8, \dots\}$ be the set of all even integers greater or equal to 4 and denote by $\mathcal{P}(\mathbf{E}_{\geq 4})$ the power set of $\mathbf{E}_{\geq 4}$. Since

$$\text{pPol}\,(\leq) \cap \text{pPol}\,(\neq) \subseteq \text{pPol}\,(\rho_k^n)$$

for every $n \geq 5$ and every $n > k \geq 1$, we have

$$\text{pPol}\,(\leq) \cap \text{pPol}\,(\neq) \subseteq \bigcap_{t \in \mathbf{E}_{\geq 4} \setminus X} \text{pPol}\,\rho^{n(t)}$$

for every subset X of $\mathbf{E}_{\geq 4}$.

So let $X \subset \mathbf{E}_{\geq 4}$ and fix $k \in X$. Then $g_k \in$ pPol $\rho^{n(t)}$ for all $t \in \mathbf{E}_{\geq 4} \setminus X$, i.e.,

$$g_k \in \bigcap_{t \in \mathbf{E}_{\geq 4}} \setminus X.$$

On the other hand, if $k \in \mathbf{E}_{\geq 4} \setminus X$, then we have

$$g_k \notin \bigcap_{t \in \mathbf{E}_{\geq 4} \setminus X} \text{pPol}\,\rho^{n(t)} \quad \text{and} \quad g_k \notin \bigcap_{t \notin X} \text{pPol}\,\rho^{n(t)},$$

since $g_k \notin$ pPol $\rho^{n(k)}$. Therefore the map

$$\chi := \mathcal{P}(\mathbf{E}_{\geq 4}) \to [\text{pPol}\,(\leq) \cap \text{pPol}\,(\neq), \text{Par}(\mathbf{2})]$$

defined by

$$\chi(X) := \bigcap_{t \in \mathbf{E}_{\geq 4} \setminus X} \mathrm{pPol}\, \rho^{n(t)}$$

is one-to-one and we have shown the following result which answers our question on cardinality of the interval $[\mathrm{pPol}\,(\leq) \cap \mathrm{pPol}\,(\neq), \mathrm{Par}(\mathbf{2})]$.

Corollary 10. *The interval of partial clones* $[\mathrm{pPol}\,(\leq) \cap \mathrm{pPol}\,(\neq), \mathrm{Par}(\mathbf{2})]$ *is of continuum cardinality on* $\mathbf{2}$.

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