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LOCAL SOLVABILITY OF THE k -HESSIAN EQUATIONS

G. TIAN, Q. WANG AND C.-J. XU

ABSTRACT. In this work, we study the existence of local solutions in \mathbb{R}^n to k -Hessian equation, for which the nonhomogeneous term f is permitted to change the sign or be non negative; if f is C^∞ , so is the local solution. We also give a classification for the second order polynomial solutions to the k -Hessian equation, it is the basis to construct the local solutions and obtain the uniform ellipticity of the linearized operators at such constructed local solutions.

1. INTRODUCTION

In this paper, we focus on the existence of local solution for the following k -Hessian equation,

$$(1.1) \quad S_k[u] = f(y, u, Du),$$

on an open domain Ω of \mathbb{R}^n , where $2 \leq k \leq n$. For a smooth function $u \in C^2$, the k -Hessian operator S_k is defined by

$$(1.2) \quad S_k[u] = S_k(D^2u) = \sigma_k(\lambda(D^2u)) = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_k},$$

where $\lambda(D^2u) = (\lambda_1, \dots, \lambda_n)$ are the eigenvalues of the Hessian matrix (D^2u) , $\sigma_k(\lambda)$ is the k -th elementary symmetric polynomial, and $S_k[u]$ is the sum of all principal minors of order k for the Hessian matrix (D^2u) . We say that a smooth function u is k -convex if the eigenvalues of the Hessian matrix (D^2u) are in the so-called Gårding cone Γ_k which is defined by

$$\Gamma_k(n) = \{\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n : \sigma_j(\lambda) > 0, 1 \leq j \leq k\}.$$

For the local solvability, Hong and Zuily [7] obtained the existence of C^∞ local solutions for Monge-Ampère equation

$$(1.3) \quad \det D^2u = f(y, u, Du) \quad y \in \Omega \subset \mathbb{R}^n,$$

when $f \in C^\infty$ is nonnegative, it is the case of $k = n$ in (1.1). The geometric background of Monge-Ampère equation can be found in [6, 4, 12]. In this work, we only consider the Hessian equation for $2 \leq k < n$, since it is classical for $k=1$. We will follow the method of [7] (see also [11, 14]) to construct the local solution by a perturbation of the polynomial-typed solution of $S_k[u] = c$ for some real constant c . Since the right hand side function in (1.1) possibly vanishes, then, the solution is in the closure of Γ_k . Thus, we need to study the closure of Γ_k , its boundary is

$$\partial\Gamma_k(n) = \{\lambda \in \mathbb{R}^n : \sigma_j(\lambda) \geq 0, \sigma_k(\lambda) = 0, 1 \leq j \leq k-1\}.$$

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From the Maclaurin's inequalities

$$\left[\frac{\sigma_k(\lambda)}{\binom{n}{k}} \right]^{1/k} \leq \left[\frac{\sigma_l(\lambda)}{\binom{n}{l}} \right]^{1/l}, \quad \lambda \in \Gamma_k, \quad k \geq l \geq 1,$$

we see that $\sigma_{k+1}(\lambda) > 0$ cannot occur for $\lambda \in \partial\Gamma_k(n)$, therefore we can express $\partial\Gamma_k$ as two parts

$$\partial\Gamma_k(n) = \mathbf{P}_1 \cup \mathbf{P}_2,$$

with

$$\begin{aligned} \mathbf{P}_1 &= \{\lambda \in \Gamma_k(n) : \sigma_j(\lambda) \geq 0, \sigma_k(\lambda) = \sigma_{k+1}(\lambda) = 0, 1 \leq j \leq k-1\} \\ \mathbf{P}_2 &= \{\lambda \in \Gamma_k(n) : \sigma_j(\lambda) \geq 0, \sigma_k(\lambda) = 0, \sigma_{k+1}(\lambda) < 0, 1 \leq j \leq k-1\}. \end{aligned}$$

Besides,

$$(1.4) \quad \mathbf{P}_2 = \emptyset, \quad \text{if } k = n.$$

In Section 2, we will prove that \mathbf{P}_1 and \mathbf{P}_2 have a more precise version, and $\mathbf{P}_2 \neq \emptyset$ if $k < n$.

In this paper, we always discuss the k -Hessian equation under the framework of ellipticity, then we follow the ideas of [8] and [9] to get the existence of the solution. Here we explain the ellipticity: in the matrix language, the ellipticity set of the k -Hessian operator, $1 \leq k \leq n$, is

$$E_k = \{S \in \mathcal{M}_s(n) : S_k(S + t\xi \times \xi) > S_k(S) > 0, \xi \in \mathbb{S}^{n-1}, \forall t \in \mathbb{R}^+\},$$

where $\mathcal{M}_s(n)$ is the space of n -symmetric real matrix. Then the Gårding cones is

$$\Gamma_k = \{S \in \mathcal{M}_s(n) : S_k(S + t\mathbf{I}) > S_k(S) > 0, \forall t \in \mathbb{R}^+\}.$$

It is possible to show that $E_k = \Gamma_k$ only for $k = 1, n$ and the example in [9] assures that $\Gamma_k \subset E_k$ and $\text{mess}(E_k \setminus \Gamma_k) > 0$. Ivochkina, Prokofeva and Yakunina [9] pointed out that the ellipticity of (1.1) is independent of the sign of f if $k < n$. But for the Monge-Ampère equation (1.3), the type of equation is determined by the sign of f , it is elliptic if $f > 0$, hyperbolic if $f < 0$ and degenerate if f vanishes at some points; it is of mixed type if f changes sign [5].

There are many results for the Dirichlet problem of (1.1) under the condition $f > 0$ (see [15] and references therein), there are also some results about $C^{1,1}$ weak solution of the Dirichlet problem of (1.1) under the degenerate condition $f \geq 0$ (see [16] and references therein). But similarly to Monge-Ampère equation, the existence of the smooth solution to Dirichlet problem of (1.1) is completely an open problem if f is not strictly positive.

In this work, for the local solution of the k -Hessian equation (1.1), we prove the following results.

Theorem 1.1. *Let $f = f(y, u, p)$ be defined and continuous near a point $Z_0 = (y_0, u_0, p_0) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$, $0 < \alpha < 1, 2 \leq k < n$. Assume that f is C^α with respect to y and $C^{2,1}$ with respect to u, p . We have that*

- (1) *If $f(Z_0) = 0$, then the equation (1.1) admits a $(k-1)$ -convex local solution $u \in C^{2,\alpha}$ near y_0 which is not $(k+1)$ -convex.*
- (2) *If $f \geq 0$ near Z_0 , then the equation (1.1) admits a k -convex local solution $u \in C^{2,\alpha}$ near y_0 which is not $(k+1)$ -convex.*
- (3) *If $f(Z_0) < 0$, the equation (1.1) admits a $(k-1)$ -convex local solution $u \in C^{2,\alpha}$ near y_0 which must not be k -convex.*

Moreover, in all the case above, the linearized operator of (1.1) at u is uniformly elliptic, and if $f \in C^\infty$ near Z_0 , then the local solution above is C^∞ near y_0 .

Remark 1.2. 1) Without loss of generality, by a translation $y \rightarrow y - y_0$ and replacing u by $u - u(0) - y \cdot Du(0)$, we can assume $Z_0 = (0, 0, 0)$ in Theorem 1.1, then the local solution in the above Theorem 1.1 is of the following form

$$(1.5) \quad u(y) = \frac{1}{2} \sum_{i=1}^n \tau_i y_i^2 + \varepsilon' \varepsilon^4 w(\varepsilon^{-2} y),$$

with arbitrarily fixed $(\tau_1, \tau_2, \dots, \tau_n) \in \mathbf{P}_2$ in the cases of (1) and (2). In the case of (3), we take some special $(\tau_1, \tau_2, \dots, \tau_n) \in \Gamma_{k-1}$. In (1.5), we always take $\varepsilon > 0$ and

$$(1.6) \quad \varepsilon' = \begin{cases} \varepsilon^\alpha, & 0 < \alpha \leq \frac{1}{2} \\ \varepsilon, & \frac{1}{2} < \alpha < 1, \end{cases}$$

then (1.5) is of the same form as the solution in [7, 11, 14], where f has good smoothness.

2) Notice that, in Case (1) of Theorem 1.1, f is permitted to change sign near Z_0 .

3) If $f = f(y, u, p)$ is independent of u and p , then the assumption on f is reduced to $f = f(y) \in C^\alpha$, which is a necessary requirement on f for the classical Schauder theory. The condition that f is $C^{2,1}$ with respect to u and p is a technical one to meet the need for tackling with the quadratic error in Nash-Moser iteration (see (3.14)).

4) In Theorem 1.1, we consider only the k -Hessian equation with $2 \leq k < n$, since the Monge-Ampère equation (1.3) is considered by [5, 7].

This article consists of three sections besides the introduction. In Section 2, we give a classification of the polynomial-typed solutions for $S_k[u] = c$ for some real constant c . Such a classification will assure the ellipticity of linearized operators at each polynomial. The results of this section given also a good understanding for the structure of solutions to k -hessian equation. In Section 3, Theorem 1.1 is proved by Nash-Moser iteration. Section 4 is an appendix in which three equivalent definitions for Gårding cone are given and proved.

2. A CLASSIFICATION OF POLYNOMIAL SOLUTIONS

For $\lambda \in \mathbb{R}^n$, set $\psi(y) = \frac{1}{2} \sum_{i=1}^n \lambda_i y_i^2$, then

$$(2.1) \quad S_k[\psi] = \sigma_k(\lambda) = c,$$

where c is a real constant. The linearized operators of $S_k[\cdot]$ at ψ is

$$(2.2) \quad \mathcal{L}_\psi = \sum_{i=1}^n \sigma_{k-1,i}(\lambda) \partial_i^2,$$

where $\sigma_{k-1,i}(\lambda)$, furthermore, $\sigma_{l,i_1,i_2,\dots,i_s}(\lambda)$, is defined in (4.7). To give a classification of polynomial solutions to equation (2.1), we recall a results of Section 2 of [15].

Proposition 2.1. (See [15]) Assume that $\lambda \in \bar{\Gamma}_k(n)$ is in descending order,

(i) then we have

$$\lambda_1 \geq \dots \lambda_k \geq \dots \lambda_p \geq 0 \geq \lambda_{p+1} \geq \dots \lambda_n$$

with $p \geq k$.

(ii) we have

$$(2.3) \quad 0 \leq \sigma_{k-1,1}(\lambda) \leq \sigma_{k-1,2}(\lambda) \leq \dots \leq \sigma_{k-1,n}(\lambda).$$

(iii) For any $\{i_1, i_2, \dots, i_s\} \subset \{1, 2, \dots, n\}$ with $l + s \leq k$, we have

$$(2.4) \quad \sigma_{l;i_1,i_2,\dots,i_s}(\lambda) \geq 0.$$

Using (ii) of the Proposition 2.1, for any $\lambda \in \overline{\Gamma}_k(n)$, the linearized operators \mathcal{L}_ψ defined in (2.2) could be degenerate elliptic. Now we study the non-strictly k -convex Garding's cone $\overline{\Gamma}_k(n)$, we will show some special uniformly elliptic case.

Theorem 2.2. *Suppose that $\lambda \in \partial\overline{\Gamma}_k(n) = \mathbf{P}_1 \cup \mathbf{P}_2$, $2 \leq k \leq n-1$. Then either*

(I) *If $\sigma_k(\lambda) = 0$ and $\sigma_{k+1}(\lambda) < 0$, then*

$$\sigma_{k-1;i}(\lambda) > 0, \quad i = 1, 2, \dots, n.$$

or

(II) *If $\sigma_k(\lambda) = 0 = \sigma_{k+1}(\lambda)$, then*

$$\sigma_j(\lambda) = 0, \quad j = k+2, \dots, n,$$

that means $\lambda \in \overline{\Gamma}_n(n)$.

In order to prove this theorem, we need several lemmas.

Lemma 2.3. *Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ and it is in the descending order. For $0 \leq s < n-k-1$, denote $\lambda^{(s)} = (\lambda_{s+1}, \dots, \lambda_n)$. Suppose that $\lambda^{(s)} \in \overline{\Gamma}_k(n-s)$ and*

$$\begin{cases} \sigma_{k-1;s+1}(\lambda^{(s)}) = 0, \\ \sigma_k(\lambda^{(s)}) = 0, \\ \sigma_{k+1}(\lambda^{(s)}) < 0. \end{cases}$$

Then $\lambda^{(s+1)} = (\lambda_{s+2}, \dots, \lambda_n) \in \overline{\Gamma}_k(n-s-1)$ and

$$\begin{cases} \sigma_{k-1;s+2}(\lambda^{(s+1)}) = 0, \\ \sigma_k(\lambda^{(s+1)}) = 0, \\ \sigma_{k+1}(\lambda^{(s+1)}) < 0. \end{cases}$$

Proof. It suffices to prove this lemma for $s=0$ since we can complete the proof by an induction on the length of $\lambda^{(s)}$. Here we call the length of $\lambda^{(n-j)}$ is j if $\lambda^{(n-j)} = (\lambda_{n-j+1}, \dots, \lambda_n)$. Thus, we suppose that

$$(2.5) \quad \sigma_{k-1;1}(\lambda) = 0, \quad \sigma_k(\lambda) = 0, \quad \sigma_{k+1}(\lambda) < 0.$$

Using (4.8)

$$\sigma_k(\lambda) = \lambda_1 \sigma_{k-1;1}(\lambda) + \sigma_{k;1}(\lambda),$$

and

$$\sigma_{k+1}(\lambda) = \lambda_1 \sigma_{k;1}(\lambda) + \sigma_{k+1;1}(\lambda),$$

we get

$$(2.6) \quad \begin{cases} \sigma_{k-1;1}(\lambda_1, \dots, \lambda_n) = \sigma_{k-1}(\lambda_2, \dots, \lambda_n) = 0, \\ \sigma_{k;1}(\lambda_1, \dots, \lambda_n) = \sigma_k(\lambda_2, \dots, \lambda_n) = 0, \\ \sigma_{k+1;1}(\lambda_1, \dots, \lambda_n) = \sigma_{k+1}(\lambda_2, \dots, \lambda_n) < 0. \end{cases}$$

By (2.4) and (2.6), we have, for $\lambda \in \overline{\Gamma}_k(n)$ satisfying (2.5),

$$\sigma_{j;1}(\lambda_1, \dots, \lambda_n) = \sigma_j(\lambda_2, \dots, \lambda_n) \geq 0, \quad \forall j \leq k,$$

which implies

$$(2.7) \quad \lambda^{(1)} := (\lambda_2, \lambda_3, \dots, \lambda_n) \in \overline{\Gamma}_k(n-1).$$

Using Proposition 2.1 for $\lambda^{(1)} \in \overline{\Gamma}_k(n-1)$, we have

$$(2.8) \quad \lambda_2 \geq \dots \geq \lambda_{1+k} \geq 0, \quad \sigma_{k-1;2}(\lambda^{(1)}) \geq 0, \quad \sigma_{k-2;2}(\lambda^{(1)}) \geq 0.$$

Using the first equation in (2.6) and the first and third inequalities in (2.8), we have

$$(2.9) \quad 0 = \sigma_{k-1;1}(\lambda) = \sigma_{k-1}(\lambda^{(1)}) = \lambda_2 \sigma_{k-2;2}(\lambda^{(1)}) + \sigma_{k-1;2}(\lambda^{(1)}) \geq \sigma_{k-1;2}(\lambda^{(1)}).$$

Then, by (2.8) and (2.9)

$$(2.10) \quad 0 \geq \sigma_{k-1;2}(\lambda^{(1)}) \geq 0.$$

Accordingly, from (2.6), (2.7) and (2.10), we get $\lambda^{(1)} \in \bar{\Gamma}_k(n-1)$ and

$$\begin{cases} \sigma_{k-1;2}(\lambda^{(1)}) = 0, \\ \sigma_k(\lambda^{(1)}) = 0, \\ \sigma_{k+1}(\lambda^{(1)}) < 0. \end{cases}$$

This completes the proof of Lemma 2.3 for $s = 0$. Then, by an induction on the length of $\lambda^{(s)}$, we finish the proof of Lemma 2.3. \square

By Proposition 2.1, if $\lambda \in \bar{\Gamma}_{m-1}(m)$, $m > 1$ and $\sigma_{m-1}(\lambda) = 0$, then

$$\sigma_{m-2;i}(\lambda) \geq 0, \quad i = 1, 2, \dots, m.$$

Under additional condition $\sigma_m(\lambda) < 0$, we have

Lemma 2.4. *Let $\lambda \in \bar{\Gamma}_{m-1}(m)$, $m > 2$ and $\sigma_{m-1}(\lambda) = 0$. If $\sigma_m(\lambda) < 0$, then we have*

$$\sigma_{m-2;i}(\lambda) > 0, \quad i = 1, 2, \dots, m.$$

Proof. By Proposition 2.1, we have $\sigma_{m-2;i}(\lambda) \geq 0$; the Maclaurin's inequalities (4.5) yields $\sigma_m(\lambda) \leq 0$. Thus, we can equivalently say, if the inequality above does not hold, that is, $\sigma_{m-2;i}(\lambda) = 0$ for some $i \in \{1, \dots, m\}$, then

$$\sigma_m(\lambda) = \lambda_1 \cdots \lambda_m = 0.$$

It is enough to prove that $\sigma_{m-1}(\lambda) = 0 = \sigma_{m-2;1}(\lambda) = 0$ imply $\sigma_m(\lambda) = 0$, since the other case can be deduced by absurd argument of this results.

Substituting $\sigma_{m-1}(\lambda) = 0 = \sigma_{m-2;1}(\lambda) = 0$ into

$$\sigma_{m-1}(\lambda) = \lambda_1 \sigma_{m-2;1}(\lambda) + \sigma_{m-1;1}(\lambda),$$

we have

$$0 = \sigma_{m-1;1}(\lambda_1, \dots, \lambda_m) = \sigma_{m-1}(\lambda_2, \dots, \lambda_m) = \prod_{i=2}^m \lambda_i.$$

Thus,

$$0 = \lambda_1 \prod_{i=2}^m \lambda_i = \sigma_m(\lambda).$$

\square

Proof of Theorem 2.2. By these two lemmas above, we prove Theorem 2.2 by an induction on k . Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n$ and λ is in the descending order.

Step 1: The case $k = 2$. We claim the following results :

Let $\lambda \in \bar{\Gamma}_2(n)$ and $\sigma_2(\lambda) = 0$, if $\sigma_3(\lambda) < 0$, then we have

$$\sigma_{1,i}(\lambda) > 0, \quad i = 1, 2, \dots, n.$$

Equivalently, for $\lambda \in \bar{\Gamma}_2(n)$ with $\sigma_2(\lambda) = 0$, if $\sigma_{1,i}(\lambda) = 0$ for some $i \in \{1, \dots, n\}$, then

$$\sigma_j(\lambda) = 0, \quad j = 3, \dots, n.$$

By assumption above, we have

$$\begin{aligned}\sigma_2(\lambda) &= \lambda_1\sigma_{1;1}(\lambda) + \sigma_{2;1}(\lambda), \\ \sigma_{1;1}(\lambda) &= \sigma_1(\lambda) - \lambda_1 = \sum_{j=2}^n \lambda_j = \sigma_1(\lambda_2, \dots, \lambda_n), \\ \sigma_{2;1}(\lambda) &= \sigma_2(\lambda_2, \dots, \lambda_n) = \frac{(\sigma_1 - \lambda_1)^2 - \sum_{i=2}^m \lambda_i^2}{2}.\end{aligned}$$

If we assume $\sigma_{1;1}(\lambda) = 0$, then $\sigma_{1;1}(\lambda) = 0$ together with $\sigma_2(\lambda) = 0$ yields

$$0 = \sigma_2(\lambda) = \lambda_1\sigma_{1;1}(\lambda) + \sigma_{2;1}(\lambda) = \sigma_{2;1}(\lambda) = \frac{(\sigma_1 - \lambda_1)^2 - \sum_{i=2}^m \lambda_i^2}{2} = \frac{-\sum_{i=2}^m \lambda_i^2}{2},$$

which implies

$$\sum_{i=2}^m \lambda_i^2 = 0,$$

and thus

$$\lambda_i = 0, \quad i = 2, \dots, n.$$

Then

$$\sigma_j(\lambda) = 0, \quad j = 2, \dots, n,$$

which contradicts with $\sigma_3(\lambda) < 0$, therefore $\sigma_{1;1}(\lambda) = 0$ is impossible.

Step 2: The case $2 < k \leq n-1$. If $k = n-1$, it is included in Lemma 2.4. Now we consider the general case $2 < k < n-1$.

The proof of part (I): We will prove that, for $2 < k < n-1$, if $\lambda \in \bar{\Gamma}_k(n)$, $\sigma_k(\lambda) = 0$ and $\sigma_{k+1}(\lambda) < 0$, then

$$\sigma_{k-1;1}(\lambda) > 0.$$

We prove this claim by absurd argument. Recall $\lambda^{(s)} = (\lambda_{s+1}, \dots, \lambda_n)$ and suppose that $\lambda = \lambda^{(0)} \in \bar{\Gamma}_k(n)$,

$$\begin{cases} \sigma_{k-1;1}(\lambda^{(0)}) = 0, \text{ (the absurd assumption),} \\ \sigma_k(\lambda^{(0)}) = 0, \\ \sigma_{k+1}(\lambda^{(0)}) < 0. \end{cases}$$

By using Lemma 2.3 and the induction assumption up to $s = n-k-1$, we have that $\lambda^{(n-k-1)} \in \bar{\Gamma}_k(k+1)$ and

$$\begin{cases} \sigma_{k-1;n-k}(\lambda^{(n-k-1)}) = \sigma_{k-1;n-k}(\lambda_{n-k}, \dots, \lambda_n) = 0, \\ \sigma_k(\lambda^{(n-k-1)}) = \sigma_k(\lambda_{n-k}, \dots, \lambda_n) = 0, \\ \sigma_{k+1}(\lambda^{(n-k-1)}) = \sigma_{k+1}(\lambda_{n-k}, \dots, \lambda_n) < 0. \end{cases}$$

This contradicts with the conclusion of Lemma 2.4 with $m = k+1$. Thus, the assumption $\sigma_{k-1;1}(\lambda) = 0$ is really absurd, and therefore $\sigma_{k-1;1}(\lambda) > 0$.

The proof of part (II): We suppose that $\lambda \in \bar{\Gamma}_k(n)$ and $\sigma_k(\lambda) = \sigma_{k+1}(\lambda) = 0$. Then $\lambda \in \bar{\Gamma}_{k+1}(n)$, and for any $\varepsilon > 0$, from the formula

$$\sigma_{k+1}(\lambda + \varepsilon) = \sum_{j=0}^{k+1} C(j, k, n) \varepsilon^j \sigma_{k+1-j}(\lambda), \quad C(j, k, n) = \frac{\binom{n}{k+1} \binom{k+1}{j}}{\binom{n}{k+1-j}}$$

with the convention $\sigma_0(\lambda) = 1$, we have

$$\lambda + \varepsilon = (\lambda_1 + \varepsilon, \lambda_2 + \varepsilon, \dots, \lambda_n + \varepsilon) \in \Gamma_{k+1}(n).$$

We see that either $\sigma_{k+2}(\lambda) \geq 0$ or $\sigma_{k+2}(\lambda) < 0$. Now we prove $\sigma_{k+2}(\lambda) = 0$. Firstly we claim that $\sigma_{k+2}(\lambda) < 0$ is impossible. Otherwise, from the assumption $\sigma_k(\lambda) = \sigma_{k+1}(\lambda) = 0$, we have

$$\begin{aligned}\sigma_k(\lambda) &= 0, \\ \sigma_{k+1}(\lambda) &= 0, \\ \sigma_{k+2}(\lambda) &< 0.\end{aligned}$$

Using the results of part (I), we obtain from both $\sigma_{k+1}(\lambda) = 0$ and $\sigma_{k+2}(\lambda) < 0$ that

$$(2.11) \quad \sigma_{k;1}(\lambda) > 0.$$

Since $\lambda \in \bar{\Gamma}_k(n)$, it is necessary by Proposition 2.1 that $\lambda_1 \geq 0$ and $\sigma_{k-1;1}(\lambda) \geq 0$. So we have

$$0 = \sigma_k(\lambda) = \lambda_1 \sigma_{k-1;1}(\lambda) + \sigma_{k;1}(\lambda) \geq \sigma_{k;1}(\lambda).$$

There is a contradiction of (2.11), thus, the case that $\sigma_{k+2}(\lambda) < 0$ does not occur and we obtain $\sigma_{k+2}(\lambda) \geq 0$ in which case $\lambda \in \bar{\Gamma}_{k+2}(n)$. Applying the Maclaurin's inequalities to $\lambda + \varepsilon \in \Gamma_{k+2}(n)$, we obtain

$$\left[\frac{1}{\binom{n}{k+1}} \sigma_{k+1}(\lambda + \varepsilon) \right]^{\frac{1}{k+1}} \geq \left[\frac{1}{\binom{n}{k+2}} \sigma_{k+2}(\lambda + \varepsilon) \right]^{\frac{1}{k+2}}.$$

Let $\varepsilon \rightarrow 0^+$, then

$$\sigma_{k+1}(\lambda) = 0 \quad \text{implies} \quad \sigma_{k+2}(\lambda) = 0.$$

Repeating the above argument for $k+1, k+2, \dots, n$, we obtain

$$\sigma_{k+j}(\lambda) = 0, \quad j = 1, 2, \dots, n-k,$$

that is, $\lambda \in \bar{\Gamma}_n(n)$ and this completes the proof. \square

Now we are back to the definitions of \mathbf{P}_1 and \mathbf{P}_2 , by Theorem 2.2, can be stated more precisely as, for $k < n$,

$$\mathbf{P}_1 = \{\lambda \in \Gamma_k : \sigma_j(\lambda) \geq 0, \sigma_k(\lambda) = \dots = \sigma_n(\lambda) = 0, 1 \leq j \leq k-1\},$$

$$\mathbf{P}_2 = \{\lambda \in \Gamma_k : \sigma_j(\lambda) > 0, \sigma_k(\lambda) = 0, \sigma_{k+1} < 0, 1 \leq j \leq k-1\}.$$

If $2 \leq k < n$, then $\mathbf{P}_2 \neq \emptyset$. Here is an example, let

$$\begin{cases} \lambda_i = 1, 1 \leq i \leq k-1 \\ \lambda_k = M, \lambda_{k+1} = -\frac{1}{M} \\ \lambda_i = 0, k+1 < i \leq n \end{cases}$$

with $M = \frac{k-1 + \sqrt{(k-1)^2 + 4}}{2} > 1$, then $M - \frac{1}{M} = k-1$, $\sigma_j(\lambda) > 0 (1 \leq j \leq k-1)$, $\sigma_k(\lambda) = 0$ and $\sigma_{k+1}(\lambda) = -1$, which means that $\mathbf{P}_2 \neq \emptyset$.

The significance of Theorem 2.2 is the breakthrough of the classical framework of the ellipticity of Hessian equations. It is well known that, $S_k[u] = f$ is elliptic if $f > 0$ and degenerate elliptic if $f \geq 0$. By the definition of \mathbf{P}_2 , the condition $f(0) = 0$ must lead to degenerate ellipticity for Monge-Ampère equation. However, it is no longer true for k -Hessian ($2 \leq k < n$) equation by Theorem 2.2. Next theorem gives a complete k -Hessian classification of second-order polynomials in the degenerate elliptic case.

Theorem 2.5. *Suppose that $\lambda \in \partial\Gamma_k(n)$, $2 \leq k \leq n-1$.*

- (1) *For any $\lambda \in \mathbf{P}_2$, we have that $\psi = \frac{1}{2} \sum_{i=1}^n \lambda_i y_i^2$ is a solution of k -Hessian equation $S_k[\psi] = 0$, and the linearized operators of $\mathcal{L}_\psi = \sum_{i=1}^n \sigma_{k-1,i}(\lambda) \partial_i^2$ is uniformly elliptic.*

(2) For any $\lambda \in \mathbf{P}_1$ with $\lambda_1 \geq \dots \geq \lambda_n$, then $\psi = \frac{1}{2} \sum_{i=1}^n \lambda_i y_i^2$ is a non-strict convex solution of k -Hessian equation $S_k[\psi] = 0$, and the linearized operators of \mathcal{L}_ψ is degenerate elliptic with

$$\sigma_{k-1,i} = 0, \quad 1 \leq i \leq k-1.$$

Moreover, if $\sigma_{k-1}(\lambda) = 0$, then $\mathcal{L}_\psi = 0$, that is, $\sigma_{k-1,i} = 0$ for $1 \leq i \leq n$; if $\sigma_{k-1}(\lambda) > 0$, then

$$(2.12) \quad \begin{cases} \lambda_i > 0, 1 \leq i \leq k-1 \\ \lambda_i = 0, k \leq i \leq n \\ \sigma_{k-1,i} = 0, 1 \leq i \leq k-1 \\ \sigma_{k-1,i} = \prod_{j=1}^{k-1} \lambda_j > 0, k \leq i \leq n \end{cases}$$

Proof. (1) is a direct consequence of (I) in Theorem 2.2. Now we prove (2). If $\lambda \in \mathbf{P}_1$, we have $\sigma_{k-1,i} \geq 0$ for $1 \leq i \leq n$ by (2.3), and $\sigma_j(\lambda) = 0$ for $k \leq j \leq n$ by Theorem 2.2 (II). From $\sigma_n(\lambda) = \prod_{i=1}^n \lambda_i = 0$, $\lambda_j \geq 0$ ($1 \leq j \leq n$) and λ is in decreasing order, it follows that $\lambda_n = 0$ and

$$\sigma_k(\lambda_1, \dots, \lambda_{n-1}) = \sigma_k(\lambda) - \lambda_n \sigma_{k-1,n}(\lambda) = \sigma_k(\lambda) = 0.$$

Similarly,

$$\sigma_j(\lambda_1, \dots, \lambda_{n-1}) = \sigma_j(\lambda) \geq 0, \sigma_{k+1}(\lambda_1, \dots, \lambda_{n-1}) = \sigma_{k+1}(\lambda) = 0, 1 \leq j \leq k-1.$$

Applying Theorem 2.2 (II) to $(\lambda_1, \dots, \lambda_{n-1}) \in \Gamma_k(n-1)$, we obtain

$$\sigma_{n-1}(\lambda_1, \dots, \lambda_{n-1}) = 0$$

and by the same reasoning in the n -dimensional case, $\lambda_{n-1} = 0$. By an induction on the dimension up to $k+1$, we see that $\lambda_i = 0, k+1 \leq i \leq n$. Since $\sigma_k(\lambda) = 0$ and

$$\sigma_k(\lambda) = \sigma_k(\lambda_1, \dots, \lambda_k, 0, \dots, 0) = \prod_{i=1}^k \lambda_i,$$

we have $\lambda_k = 0$ by recalling that λ is in descending order. By the virtue of

$$\sum_{i=1}^n \sigma_{k-1;i}(\lambda) = (n-k+1)\sigma_{k-1}(\lambda), \sigma_{k-1;i}(\lambda) \geq 0, \quad 1 \leq i \leq n,$$

we see that, if $\sigma_{k-1}(\lambda) = 0$, then $\sigma_{k-1,i} = 0$ for $1 \leq i \leq n$; if $\sigma_{k-1}(\lambda) > 0$, from

$$0 < \sigma_{k-1}(\lambda) = \sigma_{k-1}(\lambda_1, \dots, \lambda_{k-1}, 0, \dots, 0) = \prod_{i=1}^{k-1} \lambda_i,$$

we have

$$\lambda_i > 0, 1 \leq i \leq k-1.$$

By the definition of $\sigma_{k-1;i}(\lambda)$ and

$$\lambda = (\lambda_1, \dots, \lambda_{k-1}, 0, \dots, 0),$$

we obtain the last two conclusions of (2.12). \square

If $\lambda \in \Gamma_k(n)$, we certainly have $\lambda \in \Gamma_{k-1}(n)$. If $\lambda \in \mathbf{P}_2 \subset \partial\Gamma_k(n)$ with $\sigma_{k+1}(\lambda) < 0$, from $\sum_{i=1}^n \sigma_{k-1;i}(\mu) = (n-k+1)\sigma_{k-1}(\mu)$, $\mu \in \mathbb{R}^n$ and Theorem 2.2 (I), it follows that $\sigma_{k-1}(\lambda) > 0$ and $\lambda \in \Gamma_{k-1}(n)$. In those two cases above, $0 < \sigma_{k-1;1} \leq \sigma_{k-1;2} \leq \dots \leq \sigma_{k-1;n}$ implies the uniform ellipticity. Conversely, we want to know whether and how the ellipticity is true for $\lambda \in \Gamma_{k-1}(n)$. Also notice that, in the monotonicity formula (2.3), it is required that $\lambda \in \overline{\Gamma_k(n)}$ rather than $\lambda \in \overline{\Gamma_{k-1}(n)}$ as Lemma 2.6 below.

Lemma 2.6. . Suppose that $\lambda \in \Gamma_{k-1}(n)$, $2 \leq k \leq n-1$. If $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$, then

$$\sigma_{k-1;1}(\lambda) \geq 0$$

is equivalent to

$$0 \leq \sigma_{k-1;1}(\lambda) \leq \sigma_{k-1;2}(\lambda) \leq \dots \leq \sigma_{k-1;n}(\lambda).$$

Proof. We claim that $\sigma_{k-1;2}(\lambda) \geq \sigma_{k-1;1}(\lambda)$, that is

$$\sigma_{k-1}(\lambda_1, \lambda_3, \dots, \lambda_n) \geq \sigma_{k-1}(\lambda_2, \lambda_3, \dots, \lambda_n).$$

If the claim is true, by using it again, we obtain

$$\sigma_{k-1;n}(\lambda) \geq \dots \geq \sigma_{k-1;2}(\lambda) \geq \sigma_{k-1;1}(\lambda).$$

Since $\lambda \in \Gamma_{k-1}(n)$, by (4.9) we have

$$\sigma_{l;1}(\lambda) > 0, \quad l+1 \leq k-1,$$

which, together with the assumption

$$\sigma_{k-1}(\lambda_2, \lambda_3, \dots, \lambda_n) = \sigma_{k-1,1}(\lambda) \geq 0,$$

yields $(\lambda_2, \lambda_3, \dots, \lambda_n) \in \Gamma_{k-1}(n-1)$. Using (4.9) again, we obtain

$$\sigma_{k-2}(\lambda_3, \lambda_4, \dots, \lambda_n) = \sigma_{k-2,2}(\lambda_2, \lambda_3, \dots, \lambda_n) \geq 0.$$

Let $\varepsilon = \lambda_1 - \lambda_2 \geq 0$, we have

$$\begin{aligned} \sigma_{k-1;2}(\lambda) &= \sigma_{k-1}(\lambda_1, \lambda_3, \dots, \lambda_n) \\ &= (\lambda_2 + \varepsilon)\sigma_{k-2}(\lambda_3, \dots, \lambda_n) + \sigma_{k-1}(\lambda_3, \dots, \lambda_n) \\ &\geq \lambda_2\sigma_{k-2}(\lambda_3, \dots, \lambda_n) + \sigma_{k-1}(\lambda_3, \dots, \lambda_n) \\ &= \sigma_{k-1}(\lambda_2, \lambda_3, \dots, \lambda_n) = \sigma_{k-1;1}(\lambda), \end{aligned}$$

thus, the claim is proved. \square

We will give a characterization of ellipticity for the linearized operator of $S_k[\psi] = c$ with $c \in \mathbb{R}$.

Theorem 2.7. For any $c \in \mathbb{R}$, there exists $\lambda \in \Gamma_{k-1}(n)$ such that

$$(2.13) \quad 0 < \sigma_{k-1;1}(\lambda) \leq \sigma_{k-1;2}(\lambda) \leq \dots \leq \sigma_{k-1;n}(\lambda),$$

and $\psi = \frac{1}{2} \sum_{i=1}^n \lambda_i y_i^2$ is a $(k-1)$ -convex solution of k -Hessian equation $S_k[\psi] = c$, moreover, the linearized operators of $S_k[u]$ at ψ

$$(2.14) \quad \mathcal{L}_\psi = \sum_{i=1}^n \sigma_{k-1,i}(\lambda) \partial_i^2$$

is uniformly elliptic.

Proof. If $c > 0$, we take $\lambda_1 = \lambda_2 = \dots = \lambda_n = [\frac{c}{\binom{n}{k}}]^{1/k} > 0$, then $\sigma_k(\lambda) = c$; If $c = 0$, see Theorem 2.5 (1). It is left to consider the case $c < 0$.

Notice that $\mathbf{P}_2 \neq \emptyset$ when $k < n$, so we can choose $\delta = (\delta_2, \dots, \delta_n) \in \mathbf{P}_2 \subset \partial\Gamma_{k-1}(n-1)$ with $\delta_2 \geq \delta_3 \geq \dots \geq \delta_n$, that is,

$$\sigma_{k-1}(\delta_2, \dots, \delta_n) = 0, \sigma_k(\delta_2, \dots, \delta_n) < 0$$

Obviously $\delta_2 > 0$. Choosing $1 > t > 0$ small such that, for $\delta_1 = \delta_2 + 1$,

$$\begin{aligned} \sigma_{k-1}(\delta_2 + t, \dots, \delta_n + t) &> 0, \\ \delta_1 \sigma_{k-1}(\delta_2 + t, \dots, \delta_n + t) + \sigma_k(\delta_2 + t, \dots, \delta_n + t) &< 0. \end{aligned}$$

Then

$$\sigma_k(\delta_1, \delta_2 + t, \dots, \delta_n + t) = \delta_1 \sigma_{k-1}(\delta_2 + t, \dots, \delta_n + t) + \sigma_k(\delta_2 + t, \dots, \delta_n + t) < 0.$$

Since $\sigma_k(s\lambda) = s^k \sigma_k(\lambda)$ for $s > 0$, we can choose suitable $s > 0$ such that

$$\sigma_k(s\delta_1, s(\delta_2 + t), \dots, s(\delta_n + t)) = c.$$

Let $\lambda = (s\delta_1, s(\delta_2 + t), \dots, s(\delta_n + t))$, then

$$\sigma_k(\lambda) = c.$$

The fact $(\lambda_2, \lambda_3, \dots, \lambda_n) \in \Gamma_{k-1}(n-1)$ and (4.9) lead to

$$\sigma_l(\lambda_2, \lambda_3, \dots, \lambda_n) > 0, \quad 1 \leq l \leq k-1.$$

Therefore, by virtue of $\lambda_1 = s\delta_1 > 0$,

$$\sigma_1(\lambda) = \lambda_1 + \sigma_1(\lambda_2, \lambda_3, \dots, \lambda_n) > \sigma_1(\lambda_2, \lambda_3, \dots, \lambda_n) > 0$$

and

$$\sigma_l(\lambda) = \lambda_1 \sigma_{l-1}(\lambda_2, \lambda_3, \dots, \lambda_n) + \sigma_l(\lambda_2, \lambda_3, \dots, \lambda_n) > 0, \quad 2 \leq l \leq k-1.$$

By the definition of $\Gamma_{k-1}(n)$, we have proved that $\lambda \in \Gamma_{k-1}(n)$. Noticing

$$\lambda \in \Gamma_{k-1}(n), \quad \lambda_1 \geq \lambda_2 \dots \geq \lambda_n, \quad \sigma_{k-1}(\lambda_2, \lambda_3, \dots, \lambda_n) > 0$$

and applying Lemma 2.6, we see that

$$0 < \sigma_{k-1;1}(\lambda) \leq \sigma_{k-1;2}(\lambda) \leq \dots \leq \sigma_{k-1;n}(\lambda),$$

which leads to that the operator (2.14) is uniformly elliptic. This completes the proof. \square

Theorem 2.8. *For any $0 < c$, there exists $\lambda \in \Gamma_k(n)$ such that*

$$\begin{cases} 0 < \sigma_{k-1;i}(\lambda), & 1 \leq i \leq n \\ \sigma_{k+l-1}(\lambda) > 0, \sigma_{k+l}(\lambda) < 0, & 1 \leq l \leq n-k. \end{cases}$$

In particular, there exists $\lambda \in \Gamma_n(n)$ such that

$$\sigma_k(\lambda) = c.$$

Therefore, for $1 \leq l \leq n-k$, $\psi = \frac{1}{2} \sum_{i=1}^n \lambda_i y_i^2$ is a $(k+l-1)$ -convex solution of k -Hessian equation $S_k[\psi] = c$ which is not $(k+l)$ -convex. Moreover, the linearized operators of $S_k[\cdot]$ at ψ

$$\mathcal{L}_\psi = \sum_{i=1}^n \sigma_{k-1,i}(\lambda) \partial_i^2$$

is uniformly elliptic.

Proof. In the proof of Theorem 2.7, replacing σ_k by σ_{k+l} , we obtain that $\lambda \in \Gamma_{k+l-1}(n)$ with $\sigma_{k+l}(\lambda) < 0$ for $1 \leq l \leq n-k$. For this λ , choosing $s > 0$ such that $\sigma_k(s\lambda) = c$. The other part of proof is the same as those in Theorem 2.7. \square

3. EXISTENCE OF C^∞ LOCAL SOLUTIONS

In this section, by the classification of precedent section, we now prove Theorem 1.1 which is stated in the following precise version.

Theorem 3.1. *For $2 \leq k \leq n-1$, let $f = f(y, u, p)$ be defined and continuous near a point $Z_0 = (0, 0, 0) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ and $0 < \alpha < 1$. Assume that f is C^α with respect to y and $C^{2,1}$ with respect to u, p . We have the following results*

- (1) *If $f(Z_0) = 0$, then (1.1) admits a $(k-1)$ -convex local solution $C^{2,\alpha}$ near $y_0 = 0$, which is not $(k+1)$ -convex and of the following form*

$$(3.1) \quad u(y) = \frac{1}{2} \sum_{i=1}^n \tau_i y_i^2 + \varepsilon' \varepsilon^4 w(\varepsilon^{-2}y)$$

with arbitrarily fixed $(\tau_1, \dots, \tau_n) \in \mathbf{P}_2$, ε' is defined in (1.6) and $\varepsilon > 0$ very small, the function w satisfies

$$(3.2) \quad \begin{cases} \|w\|_{C^{2,\alpha}} \leq 1 \\ w(0) = 0, \nabla w(0) = 0. \end{cases}$$

- (2) *If $f \geq 0$ near Z_0 , then the equation (1.1) admits a k -convex local solution $u \in C^{2,\alpha}$ near $y_0 = 0$ which is not $(k+1)$ -convex and of the form (3.1).*

Moreover, the equation (1.1) is uniformly elliptic with respect to the solution (3.1). If $f \in C^\infty$ near Z_0 , then $u \in C^\infty$ near y_0 .

Theorem 3.1 is exactly the part (1) and (2) of Theorem 1.1.

We now proceed to take the change of unknown function $u \leftrightarrow w$ and the change of variable $y \leftrightarrow x$, the aim is to consider the equation (1.1) in the domain $B_1(0)$ with the new variable x and then the so-called "local" enter into the new equation itself, see (3.3)-(3.4). The trick is from Lin [11]. Let $\tau = (\tau_1, \dots, \tau_n) \in \mathbf{P}_2$, then $\psi(y) = \frac{1}{2} \sum_{i=1}^n \tau_i y_i^2$ is a polynomial-type solution of

$$S_k[\psi] = 0.$$

We follow Lin [11] to introduce the following function

$$u(y) = \frac{1}{2} \sum_{i=1}^n \tau_i y_i^2 + \varepsilon' \varepsilon^4 w(\varepsilon^{-2}y) = \psi(y) + \varepsilon' \varepsilon^4 w(\varepsilon^{-2}y), \quad \tau \in \mathbf{P}_2, \quad \varepsilon > 0,$$

as a candidate of solution for equation (1.2). Noting $y = \varepsilon^2 x$, we have

$$(D_{y_j} u)(x) = \tau_j \varepsilon^2 x_j + \varepsilon' \varepsilon^2 w_j(x), \quad j = 1, \dots, n,$$

and

$$(D_{y_j y_k} u)(x) = \delta_k^j \tau_j + \varepsilon' w_{jk}(x), \quad j, k = 1, \dots, n,$$

where δ_k^j is the Kronecker symbol, $w_j(x) = (D_{y_j} w)(x)$ and $w_{jk}(x) = (D_{y_j y_k}^2 w)(x)$. Then (1.1) transfers to

$$\tilde{S}_k(w) = \tilde{f}_\varepsilon(x, w(x), Dw(x)), \quad x \in B_1(0) = \{x \in \mathbb{R}^n; |x| < 1\},$$

where

$$\tilde{S}_k[w] = S_k(\delta_i^j \tau_i + \varepsilon' w_{ij}(x)) = S_k(r(w)),$$

with symmetric matrix $r(w) = (\delta_i^j \tau_i + \varepsilon' w_{ij}(x))$, and

$$\tilde{f}_\varepsilon(x, w(x), Dw(x)) = f(\varepsilon^2 x, \varepsilon^4 \psi(x) + \varepsilon' \varepsilon^4 w(x), \tau_1 \varepsilon^2 x_1 + \varepsilon' \varepsilon^2 w_1(x), \dots, \tau_n \varepsilon^2 x_n + \varepsilon' \varepsilon^2 w_n(x)).$$

We now explain the smooth condition on $f = f(x, u, p)$ and its norms, that is, f is Hölder continuous with respect to x , denoted by $f \in C_x^\alpha$, and $C^{2,1}$ with respect to u, p , denoted by $C_{u,p}^{2,1}$. We will consider $f = f(x, u, p)$ defined on

$$\mathcal{B} = \{(x, u, p) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n : x \in B_1(0), |u| \leq A, |p| \leq A\}$$

for some fixed $A > 0$. We say $f \in C_x^\alpha$ if

$$\|f\|_{C_x^\alpha} = \|f\|_{L^\infty(\mathcal{B})} + \sup_{(x,u,p) \in \mathcal{B}, (z,u,p) \in \mathcal{B}, x \neq z} \frac{|f(x, u, p) - f(z, u, p)|}{|x - z|^\alpha} < \infty.$$

When $f = f(x)$, then $f \in C_x^\alpha$ is the usual $f \in C^\alpha(B_1(0))$. When defining $f = f(z, u, p) \in C_{u,p}^{2,1}$, we regard z as a parameter as follow:

$$\|f\|_{C_{u,p}^{1,1}} = \sup_{\mathcal{B}} \{|D_{u,p}^\beta f(z, u, p)| : 0 \leq |\beta| \leq 2\} < \infty$$

and

$$\|f\|_{C_{u,p}^{2,\alpha}} = \|f\|_{C_{u,p}^{1,1}} + \sup_{(z,u,p) \in \mathcal{B}, (z,u',p') \in \mathcal{B}, (u,p) \neq (u',p')} \left\{ \frac{|D_{u,p}^\beta f(z, u, p) - D_{u,p}^\beta f(z, u', p')|}{|(u, p) - (u', p')|^\alpha}, |\beta| = 2 \right\} < \infty,$$

for $0 < \alpha < 1$ and a similar definition for $\|f\|_{C_{u,p}^{2,1}}$. Here and later on, without confusion, we will denote $\|f\|_{C_x^\alpha}$, $\|f\|_{C_{u,p}^{1,1}}$ and $\|f\|_{C_{u,p}^{2,1}}$ as $\|f\|_{C^\alpha}$, $\|f\|_{C^{1,1}}$ and $\|f\|_{C^{2,1}}$ respectively.

Similar to [11] we consider the nonlinear operators

$$(3.3) \quad G(w) = \frac{1}{\varepsilon'} [S_k(r(w)) - \tilde{f}_\varepsilon(x, w, Dw)], \quad \text{in } B_1(0).$$

The linearized operator of G at w is

$$(3.4) \quad L_G(w) = \sum_{i,j=1}^n \frac{\partial S_k(r(w))}{\partial r_{ij}} \partial_{ij}^2 + \sum_{i=1}^n a_i \partial_i + a,$$

where

$$a_i = -\frac{1}{\varepsilon'} \frac{\partial \tilde{f}_\varepsilon(x, z, p_i)}{\partial p_i}(x, w, Dw) = -\varepsilon^2 \frac{\partial f}{\partial p_i}$$

$$a = -\frac{1}{\varepsilon'} \frac{\partial \tilde{f}_\varepsilon(x, z, p_i)}{\partial z}(x, w, Dw) = -\varepsilon^4 \frac{\partial f}{\partial z}.$$

Hereafter, we denote $S_k^{ij}(r(w)) = \frac{\partial S_k(r(w))}{\partial r_{ij}}$.

Lemma 3.2. *Assume that $\tau \in \mathbf{P}_2$ and $\|w\|_{C^2(B_1(0))} \leq 1$, then the operator $L_G(w)$ is a uniformly elliptic operator if $\varepsilon > 0$ is small enough.*

Proof. In order to prove the uniform ellipticity of $L_G(w)$

$$\sum_{i,j=1}^n S_k^{ij}(r(w)) \xi_i \xi_j \geq c |\xi|^2, \quad \forall (x, \xi) \in B_1(0) \times \mathbb{R}^n,$$

it suffices to prove

$$(3.5) \quad \begin{cases} S_k^{ii}(r(w)) = \sigma_{k-1,i}(\tau_1, \tau_2, \dots, \tau_n) + O(\varepsilon'), & 1 \leq i \leq n \\ S_k^{ij}(r(w)) = O(\varepsilon'), & i \neq j, \end{cases}$$

because if it does hold, we see by (2.13) that

$$S_k^{ii}(r(w)) - \sum_{j=1, j \neq i}^n |S_k^{ij}(r(w))| > \frac{1}{2} \sigma_{k-1,i}(\tau_1, \dots, \tau_n) > 0, \quad 1 \leq i \leq n$$

if $\varepsilon > 0$ is small enough, then the matrix $(S_k^{ij}(r(w)))$ is strictly diagonally dominant and $L_G(w)$ is a uniformly elliptic operator.

Indeed, Since $S_k(r)$ is the sum all principal minors of order k of the Hessian $\det(r)$, then

$$S_k^l(r(w)) = S_{k-1}(r(w; l, l))$$

where $r(w; l, l)$ is a $(n-1) \times (n-1)$ matrix determined from r by deleting the l -th row and l -th column. But $r(w) = (\delta_i^j \tau_i + \varepsilon w_{ij}(x))$, then

$$S_{k-1}(r(w; l, l)) = \sigma_{k-1;l}(\tau_1, \tau_2, \dots, \tau_n) + O(\varepsilon')$$

and

$$S_k^{ij}(r(w)) = O(\varepsilon'), \quad i \neq j.$$

Proof is done. \square

We follows now the idea of Hong and Zuily [7] to prove the existence and a priori estimates of solution for linearized operator. In fact, if following the proof of [7] step by step, we can also obtain the existence of the local solution if f is smooth enough, the reason is that $L_G(w)$ being uniformly elliptic can regarded as a special case of $L_G(w)$ being degenerately elliptic in [7]. But if $f \in C_x^\alpha$, which is the least requirement in classical Schauder estimates, their proof [7] does not work anymore because the degeneracy results in the loss of regularity. In our case, although $L_G(w)$ is uniformly elliptic, the existence and the priori Schauder estimates of classical solutions can not be directly obtained. The difficulty lies in that we do not know whether the coefficient a of the term au in (3.4) is non-positive. After proving the existence of the linearized equation (Lemma 3.3), we can employ Nash-Moser procedure to prove the existence of local solution for (1.1) in Hölder space. We shall use the following schema :

$$(3.6) \quad \begin{cases} w_0 = 0, & w_m = w_{m-1} + \rho_{m-1}, \quad m \geq 1, \\ L_G(w_m)\rho_m = g_m, \text{ in } B_1(0), \\ \rho_m = 0 \quad \text{on } \partial B_1(0), \\ g_m = -G(w_m), \end{cases}$$

where

$$g_0(x) = \frac{1}{\varepsilon'} (\sigma_k(\tau) - f(\varepsilon^2 x, \varepsilon^4 \psi(x), \varepsilon^2(\tau_1 x_1, \tau_2 x_2, \dots, \tau_n x_n))).$$

It is pointed out on page 107, [3] that, if the operator L_G does not satisfy the condition $a \leq 0$, as is well known from simple examples, the Dirichlet problem for $L_G(w)\rho = g$ no longer has a solution in general. Notice a in (3.4) has the factor ε^4 , we will take advantage of the smallness of a to obtain the uniqueness and existence of solution for Dirichlet problem (3.7). We will assume $\|w_m\|_{C^{2,\alpha}} \leq A$ rather than $\|w_m\|_{C^{2,\alpha}} \leq 1$ as in [7], the advantage is to see how the procedure depends on A . Actually A can be taken as 1. We have uniformly Schauder estimates of its solution as follows.

Lemma 3.3. *Assume that $\|w\|_{C^{2,\alpha}(B_1(0))} \leq A$. Then there exists a unique solution $\rho \in C^{2,\alpha}(\overline{B_1(0)})$ to the following Dirichlet problem*

$$(3.7) \quad \begin{cases} L_G(w)\rho = g, & \text{in } B_1(0), \\ \rho = 0 & \text{on } \partial B_1(0) \end{cases}$$

for all $g \in C^\alpha(\overline{B_1(0)})$. Moreover,

$$(3.8) \quad \|\rho\|_{C^{2,\alpha}(\overline{B_1(0)})} \leq C \|g\|_{C^\alpha(\overline{B_1(0)})}, \quad \forall g \in C^\alpha(\overline{B_1(0)}),$$

where the constant C depends on A, τ and $\|f\|_{C^{2,1}}$. Moreover, C is independent of $0 < \varepsilon \leq \varepsilon_0$ for some $\varepsilon_0 > 0$.

By the virtue of (3.4), we write (3.7) as

$$(3.9) \quad \begin{cases} L_G(w)\rho = \sum_{i,j=1}^n \frac{\partial S_k(r(w))}{\partial r_{ij}} \partial_i \partial_j \rho + \sum_{i=1}^n a_i \partial_i \rho + a\rho = g, & \text{in } B_1(0), \\ \rho = 0 & \text{on } \partial B_1(0) \end{cases}$$

where

$$a_i = -\varepsilon^2 \frac{\partial f}{\partial p_i}, \quad a = -\varepsilon^4 \frac{\partial f}{\partial z}.$$

Noticing that for the functions, such as $\frac{\partial S_k(r(w))}{\partial r_{ij}}$, $a_i = a_i(x, w(x), Dw(x))$, $a = a(x, w(x), Dw(x))$ and $g_m = -G(w_m) = g_m(x, w_m(x), Dw_m(x), D^2 w_m(x))$ by (3.6), we regard them as the functions with variable x . In a word, we regard that all of the coefficients and non-homogeneous term in (3.9) are functions of variable x . For example,

$$\begin{aligned} & \tilde{f}_\varepsilon(x, w(x), Dw(x)) \\ & = f(\varepsilon^2 x, \varepsilon^4 \psi(x) + \varepsilon' \varepsilon^4 w(x), \tau_1 \varepsilon^2 x_1 + \varepsilon' \varepsilon^2 w_1(x), \dots, \tau_n \varepsilon^2 x_n + \varepsilon' \varepsilon^2 w_n(x)). \end{aligned}$$

Proof of Lemma 3.3. Let

$$\mu(\tau) = \inf \left\{ \sum_{i,j=1}^n S_k^{ij}(r(w)) \xi_i \xi_j, \quad \forall x \in B_1(0), |\xi| = 1, \|w\|_{C^{2,\alpha}(B_1(0))} \leq A \right\}.$$

By Lemma 3.2, $\mu(\tau) > 0$. Applying Theorem 3.7 in [3] to the solution $u \in C^0(\overline{B_1(0)}) \cap C^2(B_1(0))$ of

$$\begin{cases} L_G(w)u = \sum_{i,j=1}^n \frac{\partial S_k(r(w))}{\partial r_{ij}} \partial_i \partial_j u + \sum_{i=1}^n a_i \partial_i u = g, & \text{in } B_1(0), \\ u = 0, & \text{on } \partial B_1(0), \end{cases}$$

we have

$$(3.10) \quad \sup |u| \leq \frac{C}{\mu(\tau)} \|g\|_{C^0(\overline{B_1(0)})},$$

where $C = e^{2(\beta+1)} - 1$ and $\beta = \sup \left\{ \frac{|a_i|}{\mu(\tau)} : i = 1, 2, \dots, n \right\}$.

Let $C_1 = 1 - C \sup \frac{|a_i|}{\mu(\tau)}$ with C being the constant in (3.10). If we choose $\varepsilon_0 > 0$ small (since $a = O(\varepsilon^4)$ is small), then $C_1 > \frac{1}{2}$ is independent of $0 < \varepsilon < \varepsilon_0$. By applying Corollary 3.8 in [3] to the solution ρ to Dirichlet problem (3.9), we have

$$(3.11) \quad \sup |\rho| \leq \frac{1}{C_1} \left[\sup_{\partial B_1(0)} |\rho| + \frac{C}{\mu(\tau)} \|g\|_{C^0(\overline{B_1(0)})} \right] = \frac{C}{C_1 \mu(\tau)} \|g\|_{C^0(\overline{B_1(0)})}.$$

From (3.11) we see that the homogeneous problem

$$\begin{cases} L_G(w)\rho = \sum_{i,j=1}^n \frac{\partial S_k(r(w))}{\partial r_{ij}} \partial_i \partial_j \rho + \sum_{i=1}^n a_i \partial_i \rho + a\rho = 0, & \text{in } B_1(0), \\ \rho = 0 & \text{on } \partial B_1(0) \end{cases}$$

only possesses the trivial solution. Then we can apply a Fredholm alternative, Theorem 6.15 in [3], to the inhomogeneous problem (3.9) for which we can assert that it has a unique $C^{2,\alpha}(\overline{B_1(0)})$ solution for all $g \in C^\alpha(\overline{B_1(0)})$.

With the existence and uniqueness at hand, we can apply Theorem 6.19 [3] to obtain higher regularity up to boundary for solution to (3.9). Besides this, we have the Schauder estimates (see Problem 6.2, [3])

$$(3.12) \quad \|\rho\|_{C^{2,\alpha}} \leq C(A, \tau, \|f\|_{C^{1,1}}) \left[\|\rho_k\|_{C^0(\overline{B_1(0)})} + \|g_k\|_{C^\alpha(\overline{B_1(0)})} \right],$$

where C depends on C^α -norm of all of the coefficients; the uniform ellipticity; boundary value and boundary itself. Now we explain the dependence of $C(A, \tau, \|f\|_{C^{1,1}})$. Firstly, since

the first two derivatives of w have come into the principal coefficients $\frac{\partial S_2(r(w))}{\partial r_{ij}}$, then their C^α -norms must be involved in $\|w\|_{C^{2,\alpha}}$. That is, $\|w\|_{C^{2,\alpha}} \leq A$ arise into C . Similarly, by the virtue of the coefficients a_i and a , we have that $\|f\|_{C^{1,1}}$ and $\|w\|_{C^{2,\alpha}} \leq A$ must arise into C . Secondly, it depends on the uniform ellipticity, that is,

$$\inf \left\{ \sum_{i,j=1}^n S_k^{ij}(r(w)\xi_i\xi_j), \quad \forall x \in B_1(0), |\xi| = 1, \|w\|_{C^{2,\alpha}(B_1(0))} \leq A \right\}$$

and

$$\sup \left\{ \sum_{i,j=1}^n S_k^{ij}(r(w)\xi_i\xi_j), \quad \forall x \in B_1(0), |\xi| = 1, \|w\|_{C^{2,\alpha}(B_1(0))} \leq A \right\},$$

so $(\tau = (\tau_1, \tau_2, \dots, \tau_n))$ and A arise into C .

Thirdly, Since its boundary value is zero and boundary $\partial B_1(0)$ is C^∞ , the these two ingredients do not occur into C . Substituting (3.11) into (3.12), we obtain (3.8). \square

It follows from the standard elliptic theory (see Theorem 6.17 in [3] and Remark 2 in [1]) and an iteration argument that we obtain.

Corollary 3.4. *Assume that $u \in C^{2,\alpha}(\Omega)$ is a solution of (1.1), and the linearized operators with respect to u ,*

$$\mathcal{L}_u = \sum_{i,j=1}^n \frac{\partial S_k(u_{ij})}{\partial r_{ij}} \partial_{ij}^2 - \sum_{i=1}^n \frac{\partial f}{\partial p_i}(y, u(y), Du(y)) \partial_i - \frac{\partial f}{\partial z}(y, u(y), Du(y)),$$

is uniformly elliptic. If $f \in C^\infty$, then $u \in C^\infty(\Omega)$.

Using Lemma 3.3 above, we can use the procedure (3.6) to construct the sequence $\{w_m\}_{m \in \mathbb{N}}$. Now we study the convergence of $\{w_m\}_{m \in \mathbb{N}}$ and $\{g_m\}_{m \in \mathbb{N}}$.

Proposition 3.5. *Let $\{w_m\}_{m \in \mathbb{N}}$ and $\{g_m\}_{m \in \mathbb{N}}$ be the sequence in (3.6). Suppose that $\|w_j\|_{C^{2,\alpha}} \leq A$ for $j = 1, 2, \dots, l$. Then we have*

$$(3.13) \quad \|g_{l+1}\|_{C^\alpha} \leq C \|g_l\|_{C^\alpha}^2,$$

where C is some positive constant depends only on τ , A and $\|f\|_{C^{2,1}}$. In particular, C is independent of l .

Proof. By applying Taylor expansion with integral-typed remainder to (3.3), we have

$$\begin{aligned} -g_{l+1} &= G(w_l + \rho_l) = G(w_l) + L_G(w_l)\rho_l + Q(w_l, \rho_l) \\ &= -g_l + L_G(w_l)\rho_l + Q(w_l, \rho_l) = Q(w_l, \rho_l), \end{aligned}$$

where Q is the quadratic error of G which consists of S_k and f ,

$$\begin{aligned} (3.14) \quad Q(w_l, \rho_l) &= \sum_{i,j,st} \frac{1}{\varepsilon} \int (1-\mu) \frac{\partial^2 S_k(w_l + \mu\rho_l)}{\partial w_{ij} \partial w_{st}} d\mu(\rho_l)_{ij}(\rho_l)_{st} \\ &\quad - \sum_{i,j} \frac{1}{\varepsilon} \int (1-\mu) \frac{\partial^2 \tilde{f}_\varepsilon(w_l + \mu\rho_l)}{\partial w_i \partial w_j} d\mu(\rho_l)_i(\rho_l)_j \\ &\quad - \frac{1}{\varepsilon} \sum_i \int (1-\mu) \frac{\partial^2 \tilde{f}_\varepsilon(w_l + \mu\rho_l)}{\partial w \partial w_i} d\mu(\rho_l)_i(\rho_l) \\ &\quad - \frac{1}{\varepsilon} \int (1-\mu) \frac{\partial^2 \tilde{f}_\varepsilon(w_l + \mu\rho_l)}{\partial w^2} d\mu \cdot \rho_l^2 \\ &= I_1 + I_2 + I_3 + I_4 \end{aligned}$$

Since $S_k(r(w))$ is a k -order homogeneous polynomial with variable $r_{ij}(r(w))$ and $\tilde{f}_\varepsilon(x, w, Dw)$ is independent of r_{ij} , we see that

$$\begin{aligned} \left| \frac{\partial^2 S_k(w_l + \mu\rho_l)}{\partial w_{ij} \partial w_{st}} \right| &= \varepsilon'^2 \left| \frac{\partial^2 S_k}{\partial w_{ij} \partial w_{st}} (\delta_i^j \tau_i + \varepsilon' (w_l + \mu\rho_l)_{ij}) \right| \\ &= \varepsilon'^2 \sum_{j=2}^k C(j, \tau) [\partial^2 (w_l + \mu\rho_l)]^{j-2} \end{aligned}$$

$$\begin{aligned} \left| \frac{\partial^2 \tilde{f}_\varepsilon(w_l + \mu\rho_l)}{\partial w_i \partial w_j} \right| &= \left| \frac{\partial^2 [f(\varepsilon x, \varepsilon^4 \psi + \varepsilon' \varepsilon^4 (w_l + \mu\rho_l), \varepsilon^2 D\psi + \varepsilon' \varepsilon^2 D(w_l + \mu\rho_l))]}{\partial w_i \partial w_j} \right| \\ &\leq \varepsilon'^2 \varepsilon^4 \cdot \|f\|_{C^{1,1}}, \\ \left| \frac{\partial^2 \tilde{f}_\varepsilon(w_l + \mu\rho_l)}{\partial w \partial w_i} \right| &= \left| \frac{\partial^2 [f(\varepsilon x, \varepsilon^4 \psi + \varepsilon' \varepsilon^4 (w_l + \mu\rho_l), \varepsilon^2 D\psi + \varepsilon' \varepsilon^2 D(w_l + \mu\rho_l))]}{\partial w \partial w_i} \right| \\ &\leq \varepsilon'^2 \varepsilon^6 \|f\|_{C^{1,1}}, \\ \left| \frac{\partial^2 \tilde{f}_\varepsilon(w_l + \mu\rho_l)}{\partial w^2} \right| &= \left| \frac{\partial^2 [f(\varepsilon x, \varepsilon^4 \psi + \varepsilon' \varepsilon^4 (w_l + \mu\rho_l), \varepsilon^2 D\psi + \varepsilon' \varepsilon^2 D(w_l + \mu\rho_l))]}{\partial w^2} \right| \\ &= \varepsilon'^2 \varepsilon^8 \|f\|_{C^{1,1}}. \end{aligned}$$

Thus, $I_i (1 \leq i \leq 4)$ in Q are under control by $O(\varepsilon')$, $O(\varepsilon' \varepsilon^4)$, $O(\varepsilon' \varepsilon^6)$ and $O(\varepsilon' \varepsilon^8)$ respectively. Therefore

$$\|I_1\|_{C^\alpha} \leq C \sum_{j=1}^{k-1} \|\rho_l\|_{C^2}^j \|\rho_l\|_{C^{2,\alpha}}$$

and

$$\begin{aligned} \|I_2\|_{C^\alpha} &\leq C \|f\|_{C^\alpha} (\|w_l\|_{C^{1,\alpha}} + \|\rho_l\|_{C^{1,\alpha}}) \|\rho_l\|_{C^1}^2 + C \|f\|_{C^{1,1}} \|\rho_l\|_{C^{2,\alpha}} \|\rho_l\|_{C^1} \\ &\leq C \|\rho_l\|_{C^{2,\alpha}} \|\rho_l\|_{C^1}^2 + C \|\rho_l\|_{C^1}^2 + C \|\rho_l\|_{C^{2,\alpha}} \|\rho_l\|_{C^1} \end{aligned}$$

holds, where C depends on A and $\|f\|_{C^{2,\alpha}}$. And $\|I_3\|_{C^\alpha}$ and $\|I_4\|_{C^\alpha}$ can be estimated similarly. Accordingly,

$$\begin{aligned} \|g_{l+1}\|_{C^\alpha} &= \|Q(w_l, \rho_l)\|_{C^\alpha} \leq \sum_{i=1}^4 \|I_i\|_{C^\alpha} \\ &\leq C \sum_{j=1}^{k-1} \|\rho_l\|_{C^2}^j \|\rho_l\|_{C^{2,\alpha}} + C \|\rho_l\|_{C^{2,\alpha}} \|\rho_l\|_{C^1}^2 + \|\rho_l\|_{C^1}^2 + C \|\rho_l\|_{C^{2,\alpha}} \|\rho_l\|_{C^1}, \end{aligned}$$

where C is independent of l but dependent of A and $\|f\|_{C^{2,1}}$. Thus, by the interpolation inequalities, we have

$$\|g_{l+1}\|_{C^\alpha} \leq C \sum_{j=1}^{k-1} \|\rho_l\|_{C^{2,\alpha}}^{j+1} + C \|\rho_l\|_{C^{2,\alpha}}^3,$$

where C is independent of l . By the Schauder estimates of Lemma 3.3, we have

$$\|\rho_l\|_{C^{2,\alpha}} \leq C \|g_l\|_{C^\alpha}.$$

Recall that $\|g_l\|_{C^\alpha} = \|G(w_m)\|_{C^\alpha} \leq C(A)$ holds provided $\|w_j\|_{C^{2,\alpha}} \leq A$ for $j = 1, 2, \dots, l$. Combining the two estimates above, we obtain (3.13).

For the special case $f = f(y) = f(\varepsilon^2 x)$, then $I_2 = I_3 = I_4 = 0$ in (3.14). So the condition $f = f(y) \in C^\alpha$ is enough for the estimate (3.16)-(3.17) below. Proof is done. \square

Since C is independent of l , more exactly, A , τ and $\|f\|_{C^{2,1}}$ are independent of l . So hereafter, we can assume $A = 1$.

Proof of Theorem 3.1. Set

$$(3.15) \quad d_{l+1} = C\|g_{l+1}\|_{C^{2,\alpha}}, \quad l = 0, 1, 2, \dots$$

By (3.13) and setting $C \geq 1$ we have

$$d_{l+1} \leq d_l^2.$$

Take $\tau \in \mathbf{P}_2$ such that $\sigma_k(\tau) = f(0, 0, 0)$, we have

$$\begin{aligned} g_0(x) &= -G(0) = \frac{1}{\varepsilon'} [S_k(r(0)) - \tilde{f}(x, 0, 0)] \\ &= \frac{1}{\varepsilon'} [\sigma_k(\tau) - f(\varepsilon^2 x, \varepsilon^4 \psi(x), \varepsilon^2(\tau_1 x_1, \dots, \tau_n x_n))] \\ &= \frac{1}{\varepsilon'} [(\sigma_k(\tau) - f(0, 0, 0)) + \frac{1}{\varepsilon} [f(0, 0, 0) - f(\varepsilon^2 x, 0, 0)] \\ &\quad + \frac{1}{\varepsilon'} [f(\varepsilon^2 x, 0, 0) - f(\varepsilon^2 x, \varepsilon^4 \psi(x), \varepsilon^2(\tau_1 x_1, \dots, \tau_n x_n))] \\ &= \frac{1}{\varepsilon'} [f(0, 0, 0) - f(\varepsilon^2 x, 0, 0)] \\ &\quad - \frac{\varepsilon^4}{\varepsilon'} \int_0^1 \psi(x) (\partial_z f)(\varepsilon^2 x, t\varepsilon^4 \psi(x), t\varepsilon^2(\tau_1 x_1, \dots, \tau_n x_n)) dt \\ &\quad - \frac{\varepsilon^2}{\varepsilon'} \int_0^1 (\tau_1 x_1, \dots, \tau_n x_n) \cdot (\partial_p f)(\varepsilon^2 x, t\varepsilon^4 \psi(x), t\varepsilon^2(\tau_1 x_1, \dots, \tau_n x_n)) dt, \end{aligned}$$

where $\sigma_k(\tau) - f(0, 0, 0) = 0$ is used. Noticing

$$\frac{1}{\varepsilon'} \|f(0, 0, 0) - f(\varepsilon^2 x, 0, 0)\|_{C^\alpha(B_1(0))} \leq C \frac{\varepsilon^{2\alpha}}{\varepsilon'} \|f(\cdot, 0, 0)\|_{C^\alpha(B_{\varepsilon^2}(0))},$$

we obtain

$$(3.16) \quad \|g_0\|_{C^\alpha(B_1(0))} \leq C_1 \frac{\varepsilon^{2\alpha}}{\varepsilon'} \|f\|_{C^{1,1}}.$$

Using the definition of ε' in (1.6), we can choose $0 < \varepsilon \leq \varepsilon_0$ so small that

$$(3.17) \quad C\|g_0\|_{C^\alpha(B_1(0))} \leq \frac{1}{4}, \quad 0 < \varepsilon \leq \varepsilon_0.$$

Notice ε_0 is independent of l . Since $d_0 = C\|g_0\|_{C^\alpha}$, we have $d_1 \leq d_0^2$. Then, by an induction,

$$d_{l+1} \leq 2^{2^{l+1}} d_0^{2^{l+1}} \leq (2C)^{2^{l+1}} \|g_0\|_{C^\alpha}^{2^{l+1}}.$$

Thus, by (3.15) and (3.17)

$$(3.18) \quad \|g_{l+1}\|_{C^\alpha} \leq (2C)^{2^{l+1}-1} \|g_0\|_{C^\alpha}^{2^{l+1}} \leq \left(\frac{1}{2}\right)^{2^l} \rightarrow 0, .$$

Firstly, we claim that there exists a constant $\varepsilon_0 > 0$, depending on τ and $\|f\|_{C^{2,1}}$ such that, uniformly for $\varepsilon \in (0, \varepsilon_0]$,

$$\|w_l\|_{C^{2,\alpha}(B_1(0))} \leq 1, \quad \forall l \geq 1.$$

Indeed, set $w_0 = 0$, we have by (3.13)

$$\begin{aligned} \|w_{l+1}\|_{C^{2,\alpha}(B_1(0))} &= \left\| \sum_{i=0}^l \rho_i \right\|_{C^{2,\alpha}(B_1(0))} \leq \sum_{i=0}^l \|\rho_i\|_{C^{2,\alpha}(B_1(0))} \\ &\leq \sum_{i=0}^l C \|g_i\|_{C^\alpha(B_1(0))} \leq \sum_{i=0}^l \left(C \|g_0\|_{C^\alpha(B_1(0))} \right)^{2^i} \end{aligned}$$

where C is defined in Lemma 3.5. Thus, for any l ,

$$(3.19) \quad \|w_{l+1}\|_{C^{2,\alpha}(B_1(0))} \leq \sum_{i=0}^{\infty} \left(C \|g_0\|_{C^\alpha(B_1(0))} \right)^{2^i} \leq \sum_{i=0}^{\infty} 2^{-2^i} \leq 1.$$

Then, by Azelà-Ascoli Theorem, there is a subsequence of w_l , still denoted by w_l , such that

$$w_l \rightarrow w \quad \text{in } C^2(B_1(0)),$$

and $w \in C^{2,\alpha}(B_1(0))$. From (3.18) and $g_m = -G(w_m)$, we have

$$G(w) = \frac{1}{\varepsilon'} [S_k(r(w)) - \tilde{f}(x, w, Dw)] = 0, \quad \text{on } B_1(0).$$

That means to say the function

$$u(y) = \frac{1}{2} \sum_{i=1}^n \tau_i y_i^2 + \varepsilon' \varepsilon^4 w(\varepsilon^{-2}y) \in C^{2,\alpha}(B_{\varepsilon^2}(0)),$$

is a solution of

$$S_k[u] = f(y, u, Du), \quad \text{on } B_{\varepsilon^2}(0).$$

Now if $f(0, 0, 0) = 0$, we take $\tau \in \mathbf{P}_2$, then $\sigma_{k-1}(\tau) > 0, \sigma_k(\tau) = 0, \sigma_{k+1}(\tau) < 0$. Noticing that symmetric matrix $r(w) = (\delta_i^j \tau_i + \varepsilon' w_{ij}(x))$, we have

$$S_j[u] = \sigma_j(\lambda) = \sigma_j(\tau) + O(\varepsilon'), \quad j = 1, 2, \dots, k+1.$$

it follows that $S_j[u] > 0 (1 \leq j \leq k-1), S_{k+1}[u] < 0$ on $B_{\varepsilon^2}(0)$ for small $\varepsilon > 0$. That is, u is $(k-1)$ -convex but not $(k+1)$ -convex. Moreover if $S_k[u] = f \geq 0$ near Z_0 and $f(Z_0) = 0$, we see that u is k -convex by definition, but not $(k+1)$ -convex.

If $S_k[u] = f > 0$ near Z_0 , we use Theorem 2.8 to take $\tau \in \mathbb{R}^n$ given in $\Gamma_{k-l+1}(n) \setminus \bar{\Gamma}_{k+l}(n)$ for $1 \leq l \leq n-k$, then we can get the $(k+l-1)$ -convex but not $(k+l)$ -convex local solutions. From (3.19) and condition $Z_0 = (0, 0, 0)$, we obtain (3.2).

The C^∞ regularity of solution is given by Corollary 3.4. Thus, we have proved Theorem 3.1. \square

We also have the following elliptic results for $f(Z_0) < 0$ which is (3) in the Theorem 1.1.

Theorem 3.6. *For $2 \leq k \leq n-1$, let $f = f(y, u, p)$ be defined and continuous near a point $Z_0 = (0, 0, 0) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ and $0 < \alpha < 1$. Assume that f is C^α with respect to y and $C^{2,1}$ with respect to u, p . If $f(Z_0) < 0$, then (1.1) admits a $(k-1)$ -convex local solution $C^{2,\alpha}$ near $y_0 = 0$, which is not k -convex and of the following form*

$$u(y) = \frac{1}{2} \sum_{i=1}^n \tau_i y_i^2 + \varepsilon' \varepsilon^5 w(\varepsilon^{-2}y)$$

with $\sigma_k(\tau_1, \dots, \tau_n) = f(0, 0, 0)$, $\varepsilon > 0$ very small, ε' is defined in (1.6) and w satisfies (3.2). Moreover, the equation (1.1) is uniformly elliptic with respect to the solution above. If $f \in C^\infty$ near Z_0 , then $u \in C^\infty$ near y_0 .

Proof. For $f(0, 0, 0) < 0$, take $\tau \in \mathbb{R}^n$ as in Theorem 2.7 with $c = f(0, 0, 0) < 0$ such that

$$\sigma_{k-1}(\tau) > 0, \quad \sigma_k(\tau) = f(0, 0, 0) < 0,$$

and

$$\sigma_{k-1;n}(\tau) \geq \sigma_{k-1;n-1}(\tau) \geq \dots \sigma_{k-1,1}(\tau) > 0.$$

Now the proof is exactly the same as that of Theorem 3.1. \square

4. APPENDIX

The following results are essential in the proof of our main theorem, maybe it is classical, but we can't find a simple proof, so present here as an appendix.

4.1. The Equivalence of Three Definitions for Gårding Cone. The Gårding cone is originated from the Gårding theory of hyperbolic polynomials [2]. Each hyperbolic polynomial is essentially real. A homogeneous polynomial of k -order on a n -dimensional real vector space V is hyperbolic with respect to a direction $a \in V$ if the equation $P(sa + \lambda) = 0$ with one-variable s has k real zeros for every real $\lambda \in V$. Using a convenient solecism, we say $P(\lambda)$ is an a -hyperbolic polynomial. For an a -hyperbolic polynomial $P(\lambda)$ and $m > 1$, then by Rolle's theorem, its directional derivative in the direction a ,

$$(4.1) \quad (\nabla P(\lambda), a) = \frac{d}{ds} P(sa + \lambda) |_{s=0}$$

is also an a -hyperbolic polynomial, see Lemma 1 in [2]. For an a -hyperbolic polynomial $P(\lambda)$ with $P(a) > 0$, its Gårding cone is defined by

$$(4.2) \quad \mathcal{C}(a, P, n) = \{\lambda \in \mathbb{R}^n : P(sa + \lambda) > 0, \forall s \geq 0\}.$$

Gårding [2] has proved that $\mathcal{C}(a, P, n) = \mathcal{C}(b, P, n)$ is an open convex cone, for any $b \in \mathcal{C}(a, P, n)$ with vertex at the origin, and

$$(4.3) \quad (\nabla P(\lambda), \mu) \geq k P^{\frac{k-1}{k}}(\lambda) P^{\frac{1}{k}}(\mu), \quad \forall \lambda, \mu \in \mathcal{C}(a, P, n),$$

see (11) in [2], which is the simplest version of the general Gårding inequality, Theorem 5 in [2]. We will apply the results above to the k -th elementary symmetric polynomial

$$\sigma_k(\lambda) = \sum_{1 \leq i_1 < \dots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k},$$

which is hyperbolic with respect to

$$e = (1, 1, \dots, 1) \in \mathbb{R}^n.$$

Gårding cone is equivalently defined in the form which is easier to be verified than (4.2)

$$(4.4) \quad \Gamma_k(n) = \{\lambda \in \mathbb{R}^n | \sigma_j(\lambda) > 0, \forall j = 1, 2, \dots, k\},$$

From this definition, it follows that

$$\bar{\Gamma}_n(n) \subset \dots \subset \bar{\Gamma}_k(n) \subset \dots \subset \bar{\Gamma}_1(n).$$

Notice that Maclaurin's inequalities

$$(4.5) \quad \left[\frac{1}{\binom{n}{k}} \sigma_k(\lambda) \right]^{\frac{1}{k}} \leq \left[\frac{1}{\binom{n}{l}} \sigma_l(\lambda) \right]^{\frac{1}{l}}$$

hold for $1 \leq l \leq k$, $\lambda \in \Gamma_k(n)$; (see Lemma 15.12 in [10]). Denoting

$$\lambda + \varepsilon = (\lambda_1 + \varepsilon, \lambda_2 + \varepsilon, \dots, \lambda_n + \varepsilon)$$

and applying the formula

$$\sigma_k(\lambda + \varepsilon) = \sum_{j=0}^k C(j, k, n) \varepsilon^j \sigma_{k-j}(\lambda), \quad C(j, k, n) = \frac{\binom{n}{k} \binom{k}{j}}{\binom{n}{k-j}}, \lambda \in \mathbb{R}^n, \varepsilon \in \mathbb{R}.$$

to $\lambda + \varepsilon$; (see Section 5, [8]), we obtain

$$(4.6) \quad \lambda + \varepsilon \in \Gamma_k(n), \quad \forall \varepsilon > 0, \lambda \in \overline{\Gamma_k(n)}.$$

For any fixed t -tuple $\{i_1, i_2, \dots, i_t\} \subset \{1, 2, \dots, n\}$, we define

$$(4.7) \quad \sigma_{k; i_1, i_2, \dots, i_t}(\lambda) = \frac{\partial^t \sigma_{k+t}(\lambda)}{\partial \lambda_{i_1} \cdots \partial \lambda_{i_t}}.$$

and then for any $i \in \{1, 2, \dots, n\}$,

$$(4.8) \quad \sigma_k(\lambda) = \lambda_i \sigma_{k-1; i}(\lambda) + \sigma_{k; i}(\lambda), \quad \forall \lambda \in \mathbb{R}^n.$$

Moreover, one can verify by (4.8) that

$$\sigma_{k; i_1, i_2, \dots, i_t}(\lambda) = \sigma_k(\lambda)|_{\lambda_{i_1} = \dots = \lambda_{i_t} = 0}.$$

Ivochkina [8] introduced a cone defined by

$$(4.9) \quad \widetilde{\Gamma}_k(n) = \{\lambda \in \mathbb{R}^n \mid \sigma_{k-l; i_1, \dots, i_l} > 0, l = 0, 1, \dots, k, \}$$

and has proved that, by her own notion of "stability set of k -Hessian operator", the cones $\Gamma_k(n)$ and $\widetilde{\Gamma}_k(n)$ coincide. We will take the original definition (4.2) as the starting point to prove that the three definitions above are equivalent.

Theorem 4.1. *For the k -th elementary symmetric polynomial $\sigma_k(\lambda)$ with $e = (1, 1, \dots, 1)$, the definitions (4.2), (4.4) and (4.9) are equivalent, that is,*

$$\mathcal{C}(e, \sigma_k, n) = \Gamma_k(n) = \widetilde{\Gamma}_k(n).$$

Proof. Step 1. We will prove $\mathcal{C}(e, \sigma_k, n) = \Gamma_k(n)$. We first prove $\mathcal{C}(e, \sigma_k, n) \subset \Gamma_k(n)$. By definition of hyperbolic polynomial,

$$\sigma_n(\lambda) = \lambda_1 \lambda_2 \cdots \lambda_n$$

is hyperbolic with respect to $e = (1, 1, \dots, 1)$. Noticing

$$\sigma_n(se + \lambda) = \sum_{j=0}^n s^{n-j} \sigma_j(\lambda),$$

by the convention $\sigma_0(\lambda) = 1$, and

$$\frac{d^{n-j}}{s^{n-j}} \sigma_n(se + \lambda) \Big|_{s=0} = C(n, j) \sigma_j(\lambda).$$

By using (4.1) and Rolle's Theorem again, we see that $\sigma_k(\lambda)$, $1 \leq k \leq n$ are hyperbolic with respect to $e = (1, 1, \dots, 1)$. Accordingly, by definition (4.2) of $\mathcal{C}(e, \sigma_j, n)$, ($1 \leq j \leq k$) and letting $s = 0$, we obtain

$$\sigma_j(\lambda) > 0, \quad 1 \leq j \leq k.$$

This completes the proof $\mathcal{C}(e, \sigma_k, n) \subset \Gamma_k(n)$. Conversely, if $\lambda \in \Gamma_k(n)$, since

$$\sigma_k(se + \lambda) = \sum_{j=0}^k C(j, k, n) s^j \sigma_{k-j}(\lambda), \quad C(j, k, n) = \frac{\binom{n}{k} \binom{k}{j}}{\binom{n}{k-j}}$$

and by definition of $\Gamma_k(n)$,

$$\sigma_{k-j}(\lambda) > 0, \quad 0 \leq j \leq k,$$

we see that $\sigma_k(se + \lambda) > 0$ for all $s \geq 0$ and therefore $\lambda \in \mathcal{C}(e, \sigma_k, n)$. This completes the proof of $\Gamma_k(n) \subset \mathcal{C}(e, \sigma_k, n)$.

Step 2. We will prove that $\widetilde{\Gamma}_k(n) = \Gamma_k(n)$. Obviously $\widetilde{\Gamma}_k(n) \subset \Gamma_k(n)$. It is left to prove $\mathcal{C}(e, \sigma_k, n) \subset \widetilde{\Gamma}_k(n)$. Let $\lambda \in \mathcal{C}(e, \sigma_k, n)$, we use (4.3) with $P(\lambda) = \sigma_j(\lambda) (1 \leq j \leq k)$ and $\mu = (1, 0, \dots, 0) = e_1$ and then obtain

$$(4.10) \quad \frac{\partial \sigma_j}{\partial \lambda_1}(\lambda) = (\nabla \sigma_j, e_1) \geq 0, \quad 1 \leq j \leq k.$$

We claim that

$$(4.11) \quad \frac{\partial \sigma_k}{\partial \lambda_1}(\lambda) > 0, \quad \forall \lambda \in \mathcal{C}(e, \sigma_k, n).$$

Otherwise, we assume that $\frac{\partial \sigma_k}{\partial \lambda_1}(\lambda^0) = 0$ for some $\lambda^0 \in \mathcal{C}(e, \sigma_k, n)$. By (4.8),

$$\sigma_k(\lambda^0) = \lambda_1^0 \sigma_{k-1;1}(\lambda^0) + \sigma_{k;1}(\lambda^0) = \lambda_1^0 \frac{\partial \sigma_k}{\partial \lambda_1}(\lambda^0) + \sigma_{k;1}(\lambda^0) = \sigma_{k;1}(\lambda^0),$$

from which we obtain, by (4.4),

$$(4.12) \quad \sigma_{k-1;1}(\lambda^0) = \sigma_{k-1}(\lambda_2^0, \lambda_3^0, \dots, \lambda_n^0) = 0, \quad 0 < \sigma_k(\lambda^0) = \sigma_{k;1}(\lambda^0) = \sigma_k(\lambda_2^0, \lambda_3^0, \dots, \lambda_n^0).$$

Moreover, by virtue of (4.10), we obtain

$$0 \leq \frac{\partial \sigma_j}{\partial \lambda_1}(\lambda^0) = \sigma_{j-1}(\lambda_2^0, \lambda_3^0, \dots, \lambda_n^0), \quad 1 \leq j \leq k.$$

Therefore we have proved that $(\lambda_2^0, \lambda_3^0, \dots, \lambda_n^0) \in \overline{\Gamma}_k(n-1)$. By (4.6), for $\varepsilon > 0$,

$$(\lambda_2^0 + \varepsilon, \lambda_3^0 + \varepsilon, \dots, \lambda_n^0 + \varepsilon) \in \Gamma_k(n-1),$$

to which we apply the Maclaurin's inequalities (4.5) and obtain

$$\left[\frac{1}{\binom{n-1}{k}} \sigma_k(\lambda_2^0 + \varepsilon, \lambda_3^0 + \varepsilon, \dots, \lambda_n^0 + \varepsilon) \right]^{\frac{1}{k}} \leq \left[\frac{1}{\binom{n-1}{l}} \sigma_l(\lambda_2^0 + \varepsilon, \lambda_3^0 + \varepsilon, \dots, \lambda_n^0 + \varepsilon) \right]^{\frac{1}{l}}.$$

Letting $\varepsilon \rightarrow 0^+$, we have

$$\left[\frac{1}{\binom{n-1}{k}} \sigma_k(\lambda_2^0, \lambda_3^0, \dots, \lambda_n^0) \right]^{\frac{1}{k}} \leq \left[\frac{1}{\binom{n-1}{l}} \sigma_l(\lambda_2^0, \lambda_3^0, \dots, \lambda_n^0) \right]^{\frac{1}{l}},$$

which contradicts with (4.12) in case $l = k-1$, thus the claim (4.11) is true. Notice that

$$\frac{\partial \sigma_k}{\partial \lambda_1}(\lambda) = \sigma_{k-1}(\lambda_2, \lambda_3, \dots, \lambda_n),$$

then applying the Maclaurin's inequality to (4.10)-(4.11) leads to

$$(\lambda_2, \lambda_3, \dots, \lambda_n) \in \Gamma_{k-1}(n-1).$$

Now we can regard that $(\lambda_2, \lambda_3, \dots, \lambda_n)$ is in the same position of $(\lambda_1, \lambda_2, \dots, \lambda_n)$ as above, by an induction on k we can prove

$$\sigma_{k-l; i_1, \dots, i_l}(\lambda) > 0, \quad l = 0, 1, \dots, k.$$

That is $\Gamma_k(n) \subset \widetilde{\Gamma}_k(n)$. □

For the Gårding cone in the space of symmetric matrices, similar results to (4.11) can be seen Section 3 in [9].

The following is some by-product of (4.9). Assume that $\lambda \in \Gamma_k(n)$ is in descending order,

$$\lambda_1 \geq \cdots \lambda_{p-1} \geq \lambda_p > 0 \geq \lambda_{p+1} \geq \cdots \lambda_n,$$

then

$$(4.13) \quad \begin{cases} p \geq k \\ 0 < \sigma_{k-1,1}(\lambda) \leq \sigma_{k-1,2}(\lambda) \leq \cdots \leq \sigma_{k-1,n}(\lambda). \end{cases}$$

Otherwise, if $p < k$, we have

$$\sigma_{1;\lambda_1,\lambda_2,\dots,\lambda_{k-1}} = \sigma_1(\lambda_k, \lambda_{k+1}, \dots, \lambda_n) = \sum_{j=k}^n \lambda_j \leq 0,$$

which contradicts with (4.9). Using (4.9) again, we have $\sigma_{k-2;12}(\lambda) \geq 0$ and then

$$\begin{aligned} \frac{\partial \sigma_k}{\partial \lambda_1}(\lambda) &= \sigma_{k-1,1}(\lambda) = \sigma_{k-1,12}(\lambda) + \lambda_2 \sigma_{k-2;12}(\lambda) \\ &\leq \sigma_{k-1,12}(\lambda) + \lambda_1 \sigma_{k-2;12}(\lambda) = \sigma_{k-1,2}(\lambda) = \frac{\partial \sigma_k}{\partial \lambda_2}(\lambda), \end{aligned}$$

the remaining part of (4.13) can be proved similarly. Here the proof of (4.13) is adapted from [13].

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