Pseudo-polynomial functions over finite distributive lattices
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PSEUDO-POLYNOMIAL FUNCTIONS OVER
FINITE DISTRIBUTIVE LATTICES

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Abstract. In this paper we extend the authors previous works by considering a
multi-attribute aggregation model based on a composition of a polynomial func-
tion over a finite distributive lattice with local utility functions; these are referred
to as pseudo-polynomial functions. We present an axiomatization for this class of
pseudo-polynomial functions which differs from the previous ones both in flavour
and nature, and develop general tools which are then used to obtain all possible such
factorizations of a given pseudo-polynomial function.

1. Introduction and motivation

The Sugeno integral (introduced by Sugeno [17, 18]) remains as one of the most
noteworthy aggregation functions, and this is partially due to the fact that it provides a
meaningful way to fuse or merge values within universes where essentially no structure,
other than an order, is assumed. Even though primarily defined over real intervals,
the concept of Sugeno integral can be extended to wider domains, namely, distributive
lattices, via the notion of lattice polynomial function (i.e., a combination of variables
and constants using the lattice operations $\land$ and $\lor$). As it turned out, idempotent lattice
polynomial functions coincide with (discrete) Sugeno integrals (see e.g. [5, 15]).

 Recently, the Sugeno integral has been generalized via the notion of quasi-polynomial
function (see [3]) originally defined as a mapping $f: X^n \rightarrow X$ on a bounded chain $X$
and which can be factorized as

$$f(x_1, \ldots, x_n) = p(\varphi_1(x_1), \ldots, \varphi_n(x_n)),$$

where $p: X^n \rightarrow X$ is a polynomial function and $\varphi: X \rightarrow X$ is an order-preserving map.
This notion was later extended in two ways.

In [4], the input and output universes were allowed to be arbitrary, possibly different,
bounded distributive lattices $X$ and $Y$ so that $f: X^n \rightarrow Y$ is factorizable as in (1), where
now $p: Y^n \rightarrow Y$ and $\varphi: X \rightarrow Y$. These functions appear naturally within the scope
of decision making under uncertainty since they subsume overall preference functionals
associated with Sugeno integrals whose variables are transformed by the utility function $\varphi$.
Several axiomatizations for this function class were proposed, as well as all possible
factorizations described.

In [6] and [7] a different extension was considered, now appearing within the realm of
multicriteria decision making. Essentially, the aggregation model was based on functions
$f: X_1 \times \cdots \times X_n \rightarrow Y$ for bounded chains $X_1, \ldots, X_n$ and $Y$, which can be factorized as
compositions

$$f(x_1, \ldots, x_n) = p(\varphi_1(x_1), \ldots, \varphi_n(x_n)),$$

where $p: Y^n \rightarrow Y$ is a Sugeno integral, and each $\varphi_k: X_k \rightarrow Y$ is an order-preserving map.
Such functions were referred to as Sugeno utility functions in [6]. Pseudo-polynomial
functions were defined as functions of the form (2), where $p$ is an arbitrary (possibly
non-idempotent) lattice polynomial function, and each $\varphi_k$ satisfies a certain boundary
condition (which is weaker than order-preservation). Note that every quasi-polynomial
function (2) can be regarded as a pseudo-polynomial function, where $X_1 = \cdots = X_n = X$
and $\varphi_1 = \cdots = \varphi_n = \varphi$. Moreover, pseudo-polynomial functions naturally subsume Sugeno
utility functions, and several axiomatizations were established for this function class in

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polynomial function, axiomatization, factorization, multicriteria decision making.
was addressed in [7], where a method for producing such a factorization was presented.

In the current paper we extend the previous results by letting $X_1, \ldots, X_n$ to be arbitrary sets and $Y$ to be an arbitrary finite distributive lattice, thus subsuming the frameworks in [3, 5, 7]. Moreover, we develop general tools which allow us to produce all possible factorizations of a given pseudo-polynomial function into compositions of a lattice polynomial function $p: Y^n \to Y$ with maps $\varphi_k: X_k \to Y$.

The structure of the paper is as follows. In Section 2 we introduce the basic notions and terminology needed throughout the paper, and recall some preliminary results. For further background on aggregation functions and their use in decision making, we refer the reader to [2, 13]; for basics in the theory of lattices, see [10, 14]. In Section 3 we develop a general framework used to derive an axiomatization of pseudo-polynomial functions of somewhat different nature than those proposed in [4, 6, 7], and which will provide tools for determining all possible factorizations of given pseudo-polynomial functions in Section 4. These results are then illustrated in Section 5 by means of a concrete example, and in Section 6 we show how this new procedure can be applied to derive the algorithm provided in [7, 8].

2. Preliminaries

Throughout this paper, $Y$ is assumed to be a finite distributive lattice with meet and join operations denoted by $\land$ and $\lor$, respectively. Being finite, $Y$ has a least element and a greatest element, denoted by $0$ and $1$, respectively. By Birkhoff’s Representation Theorem [11], $Y$ can be embedded into $\mathcal{P}(U)$, the power set of a finite set $U$. Identifying $Y$ with its image under this embedding, we will consider $Y$ as being a sublattice of $\mathcal{P}(U)$ with $0 = \emptyset$ and $1 = U$. The complement of a set $S \in \mathcal{P}(U)$ will be denoted by $S$. Since $Y$ is closed under intersections, it induces a closure operator $\text{cl}$ on $U$, and since $Y$ is closed under unions, it also induces a dual closure operator $\text{int}$ (also known as “interior operator”):

$$\text{cl} (S) := \bigwedge_{y \in S} y, \quad \text{int} (S) := \bigvee_{y \in S} y, \quad \text{for all } S \in \mathcal{P}(U).$$

It is easy to verify that these two operators satisfy the following identities for any $S_1, S_2 \in \mathcal{P}(U)$:

$$\text{cl} (S_1 \lor S_2) = \text{cl} (S_1) \lor \text{cl} (S_2), \quad \text{int} (S_1 \land S_2) = \text{int} (S_1) \land \text{int} (S_2).$$

A function $p: Y^n \to Y$ is a polynomial function if it can be obtained as a composition of the lattice operations $\land$ and $\lor$ with variables and constants. Note that polynomial functions are thus order-preserving. As observed in [15], (discrete) Sugeno integrals coincide exactly with those lattice polynomial functions $p$ which are idempotent, i.e., satisfy the identity $p(y, \ldots, y) = y$. An important lattice polynomial function (in fact, a Sugeno integral) is the median function $\text{med}: Y^3 \to Y$ defined by

$$\text{med} (y_1, y_2, y_3) = (y_1 \land y_2) \lor (y_2 \land y_3) \lor (y_3 \land y_1) = (y_1 \lor y_2) \land (y_2 \lor y_3) \land (y_3 \lor y_1).$$

It is useful to observe that the above expressions for the median can be simplified when two of the arguments are comparable:

$$\text{med} (s, y, t) = s \lor (t \land y) \quad \text{whenever } s \leq t.$$  

Polynomial functions over bounded distributive lattices have very neat representations, for instance, in disjunctive normal form [12], i.e., representations by expressions of the form

$$\bigvee_{I \subseteq [n]} (a_I \land \bigwedge_{i \in I} y_i),$$

where $\bigwedge_{i \in I} y_i = 1$ when $I = \emptyset$. To provide the “canonical” expression in this disjunctive normal form, let us define $1_I$ to be the characteristic vector of $I \subseteq [n] := \{1, \ldots, n\}$, i.e., the $n$-tuple in $Y^n$ whose $i$-th component is $1$ if $i \in I$, and $0$ otherwise.
Remark 3. The function given by (4) is a Sugeno integral if and only if

\[ p(y_1, \ldots, y_n) = \bigvee_{I \subseteq [n]} (p(\mathbf{1}_I) \land \bigwedge_{i \in I} y_i). \]

Remark 2. Equation (4) is referred to as the canonical disjunctive normal form representation of \( p \).

Remark 4. Let us note that in the case \( n = 1 \), Goodstein’s theorem shows that unary polynomial functions \( p \) are exactly the functions of the form \( p(y) = s \lor (t \land y) \) with \( s = p(0) \leq p(1) = t \), and these can be written as \( p(y) = \text{med} (s, y, t) \) according to (3).

Let \( X_1, \ldots, X_n \) be arbitrary sets with at least two elements, and for each \( k \in [n] \) let us fix two distinct elements \( 0_{X_k}, 1_{X_k} \) of \( X_k \). We shall say that a mapping \( \varphi_k: X_k \rightarrow Y \) satisfies the boundary condition (for \( 0_{X_k} \) and \( 1_{X_k} \)) if for every \( x_k \in X_k \),

\[ \varphi_k(0_{X_k}) \leq \varphi_k(x_k) \leq \varphi_k(1_{X_k}). \]

Observe that if \( X_k \) is a partially ordered set with least element \( 0_{X_k} \) and greatest element \( 1_{X_k} \), and if \( \varphi_k \) is order-preserving, then it satisfies the boundary condition (cf. also Remark 22). With no danger of ambiguity, we simply write \( 0 \) and \( 1 \) instead of \( 0_{X_k} \) and \( 1_{X_k} \) in the sequel.

A function \( f: \prod_{i \in [n]} X_i \rightarrow Y \) is said to be a pseudo-polynomial function, if there is a polynomial function \( p: Y^n \rightarrow Y \) and there are unary functions \( \varphi_k: X_k \rightarrow Y \) \( (k \in [n]) \), satisfying the boundary condition, such that

\[ f(x) = p(\varphi(x)) = p(\varphi_1(x_1), \ldots, \varphi_n(x_n)) \]

holds for all \( x = (x_1, \ldots, x_n) \in \prod_{i \in [n]} X_i \). If \( p \) is a Sugeno integral, then we say that \( f \) is a pseudo-Sugeno integral. As it turns out, the notions of pseudo-polynomial function and pseudo-Sugeno integral are equivalent. This result was proved in [3] for chains \( Y \), but the proof given there actually just uses the fact that \( Y \) is a distributive lattice and that polynomial functions are “range homogeneous”, hence it applies verbatim to our setting.

Proposition 5. A function \( f: \prod_{i \in [n]} X_i \rightarrow Y \) is a pseudo-polynomial function if and only if it is a pseudo-Sugeno integral.

Clearly, if \( f \) is a pseudo-polynomial function, then it satisfies the following \( n \)-variable analogue of the boundary condition (5):

\[ f(x_k^a) \leq f(x) \leq f(x_k^b) \text{ for all } k \in [n], x \in \prod_{i \in [n]} X_i, \]

where \( x_k^a \in \prod_{i \in [n]} X_i \) denotes the \( n \)-tuple which coincides with \( x \) in all but the \( k \)-th component, whose value is \( a \).

Remark 6. Note that the particular orderings \( \varphi_k(0_{X_k}) \leq \varphi_k(1_{X_k}) \) and \( f(x_k^a) \leq f(x_k^b) \) in (5) and (7) could be reversed as the choice of \( 0_{X_k} \) and \( 1_{X_k} \) is arbitrary. Hence, the current notion of boundary condition is not more restrictive than the one used in (3).

Next we define a property that can be used to characterize pseudo-polynomial functions. We say that \( f: \prod_{i \in [n]} X_i \rightarrow Y \) is pseudo-median decomposable if for each \( k \in [n] \) there is a unary function \( \varphi_k: X_k \rightarrow Y \) satisfying (3), such that

\[ f(x) = \text{med} (f(x_k^0), \varphi_k(x_k), f(x_k^1)) \]

for every \( x \in \prod_{i \in [n]} X_i \). Note that if \( f \) is pseudo-median decomposable w.r.t. unary functions \( \varphi_k: X_k \rightarrow Y \) \( (k \in [n]) \) satisfying (4), then (5) holds.

The following theorem shows that every pseudo-median decomposable function is a pseudo-polynomial function, and provides a disjunctive normal form of a polynomial function \( p_f \) which can be used to factorize \( f \). This theorem appears in [7] [8] for the
special case of chains. We use the notation \( \hat{1}_I \) for the characteristic vector of \( I \subseteq [n] \) in \( \prod_{i \in [n]} X_i \), i.e., \( \hat{1}_I \in \prod_{i \in [n]} X_i \) is the \( n \)-tuple whose \( i \)-th component is \( 1_{X_i} \) if \( i \in I \), and \( 0_{X_i} \) otherwise.

**Theorem 7.** If \( f: \prod_{i \in [n]} X_i \rightarrow Y \) is pseudo-median decomposable w.r.t. unary functions \( \varphi_k : X_k \rightarrow Y (k \in [n]) \), then \( f(x) = p_f(\varphi(x)) \), where the polynomial function \( p_f \) is given by

\[
p_f(y_1, \ldots, y_n) = \bigvee_{I \subseteq [n]} (f(\hat{1}_I) \land \bigwedge_{i \in I} y_i).
\]

**Proof.** We need to prove that the following identity holds:

\[
f(x_1, \ldots, x_n) = \bigvee_{I \subseteq [n]} (f(\hat{1}_I) \land \bigwedge_{i \in I} \varphi_i(x_i)).
\]

We proceed by induction on \( n \). If \( n = 1 \), then the right hand side of (10) takes the form \( f(0) \lor (f(1) \land \varphi_1(x_1)) \). From (7) it follows that \( f(0) \leq f(1) \), and then, using (3), we can rewrite \( f(0) \lor (f(1) \land \varphi_1(x_1)) \) as \( \text{med}(f(0), \varphi_1(x_1), f(1)) \), which equals \( f(x_1) \) by (8).

Now suppose that the statement of the theorem is true for all pseudo-median decomposable functions in \( n-1 \) variables. Let \( f_0 \) and \( f_1 \) be the \((n-1)\)-ary functions defined by

\[
\begin{align*}
f_0(x_1, \ldots, x_{n-1}) &= f(x_1, \ldots, x_{n-1}, 0), \\
f_1(x_1, \ldots, x_{n-1}) &= f(x_1, \ldots, x_{n-1}, 1).
\end{align*}
\]

Observe that (7) implies \( f_0 \leq f_1 \). Applying the pseudo-median decomposition to \( f \) with \( k = n \) and rewriting the median using (3), we obtain

\[
f(x_1, \ldots, x_n) = \text{med}(f_0(x_1, \ldots, x_{n-1}), \varphi_n(x_n), f_1(x_1, \ldots, x_{n-1}))
\]

\[
= f_0(x_1, \ldots, x_{n-1}) \lor (f_1(x_1, \ldots, x_{n-1}) \land \varphi_n(x_n)).
\]

It is easy to verify that \( f_0 \) and \( f_1 \) are pseudo-median decomposable w.r.t. \( \varphi_1, \ldots, \varphi_{n-1} \), therefore we can apply the induction hypothesis to these functions:

\[
\begin{align*}
f_0(x_1, \ldots, x_{n-1}) &= \bigvee_{I \subseteq [n-1]} (f_0(\hat{1}_I) \land \bigwedge_{i \in I} \varphi_i(x_i)) = \bigvee_{I \subseteq [n-1]} (f(\hat{1}_I) \land \bigwedge_{i \in I} \varphi_i(x_i)), \\
f_1(x_1, \ldots, x_{n-1}) &= \bigvee_{I \subseteq [n-1]} (f_1(\hat{1}_I) \land \bigwedge_{i \in I} \varphi_i(x_i)) = \bigvee_{I \subseteq [n-1]} (f(\hat{1}_I) \land \bigwedge_{i \in I} \varphi_i(x_i)).
\end{align*}
\]

Substituting back into (11) and using distributivity we obtain the desired equality (10). \( \square \)

Now we can prove that pseudo-median decomposability actually characterizes pseudo-polynomial functions (see [3,8] for the case of chains, where the proof is slightly simpler).

**Theorem 8.** Let \( f: \prod_{i \in [n]} X_i \rightarrow Y \) be a function. Then \( f \) is a pseudo-polynomial function if and only if \( f \) is pseudo-median decomposable.

**Proof.** Sufficiency follows from Theorem 7 so we only need to show that if \( f \) is a pseudo-polynomial function, then it is pseudo-median decomposable. Suppose that \( f(x) = p(\varphi(x)) \) as in (8), and let \( k \in [n] \). We have to prove that (8) holds for all \( x \in \prod_{i \in [n]} X_i \). Regarding \( x_i \) as a fixed element of \( X_i \) for each \( i \neq k \), we can define a unary polynomial function \( u: Y \rightarrow Y \) by

\[
u(y) = p(\varphi_1(x_1), \ldots, \varphi_{k-1}(x_{k-1}), y, \varphi_{k+1}(x_{k+1}), \ldots, \varphi_n(x_n)).
\]

To simplify notation, let us write \( a := \varphi_k(0) \), \( z := \varphi_k(x_k) \), \( b := \varphi_k(1) \), and let us note that the boundary condition (3) yields \( a \leq z \leq b \). With this notation (8) reads as \( u(z) = \text{med}(u(a), z, u(b)) \). In order to verify this equality, we write \( u \) in disjunctive normal form as in Remark 4: \( u(y) = s \lor (t \land y) \), where \( s \leq t \). Now the proof is a straightforward computation, making heavy use of distributivity:
\[ \text{med}(u(a), z, u(b)) = u(a) \lor (u(b) \land z) = (s \lor (t \land a)) \lor ((s \lor (t \land b)) \land z) \]
\[ = s \lor (t \land a) \lor (s \lor (t \land b) \land z) = s \lor (t \land a) \lor (s \land z) \lor (t \land z) \]
\[ = s \lor (t \land a) \lor (t \land z) = s \lor (t \land a \lor z) = s \lor (t \land z) = u(z) \]

\[ \square \]

3. Characterization of pseudo-polynomial functions

Let \( f : \prod_{i \in [n]} X_i \to Y \) be a function satisfying (7), and for each \( k \in [n] \) let us define two auxiliary functions \( \Phi_k^-, \Phi_k^+ : X_k \to Y \) as follows:

\[ \Phi_k^-(a_k) := \bigvee_{x : x_k = a_k} \text{cl}(f(x) \land f(x_k^-)), \quad \Phi_k^+(a_k) := \bigwedge_{x : x_k = a_k} \text{int}(f(x) \lor f(x_k^+)). \]

Here the join and the meet range over all \( x \in \prod_{i \in [n]} X_i \) whose \( k \)-th component is \( a_k \).

Note that from (7) it follows that \( \Phi_k^- \) and \( \Phi_k^+ \) satisfy the boundary condition (5). With the help of these functions, we will give a necessary and sufficient condition for \( f \) to be a pseudo-polynomial function. The following lemma formulates a simple observation that allows us to solve equation (8) for \( \varphi_k(x_k) \).

**Lemma 9.** For any \( u \leq m \leq w, v \in Y \) the following two conditions are equivalent:

(i) \( \text{med}(u, v, w) = m \);

(ii) \( m \land \overline{u} \leq v \leq m \lor \overline{w} \).

**Proof.** Assuming that \( \text{med}(u, v, w) = m \), we can estimate \( m \land \overline{u} \) using (4) as follows:

\[ m \land \overline{u} = (u \lor (v \land w)) \land \overline{u} = (u \land \overline{u}) \lor (v \land w) \land \overline{u} = 0 \lor (v \land w) \land \overline{u} \leq v. \]

An analogous argument shows that \( v \leq m \lor \overline{w} \), and this establishes (i) \( \implies \) (ii).

In order to prove (ii) \( \implies \) (i), let us first compute \( \text{med}(u, m \land \overline{u}, w) \), again with the help of (4):

\[ \text{med}(u, m \land \overline{u}, w) = u \lor (m \land \overline{u} \land w) = (u \lor m) \land (u \lor \overline{u}) \land (u \lor w) = m \land 1 \land w = m. \]

Similarly, we have \( \text{med}(u, m \lor \overline{w}, w) = m \), and then, using (ii) and the monotonicity of the median function, we conclude

\[ m = \text{med}(u, m \land \overline{u}, w) \leq \text{med}(u, v, w) \leq \text{med}(u, m \lor \overline{w}, w) = m, \]

hence \( \text{med}(u, v, w) = m \).

\[ \square \]

With the help of the above lemma we derive from Theorem 8 a necessary condition for \( f \) to be a pseudo-polynomial function.

**Proposition 10.** If \( f : \prod_{i \in [n]} X_i \to Y \) is a pseudo-polynomial function, then it satisfies (7) and

\[ \Phi_k^- \leq \Phi_k^+, \quad \text{for all } k \in [n]. \]

**Proof.** Let us suppose that \( f(x) = p(\varphi(x)) \) is a pseudo-polynomial function. Then (8) holds by Theorem 8 and applying Lemma 9 with \( u = f(x_k^-), m = f(x), w = f(x_k^+), v = \varphi_k(x_k) \), we see that \( f(x) \land f(x_k^-) \leq \varphi_k(x_k) \leq f(x) \lor f(x_k^+) \). Moreover, since \( \varphi_k(x_k) \in Y \), we have

\[ \text{cl}(f(x) \land f(x_k^-)) \leq \varphi_k(x_k) \leq \text{int}(f(x) \lor f(x_k^+)). \]

Considering these inequalities for all \( x \in \prod_{i \in [n]} X_i \) with a fixed \( k \)-th component \( x_k = a_k \), it follows that

\[ \Phi_k^-(a_k) \leq \varphi_k(a_k) \leq \Phi_k^+(a_k) \]

for all \( k \in [n], a_k \in X_k \).

\[ \square \]
Remark 11. Let us note that (13) holds if and only if each joinand in the definition of $\Phi_k^- (a_k)$ is less than or equal to each meetand in the definition of $\Phi_k^+ (a_k)$. In other words, (13) is equivalent to

$$\text{cl}(f(y) \land f(x_k^0)) \leq \int(f(x) \lor f(x_k^1)) \text{ for all } x, y \in \prod_{i \in [n]} X \text{ with } x_k = y_k.$$ 

In order to prove that the necessary condition presented in the above proposition is also sufficient, we verify that (7) and (13) imply that $f$ is pseudo-median decomposable with respect to $\Phi_1^-, \ldots, \Phi_n^-$ and also with respect to $\Phi_1^+, \ldots, \Phi_n^+$. 

Proposition 12. Suppose that $f : \prod_{i \in [n]} X_i \to Y$ satisfies (7) and (13). Then, for all $x \in \prod_{i \in [n]} X_i$ and $k \in [n]$, we have

$$f(x) = \text{med}(f(x_k^0), \Phi_k^- (a_k), f(x_k^1)) = \text{med}(f(x_k^0), \Phi_k^+ (x_k), f(x_k^1)).$$

Proof. By the definition of $\Phi_k^-$, we have

$$\text{med}(f(x_k^0), \Phi_k^- (a_k), f(x_k^1)) = \text{med}\left(\text{med}(f(x_k^0), f(x_k^1)), \text{cl}(f(y) \land f(x_k^1)) \right).$$

For some order $Y$ it follows that joins distribute over medians, and thus:

$$(15) \quad \text{med}(f(x_k^0), \Phi_k^- (a_k), f(x_k^1)) = \bigvee_{y_k = a_k} \text{med}(f(x_k^0), \text{cl}(f(y) \land f(x_k^1)), f(x_k^1)).$$

We can estimate this join from below by keeping only the joinand corresponding to $y = x$ (this indeed appears in the join, since $x_k = a_k$):

$$(16) \quad \text{med}(f(x_k^0), \Phi_k^- (a_k), f(x_k^1)) \geq \text{med}\left(\text{med}(f(x_k^0), \text{cl}(f(x) \land f(x_k^1)), f(x_k^1)) \right).$$

Applying Lemma 9 with $u = f(x_k^0)$, $v = \text{cl}(f(x) \land f(x_k^1))$, $w = f(x_k^1)$ and $u = f(x)$ and taking into account that $f(x_k^0) \leq f(x) \leq f(x_k^1)$ holds by (7), we see that the right hand side of (16) equals $f(x)$. This yields the inequality

$$(17) \quad \text{med}(f(x_k^0), \Phi_k^- (a_k), f(x_k^1)) \geq f(x).$$

In order to prove the converse inequality, let us note that property (13) implies

$$\text{cl}(f(y) \land f(x_k^1)) \leq \int(f(y) \lor f(x_k^1)),$$

whenever $y_k = a_k$ (see Remark 11). Thus, replacing $\text{cl}(f(y) \land f(x_k^1))$ by $\int(f(x) \lor f(x_k^1))$ in each joinand on the right hand side of (17), we get the upper estimate

$$(18) \quad \text{med}(f(x_k^0), \Phi_k^- (a_k), f(x_k^1)) \leq \text{med}(f(x_k^0), \int(f(x) \lor f(x_k^1)), f(x_k^1)).$$

Again, Lemma 9 shows that the right hand side of (18) equals $f(x)$, hence we have

$$(19) \quad \text{med}(f(x_k^0), \Phi_k^- (a_k), f(x_k^1)) \leq f(x).$$

Combining inequalities (17) and (19), we get the desired equality

$$(20) \quad \text{med}(f(x_k^0), \Phi_k^- (a_k), f(x_k^1)) = f(x).$$

Propositions 11 and 12 together with Theorem 8 yield the following characterization of pseudo-polynomial functions.

Theorem 13. A function $f : \prod_{i \in [n]} X_i \to Y$ is a pseudo-polynomial function if and only if it satisfies conditions (7) and (13).

Remark 14. Theorem 13 is of different nature than Theorem 8 and the various characterizations obtained in 8: here the necessary and sufficient condition for $f$ being a pseudo-polynomial function is given solely in terms of $f$ itself, without referring to the existence of certain functions $\varphi_k$. 
4. Factorizations of pseudo-polynomial functions

Let us suppose that \( f: \prod_{i \in [n]} X_i \to Y \) satisfies (7) and (13). According to Theorem 13, \( f \) is a pseudo-polynomial function, i.e., it has a factorization of the form \( f(\mathbf{x}) = p(\varphi(\mathbf{x})) \), where \( p: Y^n \to Y \) is a polynomial function and each \( \varphi_k: X_k \to Y \) \((k \in [n])\) is a unary map satisfying (4). We now show how to construct such a factorization; in fact, we will find all possible factorizations. First we describe the set of possible functions \( \varphi_k \).

**Theorem 15.** For any function \( f: \prod_{i \in [n]} X_i \to Y \) satisfying (7) and unary maps \( \varphi_k: X_k \to Y \) \((k \in [n])\) satisfying (5), the following three conditions are equivalent:

(i) \( \Phi_k^- \leq \varphi_k \leq \Phi_k^+ \) holds for all \( k \in [n] \);

(ii) \( f(\mathbf{x}) = p_f(\varphi(\mathbf{x})) \) (where \( p_f \) is given by (9) in Theorem 7);

(iii) there exists a polynomial function \( p: Y^n \to Y \) such that \( f(\mathbf{x}) = p(\varphi(\mathbf{x})) \).

**Proof.** The implication (i) \( \Rightarrow \) (ii) is trivial, and (ii) \( \Rightarrow \) (iii) has been established in the course of the proof of Proposition 14 (see equation (14)).

So suppose that (i) holds. Then obviously (13) holds, and Proposition 14 shows that \( f \) is pseudo-median decomposable with respect to \( \Phi_1^-, \ldots, \Phi_n^+ \) and also with respect to \( \Phi_1^+, \ldots, \Phi_n^+ \). Since \( \Phi_k^- \leq \varphi_k \leq \Phi_k^+ \) holds for all \( k \in [n] \) by (i) we have

\[
\begin{align*}
f(\mathbf{x}) &= \text{med} \left( f(x_0^k), \Phi_k^-(x_k), f(x_1^k) \right) \\
&\leq \text{med} \left( f(x_0^k), \varphi_k(x_k), f(x_1^k) \right) \\
&\leq \text{med} \left( f(x_0^k), \Phi_k^+(x_k), f(x_1^k) \right) = f(\mathbf{x}),
\end{align*}
\]

therefore \( f \) is pseudo-median decomposable with respect to \( \varphi_1, \ldots, \varphi_n \). Now (ii) follows immediately from Theorem 7. \( \square \)

Theorem 15 describes all those unary maps \( \varphi_1, \ldots, \varphi_n \) that can occur in a factorization of \( f \), but it does not provide all possible polynomial functions \( p \). (We know that \( p_f \) can be used in any factorization, but there may be others as well.) To find all factorizations (6) of \( f \), let us fix unary functions \( \varphi_k: X_k \to Y \) \((k \in [n])\) satisfying (5), such that \( \Phi_k^- \leq \varphi_k \leq \Phi_k^+ \) for each \( k \in [n] \). To simplify notation, let \( a_k = \varphi_k(0_{X_k}) \), \( b_k = \varphi_k(1_{X_k}) \), and for each \( I \subseteq [n] \) let \( e_I \in Y^n \) be the \( n \)-tuple whose \( i \)-th component is \( a_i \) if \( i \notin I \) and \( b_i \) if \( i \in I \). If \( p: Y^n \to Y \) is a polynomial function such that \( f(\mathbf{x}) = p(\varphi(\mathbf{x})) \), then

\[
p(e_I) = f(\hat{1}_I) \quad \text{for all } I \subseteq [n],
\]

since \( e_I = \varphi(\hat{1}_I) \). We show that (20) is not only necessary but also sufficient to establish the factorization \( f(\mathbf{x}) = p(\varphi(\mathbf{x})) \).

**Lemma 16.** Let \( f: \prod_{i \in [n]} X_i \to Y \) be a function satisfying (7) and (13), and let \( \varphi_k: X_k \to Y \) \((k \in [n])\) be maps satisfying (13), such that \( \Phi_k^- \leq \varphi_k \leq \Phi_k^+ \) for all \( k \in [n] \). Then a polynomial function \( p: Y^n \to Y \) yields a factorization \( f(\mathbf{x}) = p(\varphi(\mathbf{x})) \) if and only if (20) holds.

**Proof.** As noted above, necessity is trivial. To prove the sufficiency, let us assume that \( p \) satisfies (20), and let us define a function \( f': \prod_{i \in [n]} X_i \to Y \) by \( f'(\mathbf{x}) = p(\varphi(\mathbf{x})) \). Then \( f' \) is a pseudo-polynomial function, and by Theorem 8 it is pseudo-median decomposable with respect to \( \varphi_1, \ldots, \varphi_n \). From Theorem 7 we get the following expression for \( f' \):

\[
f'(\mathbf{x}) = \bigvee_{I \subseteq [n]} (f'(\hat{1}_I) \wedge \bigwedge_{i \in I} \varphi_i(x_i)).
\]

The assumptions on \( f \) and \( \varphi_k \) guarantee that \( f(\mathbf{x}) = p_f(\varphi(\mathbf{x})) \) by Theorem 15 and hence \( f \) can be written as

\[
f(\mathbf{x}) = \bigvee_{I \subseteq [n]} (f(\hat{1}_I) \wedge \bigwedge_{i \in I} \varphi_i(x_i)).
\]

Since \( f'(\hat{1}_I) = p(\varphi(\hat{1}_I)) = p(e_I) \) and \( p(e_I) = f(\hat{1}_I) \) by (20), it follows from (21) and (22) that \( f'(\mathbf{x}) = f(\mathbf{x}) \). \( \square \)
A certain interpolation problem is this equivalent to whereas the greatest solution is problem over Y. Thus we obtain the following description of all possible factorizations of a given pseudo-polynomial function. Let \( f \) be a given function satisfying (20) gives rise to a polynomial interpolation problem over \( Y \): the values of the unknown polynomial function \( p \) are prescribed at certain (\( 2^n \) many) points in \( Y^n \). It has been shown in (9) that the least solution of this interpolation problem is

\[
p^-(y) = \bigvee_{I \subseteq [n]} (c^-_I \land \bigwedge_{i \in I} y_i), \quad \text{where } c^-_I = \text{cl}(f(\hat{1}_I) \land \bigwedge_{i \in I} \pi_i),
\]

whereas the greatest solution is

\[
p^+(y) = \bigvee_{I \subseteq [n]} (c^+_I \land \bigwedge_{i \in I} y_i), \quad \text{where } c^+_I = \text{int}(f(\hat{1}_I) \lor \bigvee_{i \in I} \bar{b}_i).
\]

In other words, a polynomial function \( p \) is a solution of (20) if and only if \( p^- \leq p \leq p^+ \). Since, by Theorem 1, \( p \) is uniquely determined by its values on the tuples \( \hat{1}_I (I \subseteq [n]) \), this is equivalent to

\[
e^-_I = p^-(\hat{1}_I) \leq p(\hat{1}_I) \leq p^+(\hat{1}_I) = e^+_I \quad \text{for all } I \subseteq [n].
\]

Thus we obtain the following description of all possible factorizations of a given pseudo-polynomial function \( f \).

**Theorem 17.** Let \( f: \prod_{i \in [n]} X_i \to Y \) be a function satisfying (4), for each \( k \in [n] \) let \( \varphi_k: X_k \to Y \) be a given function satisfying (5), and let \( p: Y^n \to Y \) be a polynomial function. Then \( f(x) = p(\varphi(x)) \) if and only if \( \Phi^-_k \leq \varphi_k \leq \Phi^+_k \) for each \( k \in [n] \), and we have \( p^- \leq p \leq p^+ \).

**Remark 18.** Note that the polynomial functions \( p^- \) and \( p^+ \) are defined in terms of the maps \( \varphi_k \), hence we have to choose these maps first, and only then we can determine \( p^- \) and \( p^+ \) (cf. the example in Section 3).

**Remark 19.** Clearly, \( e^-_I \leq f(\hat{1}_I) \leq e^+_I \) holds independently of \( a_k, b_k \), hence the polynomial function \( p_f \) can be used in any factorization of \( f \), as it was already shown in Theorem 7.

**Remark 20.** If \( X_k \) is a partially ordered set for each \( k \in [n] \) and \( f \) is order-preserving, then \( \Phi^-_k \) and \( \Phi^+_k \) are also order-preserving. This shows that every order-preserving pseudo-polynomial function has a factorization where each \( \varphi_k \) is order-preserving. Consequently, order-preserving pseudo-Sugeno integrals coincide with Sugeno utility functions (cf. Corollary 2 in [8]).

### 5. An Example

We illustrate the results of the previous section with a simple example, where preferences about travelling with four airlines \( A_1, A_2, A_3, A_4 \) in economy class (E) and first class (F) are modelled by pseudo-polynomial functions. Table (a) shows a fictitious customer’s evaluation of these eight options, where B, N, G, V stand for “bad”, “neutral”,

<table>
<thead>
<tr>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( f(x_1, x_2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>E</td>
<td>B</td>
</tr>
<tr>
<td>A</td>
<td>F</td>
<td>B</td>
</tr>
<tr>
<td>A</td>
<td>E</td>
<td>B</td>
</tr>
<tr>
<td>A</td>
<td>F</td>
<td>D</td>
</tr>
<tr>
<td>A</td>
<td>E</td>
<td>N</td>
</tr>
<tr>
<td>A</td>
<td>F</td>
<td>G</td>
</tr>
<tr>
<td>A</td>
<td>E</td>
<td>N</td>
</tr>
<tr>
<td>A</td>
<td>F</td>
<td>V</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x_1 )</th>
<th>( \Phi^-_k(x_1) )</th>
<th>( \Phi^+_k(x_1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>B</td>
<td>B</td>
</tr>
<tr>
<td>A</td>
<td>D</td>
<td>D</td>
</tr>
<tr>
<td>A</td>
<td>G</td>
<td>G</td>
</tr>
<tr>
<td>A</td>
<td>V</td>
<td>V</td>
</tr>
</tbody>
</table>

**Table 1. The airline example**
shows that in both cases one can use the polynomial function

\[ f : X_1 \times X_2 \to Y, \]

where

\[ X_1 := \{A_1, A_2, A_3, A_4\}, \quad X_2 := \{E, F\}, \quad Y := \{B, N, G, V, D\}. \]

It is plausible that D is better than B, worse than G, and incomparable with N, hence the ordering of Y is the one given in Figure 1. This is a distributive lattice that can be embedded into the power set of a three-element set U as shown in Figure 1. The figure does not indicate the representation of each element of Y as a subset of U, only the complements of the elements, since this is all we need in order to perform the computations that follow. (Note that B and V are the complements of each other.)

For \( y \in Y \) we have obviously \( \text{cl}(y) = \text{int}(y) = y \), and for the three “extra” elements the closures and interiors can be read easily from Figure 1:

\[
\begin{align*}
\text{cl}(D) &= V, & \text{cl}(N) &= V, & \text{cl}(G) &= V, \\
\text{int}(D) &= N, & \text{int}(N) &= D, & \text{int}(G) &= B.
\end{align*}
\]

It is obvious that \( 0 \times_2 = E \) and \( 1 \times_2 = F \), but \( 0 \times_1 \) and \( 1 \times_1 \) are not clear. However, from Table 1(a) we can infer that \( 0 \times_1 = A \) and \( 1 \times_1 = A \) if \( f \) satisfies \( \phi_1 \) at all. (If not, then \( f \) is not a pseudo-polynomial function.)

Table 1(b) shows the auxiliary functions \( \Phi_k^-, \Phi_k^+ \) corresponding to the function \( f \). We give the details of the computation of \( \Phi_2^+(E) \), the other values can be calculated similarly:

\[
\Phi_2^+(E) = \bigwedge_{x_1 \in X_1} \text{int}(f(x_1, E) \lor \text{int}(f(x_1, F)))
\]

\[ = \text{int}(V \lor G) \lor \text{int}(B) \lor \text{int}(N \lor G) \lor \text{int}(N \lor V) \]

\[ = \text{int}(V) \lor \text{int}(G) \lor \text{int}(B) \lor \text{int}(N) = V \lor N \lor G \lor N = N. \]

We can see that \( \Phi_k^- \leq \Phi_k^+ \) for \( k = 1, 2 \), therefore \( f \) is a pseudo-polynomial function by Theorem 13. Theorem 15 implies that in any factorization \( f(x_1, x_2) = p(\varphi_1(x_1), \varphi_2(x_2)) \) of \( f \), we must have \( \varphi_1 = \Phi_1^- \) and \( \varphi_2^+ = \Phi_2^+ \), while we have two possibilities for \( \varphi_2 \) (as \( \varphi_2(E) \) can be chosen to be B or N, and \( \varphi_2(F) \) must be V). Thus there are two pairs of functions \( (\varphi_1, \varphi_2) \), namely \( (\Phi_1^-, \Phi_2^+) \) and \( (\Phi_1^+, \Phi_2^+) \) that allow us to factorize \( f \). Theorem 15 also shows that in both cases one can use the polynomial function

\[ p_f(y_1, y_2) = B \lor (y_1 \land N) \lor (y_2 \land B) \lor (V \land y_1 \land y_2) = (y_1 \land N) \lor (y_1 \land y_2) = y_1 \land (y_2 \lor N). \]

Computing the coefficients \( c_f^- \), \( c_f^+ \) for \( (\Phi_1^-, \Phi_2^+) \), one can see that in this case \( p^- = p_f = p^+ \), i.e., \( p = p_f \) is the only polynomial function such that \( f(x_1, x_2) = p(\Phi_1^-(x_1), \Phi_2^+(x_2)) \). On the other hand, choosing \( (\varphi_1, \varphi_2) = (\Phi_1^+, \Phi_2^+) \), we obtain \( p^- = y_1 \land y_2 \) and \( p^+ = p_f \), and these are the only possibilities, since there is no polynomial function strictly between \( p^- \) and \( p^+ \). Thus \( f \) has altogether three factorizations:

\[ f(x_1, x_2) = \Phi_1^-(x_1) \lor (\Phi_2^+(x_2) \lor N) = \Phi_1^+(x_1) \lor (\Phi_2^+(x_2) \lor V) = \Phi_1^-(x_1) \land \Phi_2^+(x_2). \]

Note that these factorizations are essentially the same, since \( \Phi_2^+ \lor N = \Phi_2^+ \lor N = \Phi_2^+ \). The meaning of \( \Phi_2^+ \) is pretty obvious, and \( \Phi_1^+ \) shows the customer’s opinion about the
four airlines, either based on past experience or on information received from other sources (except for airline \( A_2 \), where the value D indicates the lack of information).
The fact that \( \Phi^+_1(x_1) \) and \( \Phi^+_2(x_2) \) are aggregated in a conjunctive manner indicates a pessimistic attitude: the customer expects an enjoyable flight only if both the airline and the travel class are good enough.

6. Pseudo-polynomial functions over chains

In this section we consider the case when \( Y \) is a finite chain. As we will see, in this case the results of Section 4 lead to a generalization of Algorithm SUFF presented in [8]. As before, we will suppose that \( Y \) is a sublattice of \( \mathcal{P}(U) \) for some finite set \( U \), with least element \( \emptyset \) and greatest element \( U \). We may assume without loss of generality that \( U = \{0, 1, \ldots, m\} \), and \( Y = \{\{0\}, \{1\}, \ldots, \{m\}\} \), where \( \emptyset = \emptyset \). The closure of a set \( S \subseteq U \) is the smallest set of the form \([k]\) that contains \( S \), while the interior of \( S \) is the largest set of the form \([k]\) that is contained in \( S \) (see Figure 2). Formally, we have

\[
\text{cl}(S) = [\max S], \quad \text{int}(S) = [\min S - 1].
\]

Let us assume that \( f : \prod_{i \in [m]} X_i \to Y \) satisfies (7). Then \( f(x_0^1) = [u], f(x) = [v], f(x_k^0) = [w] \) with \( u \leq v \leq w \), hence we have

\[
\begin{align*}
\text{cl}(f(x) \wedge f(x_k^0)) &= \{ f(x), \text{ if } f(x_k^0) < f(x); 0, \text{ if } f(x_k^0) = f(x) \}, \\
\text{int}(f(x) \lor f(x_k^0)) &= \{ f(x), \text{ if } f(x_k^0) > f(x); U, \text{ if } f(x_k^0) = f(x) \}.
\end{align*}
\]

Therefore the terms in the definition of \( \Phi^+_k \) and \( \Phi^+_k \) can be determined as follows:

\[
(23) \quad \text{cl}(f(x) \wedge f(x_k^0)) = \{ f(x), \text{ if } f(x_k^0) < f(x); 0, \text{ if } f(x_k^0) = f(x) \}, \\
(24) \quad \text{int}(f(x) \lor f(x_k^0)) = \{ f(x), \text{ if } f(x_k^0) > f(x); U, \text{ if } f(x_k^0) = f(x) \}.
\]

Thus we obtain from Theorem 13 and Remark 11 the following characterization of pseudo-polynomial functions valued in a chain.

**Theorem 21.** If \( Y \) is a finite chain, then a function \( f : \prod_{i \in [m]} X_i \to Y \) is a pseudo-polynomial function if and only if it satisfies condition (2) and

\[
f(x_0^k) < f(x_k^{a_k}) \text{ and } f(y_0^k) < f(y_k^{b_k}) \implies f(x_k^{a_k}) \leq f(y_k^{b_k})
\]

holds for all \( x, y \in \prod_{i \in [m]} X_i \) and \( k \in [n], a_k \in X_k \).

Let us now define the following three sets for any \( k \in [n], a_k \in X_k \), as in [8]:

\[
\begin{align*}
\mathcal{N}^{a_k} &= \{ f(x) : x_k = a_k \text{ and } f(x_0^k) < f(x) < f(x_k^0) \}, \\
\mathcal{L}^{a_k} &= \{ f(x) : x_k = a_k \text{ and } f(x_0^k) < f(x) = f(x_k^0) \}, \\
\mathcal{U}^{a_k} &= \{ f(x) : x_k = a_k \text{ and } f(x_0^k) = f(x) < f(x_k^0) \}.
\end{align*}
\]

**Figure 2.** The closure and interior of a subset of a chain
From (23) and (24) it follows that \( \Phi_k^- (a_k) = \bigvee L_{a_k} \lor \bigvee W_{a_k} \) and \( \Phi_k^+ (a_k) = \bigwedge U_{a_k} \land \bigwedge W_{a_k} \), hence the condition \( \Phi_k^- \leq \varphi_k \leq \Phi_k^+ \) in Theorem 15 can be reformulated as follows:

(a) either \( W_{a_k} = \{ \varphi_k (a_k) \} \) or \( W_{a_k} = \emptyset \);  
(b) \( \varphi_k (a_k) \geq \bigvee L_{a_k} \);  
(c) \( \varphi_k (a_k) \leq \bigwedge U_{a_k} \).

Thus by Theorem 15, \( f \) is a pseudo-polynomial function if and only if there are functions \( \varphi_k \) satisfying the above three conditions. If each \( X_k \) is a bounded chain and \( f \) is an order-preserving function depending on all of its variables, then (a),(b),(c) are equivalent to equation (18) in [8], and Algorithm SUFF does not return the value false if and only if (13) holds. Thus, in the finite case, Theorem 7 of [8] follows as a special case of Theorem 15. Moreover, the results of Section 4 not only generalize Algorithm SUFF to arbitrary finite distributive lattices (instead of finite chains) and to pseudo-polynomial functions (instead of Sugeno utility functions), but they provide all possible factorizations of a given pseudo-polynomial function \( f \) (whereas Algorithm SUFF constructs only one factorization).

Algorithm 1  Sugeno Utility Function Factorization (SUFF) [8]

Require: \( f \) depends on all of its variables and satisfies (7)
1: if \( f \) is not order-preserving then  
2: \quad return false  
3: end if  
4: for \( k \in [n] \) do  
5: \quad for \( a_k \in X_k \) do  
6: \quad \quad compute \( W_{a_k} \)  
7: \quad \quad if \( |W_{a_k}| \geq 2 \) then  
8: \quad \quad \quad return false  
9: \quad end if  
10: \quad \quad compute \( L_{a_k}, U_{a_k} \) and  
11: \quad \quad \quad \quad \quad \quad w_{a_k} := w \) if \( W_{a_k} = \{ w \} \)  
12: \quad \quad \quad \quad \quad \quad l_{a_k} := \bigvee L_{a_k} \)  
13: \quad \quad \quad \quad \quad \quad u_{a_k} := \bigwedge U_{a_k} \)  
14: \quad \quad \quad if \( l_{a_k} > u_{a_k} \) or \( l_{a_k} > w_{a_k} \) or \( w_{a_k} > u_{a_k} \) then  
15: \quad \quad \quad \quad return false  
16: \quad \quad \quad end if  
17: \quad \quad \quad if \( W_{a_k} \neq \emptyset \) then  
18: \quad \quad \quad \quad \varphi_k^f (a_k) := w_{a_k} \) \hspace{1cm} // (W)  
19: \quad \quad \quad else if \( L_{a_k} \neq \emptyset \) then  
20: \quad \quad \quad \quad \varphi_k^f (a_k) := l_{a_k} \) \hspace{1cm} // (L),(LU)  
21: \quad \quad \quad else if \( U_{a_k} \neq \emptyset \) then  
22: \quad \quad \quad \quad \varphi_k^f (a_k) := u_{a_k} \) \hspace{1cm} // (U)  
23: \quad \quad \quad else  
24: \quad \quad \quad \quad return false \hspace{1cm} // \( a_k \) is inessential  
25: \quad \quad \quad end if  
26: \quad \quad end for  
27: \quad end for  
28: compute \( p_f \)  
29: return \( p_f, \varphi_1^f, \ldots, \varphi_n^f \) \hspace{1cm} // \( f \) is a SUF

Remark 22. If the lattice \( Y \) is a finite chain, as it is the case in many applications, then any map \( \varphi_k : X_k \rightarrow Y \) attains its minimum and its maximum at some points in \( X_k \), hence there exist elements \( 0_{X_k} \) and \( 1_{X_k} \) such that (5) holds. Thus, the boundary condition does not impose any restriction on \( \varphi_k \); the point is that we must know the
function \( f \) in a factorization of \( X \).

argmin and argmax denote the elements of \( f \), and then we can use Theorem 17 to find the possible polynomial functions absorbed by some of the original meetands.

Similarly, each of the new meetands that we have added to the meet defining \( \Phi_k^- \) is absorbed by some of the original meetands.

With these new definitions, we can apply Theorem 13 to decide whether a given function \( f \) is a pseudo-polynomial function, and we can find all maps \( \varphi_k \) that can appear in a factorization of \( f \) with the help of Theorem 15. Once we have the maps \( \varphi_k \), we can define \( 0_{X_k} \) and \( 1_{X_k} \) as

\[
0_{X_k} := \arg\min_{x_k \in X_k} \varphi_k(x_k) \quad \text{and} \quad 1_{X_k} := \arg\max_{x_k \in X_k} \varphi_k(x_k),
\]

and then we can use Theorem 17 to find the possible polynomial functions \( p \). (Here argmin and argmax denote the elements of \( X_k \) where \( \varphi_k \) attains its minimum and its maximum, respectively.) The price that we have to pay for this generality is that the computation of \( \Phi_k^- \) and \( \Phi_k^+ \) is longer. Alternatively, we can apply “reverse engineering” to \( f \), as in Section 5, in order to find \( 0_{X_k} \) and \( 1_{X_k} \).

7. Concluding remarks

We have extended the study of Sugeno utility functions over chains developed in \cite{6, 7, 8} to the case of finite distributive lattices. We refined the axiomatization given in \cite{6, 8} by providing necessary and sufficient conditions for a function defined on a Cartesian product of arbitrary underlying sets and valued in a finite distributive lattice, to be factorizable as a pseudo-polynomial function. Moreover, in doing so, we were able to furnish all possible factorizations, if such a factorization exists, and we proposed a new procedure for constructing them, which subsumes that of \cite{6, 7, 8} in the case when the codomain is a finite chain.

Looking at directions for further research, we are inevitably drawn to the two following topics. As mentioned, pseudo-polynomial functions play an important role in multicriteria decision making since they subsume the so-called Sugeno utility functions, which in turn are used to model preference relations: Say that a preference relation \( \preceq \) on \( X_1 \times \cdots \times X_n \) is Sugeno representable if there is a Sugeno utility function \( U : X_1 \times \cdots \times X_n \to Y \) such that \( x \preceq y \) if and only if \( U(x) \leq U(y) \). Given the results of the current paper, and following the line of research developed in \cite{2, 11, 10}, it is natural to consider the following problem.

Problem 23. Axiomatize those preference relations that are Sugeno representable.

This problem was solved in the realm of decision making under uncertainty in \cite{11}.

The second emerging topic is of somewhat different nature. So far, we have played within the setting where no information is missing. However, in real-life situations this is rarely the case. Translating it into mathematical terms, it is often the case that information about the functions we deal with is incomplete. In other words, we are given partial functions in the sense that they are not everywhere defined. Taking the simplest case where the only aggregation functions considered are Sugeno integrals, we are faced with the following interpolation problem.
Problem 24. Given a partial function $f : D \rightarrow X$, where $X$ is a distributive lattice and $D \subseteq X^n$, give necessary and sufficient conditions for the existence of a Sugeno integral $p : X^n \rightarrow X$ which interpolates $f$ on all of its domain $D$, i.e., $p|_D = f$. If such a Sugeno integral exists, provide a procedure to compute it.

Again, this problem has been solved for the case of finite chains $X$ in [10].

These two problems constitute topics of current research being carried out by the authors.

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