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Abstract. We compare the solution of the generalized Boussinesq systems, for various values of a,b,c,d,

$$\begin{aligned} \eta_t + u_x + \varepsilon((\eta u)_x + a u_{xxx} - b\eta_{xxt}) &= 0 \\ u_t + \eta_x + \varepsilon(u u_x + c\eta_{xxx} - d u_{xxt}) &= 0. \end{aligned}$$

These systems describe the two-way propagation of small amplitude long waves in shallow water. We prove, using an energy method introduced by Bona, Pritchard and Scott [7], that respective solutions of Boussinesq systems, starting from the same initial datum, remain close on a time interval inversely proportional to the wave amplitude.

Keywords. Boussinesq systems, Comparison, Energy method.

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Introduction

In this paper, we are interested in the two-way propagation of small amplitude long waves in shallow water. This phenomenon is described by the amplitude η and the velocity u of the waves, which satisfy the following generalized Boussinesq systems [4, 5], for $(x,t) \in \mathbb{R} \times \mathbb{R}$,

$$\begin{aligned} \eta_t + u_x + \varepsilon((\eta u)_x + a u_{xxx} - b\eta_{xxt}) &= 0 \\ u_t + \eta_x + \varepsilon(u u_x + c\eta_{xxx} - d u_{xxt}) &= 0. \end{aligned}$$

Here $0 < \varepsilon < 1$ denotes the quotient between the amplitude and the depth of water, and ε is assumed to be small. This coefficient is also proportional to the square of the quotient between the depth of water and the wavelength.

If we focus on the one-way propagation, the Boussinesq systems are reduced to the Korteweg-de Vries equation (KdV) [9]

$$\eta_t + \eta_x + \varepsilon(\alpha\, \eta\eta_x + \beta\, \eta_{xxx}) = 0,$$

or the Benjamin-Bona-Mahony equation (BBM) [3]

$$\eta_x + \eta_t + \varepsilon(\alpha\, \eta\eta_x - \beta\, \eta_{xxt}) = 0.$$

Recall that Bona, Pritchard and Scott [7] proved the following result.

Theorem 0.1 *Let $m \geq 0$ and $f \in H^{m+5}(\mathbb{R})$. Let $\varepsilon > 0$ and η, ζ respectively the unique solution of KdV and BBM in $\mathcal{C}([0,\infty[; H^{m+5}(\mathbb{R}))$, starting from the same initial datum $\eta|_{t=0} = \zeta|_{t=0} = f$. Then, there exists a constant $M_i > 0$ with $0 \leq i \leq m$ such that for $|t| \leq \varepsilon^{-1}T$, we have*

$$||\partial_x^i \eta(.,t) - \partial_x^i \zeta(.,t)||_{L^2(\mathbb{R})} \leq M_i\, \varepsilon^2\, |t|,$$

with $T := \frac{C_{m+5}}{\|f\|_{m+5}}$.

Concerning the Boussinesq systems, our result reads as follows.

Theorem 0.2 *Let $m \geq 0$ and $(f,g) \in H^{m+5}(\mathbb{R}) \times H^{m+5}(\mathbb{R})$. Let $(\eta_1, u_1), (\eta_2, u_2)$ be solution in $\mathcal{C}([-\varepsilon^{-1/2}T, \varepsilon^{-1/2}T]; H^{m+5}(\mathbb{R}))^2$ of Boussinesq system with (a_1, b_1, c_1, d_1), respectively (a_2, b_2, c_2, d_2), starting from the same initial datum (f, g) with*

$$T := \frac{C_{m+5}}{\|(f,g)\|_{m+5}}.$$

Then, for all $0 \leq i \leq m$, there exists a constant $M_i > 0$, depending only on $\|(f,g)\|_{m+5}$, such that for $|t| \leq \varepsilon^{-1/2}T$, we have

$$\left\| \partial_x^i \begin{pmatrix} \eta_1(\cdot,t) \\ u_1(\cdot,t) \end{pmatrix} - \partial_x^i \begin{pmatrix} \eta_2(\cdot,t) \\ u_2(\cdot,t) \end{pmatrix} \right\|_{L^2} \leq M_i \varepsilon |t|.$$

This result reflects the fact that the solutions of Boussinesq systems with various coefficients are close with a maximum deviation of the order of $\varepsilon^{1/2}$. This is not as good as the order of the neglected terms in the asymptotic expansion of the Euler equation. On the other hand, the obtained time of comparison is of order $\varepsilon^{-1/2}$, although the effects of dispersion and the effects of the nonlinearity have an order 1 influence. One can certainly hope that the solutions remain close longer, and improve the result, for example by choosing a suitable velocity, or a good combination of KdV solutions. In this way, Alazman et al. [1] recently obtained a similar result and showed that the solution of the Boussinesq system with one-way propagation is well approximated by the solution of the KdV or BBM. Related papers concern the comparison of the solution of two-dimensional Euler systems with the solution of Boussinesq systems or KdV equation [6, 8, 10, 11].

Our paper is organised as follows. In the first section, the local well-posedness of the Boussinesq systems is summed up. The second section deals with the comparison of the solution of the systems. Finally, we present some numerical illustrations.

1 Summary of existence theory

We consider the initial value problem, for $(x,t) \in \mathbb{R} \times \mathbb{R}$,

$$\eta_t + u_x + \varepsilon\left((\eta u)_x + a u_{xxx} - b\eta_{xxt}\right) = 0 \tag{1.1}$$

$$u_t + \eta_x + \varepsilon\left(u u_x + c\eta_{xxx} - d u_{xxt}\right) = 0 \tag{1.2}$$

$$\eta(x,0) = f(x),\ u(x,0) = g(x). \tag{1.3}$$

The coefficients a, b, c, d satisfy one of these two conditions:

- $b \geq 0,\ d \geq 0,\ a \leq 0,\ c \leq 0$
- $b \geq 0,\ d \geq 0,\ a = c > 0.$

Recall that the Cauchy problem is locally well-posed [4, 5]. In this work, we only need smooth data, and we refer to these papers for more precise assumptions on the regularity.

Theorem 1.1 *Let $m > 3/2$ and $(f, g) \in H^m(\mathbb{R}) \times H^m(\mathbb{R})$. Then there exists $C_m > 0$, depending only on m, such that for*

$$T := \frac{C_m}{\|(f, g)\|_m},$$

there exists a unique solution $(\eta, u) \in \mathcal{C}([-\varepsilon^{-3/2}T, \varepsilon^{-3/2}T]; H^m(\mathbb{R})) \times \mathcal{C}([-\varepsilon^{-3/2}T, \varepsilon^{-3/2}T]; H^m(\mathbb{R}))$ of the Cauchy problem (1.1)-(1.2)-(1.3).
Moreover, let $(\widetilde{f}, \widetilde{g}) \in H^m(\mathbb{R}) \times H^m(\mathbb{R})$ be the initial datum of the unique solution $(\widetilde{\eta}, \widetilde{u})$ and $M > 0$ such that

$$\|(f, g)\|_m \leq M \text{ and } \|(\widetilde{f}, \widetilde{g})\|_m \leq M.$$

Then there exists $c_m > 0$ such that for $|t| \leq \varepsilon^{-3/2} C_m / M$,

$$\left\| \begin{pmatrix} \eta(\cdot, t) \\ u(\cdot, t) \end{pmatrix} - \begin{pmatrix} \widetilde{\eta}(\cdot, t) \\ \widetilde{u}(\cdot, t) \end{pmatrix} \right\|_m \leq c_m \left\| \begin{pmatrix} f \\ g \end{pmatrix} - \begin{pmatrix} \widetilde{f} \\ \widetilde{g} \end{pmatrix} \right\|_m .$$

Proof. The proof in [5] is actually done with $\varepsilon = 1$. The change of variables

$$x = \varepsilon^{-1/2} \widetilde{x}, t = \varepsilon^{-1/2} \widetilde{t}, \eta = \varepsilon^{-1} \widetilde{\eta}, u = \varepsilon^{-1} \widetilde{u}$$

provides the following initial value problem

$$\begin{aligned} \widetilde{\eta}_{\widetilde{t}} + \widetilde{u}_{\widetilde{x}} + (\widetilde{\eta}\widetilde{u})_{\widetilde{x}} + a\widetilde{u}_{\widetilde{x}\widetilde{x}\widetilde{x}} - b\widetilde{\eta}_{\widetilde{x}\widetilde{x}\widetilde{t}} &= 0 \\ \widetilde{u}_{\widetilde{t}} + \widetilde{\eta}_{\widetilde{x}} + \widetilde{u}\widetilde{u}_{\widetilde{x}} + c\widetilde{\eta}_{\widetilde{x}\widetilde{x}\widetilde{x}} - d\widetilde{u}_{\widetilde{x}\widetilde{x}\widetilde{t}} &= 0 \\ \widetilde{\eta}(\widetilde{x}, 0) = \varepsilon f(\widetilde{x}),\ \widetilde{u}(\widetilde{x}, 0) &= \varepsilon g(\widetilde{x}). \end{aligned}$$

□

2 Comparison of solutions

2.1 The linear case

We consider the linearized problem around $(0, 0)$

$$\begin{aligned} \eta_t + u_x + \varepsilon(a u_{xxx} - b\eta_{xxt}) &= 0 \\ u_t + \eta_x + \varepsilon(c\eta_{xxx} - d u_{xxt}) &= 0 \\ \eta(x, 0) = f(x), u(x, 0) = g(x). \end{aligned}$$

Let us compare the solution of the KdV-KdV system ($a = c = 1, b = d = 0$) and the solution of the BBM-BBM system ($a = c = 0, b = d = 1$).

Assume that $f, g \in \mathcal{S}(\mathbb{R})$. The solution of KdV-KdV is given by the Fourier transform in space

$$\begin{pmatrix} \widehat{\eta} \\ \widehat{u} \end{pmatrix} (k, t) = e^{-ikA_1(k)t} \begin{pmatrix} \widehat{f} \\ \widehat{g} \end{pmatrix} (k),$$

with

$$A_1(k) = \begin{pmatrix} 0 & 1 - \varepsilon k^2 \\ 1 - \varepsilon k^2 & 0 \end{pmatrix}.$$

And the solution of BBM-BBM is

$$\begin{pmatrix} \widehat{\zeta} \\ \widehat{v} \end{pmatrix} (k, t) = e^{-ikA_2(k)t} \begin{pmatrix} \widehat{f} \\ \widehat{g} \end{pmatrix} (k),$$

with

$$A_2(k) = \begin{pmatrix} 0 & \frac{1}{1+\varepsilon k^2} \\ \frac{1}{1+\varepsilon k^2} & 0 \end{pmatrix}.$$

Since $A_1(k)A_2(k) = A_2(k)A_1(k)$, it provides

$$\begin{aligned} \begin{pmatrix} \widehat{\eta} \\ \widehat{u} \end{pmatrix}(k,t) - \begin{pmatrix} \widehat{\zeta} \\ \widehat{v} \end{pmatrix}(k,t) &= \left(e^{-ikA_1(k)t} - e^{-ikA_2(k)t}\right)\begin{pmatrix} \widehat{f} \\ \widehat{g} \end{pmatrix}(k) \\ &= e^{-ikA_2(k)t}\left(e^{-ik(A_1(k)-A_2(k))t} - 1\right)\begin{pmatrix} \widehat{f} \\ \widehat{g} \end{pmatrix}(k). \end{aligned}$$

Thus

$$\begin{aligned} \left|\begin{pmatrix} \widehat{\eta} \\ \widehat{u} \end{pmatrix}(k,t) - \begin{pmatrix} \widehat{\zeta} \\ \widehat{v} \end{pmatrix}(k,t)\right| &= \left|e^{-ikA_2(k)t}\left(e^{-ik(A_1(k)-A_2(k))t} - 1\right)\begin{pmatrix} \widehat{f} \\ \widehat{g} \end{pmatrix}(k)\right| \\ &\leq \left|\left(e^{-ik(A_1(k)-A_2(k))t} - 1\right)\begin{pmatrix} \widehat{f} \\ \widehat{g} \end{pmatrix}(k)\right|. \end{aligned}$$

However $|\exp(ix) - 1| \leq |x|$, and we find

$$\left|\begin{pmatrix} \widehat{\eta} \\ \widehat{u} \end{pmatrix}(k,t) - \begin{pmatrix} \widehat{\zeta} \\ \widehat{v} \end{pmatrix}(k,t)\right| \leq \left|-k(A_1(k) - A_2(k))t\begin{pmatrix} \widehat{f} \\ \widehat{g} \end{pmatrix}(k)\right|.$$

Then, using $1 + \varepsilon k^2 \geq 1$,

$$\left|\begin{pmatrix} \widehat{\eta} \\ \widehat{u} \end{pmatrix}(.,t) - \begin{pmatrix} \widehat{\zeta} \\ \widehat{v} \end{pmatrix}(.,t)\right| \leq \left|\frac{\varepsilon^2 k^5 t}{1+\varepsilon k^2}\right|\left|\begin{pmatrix} \widehat{g} \\ \widehat{f} \end{pmatrix}(k)\right| \leq \varepsilon^2 |k|^5 |t| \left|\begin{pmatrix} \widehat{g} \\ \widehat{f} \end{pmatrix}(k)\right|,$$

and

$$\left\|\begin{pmatrix} \eta \\ u \end{pmatrix}(.,t) - \begin{pmatrix} \zeta \\ v \end{pmatrix}(.,t)\right\|_{L^\infty} \leq \frac{1}{2\pi}\left\|\begin{pmatrix} \widehat{\eta} \\ \widehat{u} \end{pmatrix}(.,t) - \begin{pmatrix} \widehat{\zeta} \\ \widehat{v} \end{pmatrix}(.,t)\right\|_{L^1} \leq \varepsilon^2 |t| \int_{\mathbb{R}} |k|^5 \left|\begin{pmatrix} \widehat{g} \\ \widehat{f} \end{pmatrix}(k)\right| dk.$$

The result can be written as follows.

Theorem 2.1 *Let f and g be real valued functions with $\widehat{\partial_x^5 f}$ and $\widehat{\partial_x^5 g}$ integrables on $\mathbb{R}$. Let (η, u) and (ζ, v) be solutions of the linear Cauchy problem associated with KdV-KdV and BBM-BBM respectively, starting from the same initial datum*

$$\begin{pmatrix} \eta(\cdot,0) \\ u(\cdot,0) \end{pmatrix} = \begin{pmatrix} \zeta(\cdot,0) \\ v(\cdot,0) \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}.$$

Then there exists a constant $C > 0$, depending only on (f, g), such that, for all time $t \in \mathbb{R}$,

$$\left\|\begin{pmatrix} \eta(\cdot,t) \\ u(\cdot,t) \end{pmatrix} - \begin{pmatrix} \zeta(\cdot,t) \\ v(\cdot,t) \end{pmatrix}\right\|_{L^\infty} \leq C\varepsilon^2 |t|.$$

2.2 The main result

To simplify the reading, we only prove the result for the KdV-KdV and BBM-BBM system. The proof of the theorem 0.2 can be done in a similar way.
We now consider the following nonlinear Cauchy problems. The first one associated with KdV-KdV:

$$\begin{aligned} \eta_t + u_x + \varepsilon\left((\eta u)_x + u_{xxx}\right) &= 0 \qquad (2.1) \\ u_t + \eta_x + \varepsilon(u u_x + \eta_{xxx}) &= 0 \qquad (2.2) \\ \eta(x,0) = f(x),\ u(x,0) &= g(x). \end{aligned}$$

And the second one is called BBM-BBM:

$$\begin{aligned}\zeta_t + v_x + \varepsilon((\zeta v)_x - \zeta_{xxt}) &= 0 &(2.3)\\ v_t + \zeta_x + \varepsilon(v v_x - v_{xxt}) &= 0 &(2.4)\\ \zeta(x,0) = f(x),\ v(x,0) &= g(x). \end{aligned}$$

We prove that the respective solutions remain close as follows.

Theorem 2.2 *Let $m \geq 5$ and $(f,g) \in H^m(\mathbb{R}) \times H^m(\mathbb{R})$. Let (η, u) and (ζ, v) be solution in $\mathcal{C}([-\varepsilon^{-1/2}T, \varepsilon^{-1/2}T]; H^m(\mathbb{R})) \times \mathcal{C}([-\varepsilon^{-1/2}T, \varepsilon^{-1/2}T]; H^m(\mathbb{R}))$ of KdV-KdV and BBM-BBM respectively, where*

$$T := \frac{C_m}{\|(f,g)\|_m}.$$

Then there exists a constant $M > 0$, depending only on $\|(f,g)\|_m$, such that for $|t| \leq \varepsilon^{-1/2}T$, we have

$$\left\| \begin{pmatrix} \eta(\cdot,t) \\ u(\cdot,t) \end{pmatrix} - \begin{pmatrix} \zeta(\cdot,t) \\ v(\cdot,t) \end{pmatrix} \right\|_{L^2} \leq M\varepsilon|t|.$$

Proof. To simplify the writings, we denote the L^2-norm as $||.||$. We assume that $t > 0$, negative time being dealt with similarly. Let us define

$$\begin{aligned} \lambda &= \eta - \zeta \\ w &= u - v. \end{aligned}$$

Then (λ, w) satisfies:

$$\begin{aligned} \lambda_t + w_x - \varepsilon\lambda_{xxt} &= -\varepsilon\left((\lambda w)_x + (\lambda u + \zeta w)_x + u_{xxx} + \eta_{xxt}\right) &(2.5)\\ w_t + \lambda_x - \varepsilon w_{xxt} &= -\varepsilon\left(w w_x + (v w)_x + \eta_{xxx} + u_{xxt}\right) &(2.6)\\ \lambda(x,0) = 0, &\qquad w(x,0) = 0. \end{aligned}$$

Equations (2.5) and (2.6) are multiplied by λ and w respectively, and are integrated over space.

$$\begin{aligned} \int_{-\infty}^{+\infty} \lambda(\lambda_t + w_x - \varepsilon\lambda_{xxt}) + w(w_t + \lambda_x - \varepsilon w_{xxt})\, dx = -\varepsilon \int_{-\infty}^{+\infty} \lambda(u_{xxx} + \eta_{xxt}) + w(\eta_{xxx} + u_{xxt}) \\ - \varepsilon \int_{-\infty}^{+\infty} \lambda((\lambda w)_x + (\lambda u + \zeta w)_x) + w(w w_x + (v w)_x)\, dx. \end{aligned} \tag{2.7}$$

We first simplify the left-hand side of (2.7). Since $\lambda, w \in H^m(\mathbb{R})$, we know that, for $0 \leq i \leq m$, $\partial_x^i \lambda$ and $\partial_x^i w$ go to zero at infinity. Thus

$$\int_{-\infty}^{+\infty} (\lambda w_x + w\lambda_x) dx = \int_{-\infty}^{+\infty} (\lambda w)_x\, dx = [\lambda w]_{-\infty}^{+\infty} = 0,$$

and an integration by parts gives

$$\int_{-\infty}^{+\infty} \lambda(\lambda_t - \varepsilon\lambda_{xxt}) + w(w_t - \varepsilon w_{xxt}) dx = \frac{1}{2}\frac{d}{dt} \int_{-\infty}^{+\infty} \lambda^2 + w^2 + \varepsilon\lambda_x^2 + \varepsilon w_x^2\, dx.$$

Integration by parts are applied to the right-hand side of (2.7), and it gets:

$$\begin{aligned} \int_{-\infty}^{+\infty} \lambda((\lambda w)_x + (\lambda u + \zeta w)_x) dx &= -\int_{-\infty}^{+\infty} \lambda_x(\lambda w + \lambda u + \zeta w)\, dx \\ &= -\int_{-\infty}^{+\infty} \left(\frac{\lambda^2}{2}\right)_x (w + u)\, dx - \int_{-\infty}^{+\infty} \lambda_x \zeta w\, dx \\ &= \int_{-\infty}^{+\infty} \frac{\lambda^2}{2}(w_x + u_x)\, dx - \int_{-\infty}^{+\infty} \lambda_x \zeta w\, dx. \end{aligned}$$

Similarly

$$\int_{-\infty}^{+\infty} w(vw)_x\,dx = -\int_{-\infty}^{+\infty} w_x vw\,dx = -\int_{-\infty}^{+\infty}\left(\frac{w^2}{2}\right)_x v\,dx = \int_{-\infty}^{+\infty}\frac{w^2}{2}v_x\,dx,$$

and

$$\int_{-\infty}^{+\infty} w^2 w_x dx = \int_{-\infty}^{+\infty}\left(\frac{w^3}{3}\right)_x dx = \left[\frac{w^3}{3}\right]_{-\infty}^{+\infty} = 0.$$

Finally, the equality (2.7) becomes

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{+\infty}(\lambda^2 + w^2 + \varepsilon\lambda_x^2 + \varepsilon w_x^2)\,dx = -\varepsilon\int_{-\infty}^{+\infty}\frac{\lambda^2}{2}(w_x+u_x) + \frac{w^2}{2}v_x + \lambda_x\zeta w\,dx$$
$$-\varepsilon\int_{-\infty}^{+\infty}\lambda u_{xxx} + \lambda\eta_{xxt} + w\eta_{xxx} + wu_{xxt}\,dx.$$

However $\lambda(x,0)=0$ and $w(x,0)=0$, we deduce

$$\begin{aligned}\int_{-\infty}^{+\infty}(\lambda^2 + w^2 + \varepsilon\lambda_x^2 + \varepsilon w_x^2)\,dx &\le 2\varepsilon\int_0^t\int_{-\infty}^{+\infty}\left|\frac{\lambda^2}{2}(w_x+u_x)\right| + \left|\frac{w^2}{2}v_x\right| + |\lambda_x\zeta w| \\ &\quad + 2\varepsilon\int_0^t\int_{-\infty}^{+\infty}|\lambda u_{xxx}| + |\lambda\eta_{xx\tau}| + |w\eta_{xxx}| + |wu_{xx\tau}|dxd\tau.\end{aligned} \tag{2.8}$$

According to theorem 1.1, we have for $t\in[0,\varepsilon^{-1/2}T]$, $h=\eta,u,\zeta$, or v, and $0\le i\le m$,

$$\|\partial_x^i h\| \le \|h\|_m \le C_0 = C_0(f,g),$$

moreover, for $0\le i\le m-1$ and $t\in[0,\varepsilon^{-1/2}T]$,

$$\|\partial_x^i h\|_\infty \le \sqrt{2}(\|\partial_x^i h\|\|\partial_x^{i+1}h\|)^{\frac{1}{2}} \le \sqrt{2}C_0.$$

Let us find an upper bound of the inequality (2.8). On one hand, we have

$$\begin{aligned}\int_0^t\int_{-\infty}^{+\infty}\left|\frac{\lambda^2}{2}(w_x+u_x)\right|dxd\tau &\le \frac{1}{2}\sup_{\tau\in[0,\varepsilon^{-1/2}T]}(\|u_x(\tau)\|_\infty + \|w_x(\tau)\|_\infty) \\ &\le \frac{1}{2}\sup_{\tau\in[0,\varepsilon^{-1/2}T]}(2\|u_x(\tau)\|_\infty + \|v_x(\tau)\|_\infty)\int_0^t\|\lambda\|^2 d\tau \\ &\le \frac{3\sqrt{2}}{2}C_0\int_0^t\|\lambda\|^2 d\tau,\end{aligned}$$

$$\int_0^t\int_{-\infty}^{+\infty}\left|\frac{w^2}{2}v_x\right|dxd\tau \le \frac{1}{2}\sup_{\tau\in[0,\varepsilon^{-1/2}T]}\|v_x(\tau)\|_\infty\int_0^t\|w\|^2 d\tau \le \frac{\sqrt{2}}{2}C_0\int_0^t\|w\|^2 d\tau,$$

and, using the Cauchy-Schwarz inequality,

$$\int_0^t\int_{-\infty}^{+\infty}|\lambda_x\zeta w|dxd\tau \le \sup_{\tau\in[0,\varepsilon^{-1/2}T]}\|\zeta_x\|_\infty\int_0^t\|\lambda_x\|\|w\|d\tau \le \sqrt{2}C_0\int_0^t\|\lambda_x\|\|w\|d\tau.$$

On the other hand, the Cauchy-Schwarz inequality provides

$$\int_0^t\int_{-\infty}^{+\infty}|\lambda u_{xxx}|dxd\tau \le \int_0^t\|\lambda\|\|u_{xxx}\|d\tau \le C_0\int_0^t\|\lambda\|d\tau$$
$$\int_0^t\int_{-\infty}^{+\infty}|w\eta_{xxx}|dxd\tau \le \int_0^t\|w\|\|\eta_{xxx}\|d\tau \le C_0\int_0^t\|w\|d\tau.$$

However,

$$\begin{aligned}\eta_{xxt} &= -\partial_x^2(u_x+\varepsilon((\eta u)_x+u_{xxx}))\\ &= -(u_{xxx}+3\varepsilon\eta_{xx}u_x+3\varepsilon\eta_x u_{xx}+\varepsilon\eta u_{xxx}+\varepsilon\eta_{xxx}u+\varepsilon u_{xxxxx}),\end{aligned}$$

thus, for $t\in[0,\varepsilon^{-1/2}T]$,

$$\begin{aligned}\|\eta_{xxt}\| &\le \|u_{xxx}\|+3\varepsilon\|\eta_{xx}\|\|u_x\|_\infty+3\varepsilon\|\eta_x\|_\infty\|u_{xx}\|+\varepsilon\|\eta\|_\infty\|u_{xxx}\|+\varepsilon\|\eta_{xxx}\|\|u\|_\infty+\varepsilon\|u_{xxxxx}\|\\ &\le (1+\varepsilon)C_0+\varepsilon 8\sqrt{2}C_0^2.\end{aligned}$$

Thanks to the Cauchy-Schwarz inequality, we obtain

$$\int_0^t\int_{-\infty}^{+\infty}|\lambda\eta_{xx\tau}|dxd\tau \le \int_0^t\|\lambda\|\|\eta_{xx\tau}\|d\tau \le \Big((1+\varepsilon)C_0+\varepsilon 8\sqrt{2}C_0^2\Big)\int_0^t\|\lambda\|d\tau.$$

In the same way, we find for $t\in[0,\varepsilon^{-1/2}T]$,

$$\begin{aligned}\|u_{xxt}\| &= \|-\partial_x^2(\eta_x+\varepsilon(uu_x+\eta_{xxx}))\|\\ &\le (\|\eta_{xxx}\|+3\varepsilon\|u_x\|_\infty\|u_{xx}\|+\varepsilon\|u\|_\infty\|u_{xxx}\|+\varepsilon\|\eta_{xxxxx}\|)\\ &\le (1+\varepsilon)C_0+\varepsilon 4\sqrt{2}C_0^2,\end{aligned}$$

and then

$$\int_0^t\int_{-\infty}^{+\infty}|wu_{xx\tau}|dxd\tau \le \int_0^t\|w\|\|u_{xx\tau}\|d\tau \le \Big((1+\varepsilon)C_0+\varepsilon 4\sqrt{2}C_0^2\Big)\int_0^t\|w\|d\tau.$$

The inequality (2.8) is finally rewritten

$$\begin{aligned}\int_{-\infty}^{+\infty}(\lambda^2+w^2+\varepsilon\lambda_x^2+\varepsilon w_x^2)\,dx &\le 2\varepsilon\left(\frac{3\sqrt{2}}{2}C_0\int_0^t\|\lambda\|^2d\tau+\frac{\sqrt{2}}{2}C_0\int_0^t\|w\|^2d\tau+\sqrt{2}C_0\int_0^t\|\lambda_x\|\|w\|d\tau\right)\\ &\quad+2\varepsilon\left(\Big((2+\varepsilon)C_0+\varepsilon 8\sqrt{2}C_0^2\Big)\int_0^t\|\lambda\|d\tau+\Big((2+\varepsilon)C_0+\varepsilon 4\sqrt{2}C_0^2\Big)\int_0^t\|w\|d\tau\right).\end{aligned}$$

We define

$$A(t):=\left[\int_{-\infty}^{+\infty}(\lambda^2+w^2+\varepsilon\lambda_x^2+\varepsilon w_x^2)dx\right]^{1/2},$$

thus

$$\|\lambda\|^2+\|w\|^2\le A^2 \text{ and } \|\lambda_x\|^2+\|w_x\|^2\le\varepsilon^{-1}A^2,$$

and the inequality (2.8) becomes, using $2ab\le a^2+b^2$,

$$A^2(t)\le 2C_1\varepsilon\int_0^t A(\tau)d\tau+C_1\varepsilon^{1/2}\int_0^t A^2(\tau)d\tau\le 3C_1\varepsilon\int_0^t A(\tau)d\tau+C_1\int_0^t A^3(\tau)d\tau$$

The following Gronwall's lemma is used [2].

Lemma 2.3
Let $\alpha,\beta>0$ and $\rho>1$. Define

$$\widetilde{T}:=\beta^{-1/\rho}\alpha^{(1-\rho)/\rho}\int_0^{+\infty}(1+z^\rho)^{-1}dz$$

Then there exists a constant $C=C(\rho)>0$, independent of α and β such that if A is a non-negative, continuous function defined on $[0,\widetilde{T}]$ with $A(0)=0$ and for all $t\in[0,\widetilde{T}]$

$$A^2(t)\le\int_0^t \alpha A(\tau)+\beta A^{\rho+1}(\tau)d\tau,$$

then for all $t\in[0,\widetilde{T}]$,

$$A(t)\le C\alpha t.$$

Even if we choose T smaller, we conclude that, for $0 \le t \le \varepsilon^{-1/2}T$,

$$\left\| \begin{pmatrix} \eta(\cdot,t) \\ u(\cdot,t) \end{pmatrix} - \begin{pmatrix} \zeta(\cdot,t) \\ v(\cdot,t) \end{pmatrix} \right\|_{L^2} \le C\varepsilon t.$$

□

We can generalize this result to the space derivatives as follows.

Theorem 2.4 *Let $m \ge 0$ and $(f,g) \in H^{m+5}(\mathbb{R}) \times H^{m+5}(\mathbb{R})$. Let (η,u) and (ζ,v) be solution in $\mathcal{C}([-\varepsilon^{-1/2}T, \varepsilon^{-1/2}T]; H^{m+5}(\mathbb{R})) \times \mathcal{C}([-\varepsilon^{-1/2}T, \varepsilon^{-1/2}T]; H^{m+5}(\mathbb{R}))$ of KdV-KdV and BBM-BBM respectively, where*

$$T := \frac{C_{m+5}}{\|(f,g)\|_{m+5}}.$$

Then, for all $0 \le i \le m$, there exists a constant $M_i > 0$, depending only on $\|(f,g)\|_{m+5}$, such that for $|t| \le \varepsilon^{-1/2}T$, we have

$$\left\| \partial_x^i \begin{pmatrix} \eta(\cdot,t) \\ u(\cdot,t) \end{pmatrix} - \partial_x^i \begin{pmatrix} \zeta(\cdot,t) \\ v(\cdot,t) \end{pmatrix} \right\|_{L^2} \le M_i \varepsilon |t|.$$

Proof. The proof is done by induction on the derivative order. The preceding theorem gives the first step $i = 0$, and we assume that, for $0 \le i \le n-1$ and $0 \le t \le \varepsilon^{-1/2}T$, there exists $M_i = M_i(\|(f,g)\|_{(m+5)\times(m+5)}) > 0$ such that

$$\|\partial_x^i \lambda\|_{L^2} + \|\partial_x^i w\|_{L^2} = \left\| \partial_x^i \begin{pmatrix} \eta(\cdot,t) \\ u(\cdot,t) \end{pmatrix} - \partial_x^i \begin{pmatrix} \zeta(\cdot,t) \\ v(\cdot,t) \end{pmatrix} \right\|_{L^2} \le M_i\, \varepsilon t.$$

The result is now shown for the step n. Multiplying equations (2.5) and (2.6) by $\partial_x^{2n}\lambda$ and $\partial_x^{2n} w$ respectively, and integrating over space:

$$\begin{aligned} \int_{-\infty}^{+\infty} \partial_x^{2n}\lambda(\lambda_t + w_x - \varepsilon\lambda_{xxt}) + \partial_x^{2n} w(w_t + \lambda_x - \varepsilon w_{xxt})\, dx = -\varepsilon \int_{-\infty}^{+\infty} \partial_x^{2n}\lambda(u_{xxx} + \eta_{xxt}) + \partial_x^{2n} w(\eta_{xxx} + u_{xxt}) \\ - \varepsilon \int_{-\infty}^{+\infty} \partial_x^{2n}\lambda((\lambda w)_x + (\lambda u + \zeta w)_x) + \partial_x^{2n} w(w w_x + (vw)_x)\, dx. \end{aligned} \tag{2.9}$$

Concerning the left-hand side of (2.9), integration by parts provides

$$\int_{-\infty}^{+\infty} (\partial_x^{2n}\lambda\, w_x + \partial_x^{2n} w\, \lambda_x) dx = (-1)^n \int_{-\infty}^{+\infty} (\partial_x^n \lambda \partial_x^n w)_x\, dx = 0,$$

and

$$\int_{-\infty}^{+\infty} \partial_x^{2n}\lambda(\lambda_t - \varepsilon\lambda_{xxt}) + \partial_x^{2n} w(w_t - \varepsilon w_{xxt}) dx = \frac{(-1)^n}{2}\frac{d}{dt}\int_{-\infty}^{+\infty} (\partial_x^n\lambda)^2 + (\partial_x^n w)^2 + \varepsilon(\partial_x^{n+1}\lambda)^2 + \varepsilon(\partial_x^{n+1} w^2) dx.$$

The right-hand side of (2.9) is rewritten as follows.

$$\begin{aligned} -(-1)^n\varepsilon \int_{-\infty}^{+\infty} \partial_x^n\lambda\, \partial_x^{n+1}(\lambda w + \lambda u + \zeta w) + \partial_x^n w\, \partial_x^n(w w_x) + \partial_x^n w\, \partial_x^{n+1}(vw)\, dx \\ - (-1)^n\varepsilon \int_{-\infty}^{+\infty} \partial_x^n\lambda\, \partial_x^{n+3} u + \partial_x^n\lambda\, \partial_x^{n+2}\eta_t + \partial_x^n w\, \partial_x^{n+3}\eta + \partial_x^n w\, \partial_x^{n+2} u_t\, dx. \end{aligned}$$

Let us define

$$B^2(t) := \int_{-\infty}^{+\infty} (\partial_x^n\lambda)^2 + (\partial_x^n w)^2 + \varepsilon(\partial_x^{n+1}\lambda)^2 + \varepsilon(\partial_x^{n+1} w)^2\, dx.$$

Since $\lambda(\cdot,0)=0$ and $w(\cdot,0)=0$, the equation (2.9) becomes

$$\begin{aligned}B^2(t)=&-2\varepsilon\int_0^t\int_{-\infty}^{+\infty}\partial_x^n\lambda\,\partial_x^{n+1}(\lambda w+\lambda u+\zeta w)+\partial_x^n w\,\partial_x^n(ww_x)+\partial_x^n w\,\partial_x^{n+1}(vw)\,dxd\tau\\&-2\varepsilon\int_0^t\int_{-\infty}^{+\infty}\partial_x^n\lambda\,\partial_x^{n+3}u+\partial_x^n\lambda\,\partial_x^{n+2}\eta_\tau+\partial_x^n w\,\partial_x^{n+3}\eta+\partial_x^n w\,\partial_x^{n+2}u_\tau\,dxd\tau.\end{aligned}$$

Thanks to the Leibniz formula, we find

$$\begin{aligned}\int_0^t\int_{-\infty}^{+\infty}\partial_x^n\lambda\,\partial_x^{n+1}(\lambda u)dxd\tau&=\int_0^t\int_{-\infty}^{+\infty}\partial_x^n\lambda\left(\sum_{i=0}^{n+1}\binom{n+1}{i}\partial_x^{n+1-i}\lambda\,\partial_x^i u\right)dxd\tau\\&=\int_0^t\int_{-\infty}^{+\infty}\partial_x^n\lambda\,\partial_x^{n+1}\lambda\,u+(n+1)(\partial_x^n\lambda)^2u_xdxd\tau\\&\quad+\int_0^t\int_{-\infty}^{+\infty}\sum_{i=2}^{n+1}\binom{n+1}{i}\partial_x^n\lambda\,\partial_x^{n+1-i}\lambda\,\partial_x^i udxd\tau.\end{aligned}$$

However

$$\int_0^t\int_{\mathbb{R}}\partial_x^n\lambda\,\partial_x^{n+1}\lambda\,udxd\tau=\int_0^t\int_{\mathbb{R}}\frac{1}{2}\partial_x(\partial_x^n\lambda)^2\,udxd\tau=-\int_0^t\int_{\mathbb{R}}\frac{1}{2}(\partial_x^n\lambda)^2\,u_xdxd\tau,$$

then we have

$$\int_0^t\int_{\mathbb{R}}\partial_x^n\lambda\,\partial_x^{n+1}(\lambda u)dxd\tau=\int_0^t\int_{\mathbb{R}}(n+\frac{1}{2})(\partial_x^n\lambda)^2\,u_x+\sum_{i=2}^{n+1}\binom{n+1}{i}\partial_x^n\lambda\,\partial_x^{n+1-i}\lambda\,\partial_x^i u\,dxd\tau.$$

In the same manner, it gets

$$\int_0^t\int_{\mathbb{R}}\partial_x^n w\,\partial_x^{n+1}(vw)dxd\tau=\int_0^t\int_{\mathbb{R}}(n+\frac{1}{2})(\partial_x^n w)^2\,v_x+\sum_{i=2}^{n+1}\binom{n+1}{i}\partial_x^n w\,\partial_x^{n+1-i}w\,\partial_x^i v\,dxd\tau,$$

and

$$\int_0^t\int_{\mathbb{R}}\partial_x^n w\,\partial_x^n(ww_x)dxd\tau=\int_0^t\int_{\mathbb{R}}(2n-\frac{1}{2})(\partial_x^n w)^2\,w_x+\sum_{i=2}^{n-1}\binom{n}{i}\partial_x^n w\,\partial_x^i w\,\partial_x^{n+1-i}w\,dxd\tau.$$

Thus,

$$\begin{aligned}B^2(t)=&-2\varepsilon\int_0^t\int_{\mathbb{R}}\Big[(n+\frac{1}{2})(\partial_x^n\lambda)^2\,(w_x+u_x)+(n+\frac{1}{2})(\partial_x^n w)^2v_x+(2n-\frac{1}{2})(\partial_x^n w)^2\,w_x\\&+\partial_x^n\lambda\,\partial_x^{n+1}\zeta\partial_x^n w+\partial_x^n\lambda\,\partial_x^{n+1}w\partial_x^n\zeta+\sum_{i=2}^{n-1}\binom{n}{i}\partial_x^n w\,\partial_x^i w\,\partial_x^{n+1-i}w\\&+\sum_{i=2}^{n+1}\binom{n+1}{i}(\partial_x^n\lambda\,\partial_x^{n+1-i}\lambda\,(\partial_x^i\,w+\partial_x^i\,u)+\partial_x^n w\partial_x^{n+1-i}w\,\partial_x^i v+\partial_x^n\lambda\partial_x^{n+1-i}w\,\partial_x^i\zeta)\\&+\partial_x^n\,\lambda\,\partial_x^{n+3}u+\partial_x^n\lambda\,\partial_x^{n+2}\eta_\tau+\partial_x^n w\,\partial_x^{n+3}\eta+\partial_x^n w\,\partial_x^{n+2}u_\tau\Big]\,dxd\tau.\end{aligned}\tag{2.10}$$

Recall that, for $t\in[0,\varepsilon^{-1/2}T]$, $h=\eta,u,\zeta,v$ and $0\le i\le m+4$, we have

$$\|\partial_x^i h\|_\infty\le\sqrt{2}(\|\partial_x^i h\|\|\partial_x^{i+1}h\|)^{\frac{1}{2}}\le\sqrt{2}C_0.$$

We deduce

$$\begin{aligned}\int_0^t\int_{\mathbb{R}}\left|(n+\frac{1}{2})(\partial_x^n\lambda)^2(w_x+u_x)\right|dxd\tau &\le (n+\frac{1}{2})\int_0^t\|\partial_x^n\lambda\|^2(\|w_x\|_\infty+\|u_x\|_\infty)d\tau\\ &\le (n+\frac{1}{2})2\sqrt{2}C_0\int_0^t\|\partial_x^n\lambda\|^2d\tau,\end{aligned}\tag{2.20-a}$$

similarly

$$\int_0^t\int_{\mathbb{R}}\left|(n+\frac{1}{2})(\partial_x^n w)^2v_x\right|dxd\tau \le (n+\frac{1}{2})\sqrt{2}C_0\int_0^t\|\partial_x^n w\|^2d\tau,\tag{2.20-b}$$

$$\int_0^t\int_{\mathbb{R}}\left|(2n-\frac{1}{2})(\partial_x^n w)^2w_x\right|dxd\tau \le (2n-\frac{1}{2})2\sqrt{2}C_0\int_0^t\|\partial_x^n w\|^2d\tau,\tag{2.20-c}$$

and

$$\int_0^t\int_{\mathbb{R}}|\partial_x^n\lambda\,\partial_x^{n+1}\zeta\partial_x^n w+\partial_x^n\lambda\,\partial_x^{n+1}w\partial_x^n\zeta|dxd\tau \le \sqrt{2}C_0\left(\int_0^t\|\partial_x^n\lambda\|\|\partial_x^n w\|d\tau+\int_0^t\|\partial_x^n\lambda\|\|\partial_x^{n+1}w\|d\tau\right).$$

Using the Cauchy-Schwarz inequality, we obtain

$$\int_0^t\int_{\mathbb{R}}\sum_{i=2}^{n+1}\binom{n+1}{i}\left|\partial_x^n\lambda\partial_x^{n+1-i}\lambda\partial_x^i u\right|dxd\tau \le \sum_{i=2}^{n+1}\binom{n+1}{i}\int_0^t\|\partial_x^n\lambda\|\|\partial_x^{n+1-i}\lambda\|\|\partial_x^i u\|_\infty d\tau,$$

and the inductive assumption provides, for $0\le t\le\varepsilon^{-1/2}T$,

$$\begin{aligned}\int_0^t\int_{\mathbb{R}}\sum_{i=2}^{n+1}\binom{n+1}{i}\left|\partial_x^n\lambda\partial_x^{n+1-i}\lambda\partial_x^i u\right|dxd\tau &\le \left(\sum_{i=2}^{n+1}\binom{n+1}{i}M_{n+1-i}\right)\int_0^t\varepsilon\tau\|\partial_x^n\lambda\|d\tau\\ &\le \left(\sum_{i=2}^{n+1}\binom{n+1}{i}M_{n+1-i}\varepsilon^{1/2}T^{1/2}\right)\int_0^t\|\partial_x^n\lambda\|d\tau.\end{aligned}$$

Same computations give

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}}\sum_{i=2}^{n+1}\binom{n+1}{i}|(\partial_x^n\lambda\,\partial_x^{n+1-i}\lambda\,(\partial_x^i w+\partial_x^i u)+\partial_x^n w\partial_x^{n+1-i}w\,\partial_x^i v+\partial_x^n\lambda\partial_x^{n+1-i}w\,\partial_x^i\zeta)|\\ &\qquad\le 3\left(\sum_{i=2}^{n+1}\binom{n+1}{i}M_{n+1-i}\varepsilon^{1/2}T^{1/2}\right)\int_0^t\|\partial_x^n\lambda\|+\|\partial_x^n w\|d\tau.\end{aligned}$$

and

$$\int_0^t\int_{\mathbb{R}}\sum_{i=2}^{n-1}\binom{n}{i}\left|\partial_x^n w\partial_x^i w\partial_x^{n+1-i}w\right|dxd\tau \le \left(\sum_{i=2}^{n-1}\binom{n}{i}M_i\varepsilon^{1/2}T^{1/2}\right)\int_0^t\|\partial_x^n w\|d\tau,\tag{2.20-f}$$

On the other hand, from the Cauchy-Schwarz inequality, it gets

$$\begin{aligned}\int_0^t\int_{\mathbb{R}}\left|\partial_x^n\lambda\partial_x^{n+3}u\right|dxd\tau &\le \int_0^t\|\partial_x^n\lambda\|\|\partial_x^{n+3}u\|d\tau\le\sqrt{2}C_0\int_0^t\|\partial_x^n\lambda\|d\tau,\\ \int_0^t\int_{\mathbb{R}}\left|\partial_x^n w\partial_x^{n+3}\eta\right|dxd\tau &\le \int_0^t\|\partial_x^n w\|\|\partial_x^{n+3}\eta\|d\tau\le\sqrt{2}C_0\int_0^t\|\partial_x^n w\|d\tau.\end{aligned}$$

Since

$$\begin{aligned}\partial_x^{n+2}\eta_t &= -\partial_x^{n+2}(u_x+\varepsilon(\eta u)_x+\varepsilon u_{xxx})\\ &= -(\partial_x^{n+3}u+\varepsilon\sum_{i=0}^{n+3}\binom{n+3}{i}\partial_x^i\eta\,\partial_x^{n+3-i}u+\varepsilon\partial_x^{n+5}u),\end{aligned}$$

it implies

$$\int_0^t \int_{\mathbb{R}} \left|\partial_x^n \lambda \partial_x^{n+2}\eta_t\right| dx d\tau \leq \int_0^t \|\partial_x^n \lambda\| \|\partial_x^{n+2}\eta_t\| d\tau \leq \left((1+\varepsilon)C_0 + \sqrt{2} C_0^2 \varepsilon \sum_{i=0}^{n+3} \binom{n+3}{i} \right) \int_0^t \|\partial_x^n \lambda\| d\tau.$$

In the same way, we have

$$\int_0^t \int_{\mathbb{R}} \varepsilon |\partial_x^n w \partial_x^{n+2} u_t| dx d\tau \leq \int_0^t \varepsilon \|\partial_x^n w\| \|\partial_x^{n+2} u_t\| d\tau \leq \left((1+\varepsilon)C_0 + \sqrt{2} C_0^2 \varepsilon \sum_{i=0}^{n+2} \binom{n+2}{i} \right) \int_0^t \|\partial_x^n w\| d\tau, \tag{2.20 - k}$$

because

$$\begin{aligned} \partial_x^{n+2} u_t &= -\partial_x^{n+2}(\eta_x + \varepsilon u u_x + \varepsilon \eta_{xxx}) \\ &= -(\partial_x^{n+3}\eta + \varepsilon \sum_{i=0}^{n+2} \binom{n+2}{i} \partial_x^i u \, \partial_x^{n+3-i} u + \varepsilon \partial_x^{n+5}\eta). \end{aligned}$$

For $t \in [0, \varepsilon^{-1/2}T]$, the function $B(t)$ being such that

$$\|\partial_x^n \lambda\|^2 + \|\partial_x^n w\|^2 \leq B^2(t), \text{ and } \|\partial_x^{n+1}\lambda\|^2 + \|\partial_x^{n+1} w\|^2 \leq \varepsilon^{-1} B^2(t),$$

the equation (2.10) finally becomes

$$B^2(t) \leq 2C_n \varepsilon \int_0^t B(\tau) d\tau + C_n \varepsilon^{1/2} \int_0^t B^2(\tau) d\tau \leq 3 C_n \varepsilon \int_0^t B(\tau) d\tau + C_n \int_0^t B^3(\tau) d\tau.$$

We conclude using the Gronwall's lemma as for the initial step. □

Corollary 2.5 *Let $m \geq 1$ and $(f,g) \in H^{m+5}(\mathbb{R}) \times H^{m+5}(\mathbb{R})$. Let (η, u) and (ζ, v) be solution in $\mathcal{C}([-\varepsilon^{-1/2}T, \varepsilon^{-1/2}T]; H^{m+5}(\mathbb{R})) \times \mathcal{C}([-\varepsilon^{-1/2}T, \varepsilon^{-1/2}T]; H^{m+5}(\mathbb{R}))$ of KdV-KdV and BBM-BBM respectively, where*

$$T := \frac{C_{m+5}}{\|(f,g)\|_{m+5}}.$$

Then, for all $0 \leq i \leq m-1$, there exists a constant $N_i > 0$, depending only on $\|(f,g)\|_{m+5}$, such that for $|t| \leq \varepsilon^{-1/2}T$, we have

$$\left\| \partial_x^i \begin{pmatrix} \eta(\cdot,t) \\ u(\cdot,t) \end{pmatrix} - \partial_x^i \begin{pmatrix} \zeta(\cdot,t) \\ v(\cdot,t) \end{pmatrix} \right\|_{L^\infty} \leq N_i \varepsilon |t|.$$

Proof. Using

$$\|\partial_x^i(\eta-\zeta)\|_\infty \leq \sqrt{2}(\|\partial_x^i(\eta-\zeta)\| \|\partial_x^{i+1}(\eta-\zeta)\|)^{1/2},$$

the preceding theorem is applied with $0 \leq i \leq m$ and $|t| \leq \varepsilon^{-1/2}T$, to give

$$\|\partial_x^i(\eta-\zeta)\|_\infty \leq \sqrt{2}(M_i \varepsilon |t| M_{i+1} \varepsilon |t|)^{1/2} = \sqrt{2 M_i M_{i+1}} \varepsilon |t|,$$

and

$$\|\partial_x^i(u-v)\|_\infty \leq \sqrt{2 M_i M_{i+1}} \varepsilon |t|.$$

It is enough to choose $N_i = 2\sqrt{2 M_i M_{i+1}}$. □

Remark 2.6 *Concerning the generalized Boussinesq systems, the difference between the respective solutions can be written as follows:*

$$\begin{aligned} \lambda_t + w_x + \varepsilon (a_2 w_{xxx} - b_2 \lambda_{xxt}) &= -\varepsilon ((\lambda w)_x + (\lambda u + \zeta w)_x + (a_1 - a_2) u_{xxx} + (b_2 - b_1)\eta_{xxt}) \\ w_t + \lambda_x + \varepsilon (c_2 \lambda_{xxx} - d_2 w_{xxt}) &= -\varepsilon (w w_x + (v w)_x + (c_2 - c_1)\eta_{xxx} + (d_2 - d_1) u_{xxt}) \\ \lambda(x,0) = 0, &\qquad w(x,0) = 0, \end{aligned}$$

where $\lambda = \eta_1 - \eta_2, w = u_1 - u_2$. The rest of the proof is similar to the previous cases.

3 Numerical comparison

The aim of this section is not to perform a complete numerical study, but rather to present some relevant simulations and to explore if the result is optimal and if the time comparison is longer than the theoretical one. Therefore we have an overview of the way in which solutions of each system evolve and differ.
To simulate the system, a Crank-Nicolson scheme is used to discretize the time-derivative, and the fast Fourier transform is applied for the space derivatives. The obtained non-linear problem is solved using fixed point iterations.
We consider a bounded domain $[-L, L], L > 0$ a large fixed value. We denote $N_x > 0$ the number of Fourier modes, $\Delta t > 0$ the time step, and for $n \in \mathbb{N}$, $\hat{\eta}_n$, resp. $\hat{u}_n$, is the approximation of $\hat{\eta}$, resp. $\hat{u}$, at time $n\Delta t$.

Algorithm: By denoting by $m \geq 1$ a maximal number of the Picard iterations, the algorithm is described as follows:

- Set $\widehat{\eta}_{0,m} := \ldots := \widehat{\eta}_{0,1} := \widehat{\eta}_{0,0} := \widehat{\eta_0}$ and $\widehat{u}_{0,m} := \ldots := \widehat{u}_{0,1} := \widehat{u}_{0,0} := \widehat{u_0}$.
- For $n = 0, 1, \ldots,$ compute:
  - For $r = 0, 1, \ldots, m-1,$

$$\begin{aligned}\left(I + \frac{\Delta t}{2}A(\xi)\right)\begin{pmatrix} \widehat{\eta}_{n+1,r+1} \\ \widehat{u}_{n+1,r+1} \end{pmatrix} &= \left(I - \frac{\Delta t}{2}A(\xi)\right)\begin{pmatrix} \widehat{\eta}_n \\ \widehat{u}_n \end{pmatrix} \\ &\quad -\frac{i\varepsilon\xi\Delta t}{4}\begin{pmatrix} \frac{1}{1+\varepsilon b\xi^2}\widehat{(\eta_{n+1,r} + \eta_n)(u_{n+1,r} + u_n)} \\ \frac{1}{2(1+\varepsilon d\xi^2)}\widehat{(u_{n+1,r} + u_n)^2} \end{pmatrix},\end{aligned} \tag{3.1}$$

where, $\xi = k\frac{\pi}{L}, -\frac{N_x}{2} \leq k \leq \frac{N_x}{2} - 1,$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \; A(\xi) = i\xi\begin{pmatrix} 0 & \frac{1-\varepsilon a\xi^2)}{1+\varepsilon d\xi^2} \\ \frac{1-\varepsilon c\xi^2}{1+\varepsilon d\xi^2} & 0 \end{pmatrix}.$$

The iterations are stopped in one of the two following cases:

- when $\left(\frac{\|\eta_{n+1,r+1}-\eta_{n+1,r}\|_{l^2}}{\|\eta_{n+1,0}\|_{l^2}} + \frac{\|u_{n+1,r+1}-u_{n+1,r}\|_{l^2}}{\|u_{n+1,0}\|_{l^2}}\right) \leq \tau$, with $\tau > 0$ a fixed tolerance.
- or when $r = m - 1$. We mention that the step Δt can be reduced in this case, in order to improve the previous relative error.

We then set $\eta_{n+1} := \eta_{n+1,r+1}$ and $u_{n+1} := u_{n+1,r+1}$.

Remark 3.1 *We choose a suitable step $\Delta t > 0$ as follows:*

$$\Delta t < \frac{\Delta x}{\pi\varepsilon(2 + \varepsilon(a + c)(\pi/\Delta x)^2)(\|\eta_n\|_{l^\infty} + \|u_n\|_{l^\infty})}. \tag{3.2}$$

Indeed, from (3.1),

$$\begin{aligned}&\left(I + \frac{\Delta t}{2}A(\xi)\right)\begin{pmatrix} \widehat{\eta}_{n+1} - \widehat{\eta}_{n+1,r+1} \\ \widehat{u}_{n+1} - \widehat{u}_{n+1,r+1} \end{pmatrix} \\ &\qquad = -\frac{i\varepsilon\xi\Delta t}{4}\begin{pmatrix} \frac{1}{1+\varepsilon b\xi^2}\left(\widehat{(\eta_{n+1} + \eta_n)(u_{n+1} + u_n)} - \widehat{(\eta_{n+1,r} + \eta_n)(u_{n+1,r})}\right) \\ \frac{1}{2(1+\varepsilon d\xi^2)}\left(\widehat{(u_{n+1} + u_n)^2} - \widehat{(u_{n+1,r} + u_n)^2}\right) \end{pmatrix}.\end{aligned}$$

The nonlinear term can be written as

$$\begin{pmatrix} \frac{(\eta_{n+1}-\eta_{n+1,r})(u_{n+1}+u_{n+1,r})}{2} + \frac{(\eta_{n+1}+\eta_{n+1,r})\widehat{(u_{n+1}-u_{n+1,r})}}{2} + u_n(\eta_{n+1}-\eta_{n+1,r}) + \eta_n(u_{n+1}-u_{n+1,r}) \\ (u_{n+1}-u_{n+1,r})\widehat{(u_{n+1}} + 2u_n + u_{n+1,r}) \end{pmatrix}$$

and a linearization, near (η_n, u_n), in the right-hand side of this relation leads to the approximation:

$$\left(I + \frac{\Delta t}{2}A(\xi)\right)\begin{pmatrix} \widehat{\eta}_{n+1} - \widehat{\eta}_{n+1,r+1} \\ \widehat{u}_{n+1} - \widehat{u}_{n+1,r+1} \end{pmatrix} \approx -\frac{i\varepsilon\xi\Delta t}{4}\begin{pmatrix} \frac{1}{1+\varepsilon b\xi^2}\left(u_n\widehat{(\eta_{n+1}-\eta_{n+1,r}) + \eta_n}(u_{n+1}-u_{n+1,r})\right) \\ \frac{1}{2(1+\varepsilon d\xi^2)}\left(u_n\widehat{(u_{n+1}-u_{n+1,r})}\right) \end{pmatrix}.$$

The matrix $\left(I + \frac{\Delta t}{2}A(\xi)\right)$ is invertible if, for all ξ,

$$\Delta t < \frac{2(1+\varepsilon b\xi^2)(1+\varepsilon d\xi^2)}{|\xi(1-\varepsilon a\xi^2)(1-\varepsilon c\xi^2)|},$$

and it follows according to the Parseval's formula,

$$\begin{aligned}
&\|\eta_{n+1,r+1}-\eta_{n+1}\|_{l^2} + \|u_{n+1,r+1}-u_{n+1}\|_{l^2} \\
&\quad\lesssim \frac{\varepsilon|\xi|\Delta t}{4}\left[\frac{1}{1+\varepsilon b\xi^2}\left(1+\frac{|1-\varepsilon c\xi^2|}{1+\varepsilon d\xi^2}\right)(\|u_n\|_{l^\infty}\|\eta_{n+1,r+1}-\eta_{n+1}\|_{l^2} + \|\eta_n\|_{l^\infty}\|u_{n+1,r+1}-u_{n+1}\|_{l^2})\right. \\
&\quad\left. + \frac{1}{1+\varepsilon d\xi^2}\left(1+\frac{|1-\varepsilon a\xi^2|}{1+\varepsilon b\xi^2}\right)(\|u_n\|_{l^\infty}\|u_{n+1,r+1}-u_{n+1}\|_{l^2})\right].
\end{aligned}$$

However $\xi = k\frac{\pi}{L}$, with $-\frac{N_x}{2} \le k \le \frac{N_x}{2} - 1$, and $N_x = \frac{2L}{\Delta x}$, it follows

$$\begin{aligned}
\|\eta_{n+1,r+1}-\eta_{n+1}\|_{l^2} + \|u_{n+1,r+1}-u_{n+1}\|_{l^2} \quad &\lesssim \quad \frac{\varepsilon\pi\Delta t}{4\Delta x}\left(2+\varepsilon(a+c)(\pi/\Delta x)^2\right)(\|\eta_n\|_{l^\infty} + \|u_n\|_{l^\infty}) \\
&\quad (\|\eta_{n+1,r}-\eta_{n+1}\|_{l^2} + \|u_{n+1,r+1}-u_{n+1}\|_{l^2}).
\end{aligned}$$

Remark 3.2 *When $b, d > 0$ and since $|\xi|/(1+\xi^2) \le 1$, a better choice for the time step $\Delta t > 0$ can be done as follows:*

$$\Delta t < \frac{1}{\varepsilon(\|\eta_n\|_{l^\infty} + \|u_n\|_{l^\infty})}.$$

Simulations are performed with such a time step Δt, and with $L = 100, N_x = 2^{10}$. Similar results were obtained with smaller space steps. We start from the initial datum:

$$\eta_0(x) = \frac{15}{4}\left(-2+\cosh(3\sqrt{2/5}x)\right)\operatorname{sech}^4\left(\frac{3}{\sqrt{10}}x\right), u_0(x) = 3w\operatorname{sech}^2\left(\frac{3}{\sqrt{10}}x\right),$$

with $w = \pm 5/2$. This initial datum provides a solitary wave solution of the BBM-BBM system with $\varepsilon = 1, a = c = 0, b = d = 1/6$, given by $\eta(x,t) = \eta_0(x - wt)$ and $u(x,t) = u_0(x - wt)$. To ensure the convergence of the scheme, the following conservation law is tested : $\forall t \in \mathbb{R}$,

$$\int_{-\infty}^{+\infty} \eta^2(x,t) + u^2(x,t) + \varepsilon b\partial_x\eta^2(x,t) + \varepsilon d\partial_x u^2(x,t)dx = \int_{-\infty}^{+\infty} \eta_0^2(x) + u_0^2(x) + \varepsilon b\partial_x\eta_0^2(x) + \varepsilon d\partial_x u_0^2(x)dx.$$

Figure 1 shows the evolution over time of the solution, the L^∞−norm, the conservation law above, and the error between the approximate solution and the solitary wave.

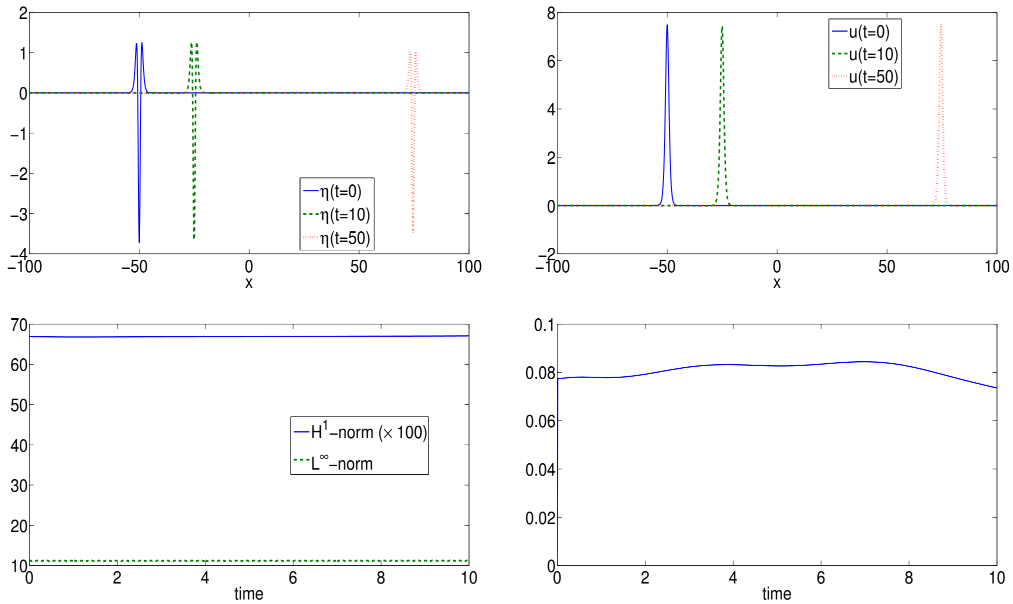

Figure 1: Result for $\varepsilon = 1, a = c = 0, b = d = 1$. On the top left, the approximated amplitude η at time $t = 0, 10, 50$. On the top right, the approximated velocity u at time $t = 0, 10, 50$. On the bottom left, the H^1 and $L^\infty-$norms. On the bottom right, the error between the approximated solution and the solitary wave.

Based on these results, the numerical scheme appears to be relevant for the simulations. The approximate solution remains close to the exact soliton, and the conservation laws are well preserved.

We represent in Figure 2 the evolution of the solution starting from a localized wave defined as follows

$$\eta_0(x) = u_0(x) = \alpha \exp(-x^2/2^2).$$

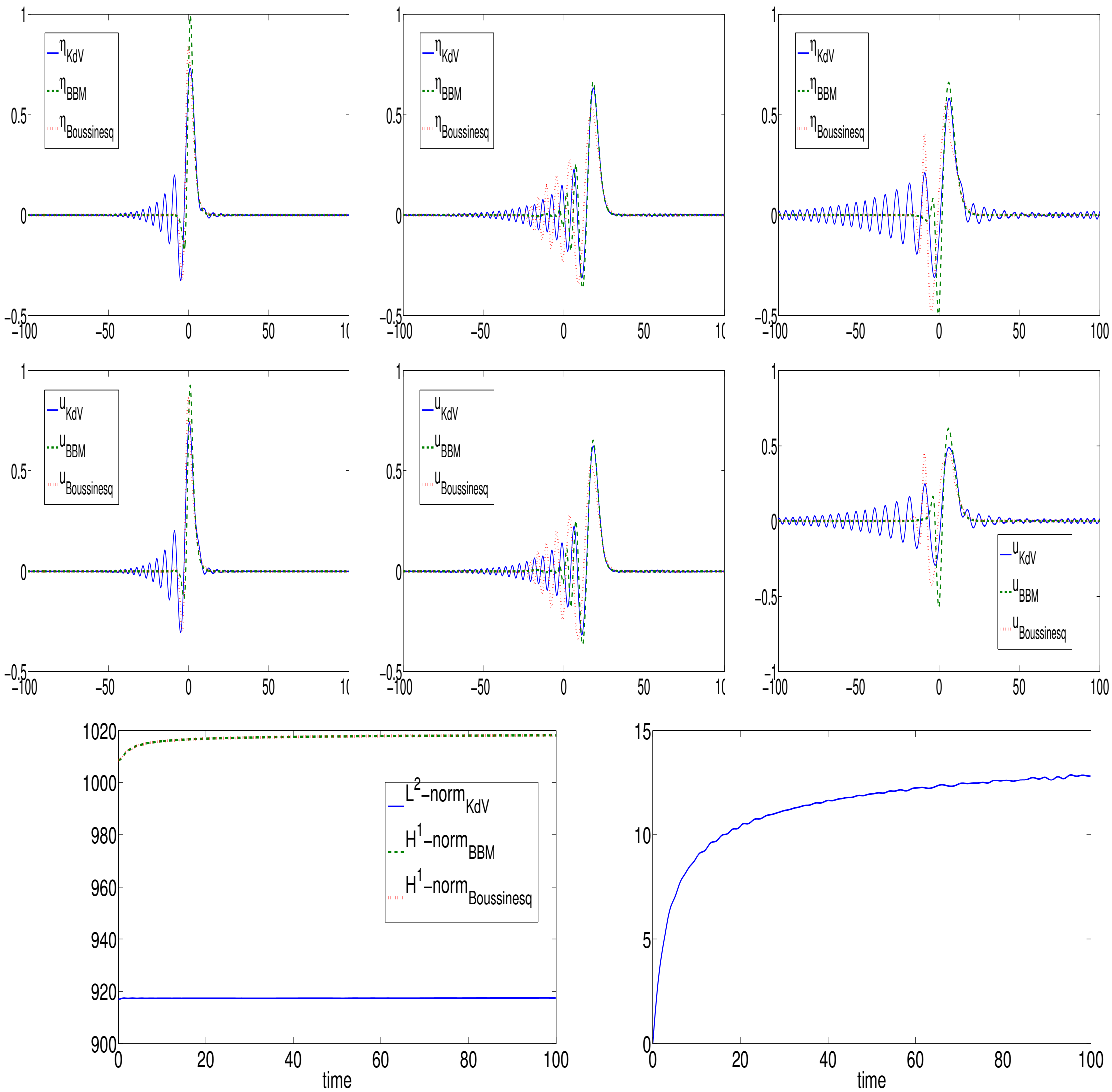

Figure 2: Result for $\varepsilon = 0.1, \alpha = 1$. On the top, the approximated amplitude η at time $t = 1/\sqrt{\varepsilon}, 1/\varepsilon, 1/\varepsilon^2$. At the center, the approximated velocity u at time $t = 1/\sqrt{\varepsilon}, 1/\varepsilon, 1/\varepsilon^2$. On the bottom left, the conservation laws. On the bottom right, the difference between the approximated solution of the Boussinesq system ($a = b = c = d = 1$), the KdV-KdV system ($a = c = 1, b = d = 0$) and the BBM-BBM system ($a = c = 0, b = d = 1$).

The solution of the KdV-KdV system disperses faster, the solution of the BBM-BBM system being the slowest. The difference between the solutions increases rapidly, the slowdown is due to the decay of solutions. Between $t = 1/\sqrt{\varepsilon}$, and $t = 1/\varepsilon$, the difference has doubled. We also noticed that conservation laws are well preserved.

We inspect in Figure 3 the influence of the parameter ε and of the amplitude of initial data.

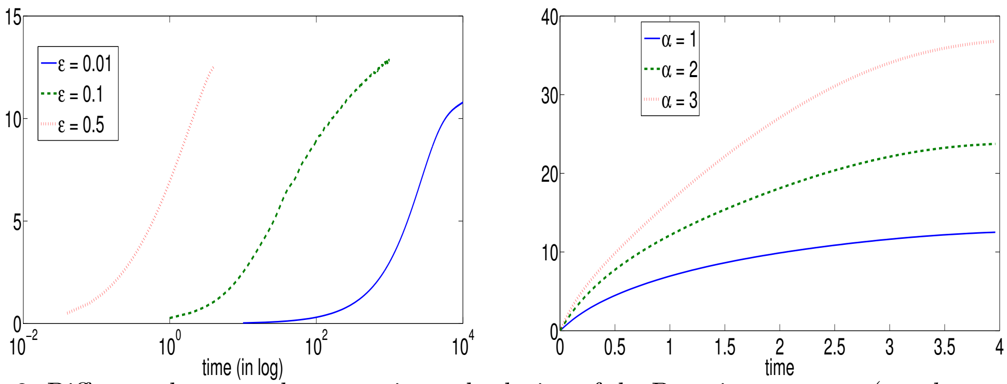

Figure 3: Difference between the approximated solution of the Boussinesq system ($a = b = c = d = 1$), the KdV-KdV system ($a = c = 1, b = d = 0$) and the BBM-BBM system ($a = c = 0, b = d = 1$). At left, result for $\varepsilon = 0.5, 0.1, 0.01$ and $\alpha = 1$. At right, result for $\varepsilon = 0.1$ and $\alpha = 1, 2, 3$.

We find that the difference between solution increases with ε. Nevertheless, the growth is quickly done during the first iterations. On the other hand, the constants depending on the norm of the initial data, it appears that the difference between the solutions is even greater than the amplitude of the solution is great and the dispersion is slow.

To sum up, when α and ε are sufficiently small, the solutions behave like the linear case. It is due to the dispersion, and thus the smallness of the solutions. Then the solutions remain close and the result can be improved. If α and ε become large, the solutions differ quickly. Nevertheless, it is possible to improve the result by selecting special solutions, e.g. by choosing one-way propagation solution [1, 7].

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