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SHARP REGULARITY PROPERTIES FOR THE NON-CUTOFF SPATIALLY HOMOGENEOUS BOLTZMANN EQUATION

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Abstract. In this work, we study the Cauchy problem for the spatially homogeneous non-cutoff Boltzamnn equation with Maxwellian molecules. We prove that this Cauchy problem enjoys Gelfand-Shilov regularizing effect, that means the smoothing properties is same as the Cauchy problem defined by the evolution equation associated to a fractional harmonic oscillator, the power of fractional is exactly the singular index of non-cutoff collisional kernel of Boltzmann equation. So that we get the sharp regularity of solution in the Gevery class and also the sharp decay of solutions with an exponential weighted. We also give a method to construct the solution of the Boltzmann equation by solving an infinite systems of ordinary differential equations. The key tools is the spectral decomposition of linear and non-linear Boltzmann operators.

Contents

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1. Introduction

In this work, we consider the spatially homogeneous Boltzmann equation

$$\begin{cases} \partial_t f = Q(f,f), \\ f|_{t=0} = f_0, \end{cases} \tag{1.1}$$

where $f = f(t,v)$ is the density distribution function depends on the variables $t \geq 0$ and $v \in \mathbb{R}^3$. The Boltzmann bilinear collision operator is given by

$$Q(g,f)(v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-v_*,\sigma)(g(v'_*)f(v') - g(v_*)f(v))dv_*d\sigma,$$

where for $\sigma \in \mathbb{S}^2$, the symbols v'_* and v' are abbreviations for the expressions,

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \quad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma,$$

which are obtained in such a way that collision preserves momentum and kinetic energy, namely

$$v'_* + v' = v + v_*, \quad |v'_*|^2 + |v'|^2 = |v|^2 + |v_*|^2.$$

The non-negative cross section $B(z,\sigma)$ depends only on $|z|$ and the scalar product $\frac{z}{|z|}\cdot\sigma$. For physical models, it usually takes the form

$$B(v-v_*,\sigma) = \Phi(|v-v_*|)b(\cos\theta), \ \cos\theta = \frac{v-v_*}{|v-v_*|}\cdot\sigma, \ 0 \leq \theta \leq \frac{\pi}{2}.$$

In this paper, we consider only the Maxwellian molecules case which is corresponded to $\Phi \equiv 1$, and we focus our attention on the angular part b satisfying

$$\beta(\theta) = 2\pi b(\cos 2\theta)|\sin 2\theta| \approx |\theta|^{-1-2s}, \ \text{when } \theta \to 0^+, \tag{1.2}$$

for some $0 < s < 1$, without loss of generality, we may assume that $b(\cos\theta)$ is supported on the set $\cos\theta \geq 0$. See for example [11] for more explanations of $\beta(\cdot)$ and [22] for general collision kernel.

We linearize the Boltzmann equation near the absolute Maxwellian distribution

$$\mu(v) = (2\pi)^{-\frac{3}{2}}e^{-\frac{|v|^2}{2}}.$$

Let $f(t,v) = \mu(v) + \sqrt{\mu}(v)g(t,v)$, we have

$$\frac{\partial g}{\partial t} + \mathcal{L}[g] = \mathbf{\Gamma}(g,g)$$

with

$$\mathbf{\Gamma}(g,h) = \frac{1}{\sqrt{\mu}}Q(\sqrt{\mu}g, \sqrt{\mu}h), \ \mathcal{L}(g) = -\frac{1}{\sqrt{\mu}}[Q(\sqrt{\mu}g,\mu) + Q(\mu,\sqrt{\mu}g)].$$

Then the Cauchy problem (1.1) can be re-writed in the form

$$\begin{cases} \partial_t g + \mathcal{L}(g) = \mathbf{\Gamma}(g,g), \\ g|_{t=0} = g_0. \end{cases} \tag{1.3}$$

The linear operator $\mathcal{L}$ is nonnegative ([11, 12, 13]), with the null space

$$\mathcal{N} = \text{span}\left\{\sqrt{\mu}, \ \sqrt{\mu}v_1, \ \sqrt{\mu}v_2, \ \sqrt{\mu}v_3, \ \sqrt{\mu}|v|^2\right\}.$$

In the present work, we study the smoothing effect for the Cauchy problem associated to the spatially homogeneous non-cutoff Boltzmann equation with Maxwellian molecules. It is well known that the non-cutoff spatially homogeneous Boltzmann equation enjoy an $\mathscr{S}(\mathbb{R}^3)$-regularizing effect for the weak solutions to the Cauchy problem (see [4, 16]). Regarding the Gevrey regularity, Ukai showed in [21] that the Cauchy problem for the Boltzmann equation has a unique local solution in Gevrey classes. Then, Desvillettes, Furioli and Terraneo proved in [3] the propagation of Gevrey regularity for solutions of the Boltzmann equation with Maxwellian molecules. For mild singularities, Morimoto and Ukai proved in [15] the Gevrey regularity of smooth Maxwellian decay solutions to the Cauchy problem of the spatially homogeneous Boltzmann equation with a modified kinetic factor, see also [26] for non modified case. On the other hand, Lekrine and Xu proved in [10] the property of Gevrey smoothing effect for the weak solutions to the Cauchy problem associated to the radially symmetric spatially homogeneous Boltzmann equation with Maxwellian molecules for $0 < s < 1/2$. This result was then completed by Glangetas and Najeme who established in [6] the analytic smoothing effect in the case when $1/2 < s < 1$. In [11, 14], the linearized non-cutoff Boltzmann operator was shown to behave essentially as a fractional harmonic oscillator $\mathcal{H}^s$, with $0 < s < 1$ and $\mathcal{H} = -\triangle + \frac{|v|^2}{4}$ ([11] for radially symmetric case and [14] for general case). The solutions of the following Cauchy problem

$$\begin{cases} \partial_t g + \mathcal{L}(g) = 0, \\ g|_{t=0} = g_0 \in L^2, \end{cases}$$

belong to the symmetric Gelfand-Shilov spaces $S^{1/2s}_{1/2s}(\mathbb{R}^3)$ for any positive time and

$$\|e^{ct\mathcal{H}^s} g(t)\|_{L^2} \leq C\|g_0\|_{L^2},$$

where the Gelfand-Shilov spaces $S^{\mu}_{\nu}(\mathbb{R}^3)$, with $\mu,\ \nu > 0, \mu + \nu \geq 1,$ is the spaces of smooth functions $f \in C^{+\infty}(\mathbb{R}^3)$ satisfying:

$$\exists\, A > 0,\ C > 0,\ \sup_{v\in\mathbb{R}^3} |v^{\beta}\partial_v^{\alpha} f(v)| \leq CA^{|\alpha|+|\beta|}(\alpha!)^{\mu}(\beta!)^{\nu},\ \ \forall\, \alpha, \beta \in \mathbb{N}^3.$$

These Gelfand-Shilov spaces can be also characterized as the sub-space of Schwartz functions $f \in \mathscr{S}(\mathbb{R}^3)$ such that,

$$\exists\, C > 0,\ \epsilon > 0,\ |f(v)| \leq Ce^{-\epsilon|v|^{\frac{1}{\nu}}},\ \ v \in \mathbb{R}^3 \text{ and } |\hat{f}(\xi)| \leq Ce^{-\epsilon|\xi|^{\frac{1}{\mu}}},\ \ \xi \in \mathbb{R}^3.$$

The symmetric Gelfand-Shilov space $S^{\nu}_{\nu}(\mathbb{R}^3)$ with $\nu \geq \frac{1}{2}$ can be also identity with

$$S^{\nu}_{\nu}(\mathbb{R}^3) = \left\{ f \in C^{\infty}(\mathbb{R}^3); \exists \tau > 0, \|e^{\tau\mathcal{H}^{\frac{1}{2\nu}}} f\|_{L^2} < +\infty \right\}.$$

See Appendix 7 for more properties of Gelfand-Shilov spaces.

From a historical point of view, the spectral analysis is a critical method of the linear Boltzmann operator([2]). In [11], the linearized non-cutoff radially symmetric Boltzmann operator is shown to be diagonal in the Hermite basis. The application of this diagonalization is appeared in their continue work [13], which showed that the Cauchy problem to the non-cutoff spatially homogeneous Boltzmann equation with the radial initial datum $g_0 \in L^2(\mathbb{R}^3)$ has a unique global radial solution and belongs to the Gelfand-Shilov class $S^{\frac{1}{2s}}_{\frac{1}{2s}}(\mathbb{R}^3)$.

In this paper, we use the spectral decomposition of linearize operator to study the Cauchy problem (1.3) in general case. The main theorem is in the following.

Theorem 1.1. *Assume that the Maxwellian collision cross-section* $b(\cdot)$ *is given in* (1.2) *with* $0 < s < 1$*, then there exists* $\varepsilon_0 > 0$ *such that for any initial datum* $g_0 \in L^2(\mathbb{R}^3) \bigcap \mathcal{N}^\perp$ *with* $\|g_0\|^2_{L^2(\mathbb{R}^3)} \le \varepsilon_0$*, the Cauchy problem* (1.3) *admits a solution belongs to the Gelfand-Shilov spaces* $S^{\frac{1}{2s}}_{\frac{1}{2s}}(\mathbb{R}^3)$ *for any* $t > 0$*. Moreover, there exists* $c_0 > 0$*, such that, for any* $t \ge 0$*,*

$$\|e^{c_0 t \mathcal{H}^s} g(t)\|_{L^2(\mathbb{R}^3)} \le Ce^{-\frac{\lambda_{2,0}}{2}t}\|g_0\|_{L^2(\mathbb{R}^3)}, \tag{1.4}$$

where

$$\lambda_{2,0} = \int_{-\pi/4}^{\pi/4} \beta(\theta)(1 - \sin^4\theta - \cos^4\theta)d\theta > 0.$$

Remark 1.2. *We have proved that for the Cauchy problem* (1.1)*, if the initial data is a small perturbation of Maxwellian in* L^2*, then the global solution return to the equilibrium with an exponential rate with respect to Gelfand-Shilov norm, that means with a exponential weighted in both sides of Fourier transformation of solution.*

The rest of the paper is arranged as follows. In Section 2, we introduce the spectral analysis of the linear and nonlinear Boltzmann operator, and transform the nonlinear Cauchy Problem of Boltzmann equation to an infinite systems of ordinary differential equation which can be solved explicitly, then we get the formal solution of the Cauchy problem for Boltzmann equation. In Section 3, we establish the upper bounded estimates of nonlinear operators with an exponential weighted, which is crucial to get the convergence of formal solution in Gelfand-Shilov space. The proof of the main Theorem 1.1 will be presented in Section 4. Finally, the Section 5 and the Section 6 is devoted to the proof of 2 propositions used in Section 4. In the Section 5, we study the spectral representation of non linear Boltzmann operators, and prove that it can be represented by an "inferior triangular matrix" of infinite dimension with three index, so that the presentation and the computations are very complicate. This inferior triangular property is essential for the construction of the formal solution by solving an infinite system of ordinary differential equations. In Section 6, we consider the eigenvalues estimate of the triangular matrix obtained in the Section 5, it is keys point to prove the convergence of the formal solution with respect to Gelfand-Shilov norm.

2. The spectral analysis of the Boltzmann operators

2.1. **Diagonalization of linear operators.** We first recall the spectral decomposition of linear Boltzmann operator. In the cutoff case, that is, when $b(\cos\theta)\sin\theta \in L^1([0, \frac{\pi}{2}])$, it was shown in [23] that

$$\mathcal{L}(\varphi_{n,l,m}) = \lambda_{n,l}\,\varphi_{n,l,m},\ \ n, l \in \mathbb{N},\ m \in \mathbb{Z}, |m| \le l.$$

This diagonalization of the linearized Boltzmann operator with Maxwellian molecules holds as well in the non-cutoff case, (see [1, 2, 5, 11, 12]). The eigenvalues are

$$\lambda_{n,l} = \int_{|\theta|\le\frac{\pi}{4}} \beta(\theta)\Big(1 + \delta_{n,0}\delta_{l,0} - (\sin\theta)^{2n+l}P_l(\sin\theta) - (\cos\theta)^{2n+l}P_l(\cos\theta)\Big)d\theta,$$

the eigenfunctions are

$$\varphi_{n,l,m}(v) = \left(\frac{n!}{\sqrt{2}\Gamma(n+l+3/2)}\right)^{1/2}\left(\frac{|v|}{\sqrt{2}}\right)^l e^{-\frac{|v|^2}{4}} L_n^{(l+1/2)}\left(\frac{|v|^2}{2}\right) Y_l^m\left(\frac{v}{|v|}\right), \tag{2.1}$$

where $\Gamma(\,\cdot\,)$ is the standard Gamma function, for any $x > 0$,

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-x} dx.$$

The l^{th}-Legendre polynomial P_l and the Laguerre polynomial $L_n^{(\alpha)}$ of order α, degree n (see [20]) read,

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l, \text{ where } |x| \leq 1;$$
$$L_n^{(\alpha)}(x) = \sum_{r=0}^{n} (-1)^{n-r} \frac{\Gamma(\alpha + n + 1)}{r!(n-r)!\Gamma(\alpha + n - r + 1)} x^{n-r}.$$

For any unit vector $\sigma = (\cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi)$ with $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$, the orthonormal basis of spherical harmonics $Y_l^m(\sigma)$ is

$$Y_l^m(\sigma) = N_{l,m} P_l^{|m|}(\cos\theta) e^{im\phi}, \ |m| \leq l,$$

where the normalisation factor is given by

$$N_{l,m} = \sqrt{\frac{2l+1}{4\pi} \cdot \frac{(l - |m|)!}{(l + |m|)!}}$$

and $P_l^{|m|}$ is the associated Legendre functions of the first kind of order l and degree $|m|$ with

$$P_l^{|m|}(x) = (1 - x^2)^{\frac{|m|}{2}} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{|m|} P_l(x). \tag{2.2}$$

The family $\left(Y_l^m(\sigma)\right)_{l \geq 0, |m| \leq l}$ constitutes an orthonormal basis of the space $L^2(\mathbb{S}^2, d\sigma)$ with $d\sigma$ being the surface measure on $\mathbb{S}^2$ (see [9], [19]). Noting that $\{\varphi_{n,l,m}(v)\}$ consist an orthonormal basis of $L^2(\mathbb{R}^3)$ composed of eigenvectors of the harmonic oscillator (see[1], [12])

$$\mathcal{H}(\varphi_{n,l,m}) = (2n + l + \frac{3}{2})\, \varphi_{n,l,m}.$$

As a special case, $\{\varphi_{n,0,0}(v)\}$ consist an orthonormal basis of $L^2_{rad}(\mathbb{R}^3)$, the radially symmetric function space (see [13]). We have that, for suitable functions g,

$$\mathcal{L}(g) = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \lambda_{n,l}\, g_{n,l,m}\, \varphi_{n,l,m},$$

where $g_{n,l,m} = (g, \varphi_{n,l,m})_{L^2(\mathbb{R}^3)}$, and

$$\mathcal{H}(g) = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (2n + l + \frac{3}{2})\, g_{n,l,m}\, \varphi_{n,l,m}\,.$$

Using this spectral decomposition, the definition of $\mathcal{L}^s, \mathcal{H}^s$, $e^{c\mathcal{H}^s}$, $e^{c\mathcal{L}^s}$ is then classical.

2.2. **Triangular effect of the non linear operators.** We study now the algebra property of the nonlinear terms

$$\mathbf{\Gamma}(\varphi_{n,l,m}, \varphi_{\tilde{n},\tilde{l},\tilde{m}}),$$

We have the following triangular effect for the nonlinear Boltzmann operators on the basis $\{\varphi_{n,l,m}\}$.

Proposition 2.1. *The following algebraic identities hold,*

$$(i_1)\qquad \mathbf{\Gamma}(\varphi_{0,0,0},\varphi_{\tilde{n},\tilde{l},\tilde{m}})=\lambda^1_{\tilde{n},\tilde{l}}\,\varphi_{\tilde{n},\tilde{l},\tilde{m}};$$

$$(i_2)\qquad \mathbf{\Gamma}(\varphi_{n,l,m},\varphi_{0,0,0})=\lambda^2_{n,l}\,\varphi_{n,l,m};$$

$$(ii_1)\qquad \mathbf{\Gamma}(\varphi_{n,0,0},\varphi_{\tilde{n},\tilde{l},\tilde{m}})=\lambda^{rad,1}_{n,\tilde{n},\tilde{l}}\,\varphi_{n+\tilde{n},\tilde{l},\tilde{m}},\ \ for\ n\geq 1;$$

$$(ii_2)\qquad \mathbf{\Gamma}(\varphi_{n,l,m},\varphi_{\tilde{n},0,0})=\lambda^{rad,2}_{n,\tilde{n},l}\,\varphi_{n+\tilde{n},l,m},\ \ for\ n\in\mathbb{N},\ \ l\geq 1;$$

$$(iii)\qquad \mathbf{\Gamma}(\varphi_{n,l,m},\varphi_{\tilde{n},\tilde{l},\tilde{m}})=\sum_{k=0}^{k_0(l,\tilde{l},m,\tilde{m})}\mu^{m,\tilde{m},m+\tilde{m}}_{n,\tilde{n},l,\tilde{l},k}\,\varphi_{n+\tilde{n}+k,l+\tilde{l}-2k,m+\tilde{m}}\ \ with$$

$$k_0(l,\tilde{l},m,\tilde{m})=\min\left(\Big[\frac{l+\tilde{l}-|m+\tilde{m}|}{2}\Big],l,\tilde{l}\right)\tag{2.3}$$

for $l\geq 1,\ \tilde{l}\geq 1, |m|\leq l,\ |\tilde{m}|\leq\tilde{l}$.

The notations in the above Proposition are as following:

$$\lambda^1_{\tilde{n},\tilde{l}}=\int_{|\theta|\leq\frac{\pi}{4}}\beta(\theta)((\cos\theta)^{2\tilde{n}+\tilde{l}}P_{\tilde{l}}(\cos\theta)-1)d\theta;$$

$$\lambda^2_{n,l}=\int_{|\theta|\leq\frac{\pi}{4}}\beta(\theta)((\sin\theta)^{2n+l}P_l(\sin\theta)-\delta_{0,n}\delta_{0,l})d\theta;$$

$$\lambda^{rad,1}_{n,\tilde{n},\tilde{l}}=\Big(\frac{2\pi^{\frac{3}{2}}(n+\tilde{n})!\Gamma(n+\tilde{n}+\tilde{l}+\frac{3}{2})}{\tilde{n}!\Gamma(\tilde{n}+\tilde{l}+\frac{3}{2})n!\Gamma(n+\frac{3}{2})}\Big)^{\frac{1}{2}}\times\int_{|\theta|\leq\frac{\pi}{4}}\beta(\theta)(\sin\theta)^{2n}(\cos\theta)^{2\tilde{n}+\tilde{l}}P_{\tilde{l}}(\cos\theta)d\theta;$$

$$\lambda^{rad,2}_{n,\tilde{n},l}=\Big(\frac{2\pi^{\frac{3}{2}}(n+\tilde{n})!\Gamma(n+\tilde{n}+l+\frac{3}{2})}{\tilde{n}!\Gamma(\tilde{n}+\frac{3}{2})n!\Gamma(n+l+\frac{3}{2})}\Big)^{\frac{1}{2}}\times\int_{|\theta|\leq\frac{\pi}{4}}\beta(\theta)(\sin\theta)^{2n+l}(\cos\theta)^{2\tilde{n}}P_l(\sin\theta)d\theta$$

and for $|m^\star|\leq l+\tilde{l}-2k$,

$$\mu^{m,\tilde{m},m^\star}_{n,\tilde{n},l,\tilde{l},k}=(-1)^k\Big(\frac{2\pi^{\frac{3}{2}}(n+\tilde{n}+k)!\Gamma(n+\tilde{n}+l+\tilde{l}-k+\frac{3}{2})}{\tilde{n}!\Gamma(\tilde{n}+\tilde{l}+\frac{3}{2})n!\Gamma(n+l+\frac{3}{2})}\Big)^{\frac{1}{2}}\times\int_{\mathbb{S}^2_\kappa}\Big[\int_{\mathbb{S}^2}b(\kappa\cdot\sigma)\Big(\frac{|\kappa-\sigma|}{2}\Big)^{2n+l}\Big(\frac{|\kappa+\sigma|}{2}\Big)^{2\tilde{n}+\tilde{l}}\times Y^m_l\Big(\frac{\kappa-\sigma}{|\kappa-\sigma|}\Big)Y^{\tilde{m}}_{\tilde{l}}\Big(\frac{\kappa+\sigma}{|\kappa+\sigma|}\Big)d\sigma\Big]\overline{Y^{m^\star}_{l+\tilde{l}-2k}}(\kappa)d\kappa.$$

We remark that $\mu^{m,\tilde{m},m^\star}_{n,\tilde{n},l,\tilde{l},k}$ vanishes to 0 if $m+\tilde{m}\neq m^\star$, so

$$\mu^{m,\tilde{m},m^\star}_{n,\tilde{n},l,\tilde{l},k}=\mu^{m,\tilde{m},m^\star}_{n,\tilde{n},l,\tilde{l},k}\delta_{m^\star,m+\tilde{m}}.\tag{2.4}$$

The coefficient $\mu^{m,\tilde{m},m^\star}_{n,\tilde{n},l,\tilde{l},k}$ satisfies the following orthogonal property.

Proposition 2.2. *For any integers* $0\leq\ k_1,\ k_2\leq\ \min(l,\tilde{l})$, $|m^\star_1|\leq\ l+\tilde{l}-2k_1,\ |m^\star_2|\leq\ l+\tilde{l}-2k_2$, *we have*

$$\sum_{|m|\leq l}\sum_{|\tilde{m}|\leq\tilde{l}}\mu^{m,\tilde{m},m^\star_1}_{n,\tilde{n},l,\tilde{l},k_1}\overline{\mu^{m,\tilde{m},m^\star_2}_{n,\tilde{n},l,\tilde{l},k_2}}=\sum_{|m|\leq l}\sum_{|\tilde{m}|\leq\tilde{l}}\Big|\mu^{m,\tilde{m},m^\star_1}_{n,\tilde{n},l,\tilde{l},k_1}\Big|^2\delta_{k_1,k_2}\delta_{m^\star_1,m^\star_2}.\tag{2.5}$$

Remark 2.3. *1) Similar to the radially symmetric case, the property* (iii) *of the Proposition 2.1 and the above Proposition 2.2 imply that we have also a "triangular effect" but with a noise of order* $k_0(l, \tilde{l}, m, \tilde{m})$. *It is not very clear in the above presentation since we are in* 3*-dimension, we will understand well in the Subsection 2.3*

2) We have also

$$\lambda_{n,l} = -\lambda^1_{n,l} - \lambda^2_{n,l}.$$

It is trivial to obtain that $\lambda_{0,0} = \lambda_{1,0} = \lambda_{0,1} = 0$ *and the others are strictly positive, since when* $l \neq 0$, *and for* $n \neq 0, 1$,

$$\lambda_{n,l} \geq 2 \int_0^{\frac{\pi}{4}} (1 - \sin^{2n}\theta - \cos^{2n}\theta)\beta(\theta)d\theta = \lambda_{n,0} > 0.$$

Moreover, we referred from Theorem 2.2 in [12] *(see also Theorem 2.3 in* [14]*) that, there exists a constant* $0 < c_1 < 1$ *dependent on* s *such that, for any* $n, l \in \mathbb{N}$ *and* $n + l \geq 2$,

$$c_1[(2n + l + \frac{3}{2})^s + l^{2s}] \leq \lambda_{n,l} \leq \frac{1}{c_1}[(2n + l + \frac{3}{2})^s + l^{2s}]\,. \tag{2.6}$$

We send the proof of this Proposition 2.1 to Section 5.

2.3. **Formal and explicit solution of the Cauchy problem.** Now we solve explicitly the Cauchy problem associated to the non-cutoff spatially homogeneous Boltzmann equation with Maxwellian molecules. Consider the solution to the Cauchy problem (1.3) in the form

$$g(t) = \sum_{n=0}^{+\infty}\sum_{l=0}^{+\infty}\sum_{|m|\leq l} g_{n,l,m}(t)\varphi_{n,l,m},$$

with initial data

$$g(0) = \sum_{n=0}^{+\infty}\sum_{l=0}^{+\infty}\sum_{|m|\leq l} g^0_{n,l,m}\varphi_{n,l,m} \in L^2(\mathbb{R}^3),$$

where

$$g_{n,l,m}(t) = \left(g(t),\, \varphi_{n,l,m}\right)_{L^2(\mathbb{R}^3)}, \qquad g^0_{n,l,m} = \left(g_0,\, \varphi_{n,l,m}\right)_{L^2(\mathbb{R}^3)}.$$

In the following, we will use the short notation

$$\sum_{n,l,m}^{+\infty} = \sum_{n=0}^{+\infty}\sum_{l=0}^{+\infty}\sum_{|m|\leq l}\ .$$

In our view, this summation is divided into three terms, which is

$$\sum_{n=0}^{+\infty}\sum_{l=0}^{+\infty}\sum_{|m|\leq l} f_{n,l,m} = f_{0,0,0} + \sum_{n=1}^{+\infty} f_{n,0,0} + \sum_{n=0}^{+\infty}\sum_{l=1}^{+\infty}\sum_{|m|\leq l} f_{n,l,m}. \tag{2.7}$$

It follows from Proposition 2.1 and the above decomposition (2.7) that, for convenable function g, we have

$$
\begin{aligned}
\mathbf{\Gamma}(g,g) = & \sum_{\tilde n=0}^{+\infty}\sum_{\tilde l=0}^{+\infty}\sum_{|\tilde m|\le \tilde l} g_{0,0,0}(t) g_{\tilde n,\tilde l,\tilde m}(t)\Big(\lambda^1_{\tilde n,\tilde l}+\lambda^2_{\tilde n,\tilde l}\Big)\varphi_{\tilde n,\tilde l,\tilde m} + g_{0,0,0}(t)g_{0,0,0}(t)\mathbf{\Gamma}(\varphi_{0,0,0},\varphi_{0,0,0})\\
&+\sum_{n=1}^{+\infty}\sum_{\tilde n=1}^{+\infty} g_{n,0,0}(t)g_{\tilde n,0,0}(t)\lambda^{rad,1}_{n,\tilde n,0}\varphi_{\tilde n+n,0,0}\\
&+\sum_{n=1}^{+\infty}\sum_{\tilde n=0}^{+\infty}\sum_{\tilde l=1}^{+\infty}\sum_{|\tilde m|\le \tilde l} g_{n,0,0}(t)g_{\tilde n,\tilde l,\tilde m}(t)\lambda^{rad,1}_{n,\tilde n,\tilde l}\varphi_{\tilde n+n,\tilde l,\tilde m}\\
&+\sum_{n=0}^{+\infty}\sum_{l=1}^{+\infty}\sum_{\tilde n=1}^{+\infty}\sum_{|m|\le l} g_{n,l,m}(t)g_{\tilde n,0,0}(t)\lambda^{rad,2}_{n,\tilde n,l}\varphi_{\tilde n+n,l,m}\\
&+\sum_{n=0}^{+\infty}\sum_{l=1}^{+\infty}\sum_{\tilde n=0}^{+\infty}\sum_{\tilde l=1}^{+\infty}\sum_{|m|\le l}\sum_{|\tilde m|\le \tilde l}\sum_{k=0}^{k_0(l,\tilde l,m,\tilde m)} g_{n,l,m}(t)g_{\tilde n,\tilde l,\tilde m}(t)\mu^{m,\tilde m,m+\tilde m}_{n,\tilde n,l,\tilde l,k}\ \varphi_{n+\tilde n+k,\,l+\tilde l-2k,\,m+\tilde m},
\end{aligned}
$$

where $k_0(l,\tilde l,m,\tilde m)$ was given in (2.3). Since for fixed $n,\tilde n,l,\tilde l\in\mathbb{N}$,

$$
\begin{aligned}
&\Big\{(m,\tilde m,k)\in\mathbb{Z}^2\times\mathbb{N}; |m|\le l, |\tilde m|\le \tilde l, 0\le k\le k_0(l,\tilde l,m,\tilde m)\Big\}\\
&=\Big\{(m,\tilde m,k)\in\mathbb{Z}^2\times\mathbb{N}; |m|\le l, |\tilde m|\le \tilde l, 0\le k\le \min(l,\tilde l), |m+\tilde m|\le l+\tilde l-2k\Big\},
\end{aligned}
$$

we obtain by change the order of the summation

$$
\sum_{|m|\le l}\sum_{|\tilde m|\le\tilde l}\sum_{k=0}^{k_0(l,\tilde l,m,\tilde m)} = \sum_{k=0}^{\min(l,\tilde l)}\sum_{\substack{|m|\le l,|\tilde m|\le\tilde l\\ |m+\tilde m|\le l+\tilde l-2k}} \tag{2.8}
$$

where $\sum_{\substack{|m|\le l,|\tilde m|\le\tilde l\\ |m+\tilde m|\le l+\tilde l-2k}}$ is the double summation of m and $\tilde m$ with constraints $|m+\tilde m|\le l+\tilde l-2k$.
Using

$$
\mathbf{\Gamma}(\varphi_{0,0,0},\varphi_{0,0,0}) = \mathbf{\Gamma}(\sqrt{\mu},\sqrt{\mu})=0,\quad \lambda^1_{\tilde n,\tilde l}+\lambda^2_{\tilde n,\tilde l} = -\lambda_{\tilde n,\tilde l}
$$

and the formula (2.8), $\mathbf{\Gamma}(g,g)$ can be rewritten as

$$
\begin{aligned}
\mathbf{\Gamma}(g,g) = & -g_{0,0,0}(t)\sum_{\tilde n=0}^{+\infty}\sum_{\tilde l=0}^{+\infty}\sum_{|\tilde m|\le\tilde l} g_{\tilde n,\tilde l,\tilde m}(t)\lambda_{\tilde n,\tilde l}\varphi_{\tilde n,\tilde l,\tilde m}\\
&+\sum_{n=1}^{+\infty}\sum_{\tilde n=1}^{+\infty} g_{n,0,0}(t)g_{\tilde n,0,0}(t)\lambda^{rad,1}_{n,\tilde n,0}\varphi_{\tilde n+n,0,0}\\
&+\sum_{n=1}^{+\infty}\sum_{\tilde n=0}^{+\infty}\sum_{\tilde l=1}^{+\infty}\sum_{|\tilde m|\le\tilde l} g_{n,0,0}(t)g_{\tilde n,\tilde l,\tilde m}(t)\left(\lambda^{rad,1}_{n,\tilde n,\tilde l}+\lambda^{rad,2}_{\tilde n,n,\tilde l}\right)\varphi_{\tilde n+n,\tilde l,\tilde m}\\
&+\sum_{n=0}^{+\infty}\sum_{l=1}^{+\infty}\sum_{\tilde n=0}^{+\infty}\sum_{\tilde l=1}^{+\infty}\sum_{k=0}^{\min(l,\tilde l)}\sum_{\substack{|m|\le l,|\tilde m|\le\tilde l\\ |m+\tilde m|\le l+\tilde l-2k}} g_{n,l,m}(t)g_{\tilde n,\tilde l,\tilde m}(t)\mu^{m,\tilde m,m+\tilde m}_{n,\tilde n,l,\tilde l,k}\ \varphi_{n+\tilde n+k,\,l+\tilde l-2k,\,m+\tilde m}.
\end{aligned}
$$

For convenable function g, we also have

$$
\mathcal{L}g = \sum_{n,l,m}^{+\infty}\lambda_{n,l}\,g_{n,l,m}(t)\,\varphi_{n,l,m}.
$$

Formally, we take inner product with $\overline{\varphi_{n^\star,l^\star,m^\star}}$ on both sides of (1.3), we find that the functions $\{g_{n^\star,l^\star,m^\star}(t)\}$ satisfy the following infinite system of the differential equations

$$\begin{aligned}
&\partial_t g_{n^\star,l^\star,m^\star}(t) + \lambda_{n^\star,l^\star}(1+g_{0,0,0}(t))g_{n^\star,l^\star,m^\star}(t)\\
&\quad= \delta_{l^\star,0}\sum_{\substack{n+\tilde{n}=n^\star\\ n\geq 1,\tilde{n}\geq 1}} \lambda^{rad,1}_{n,\tilde{n},0}\, g_{n,0,0}(t)g_{\tilde{n},0,0}(t)\\
&\qquad+\sum_{\substack{n+\tilde{n}=n^\star\\ n\geq 1,\tilde{n}\geq 0,\tilde{l}\geq 1}} \delta_{\tilde{l},l^\star}\left(\lambda^{rad,1}_{n,\tilde{n},l^\star}+\lambda^{rad,2}_{\tilde{n},n,l^\star}\right) g_{\tilde{n},l^\star,m^\star}(t)g_{n,0,0}(t)\\
&\qquad+\sum_{(n,\tilde{n},l,\tilde{l},k,m,\tilde{m})\in\Delta_{n^\star,l^\star,m^\star}} \mu^{m,\tilde{m},m^\star}_{n,\tilde{n},l,\tilde{l},k}\, g_{n,l,m}(t)g_{\tilde{n},\tilde{l},\tilde{m}}(t)
\end{aligned}$$

with initial data

$$g_{n^\star,l^\star,m^\star}(0) = g^0_{n^\star,l^\star,m^\star} \tag{2.9}$$

and where

$$\begin{aligned}
\Delta_{n^\star,l^\star,m^\star} = \Big\{&(n,\tilde{n},l,\tilde{l},k,m,\tilde{m})\in\mathbb{N}^5\times\mathbb{Z}^2;\\
&l\geq 1,\tilde{l}\geq 1, 0\leq k\leq \min(l,\tilde{l}), |m|\leq l, |\tilde{m}|\leq\tilde{l}\\
&\text{and } n+\tilde{n}+k=n^\star,\ l+\tilde{l}-2k=l^\star,\ m+\tilde{m}=m^\star\Big\},
\end{aligned}$$

which is a subset of a hyperplane of dimension 4.

Remark 2.4. *The summation in the last term of the previous equation for $\partial_t g_{n^\star,l^\star,m^\star}(t)$ is a bit complicated. For the sake of simplicity, it will be convenient to abuse the notation in this summation and write*

$$\begin{aligned}
&\partial_t g_{n^\star,l^\star,m^\star}(t) + \lambda_{n^\star,l^\star}(1+g_{0,0,0}(t))g_{n^\star,l^\star,m^\star}(t)\\
&\quad= \delta_{l^\star,0}\sum_{\substack{n+\tilde{n}=n^\star\\ n\geq 1,\tilde{n}\geq 1}} \lambda^{rad,1}_{n,\tilde{n},0}\, g_{n,0,0}(t)g_{\tilde{n},0,0}(t)\\
&\qquad+\sum_{\substack{n+\tilde{n}=n^\star\\ n\geq 1,\tilde{n}\geq 0,\tilde{l}\geq 1}} \delta_{\tilde{l},l^\star}\left(\lambda^{rad,1}_{n,\tilde{n},l^\star}+\lambda^{rad,2}_{\tilde{n},n,l^\star}\right) g_{\tilde{n},l^\star,m^\star}(t)g_{n,0,0}(t)\\
&\qquad+\sum_{\substack{n+\tilde{n}+k=n^\star\\ n\geq 0,\tilde{n}\geq 0}}\ \sum_{\substack{l+\tilde{l}-2k=l^\star\\ l\geq 1,\tilde{l}\geq 1\\ 0\leq k\leq\min(l,\tilde{l})}}\ \sum_{\substack{m+\tilde{m}=m^\star\\ |m|\leq l,|\tilde{m}|\leq\tilde{l}}} \mu^{m,\tilde{m},m^\star}_{n,\tilde{n},l,\tilde{l},k}\, g_{n,l,m}(t)g_{\tilde{n},\tilde{l},\tilde{m}}(t).
\end{aligned}\tag{2.10}$$

Here and after, we always use this notation.

Theorem 2.5. *For any initial data $\{g^0_{n^\star,l^\star,m^\star};\ n^\star,l^\star\in\mathbb{N}, |m^\star|\leq l^\star\}$ with*

$$g^0_{0,0,0}=g^0_{1,0,0}=g^0_{0,1,0}=g^0_{0,1,1}=g^0_{0,1,-1}=0, \tag{2.11}$$

the system (2.10) *admits a global solution $\{g_{n^\star,l^\star,m^\star}(t);\ n^\star,l^\star\in\mathbb{N}, |m^\star|\leq l^\star\}$.*

Remark 2.6. *Using the triangular effect property of Proposition 2.1, we solve explicitly the infinite system of differential equation* (2.10) *with any initial data $\{g^0_{n^\star,l^\star,m^\star};\ n^\star,l^\star\in\mathbb{N}, |m^\star|\leq l^\star\}$, in particulary, we don't ask it belong to ℓ^2. That means we can construct the formal solution for any initial data $g_0\in\mathscr{S}'(\mathbb{R}^3)$.*

Proof. Formally, the system (2.10) is non linear of quadratic form, but the infinite matrix of this quadratic form is in fact inferior triangular (see [13] for radially symmetric case with the simple index). Since the sequence is defined by multi-index, we prove this property by the following different case, and in each case by induction.

Induction on $n^\star \in \mathbb{N}$:

(1) the case : $n^\star = 0$. We prove now the existence of $\{g_{0,l^\star,m^\star}(t);\ l^\star \in \mathbb{N}, |m^\star| \le l^\star\}$ by induction on $l^\star \in \mathbb{N}$. From the assumption (2.11), and $\lambda_{0,0} = \lambda_{0,1} = \lambda_{1,0} = 0$, one get that

$$g_{0,0,0}(t) = g_{1,0,0}(t) = g_{0,1,0}(t) = g_{0,1,1}(t) = g_{0,1,-1}(t) = 0.$$

Let now $l^\star \ge 1$, we put the following assumption of induction:
(H-1) : *For any $l \le l^\star - 1$, $|m| \le l$, the functions $g_{0,l,m}(t)$ solve the equation* (2.10) *with initial data* (2.9)*.*

We consider the following equation

$$\partial_t g_{0,l^\star,m^\star}(t) + \lambda_{0,l^\star} g_{0,l^\star,m^\star}(t) = \sum_{\substack{l+\tilde{l}=l^\star\\ l\ge1,\tilde{l}\ge1}} \sum_{\substack{m+\tilde{m}=m^\star\\ |m|\le l,|\tilde{m}|\le\tilde{l}}} \mu^{m,\tilde{m},m^\star}_{0,0,l,\tilde{l},0}\, g_{0,l,m}(t) g_{0,\tilde{l},\tilde{m}}(t).$$

This differential equation can be solved since the functions $g_{0,l,m}(t)$ on the right hand side are only involving the functions $\{g_{0,l,m}(t)\}_{l\le l^\star-1}$ which have been already known by the assumption of induction **(H-1)**.

In particular, for any $|m| \le 2$,

$$g_{0,2,m}(t) = e^{-\lambda_{0,2}t} g_{0,2,m}(0), \tag{2.12}$$

where

$$\lambda_{0,2} = \int_{|\theta|\le\frac{\pi}{4}} \beta(\theta)\sin^2\theta\cos^2\theta d\theta > 0.$$

(2) the case : $n^\star \ge 1$. Now we put the following assumption of induction:
(H-2) : *For any $n \le n^\star - 1$, $l \in \mathbb{N}$ and $|m| \le l$, the functions $\{g_{n,l,m}(t)\}$ solve the equation* (2.10) *with initial data* (2.9)*.*

First, we want to solve the function $g_{n^\star,0,0}$ in (2.10). Since $l^\star = m^\star = 0$, (2.10) can be written as

$$\begin{aligned}
&\partial_t g_{n^\star,0,0}(t) + \lambda_{n^\star,0} g_{n^\star,0,0}(t)\\
&\qquad = \sum_{\substack{n+\tilde{n}=n^\star\\ n\ge1,\tilde{n}\ge1}} \lambda^{rad,1}_{n,\tilde{n},0}\, g_{n,0,0}(t) g_{\tilde{n},0,0}(t)\\
&\qquad\quad + \sum_{\substack{n+\tilde{n}+k=n^\star\\ n\ge0,\tilde{n}\ge0}} \sum_{\substack{l+\tilde{l}-2k=0\\ l\ge1,\tilde{l}\ge1\\ 0\le k\le\min(l,\tilde{l})}} \sum_{\substack{m+\tilde{m}=0\\ |m|\le l,|\tilde{m}|\le\tilde{l}}} \mu^{m,\tilde{m},0}_{n,\tilde{n},l,\tilde{l},k}\, g_{n,l,m}(t) g_{\tilde{n},\tilde{l},\tilde{m}}(t).
\end{aligned}$$

In the last term, when $k = 0$, we have

$$l + \tilde{l} = 0 \text{ with constraints } l \ge 1, \tilde{l} \ge 1,$$

which is impossible. Therefore, the equation (2.10) is:

$$
\begin{aligned}
&\partial_t g_{n^\star,0,0}(t) + \lambda_{n^\star,0} g_{n^\star,0,0}(t)\\
&\qquad = \sum_{\substack{n+\tilde{n}=n^\star\\ n\geq 1,\tilde{n}\geq 1}} \lambda^{rad,1}_{n,\tilde{n},0} g_{n,0,0}(t) g_{\tilde{n},0,0}(t)\\
&\qquad\quad + \sum_{\substack{n+\tilde{n}+k=n^\star\\ 0\leq n,\tilde{n}\leq n^\star-1}} \sum_{\substack{l+\tilde{l}-2k=0\\ l\geq 1,\tilde{l}\geq 1\\ 1\leq k\leq \min(l,\tilde{l})}} \sum_{\substack{m+\tilde{m}=0\\ |m|\leq l,|\tilde{m}|\leq \tilde{l}}} \mu^{m,\tilde{m},0}_{n,\tilde{n},l,\tilde{l},k} g_{n,l,m}(t) g_{\tilde{n},\tilde{l},\tilde{m}}(t).
\end{aligned}
$$

This equation can be also solved since the functions on the right hand side are only involving $\{g_{n,l,m}(t)\}_{n\leq n^\star-1,l\in\mathbb{N},|m|\leq l}$, which have been already given in the assumption of induction **(H-2)**.

Finally, let $l^\star \geq 1$, we can improve the assumption of induction as following:
(H-3) : *For any $n \leq n^\star - 1$, $l \in \mathbb{N}$, $|m| \leq l$ or $n = n^\star, l \leq l^\star - 1$, $|m| \leq l$, the functions $\{g_{n,l,m}(t)\}$ solve the equation* (2.10) *with initial data* (2.9)*.*

We want to solve the functions $g_{n^\star,l^\star,m^\star}(t)$ for all $|m^\star| \leq l^\star$ in (2.10), which is

$$
\begin{aligned}
&\partial_t g_{n^\star,l^\star,m^\star}(t) + \lambda_{n^\star,l^\star} g_{n^\star,l^\star,m^\star}(t)\\
&= \sum_{\substack{n+\tilde{n}=n^\star\\ n\geq 1,\tilde{n}\geq 0}} \left(\lambda^{rad,1}_{n,\tilde{n},l^\star} + \lambda^{rad,2}_{\tilde{n},n,l^\star}\right) g_{\tilde{n},l^\star,m^\star}(t) g_{n,0,0}(t)\\
&\quad + \sum_{\substack{n+\tilde{n}=n^\star\\ n\geq 0,\tilde{n}\geq 0}} \sum_{\substack{l+\tilde{l}=l^\star\\ l\geq 1,\tilde{l}\geq 1}} \sum_{\substack{m+\tilde{m}=m^\star\\ |m|\leq l,|\tilde{m}|\leq \tilde{l}}} \mu^{m,\tilde{m},m^\star}_{n,\tilde{n},l,\tilde{l},0} g_{n,l,m}(t) g_{\tilde{n},\tilde{l},\tilde{m}}(t)\\
&\quad + \sum_{\substack{n+\tilde{n}+k=n^\star\\ n\geq 0,\tilde{n}\geq 0}} \sum_{\substack{l+\tilde{l}-2k=l^\star\\ l\geq 1,\tilde{l}\geq 1\\ 1\leq k\leq \min(l,\tilde{l})}} \sum_{\substack{m+\tilde{m}=m^\star\\ |m|\leq l,|\tilde{m}|\leq \tilde{l}}} \mu^{m,\tilde{m},m^\star}_{n,\tilde{n},l,\tilde{l},k} g_{n,l,m}(t) g_{\tilde{n},\tilde{l},\tilde{m}}(t).
\end{aligned}
$$

Here the summation in the last two terms is understanding as Remark 2.4. This equation can be also solved since the functions on the right hand side are only involving $\{g_{n,l,m}(t)\}_{n\leq n^\star-1,l\in\mathbb{N}}$ and $\{g_{n,l,m}(t)\}_{n=n^\star,l\leq l^\star-1}$ which is given by the improved assumption of induction **(H-3)**. $\square$

Now the proof of Theorem 1.1 is reduced to prove the convergence of following series

$$
g(t) = \sum_{n,l,m}^{+\infty} g_{n,l,m}(t)\varphi_{n,l,m} \tag{2.13}
$$

in the convenable function space.

3. The upper bounded estimate of the non linear operators

3.1. **The estimate of the trilinear formula.** To prove the convergence of the formal solution obtained in the precedent section, we need to estimate the following trilinear terms

$$
(\mathbf{\Gamma}(f,g),h)_{L^2(\mathbb{R}^3)}, \quad f,g,h\in\mathscr{S}(\mathbb{R}^3)\cap\mathcal{N}^\perp.
$$

Using the spectral representation of $\mathbf{\Gamma}(\,\cdot\,,\,\cdot\,)$ given in Proposition 2.1, we need to estimate theirs coefficients.

Proposition 3.1. .
1) For $n \geq 1$, $\tilde{n}, \tilde{l} \in \mathbb{N}$, we have,

$$|\lambda^{rad,1}_{n,\tilde{n},\tilde{l}}|^2 \lesssim \tilde{n}^s(\tilde{n}+\tilde{l})^s n^{-\frac{5}{2}-2s}.$$

2) For all $\tilde{n} \geq 1$, $n, l \in \mathbb{N}$, $n+l \geq 2$, we have

$$|\lambda^{rad,2}_{n,\tilde{n},l}|^2 \lesssim \frac{\tilde{n}^{2s}}{(n+1)^s(n+l)^{\frac{5}{2}+s}}.$$

3) For any $n^\star, l^\star \in \mathbb{N}$, $|m^\star| \leq l^\star$, we have also

$$\sum_{\substack{n+\tilde{n}+k=n^\star\\ n+l\geq 2,\tilde{n}+\tilde{l}\geq 2\\ n\geq 0,\tilde{n}\geq 0}} \sum_{\substack{l+\tilde{l}-2k=l^\star\\ l\geq 1,\tilde{l}\geq 1\\ 0\leq k\leq \min(l,\tilde{l})}} \sum_{|m|\leq l} \sum_{|\tilde{m}|\leq \tilde{l}} \frac{|\mu^{m,\tilde{m},m^\star}_{n,\tilde{n},l,\tilde{l},k}|^2}{\lambda_{\tilde{n},\tilde{l}}} \lesssim \lambda_{n^\star,l^\star}.$$

The constraint of the above summation is

$$\begin{aligned}\Delta_{n^\star,l^\star} = \Big\{&(n,\tilde{n},l,\tilde{l},k,m,\tilde{m}) \in \mathbb{N}^5\times\mathbb{Z}^2;\\ &l\geq 1, \tilde{l}\geq 1, |m|\leq l, |\tilde{m}|\leq \tilde{l}, 0\leq k\leq \min(l,\tilde{l})\\ &\text{and } n+\tilde{n}+k=n^\star,\ l+\tilde{l}-2k=l^\star\Big\},\end{aligned} \tag{3.1}$$

we always write the complicated summation

$$\sum_{(n,\tilde{n},l,\tilde{l},k,m,\tilde{m})\in\Delta_{n^\star,l^\star}}$$

in a simplified form as in Remark 2.4:

$$\sum_{\substack{n+\tilde{n}+k=n^\star\\ n+l\geq 2,\tilde{n}+\tilde{l}\geq 2\\ n\geq 0,\tilde{n}\geq 0}} \sum_{\substack{l+\tilde{l}-2k=l^\star\\ l\geq 1,\tilde{l}\geq 1\\ 0\leq k\leq \min(l,\tilde{l})}} \sum_{|m|\leq l} \sum_{|\tilde{m}|\leq \tilde{l}}.$$

The proof of Proposition 3.1 is technical, so we send it to the section 6.

We prove now the following trilinear estimates for the non linear Boltzmann operator.

Proposition 3.2. *For all $f, g, h \in \mathscr{S}(\mathbb{R}^3)\bigcap \mathcal{N}^\perp$,*

$$\left|(\mathbf{\Gamma}(f,g),\ h)_{L^2(\mathbb{R}^3)}\right| \lesssim \|f\|_{L^2(\mathbb{R}^3)}\|\mathcal{L}^{\frac{1}{2}}g\|_{L^2(\mathbb{R}^3)}\|\mathcal{L}^{\frac{1}{2}}h\|_{L^2(\mathbb{R}^3)}.$$

Proof. For any $f, g, h \in \mathscr{S}(\mathbb{R}^3)\bigcap \mathcal{N}^\perp$, we use the following spectral decomposition,

$$f = \sum_{\substack{n+l\geq 2\\ n\geq 0,l\geq 0}}\sum_{|m|\leq l} f_{n,l,m}\varphi_{n,l,m},\ \ g = \sum_{\substack{n+l\geq 2\\ n\geq 0,l\geq 0}}\sum_{|m|\leq l} g_{n,l,m}\varphi_{n,l,m},\ \ h = \sum_{\substack{n+l\geq 2\\ n\geq 0,l\geq 0}}\sum_{|m|\leq l} h_{n,l,m}\varphi_{n,l,m}.$$

Using the orthogonality of basis $\{\varphi_{n,l,m}\}$ and the formula (2.8), we deduce from Proposition 2.1 that,

$$
\begin{aligned}
&(\mathbf{\Gamma}(f,g),\, h)_{L^2(\mathbb{R}^3)}\\
&= \sum_{\substack{n^\star+l^\star\geq 2\\ n^\star\geq 0,l^\star\geq 0}} \sum_{|m^\star|\leq l^\star} \overline{h_{n^\star,l^\star,m^\star}} \left(\delta_{l^\star,0} \sum_{\substack{n+\tilde{n}=n^\star\\ n\geq 2,\tilde{n}\geq 2}} \lambda^{rad,1}_{n,\tilde{n},0} f_{n,0,0}\, g_{\tilde{n},0,0} \right)\\
&\quad + \sum_{\substack{n^\star+l^\star\geq 2\\ n^\star\geq 0,l^\star\geq 0}} \sum_{|m^\star|\leq l^\star} \overline{h_{n^\star,l^\star,m^\star}} \left(\sum_{\substack{n+\tilde{n}=n^\star,\tilde{l}\geq 1\\ n\geq 2,\tilde{n}\geq 0,\tilde{n}+\tilde{l}\geq 2}} \delta_{\tilde{l},l^\star} \lambda^{rad,1}_{n,\tilde{n},l^\star} f_{n,0,0} g_{\tilde{n},l^\star,m^\star} \right)\\
&\quad + \sum_{\substack{n^\star+l^\star\geq 2\\ n^\star\geq 0,l^\star\geq 0}} \sum_{|m^\star|\leq l^\star} \overline{h_{n^\star,l^\star,m^\star}} \left(\sum_{\substack{n+\tilde{n}=n^\star,l\geq 1\\ n\geq 0,\tilde{n}\geq 2,n+l\geq 2}} \delta_{l,l^\star} \lambda^{rad,2}_{n,\tilde{n},l^\star} f_{n,l^\star,m^\star}\, g_{\tilde{n},0,0} \right)\\
&\quad + \sum_{\substack{n^\star+l^\star\geq 2\\ n^\star\geq 0,l^\star\geq 0}} \sum_{|m^\star|\leq l^\star} \overline{h_{n^\star,l^\star,m^\star}} \left(\sum_{\substack{n+l\geq 2\\ n\geq 0,l\geq 1}} \sum_{\substack{\tilde{n}+\tilde{l}\geq 2\\ \tilde{n}\geq 0,\tilde{l}\geq 1}} \sum_{k=0}^{\min(l,\tilde{l})} \sum_{\substack{|m|\leq l,|\tilde{m}|\leq \tilde{l}\\ |m+\tilde{m}|\leq l+\tilde{l}-2k}} \mu^{m,\tilde{m},m+\tilde{m}}_{n,\tilde{n},l,\tilde{l},k} f_{n,l,m} g_{\tilde{n},\tilde{l},\tilde{m}} \right.\\
&\quad \left. \times\, \delta_{n^\star,n+\tilde{n}+k} \delta_{l^\star,l+\tilde{l}-2k} \delta_{m^\star,m+\tilde{m}}. \right)
\end{aligned}
$$

For brevity, using the formula (2.4), we have

$$
\begin{aligned}
&(\mathbf{\Gamma}(f,g),\, h)_{L^2(\mathbb{R}^3)}\\
&= \sum_{n^\star=2}^{+\infty} \overline{h_{n^\star,0,0}} \left(\sum_{\substack{n+\tilde{n}=n^\star\\ n\geq 2,\tilde{n}\geq 2}} \lambda^{rad,1}_{n,\tilde{n},0} f_{n,0,0}\, g_{\tilde{n},0,0} \right)\\
&\quad + \sum_{\substack{n^\star+l^\star\geq 2\\ n^\star\geq 0,l^\star\geq 1}} \sum_{|m^\star|\leq l^\star} \overline{h_{n^\star,l^\star,m^\star}} \left(\sum_{\substack{n+\tilde{n}=n^\star\\ n\geq 2,\tilde{n}\geq 0,\tilde{n}+l^\star\geq 2}} \lambda^{rad,1}_{n,\tilde{n},l^\star} f_{n,0,0} g_{\tilde{n},l^\star,m^\star} \right)\\
&\quad + \sum_{\substack{n^\star+l^\star\geq 2\\ n^\star\geq 0,l^\star\geq 1}} \sum_{|m^\star|\leq l^\star} \overline{h_{n^\star,l^\star,m^\star}} \left(\sum_{\substack{n+\tilde{n}=n^\star\\ n\geq 0,\tilde{n}\geq 2,n+l^\star\geq 2}} \lambda^{rad,2}_{n,\tilde{n},l^\star} f_{n,l^\star,m^\star}\, g_{\tilde{n},0,0} \right)\\
&\quad + \sum_{\substack{n+l\geq 2\\ n\geq 0,l\geq 1}} \sum_{\substack{\tilde{n}+\tilde{l}\geq 2\\ \tilde{n}\geq 0,\tilde{l}\geq 1}} \sum_{|m|\leq l} \sum_{|\tilde{m}|\leq \tilde{l}} \sum_{k=0}^{\min(l,\tilde{l})} \sum_{|m^\star|\leq l+\tilde{l}-2k} \mu^{m,\tilde{m},m^\star}_{n,\tilde{n},l,\tilde{l},k} f_{n,l,m} g_{\tilde{n},\tilde{l},\tilde{m}} \overline{h_{n+\tilde{n}+k,l+\tilde{l}-2k,m^\star}}\\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

For the term I_1, since $\lambda_{n,0} \approx n^s$ in (2.6), we deduce from Cauchy-Schwarz inequality and Proposition 3.1 that,

$$
\begin{aligned}
|I_1| &\leq \sum_{n^\star \geq 2} |h_{n^\star,0,0}| \Big(\sum_{\substack{n+\tilde{n}=n^\star \\ n\geq 2, \tilde{n}\geq 2}} |\lambda^{rad,1}_{n,\tilde{n},0}|\, |f_{n,0,0}|\, |g_{\tilde{n},0,0}| \Big) \\
&\leq \|f\|_{L^2} \|\mathcal{L}^{\frac{1}{2}} g\|_{L^2} \Big(\sum_{\tilde{n}=2}^{+\infty} \sum_{n=2}^{+\infty} |h_{n+\tilde{n},0,0}|^2 \frac{|\lambda^{rad,1}_{n,\tilde{n},0}|^2}{\lambda_{\tilde{n},0}} \Big)^{\frac{1}{2}} \\
&\lesssim \|f\|_{L^2} \|\mathcal{L}^{\frac{1}{2}} g\|_{L^2} \Big[\sum_{n^\star=2}^{\infty} |h_{n^\star,0,0}|^2 \Big(\sum_{\substack{n+\tilde{n}=n^\star \\ n\geq 2, \tilde{n}\geq 2}} \frac{\tilde{n}^s}{n^{\frac{5}{2}+2s}} \Big) \Big]^{\frac{1}{2}} \lesssim \|f\|_{L^2} \|\mathcal{L}^{\frac{1}{2}} g\|_{L^2} \|\mathcal{L}^{\frac{1}{2}} h\|_{L^2}.
\end{aligned}
$$

For the term I_2, we use Cauchy-Schwarz inequality,

$$
\begin{aligned}
|I_2| &\leq \sum_{\substack{n^\star+l^\star \geq 2 \\ l^\star \geq 1}} \sum_{m^\star=-l^\star}^{l^\star} |h_{n^\star,l^\star,m^\star}| \Big(\sum_{\substack{n+\tilde{n}=n^\star \\ n\geq 2}} |\lambda^{rad,1}_{n,\tilde{n},l^\star}|\, |f_{n,0,0}|\, |g_{\tilde{n},l^\star,m^\star}| \Big) \\
&\leq \|f\|_{L^2} \|\mathcal{L}^{\frac{1}{2}} g\|_{L^2} \Big(\sum_{\tilde{n},\tilde{l},\tilde{m}} \sum_{n=2}^{+\infty} |h_{n+\tilde{n},\tilde{l},\tilde{m}}|^2 \frac{|\lambda^{rad,1}_{n,\tilde{n},\tilde{l}}|^2}{\lambda_{\tilde{n},\tilde{l}}} \Big)^{\frac{1}{2}} \\
&\leq \|f\|_{L^2} \|\mathcal{L}^{\frac{1}{2}} g\|_{L^2} \Big[\sum_{n^\star,l^\star,m^\star} |h_{n^\star,l^\star,m^\star}|^2 \Big(\sum_{\substack{n+\tilde{n}=n^\star \\ n\geq 2}} \frac{1}{\lambda_{\tilde{n},l^\star}} |\lambda^{rad,1}_{n,\tilde{n},l^\star}|^2 \Big) \Big]^{\frac{1}{2}}.
\end{aligned}
$$

We deduce from Proposition 3.1 and $\lambda_{\tilde{n},l^\star} \gtrsim \tilde{n}^s + (l^\star)^{2s}$ in (2.6) that,

$$
\sum_{\substack{n+\tilde{n}=n^\star \\ n\geq 2}} \frac{|\lambda^{rad,1}_{n,\tilde{n},l^\star}|^2}{\lambda_{\tilde{n},l^\star}} \lesssim \sum_{\substack{n+\tilde{n}=n^\star \\ n\geq 2}} \tilde{n}^s n^{-\frac{5}{2}-2s} \lesssim (n^\star)^s.
$$

Therefore,

$$
|I_2| \leq \|f\|_{L^2} \|\mathcal{L}^{\frac{1}{2}} g\|_{L^2} \|\mathcal{L}^{\frac{1}{2}} h\|_{L^2}.
$$

Similarly, for the term I_3, we use again Proposition 3.1 and $\lambda_{\tilde{n},0} \approx \tilde{n}^s$

$$
\begin{aligned}
\sum_{\substack{n+\tilde{n}=n^\star \\ n+l^\star \geq 2}} \frac{|\lambda^{rad,2}_{n,\tilde{n},l^\star}|^2}{\lambda_{\tilde{n},0}} &\lesssim \sum_{\substack{n+\tilde{n}=n^\star \\ n+l^\star \geq 2}} \frac{\tilde{n}^{2s}}{\tilde{n}^s (n+1)^s (n+l^\star)^{\frac{5}{2}+s}} \\
&\lesssim (n^\star)^s + \sum_{n=1}^{n^\star} \frac{(n^\star - n)^s}{n^2} \lesssim (n^\star)^s,
\end{aligned}
$$

which gives

$$
|I_3| \lesssim \|f\|_{L^2} \|\mathcal{L}^{\frac{1}{2}} g\|_{L^2} \|\mathcal{L}^{\frac{1}{2}} h\|_{L^2}.
$$

For the term I_4, we note that $l \geq 1$, $\tilde{l} \geq 1$,

$$
\begin{aligned}
|I_4| \leq &\sum_{\substack{\tilde{n}+\tilde{l}\geq 2 \\ \tilde{n}\geq 0, \tilde{l}\geq 1}} \sum_{|\tilde{m}|\leq \tilde{l}} (\lambda_{\tilde{n},\tilde{l}})^{\frac{1}{2}} (\lambda_{\tilde{n},\tilde{l}})^{-\frac{1}{2}} |g_{\tilde{n},\tilde{l},\tilde{m}}| \sum_{\substack{n+l\geq 2 \\ n\geq 0, l\geq 1}} \sum_{|m|\leq l} |f_{n,l,m}| \\
&\times \left| \sum_{k=0}^{\min(l,\tilde{l})} \sum_{|m^\star|\leq l+\tilde{l}-2k} \mu^{m,\tilde{m},m^\star}_{n,\tilde{n},l,\tilde{l},k} \overline{h_{n+\tilde{n}+k, l+\tilde{l}-2k, m^\star}} \right|.
\end{aligned}
$$

Applying the Cauchy-Schwarz inequality, we get

$$
\begin{aligned}
|I_4| \leq & \left(\sum_{\substack{\tilde{n}+\tilde{l}\geq 2 \\ \tilde{n}\geq 0, \tilde{l}\geq 1}} \sum_{|\tilde{m}|\leq \tilde{l}} \lambda_{\tilde{n},\tilde{l}} |g_{\tilde{n},\tilde{l},\tilde{m}}|^2 \right)^{\frac{1}{2}} \Bigg[\sum_{\substack{\tilde{n}+\tilde{l}\geq 2 \\ \tilde{n}\geq 0, \tilde{l}\geq 1}} \sum_{|\tilde{m}|\leq \tilde{l}} \\
& \left(\sum_{\substack{n+l\geq 2 \\ n\geq 0, l\geq 1}} \sum_{|m|\leq l} |f_{n,l,m}| (\lambda_{\tilde{n},\tilde{l}})^{-\frac{1}{2}} \left| \sum_{k=0}^{\min(l,\tilde{l})} \sum_{|m^\star|\leq l+\tilde{l}-2k} \mu^{m,\tilde{m},m^\star}_{n,\tilde{n},l,\tilde{l},k} \overline{h_{n+\tilde{n}+k, l+\tilde{l}-2k, m^\star}} \right| \right)^2 \Bigg]^{\frac{1}{2}} \\
\leq & \|\mathcal{L}^{\frac{1}{2}} g\|_{L^2} \|f\|_{L^2} \\
\times & \left[\sum_{\substack{\tilde{n}+\tilde{l}\geq 2 \\ \tilde{n}\geq 0, \tilde{l}\geq 1}} \sum_{\substack{n+l\geq 2 \\ n\geq 0, l\geq 1}} (\lambda_{\tilde{n},\tilde{l}})^{-1} \sum_{|m|\leq l} \sum_{|\tilde{m}|\leq \tilde{l}} \left| \sum_{k=0}^{\min(l,\tilde{l})} \sum_{|m^\star|\leq l+\tilde{l}-2k} \mu^{m,\tilde{m},m^\star}_{n,\tilde{n},l,\tilde{l},k} \overline{h_{n+\tilde{n}+k, l+\tilde{l}-2k, m^\star}} \right|^2 \right]^{\frac{1}{2}}.
\end{aligned}
$$

Expanding the summation, we have

$$
\begin{aligned}
& \sum_{|m|\leq l} \sum_{|\tilde{m}|\leq \tilde{l}} \left| \sum_{k=0}^{\min(l,\tilde{l})} \sum_{|m^\star|\leq l+\tilde{l}-2k} \mu^{m,\tilde{m},m^\star}_{n,\tilde{n},l,\tilde{l},k} \overline{h_{n+\tilde{n}+k, l+\tilde{l}-2k, m^\star}} \right|^2 \\
= & \sum_{k_1=0}^{\min(l,\tilde{l})} \sum_{|m_1^\star|\leq l+\tilde{l}-2k_1} \sum_{k_2=0}^{\min(l,\tilde{l})} \sum_{|m_2^\star|\leq l+\tilde{l}-2k_2} \overline{h_{n+\tilde{n}+k_1, l+\tilde{l}-2k_1, m_1^\star}} h_{n+\tilde{n}+k_2, l+\tilde{l}-2k_2, m_2^\star} \\
& \times \left(\sum_{|m|\leq l} \sum_{|\tilde{m}|\leq \tilde{l}} \mu^{m,\tilde{m},m_1^\star}_{n,\tilde{n},l,\tilde{l},k_1} \overline{\mu^{m,\tilde{m},m_2^\star}_{n,\tilde{n},l,\tilde{l},k_2}} \right).
\end{aligned}
$$

By using the formula (2.5) in Proposition 2.2, we obtain,

$$
\sum_{|m|\leq l} \sum_{|\tilde{m}|\leq \tilde{l}} \mu^{m,\tilde{m},m_1^\star}_{n,\tilde{n},l,\tilde{l},k_1} \overline{\mu^{m,\tilde{m},m_2^\star}_{n,\tilde{n},l,\tilde{l},k_2}} = \sum_{|m|\leq l} \sum_{|\tilde{m}|\leq \tilde{l}} \left| \mu^{m,\tilde{m},m_1^\star}_{n,\tilde{n},l,\tilde{l},k_1} \right|^2 \delta_{k_1,k_2} \delta_{m_1^\star, m_2^\star}.
$$

Then

$$
\begin{aligned}
& \sum_{|m|\leq l} \sum_{|\tilde{m}|\leq \tilde{l}} \left| \sum_{k=0}^{\min(l,\tilde{l})} \sum_{|m^\star|\leq l+\tilde{l}-2k} \mu^{m,\tilde{m},m^\star}_{n,\tilde{n},l,\tilde{l},k} \overline{h_{n+\tilde{n}+k, l+\tilde{l}-2k, m^\star}} \right|^2 \\
= & \sum_{k=0}^{\min(l,\tilde{l})} \sum_{|m^\star|\leq l+l-2k} |h_{n+\tilde{n}+k, l+\tilde{l}-2k, m^\star}|^2 \sum_{|m|\leq l} \sum_{|\tilde{m}|\leq \tilde{l}} |\mu^{m,\tilde{m},m^\star}_{n,\tilde{n},l,\tilde{l},k}|^2.
\end{aligned}
$$

It follows that

$$|I_4| \leq \|f\|_{L^2}\|\mathcal{L}^{\frac{1}{2}}g\|_{L^2}\Big(\sum_{\substack{\tilde{n}+\tilde{l}\geq 2\\ \tilde{n}\geq 0,\tilde{l}\geq 1}}\sum_{\substack{n+l\geq 2\\ n\geq 0,l\geq 1}}\sum_{0\leq k\leq \min(l,\tilde{l})}\sum_{|m^\star|\leq l+\tilde{l}-2k}\Big|h_{n+\tilde{n}+k,l+\tilde{l}-2k,m^\star}\Big|^2$$

$$\times\ (\lambda_{\tilde{n},\tilde{l}})^{-1}\sum_{|m|\leq l}\sum_{|\tilde{m}|\leq \tilde{l}}\Big|\mu^{m,\tilde{m},m^\star}_{n,\tilde{n},l,\tilde{l},k}\Big|^2\Big)^{\frac{1}{2}}$$

$$\leq \|f\|_{L^2}\|\mathcal{L}^{\frac{1}{2}}g\|_{L^2}\Big(\sum_{n^\star=0}^{+\infty}\sum_{l^\star=0}^{+\infty}\sum_{|m^\star|\leq l^\star}|h_{n^\star,l^\star,m^\star}|^2$$

$$\times\Big(\sum_{\substack{n+\tilde{n}+k=n^\star\\ n+l\geq 2,\tilde{n}+\tilde{l}\geq 2\\ n\geq 0,\tilde{n}\geq 0}}\sum_{\substack{l+\tilde{l}-2k=l^\star\\ l\geq 1,\tilde{l}\geq 1\\ 0\leq k\leq \min(l,\tilde{l})}}\Big(\lambda_{\tilde{n},\tilde{l}}\Big)^{-1}\sum_{|m|\leq l}\sum_{|\tilde{m}|\leq \tilde{l}}\Big|\mu^{m,\tilde{m},m^\star}_{n,\tilde{n},l,\tilde{l},k}\Big|^2\Big)\Big)^{\frac{1}{2}},$$

where the last summation is understanding as (3.1). Using again Proposition 3.1, we have

$$\sum_{\substack{n+\tilde{n}+k=n^\star\\ n+l\geq 2,\tilde{n}+\tilde{l}\geq 2\\ n\geq 0,\tilde{n}\geq 0}}\sum_{\substack{l+\tilde{l}-2k=l^\star\\ l\geq 1,\tilde{l}\geq 1\\ 0\leq k\leq \min(l,\tilde{l})}}\Big(\lambda_{\tilde{n},\tilde{l}}\Big)^{-1}\sum_{|m|\leq l}\sum_{|\tilde{m}|\leq \tilde{l}}\Big|\mu^{m,\tilde{m},m^\star}_{n,\tilde{n},l,\tilde{l},k}\Big|^2 \leq C\lambda_{n^\star,l^\star}.$$

We get then

$$|I_4| \lesssim |f\|_{L^2}\|\mathcal{L}^{\frac{1}{2}}g\|_{L^2}\Big[\sum_{n^\star=0}^{+\infty}\sum_{l^\star=0}^{+\infty}\sum_{|m^\star|\leq l^\star}\lambda_{n^\star,l^\star}|h_{n^\star,l^\star,m^\star}|^2\Big]^{\frac{1}{2}},$$

which ends the proof of the Proposition 3.2. □

3.2. **The trilinear formula with exponential weighted.** To prove the regularity in the Gelfand-Shilov space, we need more the upper bounded of non linear operators with exponential weighted.

Proposition 3.3. *For any $f,g,h \in \mathscr{S}(\mathbb{R}^3)\bigcap\mathcal{N}^\perp$, any $N\geq 0$, and for any $c>0$, we have*

$$\begin{aligned}(3.2)\qquad & |(\mathbf{\Gamma}(f,g), e^{c\mathcal{H}^s}\mathbf{S}_N h)_{L^2}| \\ & \leq C\|e^{\frac{c}{2}\mathcal{H}^s}\mathbf{S}_{N-2}f\|_{L^2}\|e^{\frac{c}{2}\mathcal{H}^s}\mathcal{L}^{\frac{1}{2}}\mathbf{S}_{N-2}g\|_{L^2}\|e^{\frac{c}{2}\mathcal{H}^s}\mathcal{L}^{\frac{1}{2}}\mathbf{S}_N h\|_{L^2},\end{aligned}$$

where C is a positive constant only dependent on s, and $\mathbf{S}_N$ is the orthogonal projector such that,

$$\mathbf{S}_N f = \sum_{\substack{2n+l\leq N\\ n\geq 0,l\geq 0}}\sum_{|m|\leq l}(f,\varphi_{n,l,m})_{L^2}\,\varphi_{n,l,m},$$

$$e^{c\mathcal{H}^s}\mathbf{S}_N f = \sum_{\substack{2n+l\leq N\\ n\geq 0,l\geq 0}}\sum_{|m|\leq l}e^{c\,(2n+l+\frac{3}{2})^s}(f,\varphi_{n,l,m})_{L^2}\,\varphi_{n,l,m}.$$

Remark 3.4. *1) For $h\in\mathscr{S}(\mathbb{R}^3)$, we can't use $e^{\frac{c}{2}\mathcal{H}^s}h$ as test function, since it is not belong to $\mathscr{S}(\mathbb{R}^3)$. However, for any $h\in\mathscr{S}'(\mathbb{R}^3)$, we have $e^{\frac{c}{2}\mathcal{H}^s}\mathbf{S}_N h\in\mathscr{S}(\mathbb{R}^3)$.*

2) In the right hand side of (3.2), the projector of f and g with $\mathbf{S}_{N-2}$ show more clary the triangular effect of $\mathbf{\Gamma}(\cdot,\cdot)$.

Proof. Since $f,g,h \in \mathscr{S}(\mathbb{R}^3)\bigcap \mathcal{N}^\perp$, similarly to the Proposition 3.2, we have

$$\begin{aligned}
&\Big(\mathbf{\Gamma}(f,g),\ e^{c\mathcal{H}^s}\mathbf{S}_N h\Big)_{L^2(\mathbb{R}^3)}\\
&=\sum_{n^\star=2}^{[\frac{N}{2}]} e^{c\,(2n^\star+\frac32)^s}\overline{h_{n^\star,0,0}}\left(\sum_{\substack{n+\tilde n=n^\star\\ n\geq2,\tilde n\geq2}}\lambda^{rad,1}_{n,\tilde n,0} f_{n,0,0}\, g_{\tilde n,0,0}\right)\\
&\quad+\sum_{\substack{2\leq 2n^\star+l^\star\leq N\\ n^\star\geq0,l^\star\geq1}}\sum_{|m^\star|\leq l^\star} e^{c\,(2n^\star+l^\star+\frac32)^s}\overline{h_{n^\star,l^\star,m^\star}}\left(\sum_{\substack{n+\tilde n=n^\star\\ n\geq2,\tilde n\geq0,\tilde n+l^\star\geq2}}\lambda^{rad,1}_{n,\tilde n,l^\star} f_{n,0,0}\, g_{\tilde n,l^\star,m^\star}\right)\\
&\quad+\sum_{\substack{2\leq 2n^\star+l^\star\leq N\\ n^\star\geq0,l^\star\geq1}}\sum_{|m^\star|\leq l^\star} e^{c\,(2n^\star+l^\star+\frac32)^s}\overline{h_{n^\star,l^\star,m^\star}}\left(\sum_{\substack{n+\tilde n=n^\star\\ n\geq0,\tilde n\geq2,n+l^\star\geq2}}\lambda^{rad,2}_{n,\tilde n,l^\star} f_{n,l^\star,m^\star}\, g_{\tilde n,0,0}\right)\\
&\quad+\sum_{\substack{2\leq 2n+2\tilde n+l+\tilde l\leq N\\ n+l\geq2,\tilde n+\tilde l\geq2\\ n\geq0,l\geq1,\tilde n\geq0,\tilde l\geq1}}\sum_{|m|\leq l}\sum_{|\tilde m|\leq\tilde l} f_{n,l,m}g_{\tilde n,\tilde l,\tilde m}e^{c\,(2n+2\tilde n+l+\tilde l+\frac32)^s}\\
&\qquad\qquad\times\left(\sum_{k=0}^{\min(l,\tilde l)}\sum_{|m^\star|\leq l+\tilde l-2k}\mu^{m,\tilde m,m^\star}_{n,\tilde n,l,\tilde l,k}\,\overline{h_{n+\tilde n+k,l+\tilde l-2k,m^\star}}\right)\\
&=J_1+J_2+J_3+J_4.
\end{aligned}$$

The estimate of terms J_1, J_2 and J_3 is more easy then J_4, so we only consider the term J_4,

$$\begin{aligned}
|J_4|\leq&\sum_{\substack{2\leq2\tilde n+\tilde l\leq N-2\\ \tilde n+\tilde l\geq2,\,\tilde l\geq1}}\sum_{|\tilde m|\leq\tilde l}(\lambda_{\tilde n,\tilde l})^{\frac12}e^{\frac{c(2\tilde n+\tilde l+\frac32)^s}{2}}|g_{\tilde n,\tilde l,\tilde m}|\sum_{\substack{2\leq2n+l\leq N-2\\ 4\leq2n+2\tilde n+l+\tilde l\leq N\\ n+l\geq2,\,l\geq1}}\sum_{|m|\leq l}e^{\frac{c\,(2n+l+\frac32)^s}{2}}|f_{n,l,m}|\\
&\times e^{c\,(2n+2\tilde n+l+\tilde l+\frac32)^s-\frac{c(2n+l+\frac32)^s}{2}-\frac{c(2\tilde n+\tilde l+\frac32)^s}{2}}\left|\sum_{k=0}^{\min(l,\tilde l)}\sum_{|m^\star|\leq l+\tilde l-2k}\mu^{m,\tilde m,m^\star}_{n,\tilde n,l,\tilde l,k}\big(\lambda_{\tilde n,\tilde l}\big)^{-\frac12}\,\overline{h_{n+\tilde n+k,l+\tilde l-2k,m^\star}}\right|,
\end{aligned}$$

Since for any $0<s<1$,

$$(2n+l+2\tilde n+\tilde l+\frac32)^s\leq(2n+l+\frac32)^s+(2\tilde n+\tilde l+\frac32)^s.$$

We deduce that

$$\begin{aligned}
|J_4|&\leq\|e^{\frac c2\mathcal{H}^s}\mathcal{L}^{\frac12}\mathbf{S}_{N-2}g\|_{L^2}\|e^{\frac c2\mathcal{H}^s}\mathbf{S}_{N-2}f\|_{L^2}\\
&\quad\times\Big[\sum_{\substack{4\leq2n+2\tilde n+l+\tilde l\leq N\\ 2\leq n+l,2\leq\tilde n+\tilde l\\ n\geq0,l\geq1,\tilde n\geq0,\tilde l\geq1}}\sum_{k=0}^{\min(l,\tilde l)}\sum_{|m^\star|\leq l+\tilde l-2k}e^{c\,(2n+2\tilde n+l+\tilde l+\frac32)^s}|h_{n+\tilde n+k,l+\tilde l-2k,m^\star}|^2\sum_{|\tilde m|\leq\tilde l}\sum_{|m|\leq l}\frac{|\mu^{m,\tilde m,m^\star}_{n,\tilde n,l,\tilde l,k}|^2}{\lambda_{\tilde n,\tilde l}}\Big]^{\frac12}\\
&\leq\|e^{\frac c2\mathcal{H}^s}\mathcal{L}^{\frac12}\mathbf{S}_{N-2}g\|_{L^2}\|e^{\frac c2\mathcal{H}^s}\mathbf{S}_{N-2}f\|_{L^2}\Big[\sum_{4\leq2n^\star+l^\star\leq N}\sum_{|m^\star|\leq l+\tilde l-2k}e^{c\,(2n^\star+l^\star+\frac32)^s}|h_{n^\star,l^\star,m^\star}|^2\\
&\qquad\times\Big(\sum_{\substack{n+\tilde n+k=n^\star\\ n+l\geq2,\tilde n+\tilde l\geq2\\ n\geq0,\tilde n\geq0}}\sum_{\substack{l+\tilde l-2k=l^\star\\ l\geq1,\tilde l\geq1\\ 0\leq k\leq\min(l,\tilde l)}}\big(\lambda_{\tilde n,\tilde l}\big)^{-1}\sum_{|m|\leq l}\sum_{|\tilde m|\leq\tilde l}\Big|\mu^{m,\tilde m,m^\star}_{n,\tilde n,l,\tilde l,k}\Big|^2\Big)\Big]^{\frac12}.
\end{aligned}$$

The last summation is understanding as (3.1). We can finish the proof exactly as that of Proposition 3.2. □

4. The proof of the main Theorem

In this section, we study the convergence of the formal solutions obtained on Section 2 with small L^2 initial data which end the proof of Theorem 1.1.

4.1. The uniform estimate. Let $\{g_{n,l,m}(t)\}$ be the solution of (2.10). For any $N \in \mathbb{N}$, set

$$\mathbf{S}_N g(t) = \sum_{\substack{n\geq 0, l\geq 0\\ 2n+l\leq N}} \sum_{|m|\leq l} g_{n,l,m}(t)\varphi_{n,l,m}. \tag{4.1}$$

Then $\mathbf{S}_N g(t), e^{c_0\, t\mathcal{H}^s}\mathbf{S}_N g(t) \in \mathscr{S}(\mathbb{R}^3)\bigcap \mathcal{N}^\perp$,

Multiplying $e^{c_0 t(2n^\star+l^\star+\frac{3}{2})^s}\overline{g_{n^\star,l^\star,m^\star}}(t)$ on both sides of (2.10), and take summation for $2n^\star + l^\star \leq N$, then Proposition 2.1 and the orthogonality of the basis $(\varphi_{n,l,m})_{n,l\geq 0,|m|\leq l}$ imply

$$\begin{aligned}
&\Big(\partial_t(\mathbf{S}_N g)(t), e^{c_0 t\mathcal{H}^s}\mathbf{S}_N g(t)\Big)_{L^2(\mathbb{R}^3)} + \Big(\mathcal{L}(\mathbf{S}_N g)(t), e^{c_0 t\mathcal{H}^s}\mathbf{S}_N g(t)\Big)_{L^2(\mathbb{R}^3)}\\
&\quad = \Big(\Gamma((\mathbf{S}_N g), (\mathbf{S}_N g)), e^{c_0 t\mathcal{H}^s}\mathbf{S}_N g(t)\Big)_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

Since $\mathbf{S}_N g(t) \in \mathscr{S}(\mathbb{R}^3)\bigcap \mathcal{N}^\perp$, we have

$$\Big(\mathcal{L}(\mathbf{S}_N g)(t), e^{c_0 t\mathcal{H}^s}\mathbf{S}_N g(t)\Big)_{L^2(\mathbb{R}^3)} = \|e^{\frac{c_0 t}{2}\mathcal{H}^s}\mathcal{L}^{\frac12}\mathbf{S}_N g(t)\|_{L^2(\mathbb{R}^3)},$$

we then obtain that

$$\begin{aligned}
&\frac12\frac{d}{dt}\|e^{\frac{c_0 t}{2}\mathcal{H}^s}\mathbf{S}_N g(t)\|^2_{L^2} - \frac{c_0}{2}\|e^{\frac{c_0 t}{2}\mathcal{H}^s}\mathcal{H}^{\frac{s}{2}}\mathbf{S}_N g\|^2_{L^2} + \|e^{\frac{c_0 t}{2}\mathcal{H}^s}\mathcal{L}^{\frac12}\mathbf{S}_N g(t))_{L^2}\\
&\qquad = \Big(\Gamma((\mathbf{S}_N g), (\mathbf{S}_N g)), e^{c_0 t\mathcal{H}^s}\mathbf{S}_N g(t)\Big)_{L^2}.
\end{aligned}$$

It follows from (2.6) and the Proposition 3.3 that, for $0 \leq c_0 \leq c_1$ and any $N \geq 2$, $t \geq 0$,

$$\begin{aligned}
&\frac12\frac{d}{dt}\|e^{\frac{c_0 t}{2}\mathcal{H}^s}\mathbf{S}_N g(t)\|^2_{L^2} + \frac12\|e^{\frac{c_0 t}{2}\mathcal{H}^s}\mathcal{L}^{\frac12}\mathbf{S}_N g\|^2_{L^2}\\
&\qquad \leq C\|e^{\frac{c_0 t}{2}\mathcal{H}^s}\mathbf{S}_{N-2}g\|_{L^2}\|e^{\frac{c_0 t}{2}\mathcal{H}^s}\mathcal{L}^{\frac12}\mathbf{S}_N g\|^2_{L^2}.
\end{aligned}\tag{4.2}$$

Proposition 4.1. *There exists $\epsilon_0 > 0$ and $\tilde{c}_0 > 0$ such that for all $0 < \epsilon \leq \epsilon_0, 0 \leq c_0 \leq \tilde{c}_0$, $g_0 \in L^2\bigcap \mathcal{N}^\perp$ with $\|g_0\|_{L^2} \leq \epsilon$, then,*

$$\|e^{\frac{c_0 t}{2}\mathcal{H}^s}\mathbf{S}_N g(t)\|^2_{L^2(\mathbb{R}^3)} + \frac12\int_0^t \|e^{\frac{c_0 \tau}{2}\mathcal{H}^s}\mathcal{L}^{\frac12}\mathbf{S}_N g(\tau)\|^2_{L^2}d\tau \leq \|g_0\|^2_{L^2(\mathbb{R}^3)},$$

for any $t \geq 0$, $N \geq 0$.

Proof. We prove the Proposition by induction on N.

1). For $N \leq 2$. we have $\|e^{\frac{c_0 t}{2}\mathcal{H}^s}\mathbf{S}_0 g\|^2_{L^2} = |g_{0,0,0}(t)|^2 = 0$,

$$\|e^{\frac{c_0 t}{2}\mathcal{H}^s}\mathbf{S}_1 g\|^2_{L^2} = |g_{0,0,0}(t)|^2 + \sum_{|m|\leq 1} e^{c_0\, t}|g_{0,1,m}(t)|^2 = 0,$$

and

$$\|e^{\frac{c_0 t}{2}\mathcal{H}^s}\mathbf{S}_2 g\|^2_{L^2} = |e^{c_0 t 2^s}g_{1,0,0}(t)|^2 + \sum_{|m|\leq 2} e^{c_0(2+3/2)^s t}|g_{0,2,m}(t)|^2.$$

Recall that for all $t > 0$ (see (2.12)),

$$g_{0,2,m}(t) = e^{-\lambda_{0,2}t}g_{0,2,m}(0),$$

we choose $0 < \tilde{c}_0$ small such that $\tilde{c}_0(2+3/2)^s - 2\lambda_{0,2} \leq 0$, then

$$\|e^{\frac{c_0 t}{2}\mathcal{H}^s}\mathbf{S}_2 g\|_{L^2}^2 \leq \sum_{|m|\leq 2} |g_{0,2,m}(0)|^2 \leq \|g_0\|_{L^2}^2 \leq \epsilon^2$$

for all $0 \leq c_0 \leq \tilde{c}_0, 0 < \epsilon \leq \epsilon_0$.

2). For $N > 2$**.** We want to prove that

$$\|e^{\frac{c_0 t}{2}\mathcal{H}^s}\mathbf{S}_{N-1} g\|_{L^2} \leq \epsilon \leq \epsilon_0$$

imply

$$\|e^{\frac{c_0 t}{2}\mathcal{H}^s}\mathbf{S}_N g\|_{L^2} \leq \epsilon.$$

Take now $\epsilon_0 > 0$ such that

$$0 < \epsilon_0 \leq \frac{1}{4C}.$$

Then we deduce from (4.2) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|e^{\frac{c_0 t}{2}\mathcal{H}^s}\mathbf{S}_N g(t)\|_{L^2}^2 &+ \frac{1}{2}\|e^{\frac{c_0 t}{2}\mathcal{H}^s}\mathcal{L}^{\frac{1}{2}}\mathbf{S}_N g\|_{L^2}^2 \\ &\leq C\|e^{\frac{c_0 t}{2}\mathcal{H}^s}\mathbf{S}_{N-2} g\|_{L^2}\|e^{\frac{c_0 t}{2}\mathcal{H}^s}\mathcal{L}^{\frac{1}{2}}\mathbf{S}_N g\|_{L^2}^2 \\ &\leq \frac{1}{4}\|e^{\frac{c_0 t}{2}\mathcal{H}^s}\mathcal{L}^{\frac{1}{2}}\mathbf{S}_N g\|_{L^2}^2,\end{aligned}$$

therefore,

$$\frac{d}{dt}\|e^{\frac{c_0 t}{2}\mathcal{H}^s}\mathbf{S}_N g(t)\|_{L^2}^2 + \frac{1}{2}\|e^{\frac{c_0 t}{2}\mathcal{H}^s}\mathcal{L}^{\frac{1}{2}}\mathbf{S}_N g\|_{L^2}^2 \leq 0, \tag{4.3}$$

this end the proof of the Proposition. □

4.2. **Existence of the weak solution.** We prove now the convergence of the sequence

$$g(t) = \sum_{n,l\geq 0}\sum_{|m|\leq l} g_{n,l,m}(t)\varphi_{n,l,m}$$

defined in (2.13).

Multiplying $\varphi_{n^\star,l^\star,m^\star}(v)$ on both sides of (2.10), and take summation for $2n^\star + l^\star \leq N$, then for all $N \geq 2$, $\mathbf{S}_N g(t)$ satisfies the following Cauchy problem

$$\begin{cases}\partial_t \mathbf{S}_N g + \mathcal{L}(\mathbf{S}_N g) = \mathbf{S}_N\Gamma(\mathbf{S}_N g, \mathbf{S}_N g), \\ \mathbf{S}_N g(0) = \sum\limits_{\substack{n\geq 0, l\geq 0 \\ 2n+l\leq N}}\sum\limits_{|m|\leq l} g^0_{n,l,m}\varphi_{n,l,m}.\end{cases} \tag{4.4}$$

By Proposition 4.1 with $c_0 = 0$, we have for all $t > 0$, $N \in \mathbb{N}$,

$$\|\mathbf{S}_N g(t)\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{2}\int_0^t \|\mathcal{L}^{\frac{1}{2}}\mathbf{S}_N g(\tau)\|_{L^2}^2 d\tau \leq \|g_0\|_{L^2(\mathbb{R}^3)}^2.$$

The orthogonality of the basis $(\varphi_{n,l,m})_{n,l\geq 0,|m|\leq l}$ implies that

$$\|\mathbf{S}_N g(t)\|_{L^2(\mathbb{R}^3)}^2 = \sum_{\substack{n\geq 0, l\geq 0 \\ 2n+l\leq N}}\sum_{|m|\leq l} |g_{n,l,m}(t)|^2.$$

By using the monotone convergence theorem, the sequence

$$g(t) = \sum_{n,l\geq 0}\sum_{|m|\leq l} g_{n,l,m}(t)\varphi_{n,l,m}$$

is convergent, for any $t \geq 0$,

$$\lim_{N\to\infty} \|\mathbf{S}_N g - g\|_{L^\infty(]0,t[;L^2(\mathbb{R}^3))} = 0$$

and

$$\lim_{N\to\infty} \|\mathcal{L}^{\frac{1}{2}}(\mathbf{S}_N g - g)\|_{L^2(]0,t[;L^2(\mathbb{R}^3))} = 0.$$

For any $\phi(t) \in C^1\Big(\mathbb{R}_+, \mathscr{S}(\mathbb{R}^3)\Big)$, we have

$$\begin{aligned}
&\Big|\int_0^t \Big(\mathbf{S}_N\Gamma(\mathbf{S}_N g, \mathbf{S}_N g) - \Gamma(g, g), \phi(\tau)\Big)_{L^2(\mathbb{R}^3)} d\tau\Big| \\
&\leq \Big|\int_0^t \Big(\Gamma(\mathbf{S}_N g, \mathbf{S}_N g), \mathbf{S}_N\phi(\tau) - \phi(\tau)\Big)_{L^2} d\tau\Big| \\
&\quad + \Big|\int_0^t \Big(\Gamma(\mathbf{S}_N g - g, \mathbf{S}_N g), \phi(\tau)\Big)_{L^2} d\tau\Big| + \Big|\int_0^t \Big(\Gamma(g, \mathbf{S}_N g - g), \phi(\tau)\Big)_{L^2} d\tau\Big|
\end{aligned}$$

by Proposition 3.2,

$$\begin{aligned}
&\Big|\int_0^t \Big(\mathbf{S}_N\Gamma(\mathbf{S}_N g, \mathbf{S}_N g) - \Gamma(g, g), \phi(\tau)\Big)_{L^2(\mathbb{R}^3)} d\tau\Big| \\
&\leq C\int_0^t \|\mathbf{S}_N g\|_{L^2}\|\mathcal{L}^{\frac{1}{2}}\mathbf{S}_N g\|_{L^2}\Big\|\mathcal{L}^{\frac{1}{2}}(\mathbf{S}_N\phi - \phi)\Big\|_{L^2(\mathbb{R}^3)} dt \\
&\quad + C\int_0^t \|\mathbf{S}_N g - g\|_{L^2}\|\mathcal{L}^{\frac{1}{2}}\mathbf{S}_N g\|_{L^2}\|\mathcal{L}^{\frac{1}{2}}\phi\|_{L^2(\mathbb{R}^3)} dt \\
&\quad + C\int_0^t \|g\|_{L^2}\|\mathcal{L}^{\frac{1}{2}}(\mathbf{S}_N g - g)\|_{L^2(\mathbb{R}^3)}\|\mathcal{L}^{\frac{1}{2}}\phi\|_{L^2(\mathbb{R}^3)} dt,
\end{aligned}$$

using Proposition 4.1 with $c_0 = 0$

$$\begin{aligned}
&\Big|\int_0^t \Big(\mathbf{S}_N\Gamma(\mathbf{S}_N g, \mathbf{S}_N g) - \Gamma(g, g), \phi(\tau)\Big)_{L^2(\mathbb{R}^3)} d\tau\Big| \\
&\leq C\|\mathbf{S}_N g\|_{L^\infty(]0,t[;L^2)}\|\mathcal{L}^{\frac{1}{2}}\mathbf{S}_N g\|_{L^2(]0,t[;L^2)}\Big\|\mathcal{L}^{\frac{1}{2}}(\mathbf{S}_N\phi - \phi)\Big\|_{L^2(]0,t[;L^2(\mathbb{R}^3))} \\
&\quad + C\|\mathbf{S}_N g - g\|_{L^\infty(]0,t[;L^2)}\|\mathcal{L}^{\frac{1}{2}}\mathbf{S}_N g\|_{L^2(]0,t[;L^2)}\|\mathcal{L}^{\frac{1}{2}}\phi\|_{L^2(]0,t[;L^2)} \\
&\quad + C\|g\|_{L^\infty(]0,t[;L^2)}\|\mathcal{L}^{\frac{1}{2}}(\mathbf{S}_N g - g)\|_{L^2(]0,t[;L^2)}\|\mathcal{L}^{\frac{1}{2}}\phi\|_{L^2(]0,t[;L^2)}
\end{aligned}$$

then

$$\begin{aligned}
&\Big|\int_0^t \Big(\mathbf{S}_N\Gamma(\mathbf{S}_N g, \mathbf{S}_N g) - \Gamma(g, g), \phi(\tau)\Big)_{L^2(\mathbb{R}^3)} d\tau\Big| \\
&\leq C\|g_0\|_{L^2}^2\Big\|\mathcal{L}^{\frac{1}{2}}(\mathbf{S}_N\phi - \phi)\Big\|_{L^2(]0,t[;L^2(\mathbb{R}^3))} \\
&\quad + C\|\mathbf{S}_N g - g\|_{L^\infty(]0,t[;L^2)}\|g_0\|_{L^2}\|\mathcal{L}^{\frac{1}{2}}\phi\|_{L^2(]0,t[;L^2)} \\
&\quad + C\|g_0\|_{L^2}\|\mathcal{L}^{\frac{1}{2}}(\mathbf{S}_N g - g)\|_{L^2(]0,t[;L^2)}\|\mathcal{L}^{\frac{1}{2}}\phi\|_{L^2(]0,t[;L^2)}.
\end{aligned}$$

Let $N \to +\infty$ in (4.4), we conclude that, for any $\phi(t) \in C^1\Big(\mathbb{R}_+, \mathscr{S}(\mathbb{R}^3)\Big)$,

$$\begin{aligned}
&\Big(g(t), \phi(t)\Big)_{L^2(\mathbb{R}^3)} - \Big(g(0), \phi(0)\Big)_{L^2(\mathbb{R}^3)} \\
&= -\int_0^t \Big(\mathcal{L}g(\tau), \phi(\tau)\Big)_{L^2(\mathbb{R}^3)} d\tau + \int_0^t \Big(\Gamma(g(\tau), g(\tau)), \phi(\tau)\Big)_{L^2(\mathbb{R}^3)} d\tau,
\end{aligned}$$

which shows that $g \in L^\infty(]0,+\infty[; L^2(\mathbb{R}^3))$ is a global weak solution of the Cauchy problem (1.3).

4.3. **Regularity of the solution.** For $\mathbf{S}_N g$ defined in (4.1), since

$$\lambda_{n,l} \geq \lambda_{2,0} > 0, \ \forall\, n+l \geq 2,$$

we deduce from the formulas (4.3) that

$$\begin{aligned}
&\frac{d}{dt}\|e^{\frac{c_0 t}{2}\mathcal{H}^s}\mathbf{S}_N g(t)\|_{L^2}^2 + \frac{\lambda_{2,0}}{2}\|e^{\frac{c_0 t}{2}\mathcal{H}^s}\mathbf{S}_N g\|_{L^2}^2\\
&\leq \frac{d}{dt}\|e^{\frac{c_0 t}{2}\mathcal{H}^s}\mathbf{S}_N g(t)\|_{L^2}^2 + \frac{1}{2}\|e^{\frac{c_0 t}{2}\mathcal{H}^s}\mathcal{L}^{\frac{1}{2}}\mathbf{S}_N g\|_{L^2}^2 \leq 0.
\end{aligned}$$

We have then

$$\frac{d}{dt}\Big(e^{\frac{\lambda_{2,0}t}{2}}\|e^{\frac{c_0 t}{2}\mathcal{H}^s}\mathbf{S}_N g(t)\|_{L^2}^2\Big) \leq 0,$$

it deduces that for any $t > 0$, and $N \in \mathbb{N}$,

$$\|e^{\frac{c_0 t}{2}\mathcal{H}^s}\mathbf{S}_N g(t)\|_{L^2(\mathbb{R}^3)} \leq e^{-\frac{\lambda_{2,0}t}{2}}\|g_0\|_{L^2(\mathbb{R}^3)}.$$

The orthogonal of the basis $(\varphi_{n,l,m})_{n,l\geq 0,|m|\leq l}$ implies that

$$\|e^{\frac{c_0 t}{2}\mathcal{H}^s}\mathbf{S}_N g(t)\|_{L^2(\mathbb{R}^3)}^2 = \sum_{\substack{2n+l\leq N\\ n\geq 0, l\geq 0}}\sum_{|m|\leq l} e^{\frac{c_0 t}{2}(2n+l+\frac{3}{2})^s}|g_{n,l,m}(t)|^2.$$

By using the monotone convergence theorem, we conclude that

$$\|e^{\frac{c_0 t}{2}\mathcal{H}^s} g(t)\|_{L^2(\mathbb{R}^3)} \leq e^{-\frac{\lambda_{2,0}t}{2}}\|g_0\|_{L^2(\mathbb{R}^3)}.$$

This ends the proof of Theorem 1.1.

5. The spectral representation

This section is devoted to the proof of the Proposition 2.1, the Proposition 2.2 and some propositions used in section 6.

5.1. **Harmonic identities.** We prepare some technical computation. In all this section, n, l, $\tilde{n}$, $\tilde{l}$ will be fixed integers of $\mathbb{N}$ and we will use the following notation in this section :

$$a_{l,m} = \frac{(l-|m|)!}{(l+|m|)!}.$$

For any unit vector

$$\sigma = (\sigma_1, \sigma_2, \sigma_3) = (\cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi) \in \mathbb{S}^2$$

with $\theta \in [0,\pi]$ and $\phi \in [0, 2\pi]$, the orthonormal basis of spherical harmonics $Y_l^m(\sigma)$ ($|m| \leq l$) is (see the definition (2.2) of $P_l^{|k|}$)

$$\begin{aligned}
Y_l^m(\sigma) &= \sqrt{\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}\,P_l^{|m|}(\cos\theta)e^{im\phi},\\
&= \sqrt{\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}\left(\frac{d^{|m|}P_l}{dx^{|m|}}\right)(\sigma_1)\,(\sigma_2 + i\ \mathrm{sgn}(m)\,\sigma_3)^{|m|}.
\end{aligned} \tag{5.1}$$

We recall the following properties (see [17] and [20]):
- Addition theorem: For any integer $l \geq 0$ and α_1, α_2 in $\mathbb{S}^2$,

$$P_l(\alpha_1 \cdot \alpha_2) = \frac{4\pi}{2\,l+1} \sum_{m=-l}^{l} Y_l^m(\alpha_1)\, Y_l^{-m}(\alpha_2). \tag{5.2}$$

If we set the following coordinates

$$\alpha_j = (\cos(\theta_j), sin(\theta_j)\cos(\phi_j), \sin(\theta_j)\sin(\phi_j))$$

for $j = 1, 2$, the previous addition theorem reads as follows

$$\begin{aligned} &P_l(\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos(\phi_1 - \phi_2)) \\ &= \sum_{m=-l}^{l} \frac{(l-|m|)!}{(l+|m|)!} P_l^{|m|}(\cos\theta_1)\, P_l^{|m|}(\cos\theta_2)\, e^{im(\phi_1-\phi_2)}. \end{aligned} \tag{5.3}$$

- Integral form of the addition theorem: For any integer $l \geq 0$ and m, $|m| \leq l$, any $\sigma \in \mathbb{S}^2$,

$$Y_l^m(\sigma) = \frac{2\,l+1}{4\,\pi} \int_{\mathbb{S}^2_\eta} P_l(\sigma \cdot \eta)\, Y_l^m(\eta)\, d\eta. \tag{5.4}$$

- Funk-Hecke Formula: For any continuous function $f \in C([-1, 1])$, any $\sigma \in \mathbb{S}^2$ and integers $l \geq 0$, $|m| \leq l$,

$$\int_{\mathbb{S}^2_\eta} f(\sigma \cdot \eta)\, Y_l^m(\eta)\, d\eta = \left(2\pi \int_{-1}^{1} f(x)\, P_l(x)\, dx\right) Y_l^m(\sigma). \tag{5.5}$$

For $\kappa \in \mathbb{S}^2$ fixed, we can find $\theta_0 \in [0, \pi]$, $\phi_0 \in [0, 2\pi[$) such that

$$\kappa = (\cos\theta_0, \sin\theta_0 \cos\phi_0, \sin\theta_0 \sin\phi_0).$$

Construct the orthogonal vectors with respect to κ

$$\kappa^1 = (-\sin\theta_0, \cos\theta_0 \cos\phi_0, \cos\theta_0 \sin\phi_0),\ \ \kappa^2 = (0, \sin\phi_0, -\cos\phi_0), \tag{5.6}$$

and for $\phi \in \mathbb{R}$

$$\kappa^\perp(\phi) = \kappa^1 \cos\phi + \kappa^2 \sin\phi. \tag{5.7}$$

Then $\kappa, \kappa^1, \kappa^2$ constitute an orthonormal frame in $\mathbb{R}^3$. For any unit vector σ, we can find $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$ such that

$$\sigma = \kappa \cos\theta + \kappa^1 \sin\theta \cos\phi + \kappa^2 \sin\theta \sin\phi. \tag{5.8}$$

It is easy to verify

$$\frac{\kappa+\sigma}{|\kappa+\sigma|} = \kappa\,\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\left(\kappa^1 \cos\phi + \kappa^2 \sin\phi\right), \tag{5.9}$$

$$\frac{\kappa-\sigma}{|\kappa-\sigma|} = \kappa\,\sin\frac{\theta}{2} - \cos\frac{\theta}{2}\left(\kappa^1 \cos\phi + \kappa^2 \sin\phi\right). \tag{5.10}$$

In the proof of the Proposition 2.1, we need the following lemma.

Lemma 5.1. *For any function f in $C([-1, 1])$ any $\kappa \in \mathbb{S}^2$, $l \in \mathbb{N}$ and $|m| \leq l$, we have*

$$(i)\quad \int_{\mathbb{S}^2} f(\kappa \cdot \sigma)\, Y_l^m\left(\tfrac{\kappa+\sigma}{|\kappa+\sigma|}\right) d\sigma = \left(2\,\pi \int_0^\pi f(\cos\theta)\,\sin\theta\, P_l\left(\cos\tfrac{\theta}{2}\right) d\theta\right) Y_l^m(\kappa),$$

$$(ii)\quad \int_{\mathbb{S}^2} f(\kappa \cdot \sigma)\, Y_l^m\left(\tfrac{\kappa-\sigma}{|\kappa-\sigma|}\right) d\sigma = \left(2\,\pi \int_0^\pi f(\cos\theta)\,\sin\theta\, P_l\left(\sin\tfrac{\theta}{2}\right) d\theta\right) Y_l^m(\kappa).$$

Proof. For $\kappa \in \mathbb{S}^2$ fixed, we can find $\theta_0 \in [0,\pi]$, $\phi_0 \in [0,2\pi]$ such that

$$\kappa = (\cos\theta_0, \sin\theta_0\cos\phi_0, \sin\theta_0\sin\phi_0).$$

In the orthonormal frame $(\kappa, \kappa^1, \kappa^2)$ constructed in (5.6), for any $\sigma \in \mathbb{S}^2$ we have

$$\sigma = \kappa\cos\theta + \kappa^1\sin\theta\cos\phi + \kappa^2\sin\theta\sin\phi$$

with $\theta \in [0,\pi]$ and $\phi \in [0,2\pi]$. Therefore, $\kappa\cdot\sigma = \cos\theta$, and for any $\eta \in \mathbb{S}^2$ with

$$\eta = \kappa\cos\theta_1 + \kappa^1\sin\theta_1\cos\phi_1 + \kappa^2\sin\theta_1\sin\phi_1.$$

we deduce from (5.9)-(5.10)

$$\frac{\kappa-\sigma}{|\kappa-\sigma|}\cdot\eta = \sin\frac{\theta}{2}\,\cos\theta_1 - \cos\frac{\theta}{2}\,\sin\theta_1\,\cos(\phi-\phi_1), \tag{5.11}$$

$$\frac{\kappa+\sigma}{|\kappa+\sigma|}\cdot\eta = \cos\frac{\theta}{2}\cos\theta_1 + \sin\frac{\theta}{2}\sin\theta_1\cos(\phi-\phi_1). \tag{5.12}$$

Proof of (i). Applying the formula (5.4) for $Y_l^m\left(\frac{\kappa+\sigma}{|\kappa+\sigma|}\right)$, we have

$$\begin{aligned}
&\int_{\mathbb{S}^2_\sigma} f(\kappa\cdot\sigma)\,Y_l^m\left(\frac{\kappa+\sigma}{|\kappa+\sigma|}\right)d\sigma\\
&\qquad = \int_{\mathbb{S}^2_\sigma} f(\kappa\cdot\sigma)\,\frac{2\,l+1}{4\,\pi}\int_{\mathbb{S}^2_\eta} P_l\left(\frac{\kappa+\sigma}{|\kappa+\sigma|}\cdot\eta\right)\,Y_l^m(\eta)d\eta\,d\sigma\\
&\qquad = \int_{\mathbb{S}^2_\eta} Y_l^m(\eta)\,A(\eta)d\eta
\end{aligned}$$

where

$$A(\eta) = \int_{\mathbb{S}^2} f(\kappa\cdot\sigma)\,P_l\left(\frac{\kappa+\sigma}{|\kappa+\sigma|}\cdot\eta\right)\,d\sigma.$$

Then, applying the addition theorem (5.3) and (5.12)

$$\begin{aligned}
P_l\left(\frac{\kappa+\sigma}{|\kappa+\sigma|}\cdot\eta\right) &= P_l(\cos\frac{\theta}{2}\cos\theta_1 + \sin\theta_1\sin\frac{\theta}{2}\cos(\phi-\phi_1))\\
&= \sum_{m=-l}^{l}\frac{(l-m)!}{(l+m)!}P_l^m(\cos\frac{\theta}{2})P_l^m(\cos\theta_1)e^{im(\phi-\phi_1)},
\end{aligned}$$

direct calculation shows that

$$A(\eta) = \left(\int_0^\pi f(\cos\theta)\,(2\,\pi\sin(\theta))\,P_l\left(\cos\frac{\theta}{2}\right)d\theta\right)\,P_l(\kappa\cdot\eta).$$

Henceforth, we get that

$$\begin{aligned}
&\int_{\mathbb{S}^2_\sigma} f(\kappa\cdot\sigma)\,Y_l^m\left(\frac{\kappa+\sigma}{|\kappa+\sigma|}\right)d\sigma\\
&= \left(2\,\pi\int_{\theta=0}^{\theta=\pi} f(\cos\theta)\,\sin\theta\,P_l\left(\cos\left(\frac{\theta}{2}\right)\right)d\theta\right)\int_{\mathbb{S}^2_\eta} Y_l^m(\eta)\,\frac{2\,l+1}{4\,\pi}P_l(\kappa\cdot\eta)d\eta
\end{aligned}$$

and we conclude by formula (5.4).

The proof of (*ii*) is similar by using (5.11). □

As a direct consequence of part (*i*) of the previous lemma, we have :

Corollary 5.2. *For $\tilde{l}, \tilde{m} \in \mathbb{N}$ and $|\tilde{m}| \leq \tilde{l}$, we have for the cross section b satisfying* (1.2),

$$\int_{\mathbb{S}^2} b(\kappa\cdot\sigma)\left(Y_{\tilde{l}}^{\tilde{m}}\left(\frac{\kappa+\sigma}{|\kappa+\sigma|}\right)\left(\frac{1+\kappa\cdot\sigma}{2}\right)^{\frac{2\tilde{n}+\tilde{l}}{2}} - Y_{\tilde{l}}^{\tilde{m}}(\kappa)\right)d\sigma$$

$$= \left[\int_{|\theta|\leq\frac{\pi}{4}} \beta(\theta)\left((\cos\theta)^{2\tilde{n}+\tilde{l}}P_{\tilde{l}}(\cos\theta)-1\right)d\theta\right]Y_{\tilde{l}}^{\tilde{m}}(\kappa). \tag{5.13}$$

Lemma 5.3. *Let $\kappa \in \mathbb{S}^2$ and the cross section b satisfies* (1.2)*. Assume also that $n, l, \tilde{n}, \tilde{l} \in \mathbb{N}$ with $l \geq 1$, $\tilde{l} \geq 1$, $|m| \leq l$, $|\tilde{m}| \leq \tilde{l}$. Then there exists some constants $c^k_{n,l,m,\tilde{n},\tilde{l},\tilde{m}}$ such that*

$$\int_{\mathbb{S}^2} b(\kappa\cdot\sigma)Y_l^m\left(\frac{\kappa-\sigma}{|\kappa-\sigma|}\right)Y_{\tilde{l}}^{\tilde{m}}\left(\frac{\kappa+\sigma}{|\kappa+\sigma|}\right)\left(\frac{1-\kappa\cdot\sigma}{2}\right)^{\frac{2n+l}{2}}\left(\frac{1+\kappa\cdot\sigma}{2}\right)^{\frac{2\tilde{n}+\tilde{l}}{2}} d\sigma$$

$$= \sum_{k=0}^{k_0(l,\tilde{l},m,\tilde{m})} c^k_{n,l,m,\tilde{n},\tilde{l},\tilde{m}}\, Y_{l+\tilde{l}-2k}^{m+\tilde{m}}(\kappa). \tag{5.14}$$

Proof. Without loss of generality, we set $\min(l,\tilde{l}) = \tilde{l}$. We consider the same frame $(\kappa, \kappa^1, \kappa^2)$ defined by (5.6) used in the proof of the previous lemma and the transform (5.8)

$$\sigma = \kappa\cos\theta + \kappa^1\sin\theta\cos\phi + \kappa^2\sin\theta\sin\phi$$

with $\theta \in [0, \frac{\pi}{2}]$ and $\phi \in [0, 2\pi]$. Set $\gamma = \frac{\kappa-\sigma}{|\kappa-\sigma|}$, we have

$$\kappa+\sigma = 2\kappa - 2(\kappa\cdot\gamma)\gamma,\ \ |\kappa+\sigma| = 2\sqrt{1-(\kappa\cdot\gamma)^2}.$$

Therefore,

$$\frac{\kappa+\sigma}{|\kappa+\sigma|}\cdot\eta = \frac{\kappa\cdot\eta - (\kappa\cdot\gamma)(\gamma\cdot\eta)}{\sqrt{1-(\kappa\cdot\gamma)^2}}. \tag{5.15}$$

From the integral addition theorem (5.4) we have

$$Y_{\tilde{l}}^{\tilde{m}}\left(\frac{\kappa+\sigma}{|\kappa+\sigma|}\right) = \frac{2\tilde{l}+1}{4\pi}\int_{\mathbb{S}^2_\eta} P_{\tilde{l}}\left(\frac{\kappa+\sigma}{|\kappa+\sigma|}\cdot\eta\right)Y_{\tilde{l}}^{\tilde{m}}(\eta)d\eta. \tag{5.16}$$

We now consider the formula (see (43) in Chapter III in [20])

$$x^l = \frac{1}{B_l}P_l(x) + \sum_{1\leq q}\frac{(2(l-2q)+1)l!}{(2l-2q+1)!!(2q)!!}P_{l-2q}(x) \tag{5.17}$$

where

$$B_l = \frac{1\times3\times5\cdots(2l-1)}{l!},\ \ B_0 = 1.$$

We observe that, if $q \neq \tilde{l}$, from the Funck-Hecke Formula (5.5) and the orthogonality of the polynomials $(P_l)_l$,

$$\int_{\mathbb{S}^2_\eta} P_q(\gamma\cdot\eta)\,Y_{\tilde{l}}^{\tilde{m}}(\eta)d\eta = \left(2\pi\int_{-1}^1 P_q(x)P_{\tilde{l}}(x)\,dx\right)Y_{\tilde{l}}^{\tilde{m}}(\gamma) = 0$$

and we plug the value of $P_{\tilde{l}}(x)$ from (5.17) with x equal to the value of (5.15) into the previous integral of (5.16). Expanding and using the previous orthogonality property, we then derive

$$Y_{\tilde{l}}^{\tilde{m}}\left(\frac{\kappa+\sigma}{|\kappa+\sigma|}\right) = \frac{\hat{B}_{\tilde{l}}}{(\sqrt{1-(\kappa\cdot\gamma)^2})^{\tilde{l}}}\sum_{l_1+l_2=\tilde{l}}\frac{\tilde{l}!(-\kappa\cdot\gamma)^{l_2}}{l_1!l_2!}\int_{\mathbb{S}^2_\eta}(\kappa\cdot\eta)^{l_1}(\gamma\cdot\eta)^{l_2}Y_{\tilde{l}}^{\tilde{m}}(\eta)d\eta.$$

where

$$\hat{B}_l = \frac{2l+1}{4\pi}\frac{1\times 3\times 5\cdots(2l-1)}{l!}.$$

We then remark that, from the addition Theorem (5.3), for $0 \le q_1 \le l_1$, $0 \le q_2 \le l_2$

$$\int_{\mathbb{S}^2_\eta} P_{q_1}(\kappa\cdot\eta)\, P_{q_2}(\gamma\cdot\eta)\, Y_{\tilde{l}}^{\tilde{m}}(\eta)\, d\eta = \frac{4\pi}{2q_1+1}\frac{4\pi}{2q_2+1}$$

$$\times \sum_{m_1=-q_1}^{q_1}\sum_{m_2=-q_2}^{q_2}\left(\int_{\mathbb{S}^2_\eta} Y_{\tilde{l}}^{\tilde{m}}(\eta)Y_{q_1}^{-m_1}(\eta)Y_{q_2}^{-m_2}(\eta)d\eta\right)Y_{q_1}^{m_1}(\kappa)Y_{q_2}^{m_2}(\gamma).$$

Therefore, by formula (5.17), we replace $(\kappa\cdot\eta)^{l_1}\,(\gamma\cdot\eta)^{l_2}$ by a sums of Legendre polynomials, and using the previous relation and the vanishing property (7.5), we obtain

$$Y_{\tilde{l}}^{\tilde{m}}\left(\frac{\kappa+\sigma}{|\kappa+\sigma|}\right) = \sum_{l_1+l_2=\tilde{l}}\frac{\tilde{l}!}{l_1!l_2!}\frac{\hat{B}_{\tilde{l}}}{\hat{B}_{l_1}\hat{B}_{l_2}}\frac{(-\kappa\cdot\gamma)^{l_2}}{(\sqrt{1-(\kappa\cdot\gamma)^2})^{\tilde{l}}}$$

$$\times\sum_{m_1=-l_1}^{l_1}\sum_{m_2=-l_2}^{l_2}\left(\int_{\mathbb{S}^2_\eta} Y_{\tilde{l}}^{\tilde{m}}(\eta)Y_{l_1}^{-m_1}(\eta)Y_{l_2}^{-m_2}(\eta)d\eta\right)Y_{l_1}^{m_1}(\kappa)Y_{l_2}^{m_2}(\gamma). \tag{5.18}$$

Moreover we derive from (7.5) and (7.6)

$$Y_l^m(\gamma)Y_{l_2}^{m_2}(\gamma) = \sum_{l'}\sum_{|m'|\le l'}\left(\int_{\mathbb{S}^2} Y_l^m Y_{l_2}^{m_2}\overline{Y_{l'}^{m'}}\right)Y_{l'}^{m'}(\gamma)$$

$$Y_{l'}^{m'}(\kappa)Y_{l_1}^{m_1}(\kappa) = \sum_{l''}\sum_{|m''|\le l''}\left(\int_{\mathbb{S}^2} Y_{l'}^{m'} Y_{l_1}^{m_1}\overline{Y_{l''}^{m''}}\right)Y_{l''}^{m''}(\kappa)$$

where l' and l'' are defined by $l' = l+l_2-2j_1$ and $l'' = l'+l_1-2j_2$ with $0\le j_1\le \min(l,l_2)$ and $0\le j_2\le\min(l',l_1)$. Indeed, we have

$$l'' = \tilde{l}+l-2(j_1+j_2)$$

with $0\le j_1+j_2\le\tilde{l}=\min(l,\tilde{l})$. It follows from (5.18) and part (ii) of lemma 5.1 that

$$\int_{\mathbb{S}^2} b(\kappa\cdot\sigma)Y_l^m\left(\frac{\kappa-\sigma}{|\kappa-\sigma|}\right)Y_{\tilde{l}}^{\tilde{m}}\left(\frac{\kappa+\sigma}{|\kappa+\sigma|}\right)\left(\frac{1-\kappa\cdot\sigma}{2}\right)^{\frac{2n+l}{2}}\left(\frac{1+\kappa\cdot\sigma}{2}\right)^{\frac{2\tilde{n}+\tilde{l}}{2}}d\sigma$$

$$= \sum_{l_1+l_2=\tilde{l}}\frac{\tilde{l}!(-1)^{l_2}}{l_1!l_2!}\frac{\hat{B}_{\tilde{l}}}{\hat{B}_{l_1}\hat{B}_{l_2}}\sum_{m_1=-l_1}^{l_1}\sum_{m_2=-l_2}^{l_2}\left(\int_{\mathbb{S}^2} Y_{\tilde{l}}^{\tilde{m}}(\eta)Y_{l_1}^{-m_1}(\eta)Y_{l_2}^{-m_2}(\eta)d\eta\right)$$

$$\times\sum_{l'}\sum_{m'}\left(\int_{|\theta|\le\frac{\pi}{4}}\beta(\theta)(\sin\theta)^{2n+l+l_2}P_{l'}(\sin\theta)(\cos\theta)^{2\tilde{n}}d\theta\right)$$

$$\times\left(\int_{\mathbb{S}^2} Y_l^m Y_{l_2}^{m_2}\overline{Y_{l'}^{m'}}d\eta\right)\sum_{l''}\sum_{m''}\left(\int_{\mathbb{S}^2} Y_{l'}^{m'}Y_{l_1}^{m_1}\overline{Y_{l''}^{m''}}d\eta\right)Y_{l''}^{m''}(\kappa)$$

which is nonzero when

$$m'' = m'-m_1 = m-m_2-m_1 = m+\tilde{m},$$

and $l'' = \tilde{l}+l-2(j_1+j_2)$ with $0\le j_1+j_2\le\min(l,\tilde{l})$. For l,$\tilde{l}$,m,$\tilde{m}$ fixed, we can define $l'' = l+\tilde{l}-2k$ with $0\le k\le\min(l,\tilde{l})$, then the coefficient of $Y_{l''}^{m''}(\kappa)$ is nonzero when

$$|m+\tilde{m}|\le l+\tilde{l}-2k.$$

Therefore,

$$k \le \frac{l + \tilde{l} - |m + \tilde{m}|}{2}$$

In conclusion,

$$0 \le k \le k_0(l, \tilde{l}, m, \tilde{m})$$

where $k_0(l, \tilde{l}, m, \tilde{m})$ was given in (2.3). This ends the proof of (5.14). $\square$

5.2. **The proof of the Proposition 2.1.** The spectral representation will be based on the Bobylev formula, which is the Fourier transform of the Boltzmann operator (in the Maxwellian molecules):

$$\mathcal{F}(Q(g,f))(\xi) = \int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\left[\hat{g}(\xi^-)\hat{f}(\xi^+) - \hat{g}(0)\hat{f}(\xi)\right]d\sigma,$$

where

$$\xi^- = \frac{\xi - |\xi|\sigma}{2} = \frac{|\xi|}{2}(\kappa - \sigma), \qquad \xi^+ = \frac{\xi + |\xi|\sigma}{2} = \frac{|\xi|}{2}(\kappa + \sigma)$$

with $\kappa = \frac{\xi}{|\xi|}$. Remark that

$$\kappa \cdot \sigma = \cos\theta, \quad |\xi^-| = |\xi|\,|\sin(\theta/2)|, \quad |\xi^+| = |\xi|\cos(\theta/2).$$

Let $\varphi_{n,l,m}$ be the functions defined in (2.1), then for $n, l \in \mathbb{N}$, $|m| \le l$, we have (see Lemma 7.2)

$$\widehat{\sqrt{\mu}\varphi_{n,l,m}}(\xi) = A_{n,l}\Big(\frac{|\xi|}{\sqrt{2}}\Big)^{2n+l} e^{-\frac{|\xi|^2}{2}} Y_l^m\Big(\frac{\xi}{|\xi|}\Big), \tag{5.19}$$

where

$$A_{n,l} = (-i)^l (2\pi)^{\frac{3}{4}} \left(\frac{1}{\sqrt{2}n!\Gamma(n + l + \frac{3}{2})}\right)^{\frac{1}{2}}.$$

At the special case $l = 0$, it is Hermit function,

$$\widehat{\sqrt{\mu}\varphi_{n,0,0}}(\xi) = \frac{1}{\sqrt{(2n+1)!}}|\xi|^{2n} e^{-\frac{|\xi|^2}{2}}. \tag{5.20}$$

We deduce from the Bobylev formula that, $\forall\, n,\ l,\ m,\ \tilde{n},\ \tilde{l},\ \tilde{m} \in \mathbb{N}$, with $|m| \le l, |\tilde{m}| \le \tilde{l}$,

$$\begin{aligned}
&\mathcal{F}\Big(\sqrt{\mu}\,\mathbf{\Gamma}(\varphi_{n,l,m}, \varphi_{\tilde{n},\tilde{l},\tilde{m}})\Big)(\xi) = \mathcal{F}(Q(\sqrt{\mu}\varphi_{n,l,m}, \sqrt{\mu}\varphi_{\tilde{n},\tilde{l},\tilde{m}}))(\xi)\\
&= \int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\Big[\widehat{\sqrt{\mu}\varphi_{n,l,m}}(\xi^-)\,\widehat{\sqrt{\mu}\varphi_{\tilde{n},\tilde{l},\tilde{m}}}(\xi^+) - \widehat{\sqrt{\mu}\varphi_{n,l,m}}(0)\,\widehat{\sqrt{\mu}\varphi_{\tilde{n},\tilde{l},\tilde{m}}}(\xi)\Big]d\sigma.
\end{aligned} \tag{5.21}$$

In the next propositions, we will compute the terms $\mathbf{\Gamma}(\varphi_{n,l,m}, \varphi_{\tilde{n},\tilde{l},\tilde{m}})$ and proposition 2.1 will follows.

Proposition 5.4. *The following algebraic identities hold,*

$$\begin{aligned}
(i)\quad &\mathbf{\Gamma}(\varphi_{0,0,0}, \varphi_{\tilde{n},\tilde{l},\tilde{m}})\\
&= \left(\int_{|\theta|\le\frac{\pi}{4}} \beta(\theta)\Big((\cos\theta)^{2\tilde{n}+\tilde{l}} P_{\tilde{l}}(\cos\theta) - 1\Big)d\theta\right)\varphi_{\tilde{n},\tilde{l},\tilde{m}};\\
(ii)\quad &\mathbf{\Gamma}(\varphi_{n,l,m}, \varphi_{0,0,0})\\
&= \left(\int_{|\theta|\le\frac{\pi}{4}} \beta(\theta)\Big((\sin\theta)^{2n+l} P_l(\sin\theta) - \delta_{0,n}\,\delta_{0,l}\Big)d\theta\right)\varphi_{n,l,m}.
\end{aligned}$$

This is exactly (i_1) and (i_2) of the Proposition 2.1.

Proof. Since

$$\widehat{\sqrt{\mu}\varphi_{n,l,m}}(0)=\delta_{n,0}\delta_{l,0},$$

then when $n=0, l=0$, by using (5.19) and (5.20), we have

$$\begin{aligned}
&\mathcal{F}(Q(\sqrt{\mu}\varphi_{0,0,0},\ \sqrt{\mu}\varphi_{\tilde{n},\tilde{l},\tilde{m}}))(\xi)\\
&=\int_{\mathbb{S}^2} b\left[\frac{\xi}{|\xi|}\cdot\sigma\right)\Big[\widehat{\sqrt{\mu}\varphi_{0,0,0}}(\xi^-)\,\widehat{\sqrt{\mu}\varphi_{\tilde{n},\tilde{l},\tilde{m}}}(\xi^+)-\widehat{\sqrt{\mu}\varphi_{0,0,0}}(0)\,\widehat{\sqrt{\mu}\varphi_{\tilde{n},\tilde{l},\tilde{m}}}(\xi)\Big]d\sigma.\\
&=A_{\tilde{n},\tilde{l}}e^{-\frac{|\xi|^2}{2}}\left(\frac{|\xi|}{\sqrt{2}}\right)^{2\tilde{n}+\tilde{l}}\int_{\mathbb{S}^2} b(\kappa\cdot\sigma)\left[Y_{\tilde{l}}^{\tilde{m}}\left(\frac{\kappa+\sigma}{|\kappa+\sigma|}\right)\left(\frac{|\kappa+\sigma|}{2}\right)^{2\tilde{n}+\tilde{l}}-Y_{\tilde{l}}^{\tilde{m}}(\kappa)\right]d\sigma.
\end{aligned}$$

Apply now the identity (5.13) of the Corollary 5.2, one can find that,

$$\begin{aligned}
&\mathcal{F}(Q(\sqrt{\mu}\varphi_{0,0,0},\ \sqrt{\mu}\varphi_{\tilde{n},\tilde{l},\tilde{m}}))(\xi)\\
&=\int_{|\theta|\le\frac{\pi}{4}}\beta(\theta)\Big[(\cos\theta)^{2\tilde{n}+\tilde{l}}P_{\tilde{l}}(\cos\theta)-1\Big]d\theta\,\widehat{\sqrt{\mu}\varphi_{\tilde{n},\tilde{l},\tilde{m}}}(\xi).
\end{aligned}$$

Hence by the inverse Fourier transform

$$\begin{aligned}
\mathbf{\Gamma}(\varphi_{0,0,0},\varphi_{\tilde{n},\tilde{l},\tilde{m}})&=\frac{1}{\sqrt{\mu}}Q(\sqrt{\mu}\varphi_{0,0,0},\ \sqrt{\mu}\varphi_{\tilde{n},\tilde{l},\tilde{m}})\\
&=\left(\int_{|\theta|\le\frac{\pi}{4}}\beta(\theta)\Big((\cos\theta)^{2\tilde{n}+\tilde{l}}P_{\tilde{l}}(\cos\theta)-1\Big)d\theta\right)\varphi_{\tilde{n},\tilde{l},\tilde{m}}.
\end{aligned}$$

The result of (i) follows. Similar arguments apply to the case (ii), and this ends the proof of Proposition 5.4. □

Proposition 5.5. *The following algebraic identities hold,*

$$\begin{aligned}
(i)\quad &\mathbf{\Gamma}(\varphi_{n,0,0},\varphi_{\tilde{n},\tilde{l},\tilde{m}})\\
&=\frac{A_{\tilde{n},\tilde{l}}A_{n,0}}{A_{n+\tilde{n},\tilde{l}}}\left(\int_{|\theta|\le\frac{\pi}{4}}\beta(\theta)(\sin\theta)^{2n}(\cos\theta)^{2\tilde{n}+\tilde{l}}P_{\tilde{l}}(\cos\theta)d\theta\right)\varphi_{n+\tilde{n},\tilde{l},\tilde{m}},\quad n\ge1;
\end{aligned}$$

$$\begin{aligned}
(ii)\quad &\mathbf{\Gamma}(\varphi_{n,l,m},\varphi_{\tilde{n},0,0})\\
&=\frac{A_{\tilde{n},0}A_{n,l}}{A_{n+\tilde{n},l}}\left(\int_{|\theta|\le\frac{\pi}{4}}\beta(\theta)(\sin\theta)^{2n+l}P_l(\sin\theta)(\cos\theta)^{2\tilde{n}}d\theta\right)\varphi_{n+\tilde{n},l,m},\quad l\ge1.
\end{aligned}$$

This is exactly (ii_1) and (ii_2) of the Proposition 2.1.

Proof. For $n\ge1$, using (5.20), we remark that $\widehat{\sqrt{\mu}\varphi_{n,0,0}}(0)=0$, and using (5.19) for (5.21) and get

$$\begin{aligned}
\mathcal{F}(Q(\sqrt{\mu}\varphi_{n,0,0},\ \sqrt{\mu}\varphi_{\tilde{n},\tilde{l},\tilde{m}}))(\xi)&=A_{\tilde{n},\tilde{l}}A_{n,0}e^{-\frac{|\xi|^2}{2}}\left(\frac{|\xi|}{\sqrt{2}}\right)^{2(n+\tilde{n})+\tilde{l}}\\
&\times\int_{\mathbb{S}^2} b(\kappa\cdot\sigma)\Big(\frac{1-\kappa\cdot\sigma}{2}\Big)^n\Big(\frac{|\kappa+\sigma|}{2}\Big)^{2\tilde{n}+\tilde{l}}Y_{\tilde{l}}^{\tilde{m}}\Big(\frac{\kappa+\sigma}{|\kappa+\sigma|}\Big)d\sigma.
\end{aligned}$$

We then apply the identity (5.13) of corollary 5.2 and again (5.19) and derive

$$\begin{aligned}
&\mathcal{F}(Q(\sqrt{\mu}\varphi_{n,0,0},\ \sqrt{\mu}\varphi_{\tilde{n},\tilde{l},\tilde{m}}))(\xi)\\
&=\frac{A_{\tilde{n},\tilde{l}}A_{n,0}}{A_{n+\tilde{n},\tilde{l},0}}\Big[\int_{|\theta|\le\frac{\pi}{4}}\beta(\theta)\Big((\sin\theta)^{2n}(\cos\theta)^{2\tilde{n}+\tilde{l}}P_{\tilde{l}}(\cos\theta)\Big)d\theta\Big]\,\widehat{\sqrt{\mu}\varphi_{\tilde{n}+n,\tilde{l},\tilde{m}}}(\xi).
\end{aligned}$$

We obtain that by the inverse Fourier transform

$$\begin{aligned}\mathbf{\Gamma}(\varphi_{n,0,0},\varphi_{\tilde n,\tilde l,\tilde m}) &= \frac{1}{\sqrt{\mu}}Q(\sqrt{\mu}\varphi_{n,0,0},\ \sqrt{\mu}\varphi_{\tilde n,\tilde l,\tilde m})\\ &= \frac{A_{\tilde n,\tilde l}A_{n,0}}{A_{n+\tilde n,\tilde l,0}}\Big[\int_{|\theta|\le\frac{\pi}{4}}\beta(\theta)\Big((\sin\theta)^{2n}(\cos\theta)^{2\tilde n+\tilde l}P_{\tilde l}(\cos\theta)\Big)d\theta\Big]\varphi_{\tilde n+n,\tilde l,\tilde m}.\end{aligned}$$

Thus (i) follows. Analogously, (ii) holds true. This ends the proof of Proposition 5.5. □

Proposition 5.6. *The following algebraic identities hold for $l\ge 1$, $\tilde l\ge 1$:*

$$\begin{aligned}&\mathbf{\Gamma}(\varphi_{n,l,m},\varphi_{\tilde n,\tilde l,\tilde m})\\ &= \sum_{k=0}^{k_0(l,\tilde l,m,\tilde m)}\frac{A_{\tilde n,\tilde l}A_{n,l}}{A_{n+\tilde n+k,l+\tilde l-2k}}\left(\int_{\mathbb{S}^2_\kappa}G^{m,\tilde m}_{n,\tilde n,l,\tilde l}(\kappa)\overline{Y^{m+\tilde m}_{l+\tilde l-2k}}(\kappa)d\kappa\right)\varphi_{n+\tilde n+k,l+\tilde l-2k,m+\tilde m},\end{aligned}$$

where $k_0(l,\tilde l,m,\tilde m)$ is given in (2.3) *and $G^{m,\tilde m}_{n,\tilde n,l,\tilde l}$ is defined by*

$$\begin{aligned}G^{m,\tilde m}_{n,\tilde n,l,\tilde l}(\kappa) = \int_{\mathbb{S}^2}&b(\kappa\cdot\sigma)\Big(|\kappa-\sigma|/2\Big)^{2n+l}\Big(|\kappa+\sigma|/2\Big)^{2\tilde n+\tilde l}\\ &\times Y^m_l\Big(\frac{\kappa-\sigma}{|\kappa-\sigma|}\Big)Y^{\tilde m}_{\tilde l}\Big(\frac{\kappa+\sigma}{|\kappa+\sigma|}\Big)d\sigma.\end{aligned}\tag{5.22}$$

This is exactly (iii) of the Proposition 2.1.

Proof. Now we consider the case when $l\ge 1$ and $\tilde l\ge 1$. Since $\widehat{\sqrt{\mu}\varphi_{n,l,m}}(0)=0$, we get from (5.19)-(5.21)

$$\begin{aligned}&\mathcal{F}(Q(\sqrt{\mu}\varphi_{n,l,m},\ \sqrt{\mu}\varphi_{\tilde n,\tilde l,\tilde m}))(\xi) = A_{\tilde n,\tilde l}A_{n,l}e^{-\frac{|\xi|^2}{2}}\left(\frac{|\xi|}{\sqrt2}\right)^{2(n+\tilde n)+\tilde l+l}\\ &\quad\times\int_{\mathbb{S}^2}b(\kappa\cdot\sigma)Y^m_l\left(\frac{\kappa-\sigma}{|\kappa-\sigma|}\right)Y^{\tilde m}_{\tilde l}\left(\frac{\kappa+\sigma}{|\kappa+\sigma|}\right)\left(\frac{|\kappa-\sigma|}{2}\right)^{2n+l}\left(\frac{|\kappa+\sigma|}{2}\right)^{2\tilde n+\tilde l}d\sigma\\ &= A_{\tilde n,\tilde l}A_{n,l}e^{-\frac{|\xi|^2}{2}}\Big(\frac{|\xi|}{\sqrt2}\Big)^{2(n+\tilde n)+\tilde l+l}G^{m,\tilde m}_{\tilde n,n,\tilde l,l}(\kappa).\end{aligned}$$

From the lemma 5.3, $G^{m,\tilde m}_{n,\tilde n,l,\tilde l}(\kappa)$ can be decomposed as a finite Laplace series

$$G^{m,\tilde m}_{n,\tilde n,l,\tilde l}(\kappa) = \sum_{k=0}^{k_0(l,\tilde l,m,\tilde m)}\left(\int_{\mathbb{S}^2_\kappa}G^{m,\tilde m}_{n,\tilde n,l,\tilde l}(\kappa)\overline{Y^{m+\tilde m}_{l+\tilde l-2k}}d\kappa\right)Y^{m+\tilde m}_{l+\tilde l-2k}(\kappa),$$

where $k_0(l,\tilde l,m,\tilde m)$ was given in (2.3). By using this expansion, we derive

$$\begin{aligned}&\mathcal{F}(Q(\sqrt{\mu}\varphi_{n,l,m},\ \sqrt{\mu}\varphi_{\tilde n,\tilde l,\tilde m}))(\xi)\\ &= \sum_{k=0}^{k_0(l,\tilde l,m,\tilde m)}\frac{A_{\tilde n,\tilde l}A_{n,l}}{A_{n+\tilde n+k,l+\tilde l-2k}}\left(\int_{\mathbb{S}^2_\kappa}G^{m,\tilde m}_{n,\tilde n,l,\tilde l}(\kappa)\overline{Y^{m+\tilde m}_{l+\tilde l-2k}}(\kappa)d\kappa\right)\mathcal{F}(\sqrt{\mu}\varphi_{n+\tilde n+k,l+\tilde l-2k,m+\tilde m})\end{aligned}$$

and we conclude by taking the inverse Fourier transform. This ends the proof of Proposition 5.6. □

5.3. **The proof of the Proposition 2.2.** We now prove the following identity of Proposition 2.2

$$\sum_{|m|\le l}\sum_{|\tilde m|\le \tilde l}\mu^{m,\tilde m,m'_1}_{n,\tilde n,l,\tilde l,k_1}\overline{\mu^{m,\tilde m,m'_2}_{n,\tilde n,l,\tilde l,k_2}}=\sum_{|m|\le l}\sum_{|\tilde m|\le \tilde l}\left|\mu^{m,\tilde m,m'_1}_{n,\tilde n,l,\tilde l,k_1}\right|^2\delta_{k_1,k_2}\delta_{m'_1,m'_2}.$$

We state it in the following proposition with the notations $G^{m,\tilde m}_{n,\tilde n,l,\tilde l}(\kappa)$ given in (5.22), since from the Proposition 2.1, we have, for $|m'|\le l+\tilde l-2k$

$$\mu^{m,\tilde m,m'}_{n,\tilde n,l,\tilde l,k}=(-1)^k\Big(\frac{2\pi^{\frac32}(n+\tilde n+k)!\Gamma(n+\tilde n+l+\tilde l-k+\frac32)}{\tilde n!\Gamma(\tilde n+\tilde l+\frac32)n!\Gamma(n+l+\frac32)}\Big)^{\frac12}$$
$$\times\int_{\mathbb{S}^2_\kappa}G^{m,\tilde m}_{n,\tilde n,l,\tilde l}(\kappa)\overline{Y^{m'}_{l+\tilde l-2k}}(\kappa)d\kappa.$$

Proposition 5.7. *For $G^{m,\tilde m}_{n,\tilde n,l,\tilde l}(\kappa)$ given in* (5.22) *and any integers $n,\tilde n\ge 0$, $|m|\le l$, $|m'|\le l'$, $|m^\star|\le l^\star$, we have*

$$\sum_{|m|\le l}\sum_{|\tilde m|\le \tilde l}\left(\int_{\mathbb{S}^2_\kappa}G^{m,\tilde m}_{n,\tilde n,l,\tilde l}(\kappa)\overline{Y^{m'}_{l'}}(\kappa)d\kappa\right)\left(\int_{\mathbb{S}^2_\kappa}\overline{G^{m,\tilde m}_{n,\tilde n,l,\tilde l}(\kappa)}Y^{m^\star}_{l^\star}(\kappa)d\kappa\right)$$
$$=\sum_{|m|\le l}\sum_{|\tilde m|\le \tilde l}\left|\int_{\mathbb{S}^2_\kappa}G^{m,\tilde m}_{n,\tilde n,l,\tilde l}(\kappa)\overline{Y^{m'}_{l'}}(\kappa)d\kappa\right|^2\delta_{l',l^\star}\delta_{m',m^\star}.$$

Proof. We recall the definition (5.22) of $G^{m,\tilde m}_{n,\tilde n,l,\tilde l}(\kappa)$

$$G^{m,\tilde m}_{n,\tilde n,l,\tilde l}(\kappa)=\int_{\mathbb{S}^2}b(\kappa\cdot\sigma)\,(|\kappa-\sigma|/2)^{2n+l}\,(|\kappa+\sigma|/2)^{2\tilde n+\tilde l}$$
$$\times Y^m_l\left(\frac{\kappa-\sigma}{|\kappa-\sigma|}\right)Y^{\tilde m}_{\tilde l}\left(\frac{\kappa+\sigma}{|\kappa+\sigma|}\right)d\sigma$$

and we consider the transform (5.6)-(5.8) for a unit vector σ

$$\sigma=\kappa\cos\theta+\kappa^1\sin\theta\cos\phi+\kappa^2\sin\theta\sin\phi$$
$$=\kappa\cos\theta+\kappa^\perp(\phi)\,\sin\theta$$

with $\theta\in[0,\frac{\pi}{2}]$ and $\phi\in[0,2\pi]$. Therefore, using (5.9)-(5.10), the change of variable $\theta=2\,\theta_1$, the odd-even parity of P^m_l and the definition (1.2) of β we find

$$G^{m,\tilde m}_{n,\tilde n,l,\tilde l}(\kappa)=\int_{|\theta_1|\le\frac{\pi}{4}}\beta(\theta_1)(\sin\theta_1)^{2n+l}(\cos\theta_1)^{2\tilde n+\tilde l}\quad\times$$
$$\int_0^{2\pi}Y^m_l(\kappa\sin\theta_1-\kappa^\perp(\phi_1)\cos\theta_1)Y^{\tilde m}_{\tilde l}(\kappa\cos\theta_1+\kappa^\perp(\phi_1)\sin\theta_1)\frac{d\phi_1}{2\pi}d\theta_1\tag{5.23}$$

and

$$\int_{\mathbb{S}^2_\kappa}G^{m,\tilde m}_{n,\tilde n,l,\tilde l}(\kappa)\overline{Y^{m'}_{l'}}(\kappa)d\kappa=\int_{|\theta_1|\le\frac{\pi}{4}}\beta(\theta_1)(\sin\theta_1)^{2n+l}(\cos\theta_1)^{2\tilde n+\tilde l}\quad\times$$
$$\int_0^{2\pi}\left(\int_{\mathbb{S}^2_\kappa}Y^m_l(\kappa\sin\theta_1-\kappa^\perp(\phi_1)\cos\theta_1)\quad\times\right.\tag{5.24}$$
$$\left.Y^{\tilde m}_{\tilde l}(\kappa\cos\theta_1+\kappa^\perp(\phi_1)\sin\theta_1)\overline{Y^{m'}_{l'}}(\kappa)d\kappa\right)\frac{d\phi_1}{2\pi}d\theta_1.$$

Using equivalent notations of (5.6)-(5.8) for an another unit vector γ

$$\begin{aligned}\gamma &= \gamma\cos\theta + \gamma^1\sin\theta\cos\phi + \gamma^2\sin\theta\sin\phi \\ &= \gamma\cos\theta + \gamma^\perp(\phi)\sin\theta,\end{aligned}$$

we have also

$$\int_{\mathbb{S}^2_\gamma} \overline{G^{m,\tilde m}_{n,\tilde n,l,\tilde l}(\gamma)} Y^{m^\star}_{l^\star}(\gamma) d\gamma = \int_{|\theta_2|\le\frac{\pi}{4}} \beta(\theta_2)(\sin\theta_2)^{2n+l}(\cos\theta_2)^{2\tilde n+\tilde l} \quad \times \int_0^{2\pi}\int_{\mathbb{S}^2_\gamma}$$
$$\overline{Y^m_l}(\gamma\sin\theta_2 - \gamma^\perp(\phi_2)\cos\theta_2)\overline{Y^{\tilde m}_{\tilde l}}(\gamma\cos\theta_2 + \gamma^\perp(\phi_1)\sin\theta_2) Y^{m^\star}_{l^\star}(\gamma) d\gamma \frac{d\phi_2}{2\pi} d\theta_2.$$

The goal of this proposition is to compute the following term :

$$\mathbf{A} = \sum_{|m|\le l}\sum_{|\tilde m|\le\tilde l}\left(\int_{\mathbb{S}^2_\kappa} G^{m,\tilde m}_{n,\tilde n,l,\tilde l}(\kappa)\overline{Y^{m'}_{l'}}(\kappa)d\kappa\right)\left(\int_{\mathbb{S}^2_\gamma}\overline{G^{m,\tilde m}_{n,\tilde n,l,\tilde l}(\gamma)} Y^{m^\star}_{l^\star}(\gamma)d\gamma\right)$$

From the addition theorem (5.2), we find

$$\begin{aligned}&\sum_{|m|\le l} Y^m_l(\kappa\sin\theta_1 - \kappa^\perp(\phi_1)\cos\theta_1)\overline{Y^m_l}(\gamma\sin\theta_2 - \gamma^\perp(\phi_2)\cos\theta_2) \\ &= \frac{2l+1}{4\pi} P_l((\kappa\sin\theta_1 - \kappa^\perp(\phi_1)\cos\theta_1)\cdot(\gamma\sin\theta_2 - \gamma^\perp(\phi_2)\cos\theta_2))\end{aligned}$$

and

$$\begin{aligned}&\sum_{|\tilde m|\le \tilde l} Y^{\tilde m}_{\tilde l}(\kappa\sin\theta_1 - \kappa^\perp(\phi_1)\cos\theta_1)\overline{Y^{\tilde m}_{\tilde l}}(\gamma\sin\theta_2 - \gamma^\perp(\phi_2)\cos\theta_2) \\ &= \frac{2\tilde l+1}{4\pi} P_{\tilde l}((\kappa\sin\theta_1 - \kappa^\perp(\phi_1)\cos\theta_1)\cdot(\gamma\sin\theta_2 - \gamma^\perp(\phi_2)\cos\theta_2)).\end{aligned}$$

We then plug the two previous identities into the expression of $\mathbf{A}$ and we directly derive

$$\begin{aligned}\mathbf{A} = &\frac{2l+1}{4\pi}\frac{2\tilde l+1}{4\pi}\int_{|\theta_1|\le\frac{\pi}{4}}\left(\beta(\theta_1)(\sin\theta_1)^{2n+l}(\cos\theta_1)^{2\tilde n+\tilde l}\right) \\ &\times\int_{|\theta_2|\le\frac{\pi}{4}}\left(\beta(\theta_2)(\sin\theta_2)^{2n+l}(\cos\theta_2)^{2\tilde n+\tilde l}\right)\mathbf{B_1}(\theta_1,\theta_2)d\theta_1\, d\theta_2 \end{aligned} \tag{5.25}$$

where

$$\mathbf{B_1}(\theta_1,\theta_2) = \int_{\mathbb{S}^2_\gamma}\int_{\mathbb{S}^2_\kappa}\int_0^{2\pi}\mathbf{B_2}(\kappa,\gamma,\theta_1,\theta_2,\phi_2)\frac{d\phi_2}{2\pi}\, Y^{m^\star}_{l^\star}(\kappa)\overline{Y^{m'}_{l'}}(\gamma)d\kappa d\gamma \tag{5.26}$$

and

$$\begin{aligned}\mathbf{B_2}(\kappa,\gamma,\theta_1,\theta_2,\phi_2) = &\int_0^{2\pi} P_l\left((\kappa\sin\theta_1 - \kappa^\perp(\phi_1)\cos\theta_1)\cdot\gamma^-\right) \\ &\times P_{\tilde l}\left((\kappa\cos\theta_1 + \kappa^\perp(\phi_1)\sin\theta_1)\cdot\gamma^+\right)\frac{d\phi_1}{2\pi}.\end{aligned}$$

Here, γ^+ and γ^- are defined by (and depend on γ, θ_2, ϕ_2)

$$\gamma^+ = \gamma\cos\theta_2 + \gamma^\perp(\phi_2)\sin\theta_2, \quad \gamma^- = \gamma\sin\theta_2 - \gamma^\perp(\phi_2)\cos\theta_2. \tag{5.27}$$

From the lemma 5.9 (proved after the proposition), we have

$$\mathbf{B_2}(\kappa,\gamma,\theta_1,\theta_2,\phi_2) = \sum_{0\le q\le l}\sum_{0\le\tilde q\le\tilde l} b^{q,\tilde q}_{l,\tilde l}(\theta_1)P_q(\kappa\cdot\gamma^-)P_{\tilde q}(\kappa\cdot\gamma^+)$$

where $b^{q,\tilde q}_{l,\tilde l}(\theta_1)$ is a continuous function dependent on θ_1.

Therefore from (5.26) we deduce

$$\mathbf{B_1}(\theta_1,\theta_2) = \sum_{0\le q\le l}\sum_{0\le\tilde q\le\tilde l} b_{l,\tilde l}^{q,\tilde q}(\theta_1)\,\mathbf{B}_{q,\tilde q}(\theta_2) \tag{5.28}$$

where

$$\mathbf{B}_{q,\tilde q}(\theta_2) = \int_{\mathbb{S}^2_\gamma}\int_{\mathbb{S}^2_\kappa}\left(\int_0^{2\pi} P_q(\kappa\cdot\gamma^-)\,P_{\tilde q}(\kappa\cdot\gamma^+)\,\frac{d\phi_2}{2\pi}\right) Y_{l^\star}^{m^\star}(\kappa)\,\overline{Y_{l'}^{m'}}(\gamma)\,d\kappa d\gamma. \tag{5.29}$$

We use again two times the addition theorem (5.3) and compute (recall that $\gamma^\pm$ defined in (5.27) depend on γ, θ_2, ϕ_2)

$$\int_0^{2\pi} P_q(\kappa\cdot\gamma^-(\phi_2))P_{\tilde q}(\kappa\cdot\gamma^+(\phi_2))\frac{d\phi_2}{2\pi} =$$
$$\sum_{|k|\le\min(q,\tilde q)} a_{q,k}a_{\tilde q,k}P_q^{|k|}(\kappa\cdot\gamma)P_q^{|k|}(\sin\theta_2)(-1)^k P_{\tilde q}^{|k|}(\kappa\cdot\gamma).P_{\tilde q}^{|k|}(\cos\theta_2)$$

Since $P_q^{|k|}(\kappa\cdot\gamma)P_{\tilde q}^{|k|}(\kappa\cdot\gamma)$ is a continuous function of $\kappa\cdot\gamma$,we apply the Funk-Hecke Formula (5.5) and obtain

$$\int_{\mathbb{S}^2} P_q^{|k|}(\kappa\cdot\gamma)P_{\tilde q}^{|k|}(\kappa\cdot\gamma)Y_{l^\star}^{m^\star}(\kappa)d\kappa = \left(2\pi\int_{-1}^1 P_q^{|k|}(x)P_{\tilde q}^{|k|}(x)P_{l^\star}(x)dx\right)Y_{l^\star}^{m^\star}(\gamma).$$

Combining the two previous relations into (5.29), we obtain

$$\begin{aligned}\mathbf{B}_{q,\tilde q}(\theta_2) = &\sum_{|k|\le\min(q,\tilde q)} a_{q,k}a_{\tilde q,k}P_q^{|k|}(\sin\theta_2)(-1)^k P_{\tilde q}^{|k|}(\cos\theta_2)\\ &\times\left(2\pi\int_{-1}^1 P_q^{|k|}(x)P_{\tilde q}^{|k|}(x)P_{l^\star}(x)dx\right)\int_{\mathbb{S}^2_\gamma} Y_{l^\star}^{m^\star}(\gamma)\overline{Y_{l'}^{m'}}(\gamma)d\gamma.\end{aligned} \tag{5.30}$$

Finely, if $(l^\star,m^\star)\neq(l',m')$, the orthogonality of the spherical harmonics implies that $\mathbf{B}_{q,\tilde q}=0$ for all q and $\tilde q$, and so on for $\mathbf{B_1}$ and $\mathbf{A}$. This concludes the proof of the proposition 5.7. □

Remark 5.8. *From the previous proof, in the special case $(l^\star,m^\star)=(l',m')$, we have from* (5.25)*,* (5.28) *and* (5.30)

$$\begin{aligned}&\sum_{|m|\le l}\sum_{|\tilde m|\le\tilde l}\left|\int_{\mathbb{S}^2_\kappa} G_{n,\tilde n,l,\tilde l}^{m,\tilde m}(\kappa)\overline{Y_{l'}^{m'}}(\kappa)d\kappa\right|^2\delta_{l',l^\star}\\ &=\frac{2l+1}{4\pi}\frac{2\tilde l+1}{4\pi}\int_{|\theta_1|\le\frac{\pi}{4}}\left(\beta(\theta_1)(\sin\theta_1)^{2n+l}(\cos\theta_1)^{2\tilde n+\tilde l}\right)\\ &\quad\times\int_{|\theta_2|\le\frac{\pi}{4}}\left(\beta(\theta_2)(\sin\theta_2)^{2n+l}(\cos\theta_2)^{2\tilde n+\tilde l}\right)B_{l,\tilde l,l'}(\theta_1,\theta_2)d\theta_1\,d\theta_2\end{aligned}$$

where $B_{l,\tilde l,l'}(\theta_1,\theta_2)$ is defined by

$$B_{l,\tilde l,l'}(\theta_1,\theta_2) = \sum_{0\le q\le l}\sum_{0\le\tilde q\le\tilde l} b_{l,\tilde l}^{q,\tilde q}(\theta_1)\quad\times$$
$$\sum_{|k|\le\min(q,\tilde q)} a_{q,k}a_{\tilde q,k}P_q^{|k|}(\sin\theta_2)\,(-1)^k P_{\tilde q}^{|k|}(\cos\theta_2)\,2\pi\int_{-1}^1 P_q^{|k|}(x)\,P_{\tilde q}^{|k|}(x)\,P_{l'}(x)\,dx.$$

We now prove the following lemma used in the previous proposition 5.7.

Lemma 5.9. *For any l, $\tilde{l}$, q, $\tilde{q} \in \mathbb{N}$, there exists a continuous function $b^{q,\tilde{q}}_{l,\tilde{l}}(\theta)$ such that for any real θ, ϕ and unit vectors κ, η^+, η^- we have*

$$\begin{aligned}\int_0^{2\pi} & P_l\left((\kappa\sin\theta - \kappa^\perp(\phi)\cos\theta)\cdot\eta^-\right) \\ & \times P_{\tilde{l}}\left((\kappa\cos\theta + \kappa^\perp(\phi)\sin\theta)\cdot\eta^+\right)\frac{d\phi}{2\pi} \\ &= \sum_{0\le q\le l}\sum_{0\le\tilde{q}\le\tilde{l}} b^{q,\tilde{q}}_{l,\tilde{l}}(\theta)\, P_q(\kappa\cdot\eta^-)P_{\tilde{q}}(\kappa\cdot\eta^+)\end{aligned}$$

where the coefficients $b_{q,\tilde{q}} \equiv 0$ if $(q,\tilde{q}) \neq (l-2q_1, \tilde{l}-2q_2)$ for any integers q_1, q_2.

Proof. We apply the addition theorem (5.3) in the frame $(\kappa, \kappa^1, \kappa^2)$ and the relation (5.1).

$$\begin{aligned}P_{\tilde{l}}\left((\kappa\cos\theta + \kappa^\perp(\phi)\sin\theta)\cdot\eta^+\right) &= \sum_{|\tilde{k}|\le\tilde{l}} a_{\tilde{l},\tilde{k}} P^{|\tilde{k}|}_{\tilde{l}}(\cos\theta)\left(\frac{\mathrm{d}^{|\tilde{k}|}P_{\tilde{l}}}{\mathrm{d}x^{|\tilde{k}|}}\right)(\kappa\cdot\eta^+)\, e^{i\tilde{k}\phi}\, U^+_{\tilde{k}}, \\ P_l\left((\kappa\sin\theta - \kappa^\perp(\phi)\cos\theta)\cdot\eta^-\right) &= P_l\left(\left(\kappa\sin(\theta) + \kappa^\perp(\phi+\pi)\cos(\theta)\right)\cdot\eta^-\right) \\ &= \sum_{|k|\le l} a_{l,k} P^{|k|}_l(\sin\theta)\left(\frac{\mathrm{d}^{|k|}P_k}{\mathrm{d}x^{|k|}}\right)(\kappa\cdot\eta^-)\, e^{ik(\phi+\pi)}\, U^-_k\end{aligned}$$

where

$$U^+_k = \left((\kappa^1\cdot\eta^+) - i\,\mathrm{sgn}(k)(\kappa^2\cdot\eta^+)\right)^{|k|},\ \ U^-_k = \left((\kappa^1\cdot\eta^-) - i\,\mathrm{sgn}(k)(\kappa^2\cdot\eta^-)\right)^{|k|}.$$

We derive

$$\begin{aligned}\mathbf{I} &= \int_0^{2\pi} P_l\left((\kappa\sin\theta - \kappa^\perp(\phi)\cos\theta)\cdot\eta^-\right)\, P_{\tilde{l}}\left((\kappa\cos\theta + \kappa^\perp(\phi)\sin\theta)\cdot\eta^+\right)\frac{d\phi_1}{2\pi} \\ &= \sum_{|k|\le\min(l,\tilde{l})} c^k_1(\theta_1)P^{|k|}_l(\kappa\cdot\eta^-)P^{|k|}_{\tilde{l}}(\kappa\cdot\eta^+)(-1)^k U^-_{-k}U^+_k\end{aligned}$$

where

$$c^k_1(\theta_1) = a_{l,k}a_{\tilde{l},k}P^{|k|}_l(\sin\theta_1)P^{|k|}_{\tilde{l}}(\cos\theta_1).$$

We then write

$$\mathbf{I} = \sum_{0\le k\le\min(l,\tilde{l})} c^k_1(\theta_1)P^{|k|}_l(\kappa\cdot\eta^-)P^{|k|}_{\tilde{l}}(\kappa\cdot\eta^+)V_k$$

where $V_0 = 1$ and V_k is defined for $k \ge 1$ by

$$\begin{aligned}V_k = \Big\{ & \left[(-\kappa^1\cdot\eta^- - i\kappa^2\cdot\eta^-)(\kappa^1\cdot\eta^+ - i\kappa^2\cdot\eta^+)\right]^k \\ & + \left[(-\kappa^1\cdot\eta^- + i\kappa^2\cdot\eta^-)(\kappa^1\cdot\eta^+ + i\kappa^2\cdot\eta^+)\right]^k \Big\}.\end{aligned}$$

V_k is polynomial type : $V_k = p_k((\kappa\cdot\eta^+)\,(\kappa\cdot\eta^-))$ where

$$\begin{aligned}p_k(x) &= \left(x + i\sqrt{1-x^2}\right)^k + \left(x - i\sqrt{1-x^2}\right)^k \\ &= 2\sum_{0\le 2r\le k}(-1)^k\binom{k}{r}(x)^{k-2r}(1-x^2)^r.\end{aligned}$$

Indeed, direct calculations show that

$$
\begin{aligned}
&(\kappa^1 \cdot \eta^+ - i\kappa^2 \cdot \eta^+)(-\kappa^1 \cdot \eta^- - i\kappa^2 \cdot \eta^-) \\
&\qquad = (\kappa \cdot \eta^-)(\kappa \cdot \eta^+) + i(\kappa \cdot (\gamma^1 \sin\phi_2 - \gamma^2 \cos\phi_2)), \\
&(\kappa \cdot (\eta^1 \sin\phi_2 - \eta^2 \cos\phi_2))^2 = 1 - (\kappa \cdot \eta^-)^2 - (\kappa \cdot \eta^+)^2.
\end{aligned}
$$

Expanding the polynomials $P_l^{|k|}(\kappa\cdot\eta^-)\, P_{\tilde{l}}^{|k|}(\kappa\cdot\eta^+)\, p_k((\kappa\cdot\eta^-)\,(\kappa\cdot\eta^+))$ in the basis $(P_q)_{q\geq 0}$ (and taking in account of the parity), one can verify that there exists a continuous coefficients $b_{l,\tilde{l}}^{q,\tilde{q}}(\theta)$ such that

$$
\mathbf{I} = \sum_{0\leq 2q_1\leq l}\ \sum_{0\leq 2q_2\leq \tilde{l}} b_{l,\tilde{l}}^{l-2q_1,\tilde{l}-2q_2}(\theta)\, P_{l-2q_1}(\kappa\cdot\eta^-) P_{\tilde{l}-2q_2}(\kappa\cdot\eta^+).
$$

This conclude the proof of the lemma 5.9 and the Proposition 5.7. □

5.4. **Reduction of the expression of the non-linear eigenvalue.** We derive in the following propositions 5.10 and 5.12 some simplifications of the expression of the non-linear eigenvalue $\mu_{n,\tilde{n},l,\tilde{l},k_1}^{m,\tilde{m},m_1'}$, which will be used in the next section 6.

Proposition 5.10. *For $G_{n,\tilde{n},l,\tilde{l}}^{m,\tilde{m}}(\kappa)$ given in* (5.22) *and any integers $n, \tilde{n} \geq 0$, $|m| \leq l$, $|m'| \leq l'$, we have*

$$
\begin{aligned}
\sum_{|m|\leq l}\sum_{|\tilde{m}|\leq \tilde{l}} \left| \int_{\mathbb{S}^2_\kappa} G_{n,\tilde{n},l,\tilde{l}}^{m,\tilde{m}}(\kappa)\overline{Y_{l'}^{m'}}(\kappa)d\kappa \right|^2 &= \frac{2l+1}{4\pi}\frac{2\tilde{l}+1}{4\pi} \\
&\int_{|\theta_1|\leq\frac{\pi}{4}} \beta(\theta_1)(\sin\theta_1)^{2n+l}(\cos\theta_1)^{2\tilde{n}+\tilde{l}} \int_{|\theta_2|\leq\frac{\pi}{4}} \beta(\theta_2)(\sin\theta_2)^{2n+l}(\cos\theta_2)^{2\tilde{n}+\tilde{l}} \\
&\times\left(2\pi \int_{-1}^1 F_{l,\tilde{l}}(x,\theta_1,\theta_2)P_{l'}(x)dx\right) d\theta_2 d\theta_1
\end{aligned}
\tag{5.31}
$$

where

$$
\begin{aligned}
F_{l,\tilde{l}}(x,\theta_1,\theta_2) = \int_0^{2\pi}\int_0^{2\pi} &P_l(\tau^1(\theta_1,\phi_1)J(x)\cdot\tau^1(\theta_2,\phi_2)) \\
&\times P_{\tilde{l}}(\tau(\theta_1,\phi_1)J(x)\cdot\tau(\theta_2,\phi_2))\frac{d\phi_1}{2\pi}\frac{d\phi_2}{2\pi},
\end{aligned}
\tag{5.32}
$$

$J(x)$ is the matrix function

$$
J(x) = \begin{pmatrix} x & \sqrt{1-x^2} & 0 \\ -\sqrt{1-x^2} & x & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

and $\tau(\theta,\phi)$, $\tau^1(\theta,\phi)$ are the vectors

$$
\begin{aligned}
\tau(\theta,\phi) &= (\cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi), \\
\tau^1(\theta,\phi) &= (\sin\theta, -\cos\theta\cos\phi, -\cos\theta\sin\phi).
\end{aligned}
$$

Remark 5.11. *We remark that in the formula* (5.31)*, the right hand side is independent of* m'*. Therefore this implies*

$$\sum_{|m|\le l}\sum_{|\tilde m|\le \tilde l}\left|\int_{\mathbb{S}^2_\kappa} G^{m,\tilde m}_{n,\tilde n,l,\tilde l}(\kappa)\overline{Y^{m'}_{l'}}(\kappa)d\kappa\right|^2 = \sum_{|m|\le l}\sum_{|\tilde m|\le \tilde l}\left|\int_{\mathbb{S}^2_\kappa} G^{m,\tilde m}_{n,\tilde n,l,\tilde l}(\kappa)\overline{Y^{0}_{l'}}(\kappa)d\kappa\right|^2$$

$$= \sum_{|q|\le \min(l,\tilde l)}\left|\int_{\mathbb{S}^2_\kappa} G^{q,-q}_{n,\tilde n,l,\tilde l}(\kappa)\overline{Y^{0}_{l'}}(\kappa)d\kappa\right|^2, \tag{5.33}$$

since from (5.14) *the integral vanishes if* $m+\tilde m\neq 0$.

Proof. We will prove that

$$2\pi\int_{-1}^{1} F_{l,\tilde l}(x,\theta_1,\theta_2)P_{l'}(x)dx = B_{l,\tilde l,l'}(\theta_1,\theta_2)$$

where $B_{l,\tilde l,l'}(\theta_1,\theta_2)$ is given in the Remark 5.8 and we will conclude.

We express the terms of $F_{l,\tilde l}(x,\theta_1,\theta_2)$ given in (5.32). We note from (5.7)

$$\kappa_x = (x,\ \sqrt{1-x^2},0),\quad \kappa_x^\perp(\phi) = (-\sqrt{1-x^2}\cos\phi, x\cos\phi, -\sin\phi).$$

We compute

$$\begin{aligned}\tau^1(\theta_1,\phi_1)J(x) &= \begin{pmatrix}\sin\theta_1\\ -\cos\theta_1\cos\phi_1\\ -\cos\theta_1\sin\phi_1\end{pmatrix}^T\times\begin{pmatrix} x & \sqrt{1-x^2} & 0\\ -\sqrt{1-x^2} & x & 0\\ 0 & 0 & 1\end{pmatrix}\\ &= \begin{pmatrix} x\sin\theta_1 - \sqrt{1-x^2}\cos\theta_1\cos\phi_1\\ -\sqrt{1-x^2}\sin\theta_1 - x\cos\theta_1\cos\phi_1\\ -\cos\theta_1\sin\phi_1\end{pmatrix}^T\\ &= \sin(\theta_1)\,\kappa_x - \cos(\theta_1)\,\kappa_x^\perp(\phi_1).\end{aligned}$$

Similary we have

$$\tau(\theta_1,\phi_1)J(x) = \cos(\theta_1)\,\kappa_x + \sin(\theta_1)\,\kappa_x^\perp(\phi_1).$$

From lemma 5.9, we deduce

$$\begin{aligned}&\int_0^{2\pi} P_l(\tau^1(\theta_1,\phi_1)J(x)\cdot\tau^1(\theta_2,\phi_2))P_{\tilde l}(\tau(\theta_1,\phi_1)J(x)\cdot\tau(\theta_2,\phi_2))\frac{d\phi_1}{2\pi}\\ &= \sum_{0\le q\le l}\ \sum_{0\le\tilde q\le\tilde l} b^{q,\tilde q}_{l,\tilde l}(\theta_1)\,P_q(\kappa_x\cdot\tau^1(\theta_2,\phi_2))P_{\tilde q}(\kappa_x\cdot\tau(\theta_2,\phi_2)).\end{aligned}$$

Therefore using (5.32)

$$\begin{aligned}2\pi\int_{-1}^{1} F_{l,\tilde l}(x,\theta_1,\theta_2)P_{l'}(x)dx &= \sum_{0\le q\le l}\ \sum_{0\le\tilde q\le\tilde l} b^{q,\tilde q}_{l,\tilde l}(\theta_1)\times\\ 2\pi\int_{-1}^{1}\int_0^{2\pi} P_q(\kappa_x\cdot\tau^1(\theta_2,\phi_2))&P_{\tilde q}(\kappa_x\cdot\tau(\theta_2,\phi_2))P_{l'}(x)\frac{d\phi_2}{2\pi}dx.\end{aligned}$$

We then apply the addition theorem (5.3)

$$P_{\tilde q}(\kappa_x\cdot\tau(\theta_2,\phi_2)) = \sum_{|\tilde k|\le\tilde q} a_{\tilde q,k}P^{|\tilde k|}_{\tilde q}(x)\,P^{|\tilde k|}_{\tilde q}(\cos\theta_2)\,e^{-i\tilde k\phi_2},$$

$$P_q(\kappa_x\cdot\tau(\theta_2,\phi_2)) = \sum_{|k|\le q} a_{q,k}P^{|k|}_q(x)\,P^{|k|}_q(\sin\theta_2)\,e^{-ik\phi_2}(-1)^k$$

and we find

$$2\pi \int_{-1}^{1} F_{l,\tilde{l}}(x,\theta_1,\theta_2)P_{l'}(x)dx = \sum_{0\le q\le l}\sum_{0\le \tilde{q}\le \tilde{l}} b_{l,\tilde{l}}^{q,\tilde{q}}(\theta_1) \quad \times$$

$$\sum_{|k|\le \min(q,\tilde{q})} a_{q,k}a_{\tilde{q},k}P_q^{|k|}(\sin\theta_2)\,(-1)^k P_{\tilde{q}}^{|k|}(\cos\theta_2)\,2\pi\int_{-1}^{1} P_q^{|k|}(x)\,P_{\tilde{q}}^{|k|}(x)\,P_{l'}(x)\,dx,$$

$$= B_{l,\tilde{l},l'}(\theta_1,\theta_2)$$

from the remark 5.8. This ends the proof of the formula (5.31). □

The following proposition will provide a convenient expression to estimate the nonlinear eigenvalue $\mu_{n,\tilde{n},l,\tilde{l},k_1}^{m,\tilde{m},m'_1}$ in section 6.

Proposition 5.12. *For $G_{n,\tilde{n},l,\tilde{l}}^{m,\tilde{m}}(\kappa)$ given in (5.22), and any integers $n,\tilde{n}\ge 0$, $|m|\le l$, $|m'|\le l'$, we have*

$$\sum_{|m|\le l}\sum_{|\tilde{m}|\le \tilde{l}}\left|\int_{\mathbb{S}_\kappa^2} G_{n,\tilde{n},l,\tilde{l}}^{m,\tilde{m}}(\kappa)\overline{Y_{l'}^{m'}}(\kappa)d\kappa\right|^2$$

$$= \sum_{|q|\le \min(l,\tilde{l})}\left((-1)^q\left(\frac{4\pi}{2l'+1}\right)^{\frac12} G_{n,\tilde{n},l,\tilde{l}}^{q,-q}(e_1)\right)\left(\int_{\mathbb{S}_\kappa^2} G_{n,\tilde{n},l,\tilde{l}}^{q,-q}(\kappa)\overline{Y_{l'}^{0}}(\kappa)d\kappa\right), \tag{5.34}$$

where $e_1=(1,0,0)$ and

$$G_{n,\tilde{n},l,\tilde{l}}^{q,-q}(e_1) = \left(\frac{2l+1}{4\pi}\right)^{\frac12}\left(\frac{2\tilde{l}+1}{4\pi}\right)^{\frac12}\left(\frac{(l-|q|)!}{(l+|q|)!}\right)^{\frac12}\left(\frac{(\tilde{l}-|q|)!}{(\tilde{l}+|q|)!}\right)^{\frac12} \times$$

$$\int_{|\theta_2|\le\frac{\pi}{4}} \beta(\theta_2)(\sin\theta_2)^{2n+l}(\cos\theta_2)^{2\tilde{n}+\tilde{l}}P_l^{|q|}(\sin\theta_2)(-1)^q P_{\tilde{l}}^{|q|}(\cos\theta_2)d\theta_2.$$

Proof. For $0\le k\le \min(l,\tilde{l})$ and $|m'|\le l'$, we deduce from (5.31) that,

$$\mathbf{I} = \sum_{|m|\le l}\sum_{|\tilde{m}|\le \tilde{l}}\left|\int_{\mathbb{S}_\kappa^2} G_{n,\tilde{n},l,\tilde{l}}^{m,\tilde{m}}(\kappa)\overline{Y_{l'}^{m'}}(\kappa)d\kappa\right|^2 = \frac{2l+1}{4\pi}\frac{2\tilde{l}+1}{4\pi} \quad\times$$

$$\int_{|\theta_1|\le\frac{\pi}{4}} \beta(\theta_1)(\sin\theta_1)^{2n+l}(\cos\theta_1)^{2\tilde{n}+\tilde{l}}d\theta_1 \quad \times \tag{5.35}$$

$$\int_{|\theta_2|\le\frac{\pi}{4}} \beta(\theta_2)(\sin\theta_2)^{2n+l}(\cos\theta_2)^{2\tilde{n}+\tilde{l}}\left(2\pi\int_{-1}^{1} F_{l,\tilde{l}}(x,\theta_1,\theta_2)\,P_{l'}(x)dx\right)d\theta_2,$$

where $F_{l,\tilde{l}}(x,\theta_1,\theta_2)$ was defined in (5.32), such that

$$F_{l,\tilde{l}}(x,\theta_1,\theta_2) = \int_0^{2\pi}\int_0^{2\pi} P_l(\tau^1(\theta_1,\phi_1)J(x)\cdot\tau^1(\theta_2,\phi_2))$$

$$\times P_{\tilde{l}}(\tau(\theta_1,\phi_1)J(x)\cdot\tau(\theta_2,\phi_2))\frac{d\phi_1}{2\pi}\frac{d\phi_2}{2\pi}.$$

We apply the addition theorem (5.2)

$$P_l(\tau^1(\theta_1,\phi_1)J(x)\cdot\tau^1(\theta_2,\phi_2)) = \tfrac{4\pi}{2\,l+1}\sum_{q=-l}^{l} Y_l^q(\tau^1(\theta_1,\phi_1)J(x))\, Y_l^{-q}(\tau^1(\theta_2,\phi_2))$$

$$P_{\tilde{l}}(\tau(\theta_1,\phi_1)J(x)\cdot\tau(\theta_2,\phi_2)) = \tfrac{4\pi}{2\,\tilde{l}+1}\sum_{\tilde{k}=-\tilde{l}}^{\tilde{l}} Y_{\tilde{l}}^{\tilde{q}}(\tau(\theta_1,\phi_1)J(x))\, Y_{\tilde{l}}^{-\tilde{q}}(\tau(\theta_2,\phi_2)).$$

Since

$$Y_l^{-q}(\tau^1(\theta_2,\phi_2)) = \left(\frac{2l+1}{4\pi}\right)^{\frac{1}{2}}\left(\frac{(l-|q|)!}{(l+|q|)!}\right)^{\frac{1}{2}} P_l^{|q|}(\sin\theta_2)e^{-iq(\phi_2+\pi)}$$

$$Y_{\tilde{l}}^{-\tilde{q}}(\tau(\theta_2,\phi_2) = \left(\frac{4\tilde{l}+1}{2\pi}\right)^{\frac{1}{2}}\left(\frac{(\tilde{l}-|\tilde{q}|)!}{(\tilde{l}+|\tilde{q}|)!}\right)^{\frac{1}{2}} P_{\tilde{l}}^{|\tilde{q}|}(\cos\theta_2)e^{-i\tilde{q}\phi_2}$$

we find that

$$\begin{aligned} F_{l,\tilde{l}}(x,\theta_1,\theta_2) = \sum_{|q|\le\min(l,\tilde{l})} &\left(\frac{4\pi}{2l+1}\right)^{\frac{1}{2}}\left(\frac{4\pi}{2\tilde{l}+1}\right)^{\frac{1}{2}}\left(\frac{(l-|q|)!}{(l+|q|)!}\right)^{\frac{1}{2}}\left(\frac{(\tilde{l}-|q|)!}{(\tilde{l}+|q|)!}\right)^{\frac{1}{2}}\times \\ &P_l^{|q|}(\sin\theta_2)(-1)^q P_{\tilde{l}}^{|q|}(\cos\theta_2)\times \\ &\int_0^{2\pi} Y_l^q(\tau^1(\theta_1,\phi_1)J(x))Y_{\tilde{l}}^{-q}(\tau(\theta_1,\phi_1)J(x))\frac{d\phi_1}{2\pi}. \end{aligned}$$

We plug the previous relation into (5.35) and we get

$$\begin{aligned} &\mathbf{I} = \sum_{|q|\le\min(l,\tilde{l})}\int_{|\theta_2|\le\frac{\pi}{4}}\beta(\theta_2)(\sin\theta_2)^{2n+l}(\cos\theta_2)^{2\tilde{n}+\tilde{l}}P_l^{|q|}(\sin\theta_2)P_{\tilde{l}}^{|q|}(\cos\theta_2)d\theta_2, \\ &(-1)^q\left(\frac{2l+1}{4\pi}\right)^{\frac{1}{2}}\left(\frac{2\tilde{l}+1}{4\pi}\right)^{\frac{1}{2}}\left(\frac{(l-|q|)!}{(l+|q|)!}\right)^{\frac{1}{2}}\left(\frac{(\tilde{l}-|q|)!}{(\tilde{l}+|q|)!}\right)^{\frac{1}{2}}\times \\ &\int_{|\theta_1|\le\frac{\pi}{4}}\beta(\theta_1)(\sin\theta_1)^{2n+l}(\cos\theta_1)^{2\tilde{n}+\tilde{l}}\quad\times \\ &\int_0^{2\pi}2\pi\int_{-1}^{1}Y_l^q(\tau^1(\theta_1,\phi_1)J(x))Y_{\tilde{l}}^{-q}(\tau(\theta_1,\phi_1)J(x))P_{l'}(x)dx\frac{d\phi_1}{2\pi}d\theta_1. \end{aligned}$$

On one hand, from (5.23)

$$\begin{aligned} G_{n,\tilde{n},l,\tilde{l}}^{q,-q}(e_1) &= \int_{|\theta_2|\le\frac{\pi}{4}}\beta(\theta_2)(\sin\theta_2)^{2n+l}(\cos\theta_2)^{2\tilde{n}+\tilde{l}}\quad\times \\ &\int_0^{2\pi}Y_l^q(e_1\sin\theta_2-e_1^\perp(\phi_2)\cos\theta_2)Y_{\tilde{l}}^{-q}(e_1\cos\theta_2+e_1^\perp(\phi_2)\sin\theta_2)\frac{d\phi_2}{2\pi}d\theta_2 \\ &= \int_{|\theta_2|\le\frac{\pi}{4}}\beta(\theta_2)(\sin\theta_2)^{2n+l}(\cos\theta_2)^{2\tilde{n}+\tilde{l}}\left(\tfrac{2l+1}{4\pi}\right)^{\frac{1}{2}}\left(\tfrac{2\tilde{l}+1}{4\pi}\right)^{\frac{1}{2}}\left(\tfrac{(l-|q|)!}{(l+|q|)!}\right)^{\frac{1}{2}}\left(\tfrac{(\tilde{l}-|q|)!}{(\tilde{l}+|q|)!}\right)^{\frac{1}{2}} \\ &\quad P_l^{|q|}(\sin\theta_2)(-1)^q P_{\tilde{l}}^{|q|}(\cos\theta_2)d\theta_2, \end{aligned}$$

On the other hand, from (5.24) and from (5.36) of the next lemma 5.13,

$$\int_{\mathbb{S}^2_\kappa} G^{q,-q}_{n,\tilde n,l,\tilde l}(\kappa)\overline{Y^0_{l'}}(\kappa)d\kappa = \int_{|\theta_1|\le\frac{\pi}{4}} \beta(\theta_1)(\sin\theta_1)^{2n+l}(\cos\theta_1)^{2\tilde n+\tilde l}\int_0^{2\pi}\int_{\mathbb{S}^2_\kappa}$$
$$Y^q_l(\kappa\sin\theta_1-\kappa^\perp(\phi_1)\cos\theta_1)Y^{-q}_{\tilde l}(\kappa\cos\theta_1+\kappa^\perp(\phi_1)\sin\theta_1)\overline{Y^0_{l'}}(\kappa)d\kappa\frac{d\phi_1}{2\pi}d\theta_1$$
$$=\left(\frac{2l'+1}{4\pi}\right)^{\frac12}\int_{|\theta_1|\le\frac{\pi}{4}}\beta(\theta_1)(\sin\theta_1)^{2n+l}(\cos\theta_1)^{2\tilde n+\tilde l}\quad\times$$
$$\int_0^{2\pi}2\pi\int_{-1}^1 Y^q_l(\tau^1(\theta_1,\phi_1)J(x))Y^{-q}_{\tilde l}(\tau(\theta_1,\phi_1)J(x))P_{l'}(x)dx\frac{d\phi_1}{2\pi}d\theta_1.$$

Combining the three previous relations leads to (5.34), and this concludes the proof of the Proposition. □

We now prove the following technical lemma.

Lemma 5.13. *For any any integers $l,\tilde l,l'\ge 0$ and $|q|\le l$, we have*

$$\int_{\mathbb{S}^2_\kappa} Y^q_l(\kappa\sin\theta_1-\kappa^\perp(\phi_1)\cos\theta_1)Y^{-q}_{\tilde l}(\kappa\cos\theta_1+\kappa^\perp(\phi_1)\sin\theta_1)\overline{Y^0_{l'}}(\kappa)d\kappa$$
$$=2\pi\int_{-1}^1 Y^q_l(\tau^1(\theta_1,\phi_1)J(x))Y^{-q}_{\tilde l}(\tau(\theta_1,\phi_1)J(x))\overline{Y^0_{l'}}(e_1J(x))dx. \tag{5.36}$$

Proof. We consider

$$\mathbf{I}=\int_{\mathbb{S}^2_\kappa} Y^q_l(\kappa\sin\theta_1-\kappa^\perp(\phi_1)\cos\theta_1)Y^{-q}_{\tilde l}(\kappa\cos\theta_1+\kappa^\perp(\phi_1)\sin\theta_1)\overline{Y^0_{l'}}(\kappa)d\kappa.$$

From (5.1) we have

$$Y^q_l(\kappa\sin\theta_1-\kappa^\perp(\phi_1)\cos\theta_1)=N_{l,q}\left(\frac{d^{|q|}P_l}{dx^{|q|}}\right)(\sigma^-_1)\,(\sigma^-_2+i\ \mathrm{sgn}(q)\,\sigma^-_3)^{|q|},$$
$$Y^{-q}_{\tilde l}(\kappa\cos\theta_1+\kappa^\perp(\phi_1)\sin\theta_1)=N_{\tilde l,q}\left(\frac{d^{|q|}P_{\tilde l}}{dx^{|q|}}\right)(\sigma^+_1)\,(\sigma^+_2-i\ \mathrm{sgn}(q)\,\sigma^+_3)^{|q|},$$

where we have set

$$(\sigma^-_1,\sigma^-_2,\sigma^-_3)=\kappa\sin\theta_1-\kappa^\perp(\phi_1)\cos\theta_1$$
$$(\sigma^+_1,\sigma^+_2,\sigma^+_3)=\kappa\cos\theta_1+\kappa^\perp(\phi_1)\sin\theta_1.$$

Noting $\kappa=\kappa_{\theta,\phi}=\tau(\theta,\phi)$, direct computations lead to

$$(\sigma^-_2+i\,\sigma^-_3)(\sigma^+_2-i\,\sigma^+_3)=\cos\theta_1\sin\theta_1\left(\sin^2\theta-\cos^2\phi_1\cos^2\theta-\sin^2\phi_1\right)$$
$$+(\sin^2\theta_1-\cos^2\theta_1)\cos\phi_1\sin\theta\cos\theta+i\ \sin\phi_1\sin\theta,$$

which does not depend of ϕ. Since $\sigma^\pm_1$ do not depend also on ϕ, we get with the change of variable $x=\cos\theta$

$$\mathbf{I}=2\pi\int_0^\pi Y^q_l(\kappa_{\theta,0}\sin\theta_1-\kappa^\perp_{\theta,0}(\phi_1)\cos\theta_1)$$
$$Y^{-q}_{\tilde l}(\kappa_{\theta,0}\cos\theta_1+\kappa^\perp_{\theta,0}(\phi_1)\sin\theta_1)\overline{Y^0_{l'}}(\kappa_{\theta,0})\ \sin\theta\,d\theta$$
$$=2\pi\int_{-1}^1 Y^q_l(\tau^1(\theta_1,\phi_1)J(x))Y^{-q}_{\tilde l}(\tau(\theta_1,\phi_1)J(x))\overline{Y^0_{l'}}(e_1J(x))dx.$$

This concludes the proof of the lemma 5.13 and proposition 5.12. □

6. Estimates of the non linear eigenvalues

In this section, we prove the Proposition 3.1, we need the following fundamental result of Gamma function. It is well known of the stirling's formula (see 12.33 in [25], [18]) that,

$$\Gamma(x+1) = \sqrt{2\pi x}\Big(\frac{x}{e}\Big)^x e^{\frac{\nu(x)}{12x}}, \text{ for } x \geq 1,$$

where $0 < \nu(x) < 1$. Then we can introduce an estimate in the following.

Let $a,\ b$ be two fixed constant, for any $x > 0$, with $|b-a| \leq x+b$, $x+a \geq 1$, $x+b \geq 1$, we have

$$\frac{\Gamma(x+a+1)}{\Gamma(x+b+1)} \leq C_{a,b}(x+a)^{a-b}, \tag{6.1}$$

where $C_{a,b}$ is dependent only on $a,\ b$. We also recall the definition of the Beta function

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{6.2}$$

6.1. **The estimate for the radially symmetric terms.**

We first give the estimate of $|\lambda^{rad,1}_{n,\tilde{n},\tilde{l}}|^2$, and $|\lambda^{rad,2}_{n,\tilde{n},l}|^2$, which is 1), 2) in Proposition 3.1. Recall that

$$\begin{aligned}
|\lambda^{rad,1}_{n,\tilde{n},\tilde{l}}|^2 &= \frac{A_{\tilde{n},\tilde{l}}A_{n,0}}{A_{n+\tilde{n},\tilde{l}}}\int_{|\theta|\leq\frac{\pi}{4}}\beta(\theta)(\sin\theta)^{2n}(\cos\theta)^{2\tilde{n}+\tilde{l}}P_{\tilde{l}}(\cos\theta)d\theta,\\
|\lambda^{rad,2}_{n,\tilde{n},l}|^2 &= \frac{A_{\tilde{n},0}A_{n,l}}{A_{n+\tilde{n},l}}\int_{|\theta|\leq\frac{\pi}{4}}\beta(\theta)(\sin\theta)^{2n+l}(\cos\theta)^{2\tilde{n}}P_{l}(\sin\theta)d\theta
\end{aligned}$$

where

$$A_{n,l} = (-i)^l(2\pi)^{\frac{3}{4}}\left(\frac{1}{\sqrt{2}n!\Gamma(n+l+\frac{3}{2})}\right)^{\frac{1}{2}}.$$

Lemma 6.1. *For $n \geq 1$, $\tilde{n}, \tilde{l} \in \mathbb{N}$,*

$$|\lambda^{rad,1}_{n,\tilde{n},\tilde{l}}|^2 \lesssim \tilde{n}^s(\tilde{n}+\tilde{l})^s n^{-\frac{5}{2}-2s}. \tag{6.3}$$

For all $\tilde{n} \geq 1$, $n, l \in \mathbb{N}$, $n+l \geq 2$,

$$|\lambda^{rad,2}_{n,\tilde{n},l}|^2 \lesssim \frac{\tilde{n}^{2s}}{(n+1)^s(n+l)^{\frac{5}{2}+s}}. \tag{6.4}$$

Proof. We estimate $|\lambda^{rad,1}_{n,\tilde{n},\tilde{l}}|^2$. Recalled from the definition of $\beta(\theta)$ that

$$\beta(\theta) \approx |\sin\theta|^{-1-2s},$$

and $P_{\tilde{l}}(x) \leq 1$, we have,

$$|\lambda^{rad,1}_{n,\tilde{n},\tilde{l}}|^2 \lesssim \frac{(n+\tilde{n})!\Gamma(n+\tilde{n}+\tilde{l}+\frac{3}{2})}{n!\tilde{n}!\Gamma(n+\frac{3}{2})\Gamma(\tilde{n}+\tilde{l}+\frac{3}{2})}\left(\int_0^{\frac{\pi}{4}}(\sin\theta)^{2n-1-2s}(\cos\theta)^{2\tilde{n}+\tilde{l}}d\theta\right)^2.$$

By using the Cauchy Schwarz inequality and the Beta Function (6.2), we derive that

$$\Big(\int_0^{\frac{\pi}{4}} (\sin\theta)^{2n-1-2s}(\cos\theta)^{2\tilde{n}+\tilde{l}}\,d\theta\Big)^2 = \Big(\int_0^{\frac{1}{2}} t^{n-1-s}(1-t)^{\tilde{n}+\frac{\tilde{l}}{2}-\frac{1}{2}}dt\Big)^2$$
$$\lesssim \int_0^{\frac{1}{2}} t^{n-1-s}(1-t)^{\tilde{n}+s}dt \times \int_0^{\frac{1}{2}} t^{n-1-s}(1-t)^{\tilde{n}+\tilde{l}+\frac{1}{2}+s}dt$$
$$\lesssim \frac{(\Gamma(n-s))^2\Gamma(\tilde{n}+1+s)\Gamma(\tilde{n}+\tilde{l}+\frac{3}{2}+s)}{(n+\tilde{n})!\Gamma(n+\tilde{n}+\tilde{l}+\frac{3}{2})}.$$

Then,

$$|\lambda^{rad,1}_{n,\tilde{n},\tilde{l}}|^2 \lesssim \frac{(n+\tilde{n})!\Gamma(n+\tilde{n}+\tilde{l}+\frac{3}{2})}{n!\tilde{n}!\Gamma(n+\frac{3}{2})\Gamma(\tilde{n}+\tilde{l}+\frac{3}{2})} \times \frac{(\Gamma(n-s))^2\Gamma(\tilde{n}+1+s)\Gamma(\tilde{n}+\tilde{l}+\frac{3}{2}+s)}{(n+\tilde{n})!\Gamma(n+\tilde{n}+\tilde{l}+\frac{3}{2})}$$
$$\lesssim \frac{\Gamma(n-s)\Gamma(n-s)\Gamma(\tilde{n}+1+s)\Gamma(\tilde{n}+\tilde{l}+\frac{3}{2}+s)}{n!\Gamma(n+\frac{3}{2})\tilde{n}!\Gamma(\tilde{n}+\tilde{l}+\frac{3}{2})}. \tag{6.5}$$

We deduce from the formula (6.1) with $x=n$, $a=-s$, $b=0$,

$$\frac{\Gamma(n-s+1)}{n!} \lesssim \frac{1}{(n-s)^s}.$$

By the recurrence formula for Gamma function that

$$\Gamma(n+1-s) = (n-s)\Gamma(n-s),$$

we obtain

$$\frac{\Gamma(n-s)}{n!} = \frac{1}{(n-s)}\frac{\Gamma(n+1-s)}{\Gamma(n+1)} \lesssim \frac{1}{(n-s)^{s+1}} \lesssim \frac{1}{n^{1+s}}.$$

Using $x=n$, $a=-s$, $b=\frac{1}{2}$ in (6.1),

$$\frac{\Gamma(n+1-s)}{\Gamma(n+\frac{3}{2})} \lesssim \frac{1}{n^{\frac{1}{2}+s}},$$

and recurrence formula $\Gamma(n+1-s) = (n-s)\Gamma(n-s)$,

$$\frac{\Gamma(n-s)}{\Gamma(n+\frac{3}{2})} = \frac{1}{n-s}\frac{\Gamma(n+1-s)}{\Gamma(n+\frac{3}{2})} \lesssim \frac{1}{(n-s)n^{\frac{1}{2}+s}} \lesssim \frac{1}{n^{\frac{3}{2}+s}}.$$

Using $x=\tilde{n}+1$, $a=s, b=0$ in (6.1), we have

$$\frac{\Gamma(\tilde{n}+1+s)}{\tilde{n}!} = \frac{\tilde{n}+1}{\tilde{n}+1+s}\frac{\Gamma(\tilde{n}+2+s)}{(\tilde{n}+1)!} \lesssim \tilde{n}^s.$$

Using $x=\tilde{n}+\tilde{l}+\frac{1}{2}, a=s, b=0$,

$$\frac{\Gamma(\tilde{n}+\tilde{l}+\frac{3}{2}+s)}{\Gamma(\tilde{n}+\tilde{l}+\frac{3}{2})} \lesssim (\tilde{n}+\tilde{l}+\frac{1}{2}+s)^s \lesssim (\tilde{n}+\tilde{l})^s.$$

Substitute these estimate into (6.5), we obtain

$$|\lambda^{rad,1}_{n,\tilde{n},\tilde{l}}|^2 \lesssim \tilde{n}^s(\tilde{n}+\tilde{l})^s n^{-\frac{5}{2}-2s}.$$

This is the formula of (6.3)

Analogously, for the term $|\lambda^{rad,2}_{n,\tilde n,l}|^2$,

$$|\lambda^{rad,2}_{n,\tilde n,l}|^2 \approx \frac{(n+\tilde n)!\Gamma(n+\tilde n+l+\frac{3}{2})}{n!\tilde n!\Gamma(n+l+\frac{3}{2})\Gamma(\tilde n+\frac{3}{2})}\left(\int_0^{\frac{\pi}{4}}(\sin\theta)^{2n+l-1-2s}(\cos\theta)^{2\tilde n}d\theta\right)^2$$
$$\lesssim \frac{\Gamma(n+1-s)\Gamma(n+l-1-s)\Gamma(\tilde n+1+s)\Gamma(\tilde n+\frac{5}{2}+s)}{(n+\tilde n+1)n!\Gamma(n+l+\frac{3}{2})\tilde n!\Gamma(\tilde n+\frac{3}{2})}.$$

Applying the estimate (6.1) and recurrence formula for Gamma function

$$\Gamma(x+1)=x\Gamma(x),$$

which gives

$$|\lambda^{rad,2}_{n,\tilde n,l}|^2 \le \frac{\tilde n^{1+2s}}{(n+1)^s(n+l)^{\frac{5}{2}+s}(n+\tilde n+1)} \le \frac{\tilde n^{2s}}{(n+1)^s(n+l)^{\frac{5}{2}+s}}.$$

This ends the proof of (6.4). □

6.2. **The estimate for the general terms.** In the proof of 3) in Proposition 3.1, we need the following technical Lemma. Recall that

$$A_{n,l}=(-i)^l(2\pi)^{\frac{3}{4}}\left(\frac{1}{\sqrt{2}n!\Gamma(n+l+\frac{3}{2})}\right)^{\frac{1}{2}},$$

and

$$G^{m,\tilde m}_{n,\tilde n,l,\tilde l}(\kappa)=\int_{\mathbb{S}^2}b(\kappa\cdot\sigma)\Big(|\kappa-\sigma|/2\Big)^{2n+l}\Big(|\kappa+\sigma|/2\Big)^{2\tilde n+\tilde l}$$
$$\times Y^m_l\Big(\frac{\kappa-\sigma}{|\kappa-\sigma|}\Big)Y^{\tilde m}_{\tilde l}\Big(\frac{\kappa+\sigma}{|\kappa+\sigma|}\Big)d\sigma.$$

Then recall the notation in Proposition 2.1, we have

$$\mu^{m,\tilde m,m'}_{n,\tilde n,l,\tilde l,k}=\frac{A_{\tilde n,\tilde l}A_{n,l}}{A_{n+\tilde n+k,l+\tilde l-2k}}\left(\int_{S^2_\kappa}G^{m,\tilde m}_{n,\tilde n,l,\tilde l}(\kappa)\overline{Y^{m'}_{l+\tilde l-2k}}(\kappa)d\kappa\right),$$

It follows that,

$$\sum_{|m|\le l}\sum_{|\tilde m|\le\tilde l}\left|\mu^{m,\tilde m,m'}_{n,\tilde n,l,\tilde l,k}\right|^2=\left|\frac{A_{\tilde n,\tilde l}A_{n,l}}{A_{n+\tilde n+k,l+\tilde l-2k}}\right|^2$$
$$\times\sum_{|m|\le l}\sum_{|\tilde m|\le\tilde l}\left|\int_{S^2_\kappa}G^{m,\tilde m}_{n,\tilde n,l,\tilde l}(\kappa)\overline{Y^{m'}_{l+\tilde l-2k}}(\kappa)d\kappa\right|^2. \tag{6.6}$$

In the next Lemma we estimate

$$\sum_{|m|\le l}\sum_{|\tilde m|\le\tilde l}\left|\int_{\mathbb{S}^2_\kappa}G^{m,\tilde m}_{n,\tilde n,l,\tilde l}(\kappa)\overline{Y^{m'}_{l+\tilde l-2k}}(\kappa)d\kappa\right|^2.$$

Lemma 6.2. *For $0\le k\le\min(l,\tilde l)$, $|m'|\le l+\tilde l-2k$, we have*

$$\sum_{|m|\le l}\sum_{|\tilde m|\le\tilde l}\left|\int_{\mathbb{S}^2_\kappa}G^{m,\tilde m}_{n,\tilde n,l,\tilde l}(\kappa)\overline{Y^{m'}_{l+\tilde l-2k}}(\kappa)d\kappa\right|^2$$
$$\lesssim\frac{\tilde l\sqrt{l}}{l+\tilde l-2k+1}\left(\int_0^{\frac{\pi}{4}}\beta(\theta)(\sin\theta)^{2n+l}(\cos\theta)^{2\tilde n+\tilde l}d\theta\right)^2. \tag{6.7}$$

Proof. For $0 \le k \le \min(l, \tilde{l})$ and $|m'| \le l' = l + \tilde{l} - 2k$, we deduce from the Proposition 5.12 that

$$\sum_{|m|\le l}\sum_{|\tilde{m}|\le \tilde{l}}\left|\int_{\mathbb{S}^2_\kappa} G^{m,\tilde{m}}_{n,\tilde{n},l,\tilde{l}}(\kappa)\overline{Y^{m'}_{l+\tilde{l}-2k}}(\kappa)d\kappa\right|^2$$

$$= \sum_{|q|\le\min(l,\tilde{l})} \frac{(-1)^q}{\sqrt{4\pi}}\sqrt{\frac{(2l+1)(2\tilde{l}+1)}{2(l+\tilde{l}-2k)+1}}H_q \int_{\mathbb{S}^2_\kappa} G^{q,-q}_{n,\tilde{n},l,\tilde{l}}(\kappa)\overline{Y^0_{l+\tilde{l}-2k}}(\kappa)d\kappa,$$

where

$$H_q = \sqrt{\tfrac{(l-|q|)!}{(l+|q|)!}\tfrac{(\tilde{l}-|q|)!}{(\tilde{l}+|q|)!}}\int_{|\theta|\le\frac{\pi}{4}} \beta(\theta)(\sin\theta)^{2n+l}(\cos\theta)^{2\tilde{n}+\tilde{l}}P^{|q|}_l(\sin\theta)P^{|q|}_{\tilde{l}}(\cos\theta)d\theta.$$

We observe from (5.33) that, for any $|m'| \le l + \tilde{l} - 2k$,

$$\sum_{|m|\le l}\sum_{|\tilde{m}|\le \tilde{l}}\left|\int_{\mathbb{S}^2_\kappa} G^{m,\tilde{m}}_{n,\tilde{n},l,\tilde{l}}(\kappa)\overline{Y^{m'}_{l+\tilde{l}-2k}}(\kappa)d\kappa\right|^2 = \sum_{|m|\le\min(l,\tilde{l})}\left|\int_{\mathbb{S}^2_\kappa} G^{-m,m}_{n,\tilde{n},l,\tilde{l}}(\kappa)\overline{Y^{0}_{l+\tilde{l}-2k}}(\kappa)d\kappa\right|^2.$$

Then it follows from the Cauchy-Schwarz inequality that

$$\sum_{|m|\le l}\sum_{|\tilde{m}|\le \tilde{l}}\left|\int_{\mathbb{S}^2_\kappa} G^{m,\tilde{m}}_{n,\tilde{n},l,\tilde{l}}(\kappa)\overline{Y^{m'}_{l+\tilde{l}-2k}}(\kappa)d\kappa\right|^2 \le \frac{1}{4\pi}\frac{(2l+1)(2\tilde{l}+1)}{2(l+\tilde{l}-2k)+1}\sum_{|q|\le\min(l,\tilde{l})}\Big(H_q\Big)^2. \tag{6.8}$$

By using the Cauchy-Schwarz inequality and the addition theorem (5.3), we have

$$\sum_{|q|\le\min(l,\tilde{l})}\Big(H_q\Big)^2 \le \left(\int_{|\theta|\le\frac{\pi}{4}}\beta(\theta)|\sin\theta|^{2n+l}(\cos\theta)^{2\tilde{n}+\tilde{l}}d\theta\right)\left(\int_{|\theta|\le\frac{\pi}{4}}\beta(\theta)|\sin\theta|^{2n+l}\right.$$

$$\left.\times(\cos\theta)^{2\tilde{n}+\tilde{l}}\left[\sum_{|q|\le\min(l,\tilde{l})}\frac{(l-|q|)!}{(l+|q|)!}\Big(P^{|q|}_l(\sin\theta)\Big)^2\frac{(\tilde{l}-|q|)!}{(\tilde{l}+|q|)!}\Big(P^{|q|}_{\tilde{l}}(\cos\theta)\Big)^2\right]d\theta\right)$$

$$= \left(\int_{|\theta|\le\frac{\pi}{4}}\beta(\theta)|\sin\theta|^{2n+l}(\cos\theta)^{2\tilde{n}+\tilde{l}}d\theta\right)\left(\int_{|\theta|\le\frac{\pi}{4}}\beta(\theta)|\sin\theta|^{2n+l}(\cos\theta)^{2\tilde{n}+\tilde{l}}\right.$$

$$\left.\times\left[\int_0^{2\pi}P_l\Big((\sin\theta)^2+(\cos\theta)^2\cos\phi\Big)P_{\tilde{l}}\Big((\cos\theta)^2+(\sin\theta)^2\cos\phi\Big)\frac{d\phi}{2\pi}\right]d\theta\right).$$

From the formula (14) of Sec.10.3 in Chap.III in [20]

$$|\sqrt{l}\sqrt[4]{1-x^2}P_l(x)| \le 4\sqrt{\frac{2}{\pi}},\ \forall\ -1 \le x \le 1,$$

and $|P_{\tilde{l}}(x)| \le 1$, we derive that

$$\frac{1}{2\pi}\int_0^{2\pi}P_l\Big((\sin\theta)^2+(\cos\theta)^2\cos\phi\Big)P_{\tilde{l}}\Big((\cos\theta)^2+(\sin\theta)^2\cos\phi\Big)d\phi\Big]$$

$$\lesssim \frac{1}{\sqrt{l}}\int_0^{2\pi}\frac{1}{\sqrt[4]{1-\Big((\sin\theta)^2+(\cos\theta)^2\cos\phi\Big)^2}}d\phi$$

$$\lesssim \frac{1}{\sqrt{l}}\frac{1}{\sqrt{|\cos\theta|^2}}\int_0^{2\pi}\frac{1}{\sqrt[4]{1-(\cos\phi)^2}}d\phi$$

$$\lesssim \frac{1}{\sqrt{l}}\frac{1}{\sqrt{|\cos\theta|^2}}.$$

Since $|\theta| \le \frac{\pi}{4}$, we have $\cos\theta \ge \frac{\sqrt{2}}{2}$ and it follows that,

$$\sum_{|q|\le \min(l,\tilde{l})} \Big(H_q\Big)^2 \lesssim \frac{1}{\sqrt{l}}\Big(\int_0^{\frac{\pi}{4}} \beta(\theta)(\sin\theta)^{2n+l}(\cos\theta)^{2\tilde{n}+\tilde{l}}d\theta\Big)^2.$$

Substitute it into the formula (6.8), we conclude that, for $l \ge 1, \tilde{l} \ge 1$ with $0 \le k \le \min(l,\tilde{l})$,

$$\begin{aligned}&\sum_{|m|\le l}\sum_{|\tilde{m}|\le \tilde{l}}\left(\int_{\mathbb{S}^2_\kappa} G^{m,\tilde{m}}_{n,\tilde{n},l,\tilde{l}}(\kappa)\overline{Y^{m'}_{l+\tilde{l}-2k}}(\kappa)d\kappa\right)^2\\ &\lesssim \frac{\tilde{l}\sqrt{l}}{l+\tilde{l}-2k+1}\left(\int_0^{\frac{\pi}{4}}\beta(\theta)(\sin\theta)^{2n+l}(\cos\theta)^{2\tilde{n}+\tilde{l}}d\theta\right)^2.\end{aligned}$$

This ends the proof of (6.7). □

For $l \ge 1, \tilde{l} \ge 1$ with $0 \le k \le \min(l,\tilde{l})$, we denote $\lambda^k_{n,\tilde{n},\tilde{l},l}$

$$\begin{aligned}\lambda^k_{n,\tilde{n},\tilde{l},l} &= \frac{\tilde{l}\sqrt{l}}{l+\tilde{l}-2k+1}\left(\frac{A_{\tilde{n},\tilde{l}}A_{n,l}}{A_{n+\tilde{n}+k,l+\tilde{l}-2k}}\right)^2\\ &\times\left(\int_0^{\frac{\pi}{4}}\beta(\theta)(\sin\theta)^{2n+l}(\cos\theta)^{2\tilde{n}+\tilde{l}}d\theta\right)^2.\end{aligned}\tag{6.9}$$

It follows from (6.6) and (6.7) that, for $0 \le k \le \min(l,\tilde{l})$, with $|m'| \le l+\tilde{l}-2k$,

$$\sum_{|m|\le l}\sum_{|\tilde{m}|\le \tilde{l}}\left|\mu^{m,\tilde{m},m'}_{n,\tilde{n},l,\tilde{l},k}\right|^2 \lesssim \lambda^k_{n,\tilde{n},\tilde{l},l}.$$

Then we obtain

$$\begin{aligned}&\sum_{\substack{n+\tilde{n}+k=n^\star\\ n+l\ge2,\tilde{n}+\tilde{l}\ge2\\ n\ge0,\tilde{n}\ge0}}\ \sum_{\substack{l+l-2k=l^\star\\ l\ge1,\tilde{l}\ge1\\ 0\le k\le\min(l,\tilde{l})}}\left(\sum_{|m|\le l}\sum_{|\tilde{m}|\le \tilde{l}}\frac{|\mu^{m,\tilde{m},m^\star}_{n,\tilde{n},l,\tilde{l},k}|^2}{\lambda_{\tilde{n},\tilde{l}}}\right)\\ &\lesssim \sum_{\substack{n+\tilde{n}+k=n^\star\\ n+l\ge2,\tilde{n}+\tilde{l}\ge2\\ n\ge0,\tilde{n}\ge0}}\ \sum_{\substack{l+l-2k=l^\star\\ l\ge1,\tilde{l}\ge1\\ 0\le k\le\min(l,\tilde{l})}}\frac{\lambda^k_{n,\tilde{n},\tilde{l},l}}{\lambda_{\tilde{n},\tilde{l}}}.\end{aligned}$$

The proof of 3) in Proposition 3.1 is reduced to prove

$$\sum_{\substack{n+\tilde{n}+k=n^\star\\ n+l\ge2,\tilde{n}+\tilde{l}\ge2\\ n\ge0,\tilde{n}\ge0}}\ \sum_{\substack{l+l-2k=l^\star\\ l\ge1,\tilde{l}\ge1\\ 0\le k\le\min(l,\tilde{l})}}\frac{\lambda^k_{n,\tilde{n},\tilde{l},l}}{\lambda_{\tilde{n},\tilde{l}}} \le C\lambda_{n^\star,l^\star}.\tag{6.10}$$

Lemma 6.3. *For $n,\tilde{n},\tilde{l},l \in \mathbb{N}$ with $n+l \ge 2$ and $\tilde{n}+\tilde{l} \ge 2$, let $s_0 = \min(1-s,s)$, we have*

$$\lambda^0_{n,\tilde{n},l,\tilde{l}} \lesssim \frac{\tilde{l}\sqrt{l}}{l+\tilde{l}+1}\frac{(\tilde{n}+\tilde{l})^{2s+s_0}}{\tilde{n}^{s_0}(n+s_0)^{1-s_0}(n+l)^{\frac{3}{2}+2s+s_0}}.\tag{6.11}$$

In addition, for $1 \le k \le \min(l,\tilde{l})$, we have the following estimate

$$\lambda^k_{n,\tilde{n},l,\tilde{l}} \lesssim \frac{\tilde{l}\sqrt{l}}{l+\tilde{l}-2k+1}\frac{\tilde{n}^s(\tilde{n}+\tilde{l})^s}{(n+1)^s(n+l+1)^{\frac{5}{2}+s}}.\tag{6.12}$$

Remark 6.4. *We divided the estimate of $\lambda^k_{n,\tilde n,l,\tilde l}$ into two cases, $k = 0$ and $k \geq 1$. That is because, when we estimate the formula* (6.12)*, there exists a term*

$$\frac{(n+\tilde n+k)!\Gamma(n+n+l+l-k+\frac{3}{2})}{(n+\tilde n+1)!\Gamma(n+\tilde n+l+\tilde l+\frac{1}{2})},$$

which is big term when $k = 0$ and $l + \tilde l \gg n + \tilde n$.

Proof. By using the Cauchy-Schwarz inequality and the Beta Function (6.2) we derive that, for $n + l \geq 2$,

$$\begin{aligned}
&|\int_0^{\frac{\pi}{4}} \beta(\theta)(\sin\theta)^{2n+l}(\cos\theta)^{2\tilde n+\tilde l}d\theta|^2 \approx |\int_0^{\frac{1}{2}} t^{n+\frac{l}{2}-1-s}(1-t)^{\tilde n+\frac{\tilde l}{2}+s}dt|^2\\
&\leq \frac{\Gamma(n+1-s)\Gamma(\tilde n+1+s)}{(n+\tilde n+1)!}\frac{\Gamma(n+l-1-s)\Gamma(\tilde n+\tilde l+\frac{3}{2}+s)}{\Gamma(n+\tilde n+l+\tilde l+\frac{1}{2})},
\end{aligned}$$

Then, we can express

$$\begin{aligned}
\lambda^k_{n,\tilde n,l,\tilde l} &\leq \frac{\tilde l\sqrt{l}}{l+\tilde l-2k+1}\frac{(n+\tilde n+k)!\Gamma(n+n+l+l-k+\frac{3}{2})}{n!\Gamma(n+l+\frac{3}{2})\tilde n!\Gamma(\tilde n+\tilde l+\frac{3}{2})}\\
&\quad\times\frac{\Gamma(n+1-s)\Gamma(\tilde n+1+s)}{(n+\tilde n+1)!}\frac{\Gamma(n+l-1-s)\Gamma(\tilde n+\tilde l+\frac{3}{2}+s)}{\Gamma(n+\tilde n+l+\tilde l+\frac{1}{2})}\\
&= \frac{\tilde l\sqrt{l}}{l+\tilde l-2k+1}\frac{(n+\tilde n+k)!\Gamma(n+n+l+l-k+\frac{3}{2})}{(n+\tilde n+1)!\Gamma(n+\tilde n+l+\tilde l+\frac{1}{2})}\\
&\quad\times\frac{\Gamma(n+1-s)\Gamma(\tilde n+1+s)\Gamma(n+l-1-s)\Gamma(\tilde n+\tilde l+\frac{3}{2}+s)}{n!\Gamma(n+l+\frac{3}{2})\tilde n!\Gamma(\tilde n+\tilde l+\frac{3}{2})}.
\end{aligned}$$

We deduce from the formula (6.1) with $x = n+1$, $a = -s$, $b = 0$,

$$\frac{\Gamma(n+1-s+1)}{(n+1)!} \lesssim \frac{1}{(n+1-s)^s},$$

and the recurrence formula $\Gamma(n+1-s) = \frac{1}{n+1-s}\Gamma(n+2-s)$,

$$\frac{\Gamma(n+1-s)}{n!} = \frac{n+1}{(n+1-s)}\frac{\Gamma(n+1-s+1)}{(n+1)!} \lesssim \frac{1}{(n+1)^s}.$$

Using $x = n+l-1$, $a = -s$, $b = \frac{3}{2}$ in (6.1),

$$\frac{\Gamma(n+l-s)}{\Gamma(n+l+\frac{3}{2})} = \frac{\Gamma(n+l-1-s+1)}{\Gamma(n+l-1+\frac{3}{2}+1)} \lesssim \frac{1}{(n+l-1)^{\frac{3}{2}+s}},$$

and recurrence formula $\Gamma(n+l-1-s) = \frac{\Gamma(n+l-s)}{n+l-1-s}$,

$$\frac{\Gamma(n+l-1-s)}{\Gamma(n+l+\frac{3}{2})} = \frac{1}{n+l-1-s}\frac{\Gamma(n+l-s)}{\Gamma(n+l+\frac{3}{2})} \lesssim \frac{1}{(n+l)^{\frac{5}{2}+s}}.$$

Using $x = \tilde n+1$, $a = s, b = 0$ in (6.1), we have

$$\frac{\Gamma(\tilde n+1+s)}{\tilde n!} = \frac{\tilde n+1}{\tilde n+1+s}\frac{\Gamma(\tilde n+2+s)}{(\tilde n+1)!} \lesssim \tilde n^s.$$

Using $x = \tilde n+\tilde l+\frac{1}{2}, a = s, b = 0$ in (6.1), and $\tilde n+\tilde l \geq 2$,

$$\frac{\Gamma(\tilde n+\tilde l+\frac{3}{2}+s)}{\Gamma(\tilde n+\tilde l+\frac{3}{2})} \lesssim (\tilde n+\tilde l+\frac{1}{2}+s)^s \lesssim (\tilde n+\tilde l)^s.$$

Therefore, we obtain that, $n+l\ge 2$, $\tilde n+\tilde l\ge 2$

$$\lambda^k_{n,\tilde n,l,\tilde l}\lesssim \frac{\tilde l\sqrt{l}}{l+\tilde l-2k+1}\frac{\tilde n^s(\tilde n+\tilde l)^s}{(n+1)^s(n+l+1)^{\frac52+s}}$$
$$\times\frac{(n+\tilde n+k)!\Gamma(n+n+l+l-k+\frac32)}{(n+\tilde n+1)!\Gamma(n+\tilde n+l+\tilde l+\frac12)}.$$

Consider that, when $k\ge 1$

$$\frac{(n+\tilde n+k)!\Gamma(n+n+l+l-k+\frac32)}{(n+\tilde n+1)!\Gamma(n+\tilde n+l+\tilde l+\frac12)}\le 1,$$

We obtain the formula (6.12).

For the estimate (6.11), we assume that $n+l\ge 2$, and $s_0=\min(1-s,s)$. By using the Cauchy-Schwarz inequality and the Beta Function (6.2), we obtain

$$\begin{aligned}
&|\int_0^{\frac{\pi}{4}}\beta(\theta)(\sin\theta)^{2n+l}(\cos\theta)^{2\tilde n+\tilde l}d\theta|^2\approx|\int_0^{\frac12}t^{n+\frac{l}{2}-1-s}(1-t)^{\tilde n+\frac{\tilde l}{2}+\frac14+s}dt|^2\\
&\le\int_0^{\frac12}t^{n-1+s_0}(1-t)^{\tilde n-s_0}dt\times\int_0^{\frac12}t^{n+l-2s-1-s_0}(1-t)^{\tilde n+\tilde l+\frac12+2s+s_0}dt\\
&\le\frac{\Gamma(n+s_0)\Gamma(\tilde n+1-s_0)}{(n+\tilde n)!}\frac{\Gamma(n+l-2s-s_0)\Gamma(\tilde n+\tilde l+\frac32+2s+s_0)}{\Gamma(n+\tilde n+l+\tilde l+\frac32)},
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lambda^0_{n,\tilde n,l,\tilde l}&=\frac{\tilde l\sqrt{l}}{l+\tilde l+1}\frac{(n+\tilde n)!\Gamma(n+\tilde n+l+\tilde l+\frac32)}{n!\tilde n!\Gamma(n+l+\frac32)\Gamma(\tilde n+\tilde l+\frac32)}\\
&\quad\times\frac{\Gamma(n+s_0)\Gamma(\tilde n+1-s_0)}{(n+\tilde n)!}\frac{\Gamma(n+l-2s-s_0)\Gamma(\tilde n+\tilde l+\frac32+2s+s_0)}{\Gamma(n+\tilde n+l+\tilde l+\frac32)}\\
&\lesssim\frac{\tilde l\sqrt{l}}{l+\tilde l+1}\frac{\Gamma(n+s_0)\Gamma(\tilde n+1-s_0)\Gamma(n+l-2s-s_0)\Gamma(\tilde n+\tilde l+\frac32+2s+s_0)}{n!\tilde n!\Gamma(n+l+\frac32)\Gamma(\tilde n+\tilde l+\frac32)}.
\end{aligned}$$

Applying the estimate (6.1) and recurrence formula for Gamma function

$$\Gamma(x+1)=x\Gamma(x),$$

which gives

$$\lambda^0_{n,\tilde n,l,\tilde l}\lesssim\frac{\tilde l\sqrt{l}}{l+\tilde l+1}\frac{(\tilde n+\tilde l)^{2s+s_0}}{\tilde n^{s_0}(n+s_0)^{1-s_0}(n+l)^{\frac32+2s+s_0}}.$$

We end the proof of Lemma 6.3. □

The estimate in (6.12) is not enough in proof of 3) in the Proposition 3.1. To this end, we provide a more optimal estimate of $\lambda^k_{n,\tilde n,l,\tilde l}$ in the following Lemma.

Lemma 6.5. *For any $0 < \omega < 1$, and $k \geq 1$, we have the following estimates*

$$(i)\ \lambda^k_{n,\tilde{n},l,\tilde{l}} \lesssim \frac{\tilde{l}\sqrt{l}}{l+\tilde{l}-2k+1}\frac{\tilde{n}^s(\tilde{n}+\tilde{l})^s}{(n+1)^s(n+l)^{\frac{5}{2}+s}},\ 1 \leq k < 20;$$

$$(ii)\ \lambda^k_{n,\tilde{n},l,\tilde{l}} \lesssim \frac{\tilde{l}\sqrt{l}}{l+\tilde{l}-2k+1}\frac{(\tilde{n}+\tilde{l})^{2s}}{(n+l)^{\frac{5}{2}+2s}}e^{-\frac{1}{4}k^{\omega}}\ \text{when}\ n+l \leq k^{1-\omega}l,\ \text{and}\ k \geq 20;$$

$$(iii)\ \lambda^k_{n,\tilde{n},l,\tilde{l}} \lesssim \frac{\tilde{l}\sqrt{l}}{l+\tilde{l}-2k+1}\frac{(\tilde{n}+\tilde{l})^{2s}}{(n+l)^{\frac{5}{2}+2s}}\qquad \text{when}\ n+l \geq k^{1-\omega}l\ \text{and}\ k \geq 20.$$

Remark 6.6. *The coefficient 20 in Lemma 6.5 can be replaced by any big positive constant, this coefficient make the estimate available.*

Proof. The estimate (i) is a direct consequence of the estimate (6.12).
Now we estimate for $k \geq 20$. Recall that

$$\lambda^k_{n,\tilde{n},l,\tilde{l}} = \frac{\tilde{l}\sqrt{l}}{l+\tilde{l}-2k+1}\frac{(n+\tilde{n}+k)!\Gamma(n+\tilde{n}+l+\tilde{l}-k+\frac{3}{2})}{n!\Gamma(n+l+\frac{3}{2})\tilde{n}!\Gamma(\tilde{n}+\tilde{l}+\frac{3}{2})}$$
$$\times\Big(\int_0^{\frac{1}{2}} t^{n+\frac{l}{2}-1-s}(1-t)^{\tilde{n}+\frac{\tilde{l}}{2}}dt\Big)^2.$$

By using the Beta Function (6.2) and the Cauchy -Schwarz inequality,

$$\begin{aligned}\Big(\int_0^{\frac{1}{2}} t^{n+\frac{l}{2}-1-s}(1-t)^{\tilde{n}+\frac{\tilde{l}}{2}}dt\Big)^2 &\leq 2^{\frac{1}{2}+2s}\Big(\int_0^{\frac{1}{2}} t^{n+\frac{l}{2}-1-s}(1-t)^{\tilde{n}+\frac{\tilde{l}}{2}+\frac{1}{4}+s}dt\Big)^2\\ &\leq 2^{\frac{1}{2}+2s}\frac{\Gamma(n+\frac{k}{2})\Gamma(\tilde{n}+\frac{k}{2}+1)}{\Gamma(n+\tilde{n}+k+1)}\quad\times\\ &\quad\frac{\Gamma(n+l-\frac{k}{2}-2s)\Gamma(\tilde{n}+\tilde{l}-\frac{k}{2}+\frac{3}{2}+2s)}{\Gamma(n+\tilde{n}+l+\tilde{l}-k+\frac{3}{2})},\end{aligned}$$

we obtain

$$\begin{aligned}\lambda^k_{n,\tilde{n},l,\tilde{l}} &\lesssim \frac{\tilde{l}\sqrt{l}}{l+\tilde{l}-2k+1}\frac{(n+\tilde{n}+k)!\Gamma(n+\tilde{n}+l+\tilde{l}-k+\frac{3}{2})}{n!\Gamma(n+l+\frac{3}{2})\tilde{n}!\Gamma(\tilde{n}+\tilde{l}+\frac{3}{2})}\\ &\quad\times\frac{\Gamma(n+\frac{k}{2})\Gamma(\tilde{n}+\frac{k}{2}+1)}{\Gamma(n+\tilde{n}+k+1)}\frac{\Gamma(n+l-\frac{k}{2}-2s)\Gamma(\tilde{n}+\tilde{l}-\frac{k}{2}+\frac{3}{2}+2s)}{\Gamma(n+\tilde{n}+l+\tilde{l}-k+\frac{3}{2})}\\ &= \frac{\sqrt{l}\tilde{l}\Gamma(n+\frac{k}{2})\Gamma(\tilde{n}+\frac{k}{2}+1)\Gamma(n+l-\frac{k}{2}-2s)\Gamma(\tilde{n}+\tilde{l}-\frac{k}{2}+\frac{3}{2}+2s)}{(l+\tilde{l}-2k+1)n!\Gamma(n+l+\frac{3}{2})\tilde{n}!\Gamma(\tilde{n}+\tilde{l}+\frac{3}{2})}.\end{aligned}$$

Consider that, $l \geq k$, and $k \geq 20$,

$$n+l-\frac{k}{2}-1 \geq \frac{k}{2}-1 \geq 9,$$

Let $x = n+l-\frac{k}{2}-1$, $a = -2s$, $b = \frac{5}{2}$ in formula (6.1), we have

$$\frac{\Gamma(n+l-\frac{k}{2}-2s)}{\Gamma(n+l-\frac{k}{2}+\frac{5}{2})} \lesssim \frac{1}{(n+l-\frac{k}{2}-1-2s)^{\frac{5}{2}+2s}} \lesssim \frac{1}{(n+l)^{\frac{5}{2}+2s}}.$$

When we choose $x = \tilde{n}+\tilde{l}-\frac{k}{2}+\frac{1}{2}, a = 2s, b = 0$ in formula (6.1), then

$$\frac{\Gamma(\tilde{n}+\tilde{l}-\frac{k}{2}+\frac{3}{2}+2s)}{\Gamma(\tilde{n}+\tilde{l}-\frac{k}{2}+\frac{3}{2})} \lesssim (\tilde{n}+\tilde{l})^{2s}.$$

Therefore, we can verify that

$$\lambda^k_{n,\tilde n,l,\tilde l} \lesssim \frac{\tilde l\sqrt{l}}{l+\tilde l-2k+1}\frac{(\tilde n+\tilde l)^{2s}}{(n+l)^{\frac52+2s}}\frac{\Gamma(n+\frac k2)\Gamma(n+l-\frac k2+\frac52)}{n!\Gamma(n+l+\frac32)}$$
$$\times\frac{\Gamma(\tilde n+\frac k2+1)\Gamma(\tilde n+\tilde l-\frac k2+\frac32)}{\tilde n!\Gamma(\tilde n+\tilde l+\frac32)}.$$

Because

$$\frac{\Gamma(\tilde n+\frac k2+1)\Gamma(\tilde n+\tilde l-\frac k2+\frac32)}{\tilde n!\Gamma(\tilde n+\tilde l+\frac32)}\le 1,$$

we have

$$\lambda^k_{n,\tilde n,l,\tilde l} \lesssim \frac{\tilde l\sqrt{l}}{l+\tilde l-2k+1}\frac{(\tilde n+\tilde l)^{2s}}{(n+l)^{\frac52+2s}}\frac{\Gamma(n+\frac k2)\Gamma(n+l-\frac k2+\frac52)}{n!\Gamma(n+l+\frac32)}.$$

We consider the formula

$$\frac{\Gamma(n+\frac k2)\Gamma(n+l-\frac k2+2)}{n!\Gamma(n+l+1)}.$$

Remind that $k\le\min(l,\tilde l)$, without loss of generality, set $k=2q$ and $l=l_1+2q$ with $l_1\ge 0$, using the elementary induction,

$$\Gamma(n+\frac k2)=(n+q-1)!,$$

we have

$$\begin{aligned}\frac{\Gamma(n+\frac k2)\Gamma(n+l-\frac k2+2)}{n!\Gamma(n+l+1)}&=\frac{(n+q-1)!(n+l_1+q+1)!}{n!(n+l_1+2q)!}\\
&=\frac{(n+q-1)(n+q-2)\times\cdots\times(n+1)}{(n+2q+l_1)(n+l_1+2q-1)\times\cdots\times(n+l_1+q+2)}\\
&\le\Big(\frac{n+q}{n+2q+l_1}\Big)^q=\Big(\frac{n+\frac k2}{n+l}\Big)^{\frac k2}=\Big(1-\frac{\frac k2+l_1}{n+l}\Big)^{\frac k2}\\
&=\Big[\Big(1-\frac{\frac k2+l_1}{n+l}\Big)^{n+l}\Big]^{\frac{k}{2(n+l)}}\le e^{-\frac{lk}{4(n+l)}}.\end{aligned}$$

where we use the elementary inequality

$$(1-\frac kn)^n\le e^{-k},\ \text{when}\,k>-n.$$

For $0<\omega<1$, if

$$\frac{k^{1-\omega}l}{n+l}>1,$$

then

$$\frac{\Gamma(n+\frac k2)\Gamma(n+l-\frac k2+2)}{n!\Gamma(n+l+1)}\lesssim e^{-\frac14k^{\omega}}.$$

This implies that, when $n+l<k^{1-\omega}l$

$$\lambda^k_{n,\tilde n,l,\tilde l} \lesssim \frac{\tilde l\sqrt{l}}{l+\tilde l-2k+1}\frac{(\tilde n+\tilde l)^{2s}}{(n+l)^{\frac52+2s}}e^{-\frac14k^{\omega}},$$

This is the result of the estimate (ii). If $n+l\ge k^{1-\omega}l$, since when $k\ge 20$,

$$\frac{\Gamma(n+\frac k2)\Gamma(n+l-\frac k2+2)}{n!\Gamma(n+l+1)}\le 1.$$

Therefore, the estimate (*iii*) follows. This ends the proof of Lemma 6.5. $\square$

6.3. **The proof of** 3) **in the Proposition 3.1.**

Proof. For $\lambda^k_{n,\tilde n,l,\tilde l}$ defined in (6.9), by the analysis in (6.10), we have only need to prove

$$\sum_{\substack{n+\tilde n+k=n^\star\\ n+l\geq 2,\tilde n+\tilde l\geq 2\\ n\geq 0,\tilde n\geq 0}}\ \sum_{\substack{l+l-2k=l^\star\\ l\geq 1,\tilde l\geq 1\\ 0\leq k\leq \min(l,\tilde l)}} \frac{\lambda^k_{n,\tilde n,\tilde l,l}}{\lambda_{\tilde n,\tilde l}} \leq C\lambda_{n^\star,l^\star}.$$

By using Lemma (6.3) with $s_0=\min(s,1-s)$, Lemma 6.5 with $\omega=\frac{1}{4}$, and $\lambda_{\tilde n,\tilde l}\gtrsim \tilde n^s+\tilde l^{2s}$ in (2.6), we can divided the above summation into four terms

$$\begin{aligned}
&\sum_{\substack{n+\tilde n+k=n^\star\\ n+l\geq 2,\tilde n+\tilde l\geq 2\\ n\geq 0,\tilde n\geq 0}}\ \sum_{\substack{l+l-2k=l^\star\\ l\geq 1,\tilde l\geq 1\\ 0\leq k\leq \min(l,\tilde l)}} \frac{\lambda^k_{n,\tilde n,\tilde l,l}}{\lambda_{\tilde n,\tilde l}}\\
&=\sum_{\substack{n+\tilde n=n^\star\\ n+l\geq 2,\tilde n+\tilde l\geq 2\\ n\geq 0,\tilde n\geq 0}}\ \sum_{\substack{l+l=l^\star\\ l\geq 1,\tilde l\geq 1}} \frac{\tilde l\sqrt{l}}{l+\tilde l+1}\frac{\tilde n^s(\tilde n+\tilde l)^{s_0}}{(\tilde n+1)^{s_0}(n+s_0)^{1-s_0}(n+l)^{\frac{3}{2}+2s+s_0}}\\
&\quad+\sum_{k=1}^{\min(19,n^\star)}\ \sum_{\substack{n+\tilde n=n^\star-k\\ n+l\geq 2,\tilde n+\tilde l\geq 2\\ n\geq 0,\tilde n\geq 0}}\ \sum_{\substack{l+\tilde l=l^\star+2k\\ l\geq k,\tilde l\geq k}} \frac{\tilde l\sqrt{l}}{l+\tilde l-2k+1}\frac{\tilde n^s}{(n+1)^s(n+l)^{\frac{5}{2}+s}}\\
&\quad+\sum_{k=20}^{n^\star}\ \sum_{\substack{n+\tilde n+k=n^\star\\ n+l<k^{\frac{3}{4}}l,\tilde n+\tilde l\geq 2\\ n\geq 0,\tilde n\geq 0}}\ \sum_{\substack{l+\tilde l=l^\star+2k\\ l\geq k,\tilde l\geq k}} \frac{\tilde l\sqrt{l}}{l+\tilde l-2k+1}\frac{(\tilde n+\tilde l)^s}{(n+l)^{\frac{5}{2}+2s}}e^{-\frac{1}{4}k^{\frac{1}{4}}}\\
&\quad+\sum_{k=20}^{n^\star}\ \sum_{\substack{n+\tilde n+k=n^\star\\ n+l\geq k^{\frac{3}{4}}l,\tilde n+\tilde l\geq 2\\ n\geq 0,\tilde n\geq 0}}\ \sum_{\substack{l+\tilde l=l^\star+2k\\ l\geq k,\tilde l\geq k}} \frac{\tilde l\sqrt{l}}{l+\tilde l-2k+1}\frac{(\tilde n+\tilde l)^s}{(n+l)^{\frac{5}{2}+2s}}\\
&=\mathbf{K_1}+\mathbf{K_2}+\mathbf{K_3}+\mathbf{K_4}.
\end{aligned}$$

The estimate of $\mathbf{K_1}$:

$$\begin{aligned}
\mathbf{K_1}&=\sum_{\substack{n+\tilde n=n^\star\\ n+l\geq 2,\tilde n+\tilde l\geq 2\\ n\geq 0,\tilde n\geq 0}}\ \sum_{\substack{l+l=l^\star\\ l\geq 1,\tilde l\geq 1}} \frac{\tilde l\sqrt{l}}{l+\tilde l+1}\frac{\tilde n^s(\tilde n+\tilde l)^{s_0}}{(\tilde n+1)^{s_0}(n+s_0)^{1-s_0}(n+l)^{\frac{3}{2}+2s+s_0}}\\
&\lesssim\sum_{\substack{n+\tilde n=n^\star\\ n\geq 0,\tilde n\geq 0}}\ \sum_{\substack{l+\tilde l=l^\star\\ l\geq 1,\tilde l\geq 1}} \frac{\tilde n^s}{(n+s_0)^{1-s_0}(n+l)^{1+2s+s_0}}\Big(1+\frac{\tilde l}{\tilde n}\Big)^{s_0}.
\end{aligned}$$

Consider that $0<s_0=\min(1-s,s)<1$, using the elementary inequality,

$$\Big(1+\frac{\tilde l}{\tilde n}\Big)^{s_0}\leq 1+\Big(\frac{\tilde l}{\tilde n}\Big)^{s_0},$$

we have

$$\mathbf{K_1} \lesssim (s_0)^{1-s_0}\Big[(n^\star)^s + (l^\star)^{s_0}(n^\star)^{s-s_0}\Big]\sum_{l=1}^{l^\star-1}\frac{1}{l^{1+2s}}$$

$$+\Big[(n^\star)^s + (l^\star)^{s_0}(n^\star)^{s-s_0}\Big]\sum_{n=1}^{n^\star}\frac{1}{n^{1+s}}\sum_{l=1}^{l^\star-1}\frac{1}{l^{1+s}}$$

$$\lesssim \Big[(n^\star)^s + (l^\star)^{s_0}(n^\star)^{s-s_0}\Big].$$

We need only consider $s_0 < s$, by using the Young inequality,

$$(l^\star)^{s_0}(n^\star)^{s-s_0} \le \frac{s}{s-s_0}(n^\star)^s + \frac{s}{s_0}(l^\star)^s,$$

Then,

$$\mathbf{K_1} \lesssim (n^\star)^s + (l^\star)^s.$$

The estimate of the second term $\mathbf{K_2}$,

$$\mathbf{K_2} = \sum_{k=1}^{\min(19,n^\star)}\sum_{\substack{n+\tilde{n}=n^\star-k\\ n+l\ge 2,\tilde{n}+\tilde{l}\ge 2\\ n\ge 0,\tilde{n}\ge 0}}\sum_{\substack{l+\tilde{l}=l^\star+2k\\ l\ge k,\tilde{l}\ge k}}\frac{\tilde{l}\sqrt{l}}{l+\tilde{l}-2k+1}\frac{\tilde{n}^s}{(n+1)^s(n+l)^{\frac{5}{2}+s}}$$

$$\lesssim (n^\star)^s\sum_{k=1}^{19}\sum_{n=0}^{n^\star-k}\sum_{l=k}^{l^\star+k}\frac{(l^\star+2k-l)}{l^\star}\frac{1}{(n+1)^s(n+l)^{2+s}}$$

$$\lesssim (n^\star)^s\sum_{k=1}^{19}\sum_{n=0}^{n^\star-k}\frac{(l^\star+2k-l)}{l^\star}\Big(\frac{1}{(n+k)^{1+s}}-\frac{1}{(n+k+l^\star)^{1+s}}\Big)$$

$$\lesssim (n^\star)^s\sum_{k=1}^{19}\frac{1}{k^s} \lesssim (n^\star)^s.$$

Now we consider $k \ge 20$, for the third term $\mathbf{K_3}$

$$\mathbf{K_3} = \sum_{k=20}^{n^\star}\sum_{\substack{n+\tilde{n}+k=n^\star\\ n+l<k^{\frac{3}{4}}l,\tilde{n}+\tilde{l}\ge 2\\ n\ge 0,\tilde{n}\ge 0}}\sum_{\substack{l+\tilde{l}=l^\star+2k\\ l\ge k,\tilde{l}\ge k}}\frac{\tilde{l}\sqrt{l}}{l+\tilde{l}-2k+1}\frac{(\tilde{n}+\tilde{l})^s}{(n+l)^{\frac{5}{2}+2s}}e^{-\frac{1}{4}k^{\frac{1}{4}}}$$

$$\lesssim \sum_{k=20}^{n^\star}\sum_{n=0}^{n^\star-k}\sum_{l=k}^{l^\star+k}\frac{(l^\star+k)}{l^\star}\frac{(n^\star+l^\star)^s}{(n+l)^{2+2s}}e^{-\frac{1}{4}k^{\frac{1}{4}}}$$

$$\lesssim (n^\star+l^\star)^s\sum_{k=1}^{n^\star}e^{-\frac{1}{4}k^{\frac{1}{4}}}\sum_{n=0}^{n^\star-k}\Big(\frac{(n+l^\star+k)^{1+2s}-(n+k)^{1+2s}}{(n+k)^{1+2s}(n+l^\star+k)^{1+2s}}\Big)\frac{(l^\star+k)}{l^\star}$$

$$\lesssim (n^\star+l^\star)^s\sum_{k=1}^{n^\star}e^{-\frac{1}{4}k^{\frac{1}{4}}}\sum_{n=0}^{n^\star-k}\frac{1}{(n+k)^{1+2s}}$$

$$\lesssim (n^\star+l^\star)^s.$$

Now we estimate the remaining term $\mathbf{K_4}$. Consider the condition that

$$n+l \ge k^{\frac{3}{4}}l,$$

we obtain

$$\frac{1}{(n+l)^{\frac{3}{2}}} \leq \frac{1}{k^{\frac{9}{8}} l^{\frac{3}{2}}}.$$

$\mathbf{K_4}$ can be rewritten as

$$\begin{aligned}
\mathbf{K_4} &= \sum_{k=20}^{n^\star} \sum_{\substack{n+\tilde{n}+k=n^\star \\ n+l\geq k^{\frac{3}{4}} l, \tilde{n}+\tilde{l}\geq 2 \\ n\geq 0, \tilde{n}\geq 0}} \sum_{\substack{l+\tilde{l}=l^\star+2k \\ l\geq k, \tilde{l}\geq k}} \frac{\tilde{l}\sqrt{l}}{l+\tilde{l}-2k+1} \frac{(\tilde{n}+\tilde{l})^s}{(n+l)^{\frac{5}{2}+2s}} \\
&\lesssim \left(n^\star + l^\star\right)^s \sum_{k=20}^{n^\star} \frac{1}{k^{\frac{9}{8}}} \frac{l^\star + k}{l^\star} \sum_{n=0}^{n^\star - k} \sum_{l=k}^{l^\star + k} \frac{1}{(n+l)^{1+2s} l} \frac{l^\star + k}{l^\star} \\
&\lesssim \left(n^\star + l^\star\right)^s \sum_{k=20}^{n^\star} \frac{1}{k^{\frac{9}{8}}} \sum_{n=0}^{n^\star - k} \frac{1}{(n+k)^{1+s}} \\
&\lesssim \left(n^\star + l^\star\right)^s \sum_{k=10}^{n^\star} \frac{1}{k^{\frac{9}{8}+s}} \\
&\lesssim \left(n^\star + l^\star\right)^s.
\end{aligned}$$

Combine with the estimate of $\mathbf{K_1}$, $\mathbf{K_2}$, $\mathbf{K_3}$ and $\mathbf{K_4}$, using (2.6) again that

$$\lambda_{n^\star, l^\star} \gtrsim (n^\star)^s + (l^\star)^{2s},$$

we have

$$\sum_{\substack{n+\tilde{n}+k=n^\star \\ n+l\geq 2, \tilde{n}+\tilde{l}\geq 2 \\ n\geq 0, \tilde{n}\geq 0}} \sum_{\substack{l+l-2k=l^\star \\ l\geq 1, \tilde{l}\geq 1 \\ 0\leq k\leq \min(l,\tilde{l})}} \frac{\lambda^k_{n,\tilde{n},\tilde{l},l}}{\lambda_{\tilde{n},\tilde{l}}} \leq C\lambda_{n^\star, l^\star}.$$

This ends the proof of 3) in Proposition 3.1. □

7. Appendix

The important known results but really needed for this paper are presented in this section. For the self-content of paper, we will present some proof of those properties.

7.1. Gelfand-Shilov space. The symmetric Gelfand-Shilov space $S^\nu_\nu(\mathbb{R}^3)$ can be characterized through the decomposition into the Hermite basis H_α and the harmonic oscillator $\mathcal{H} = -\triangle + \frac{|v|^2}{4}$, for more details, see Theorem 2.1 in [7],

$$\begin{aligned}
f \in S^\nu_\nu(\mathbb{R}^3) &\Leftrightarrow\ f \in C^\infty(\mathbb{R}^3), \exists\, \tau > 0, \|e^{\tau \mathcal{H}^{\frac{1}{2\nu}}} f\|_{L^2} < +\infty; \\
&\Leftrightarrow\ f \in L^2(\mathbb{R}^3), \exists\, \epsilon_0 > 0,\ \left\| \left(e^{\epsilon_0 |\alpha|^{\frac{1}{2\nu}}} (f,\, H_\alpha)_{L^2}\right)_{\alpha\in\mathbb{N}^3} \right\|_{l^2} < +\infty; \\
&\Leftrightarrow\ \exists\, C > 0,\ A > 0,\ \|(-\triangle + \frac{|v|^2}{4})^{\frac{k}{2}} f\|_{L^2(\mathbb{R}^3)} \leq AC^k (k!)^\nu,\ \ k \in \mathbb{N}
\end{aligned}$$

where

$$H_\alpha(v) = H_{\alpha_1}(v_1) H_{\alpha_2}(v_2) H_{\alpha_3}(v_3),\ \alpha \in \mathbb{N}^3,$$

and for $x \in \mathbb{R}$,

$$H_n(x) = \frac{(-1)^n}{\sqrt{2^n n! \pi}} e^{\frac{x^2}{2}} \frac{d^n}{dx^n}(e^{-x^2}) = \frac{1}{\sqrt{2^n n! \pi}} \Big(x - \frac{d}{dx}\Big)^n (e^{-\frac{x^2}{2}}).$$

For the harmonic oscillator $\mathcal{H} = -\triangle + \frac{|v|^2}{4}$ of 3-dimension and $s > 0$, we have

$$\mathcal{H}^{\frac{k}{2}} H_\alpha = (\lambda_\alpha)^{\frac{k}{2}} H_\alpha,\ \lambda_\alpha = \sum_{j=1}^{3}(\alpha_j + \frac{1}{2}),\ k \in \mathbb{N},\ \alpha \in \mathbb{N}^3.$$

7.2. **Fourier Transform of special functions.** For the eigenvector $\varphi_{n,l,m}$ given in (2.1), Lerner, Morimoto, Pravda-Starov and Xu in [13] shows in Lemma 3.2 the Fourier transform of $\sqrt{\mu}\varphi_{n,0,0}$. Following this work, we provide the Fourier transform of $\sqrt{\mu}\varphi_{n,l,m}$.

Lemma 7.1. *Let α, $\kappa \in S^2$ and $r > 0$, then*

$$\int_{S^2_\kappa} e^{ir\kappa\cdot\alpha} Y_l^m(\kappa) d\kappa = (2\pi)^{\frac{3}{2}} i^l \Big(\frac{1}{r}\Big)^{\frac{1}{2}} J_{l+\frac{1}{2}}(r) Y_l^m(\alpha), \tag{7.1}$$

where $\kappa \cdot \alpha$ denote the scalar product and $J_{l+\frac{1}{2}}$ is the Bessel function of $l + \frac{1}{2}$ order.

Proof. Since for any real r and $|z| \leq 1$,(cf. Section 11.5 of [24])

$$\sqrt{2r} e^{irz} = \sqrt{\pi} \sum_{k=0}^{\infty} (2k+1) J_{k+\frac{1}{2}}(r)(i)^k P_k(z). \tag{7.2}$$

Substituting $z = \kappa \cdot \alpha$ into (7.2),

$$e^{ir\kappa\cdot\alpha} = \sqrt{\frac{\pi}{2r}} \sum_{k=0}^{\infty} (2k+1) J_{k+\frac{1}{2}}(r) i^k P_k(\kappa\cdot\alpha).$$

Now since $|P_k(\kappa \cdot \alpha)| \leq 1$ and for every $r > 0$,

$$\sqrt{\frac{1}{r}} \sum_{k=0}^{\infty} (2k+1) |J_{k+\frac{1}{2}}(r)| \leq \sum_{k=0}^{\infty} \frac{(\frac{r}{2})^k}{k!} e^{\frac{r^2}{4}} \leq e^{\frac{r}{2} + \frac{r^2}{4}},$$

we obtain that

$$\int_{S^2_\kappa} e^{ir\kappa\cdot\alpha} Y_l^m(\kappa) d\kappa = \sqrt{\frac{\pi}{2r}} \sum_{k=0}^{\infty} (2k+1) J_{k+\frac{1}{2}}(r) i^k \int_{S^2_\kappa} P_k(\kappa\cdot\alpha) Y_l^m(\kappa) d\kappa. \tag{7.3}$$

Consider the addition theorem of spherical harmonics $(7-34)$ in Chapter 7 of [19] that

$$P_k(\kappa\cdot\alpha) = \frac{4\pi}{2k+1} \sum_{m=-k}^{k} [Y_k^m(\kappa)]^* Y_k^m(\alpha),$$

where $[Y_k^m(\kappa)]^*$ is the conjugate of $Y_k^m(\kappa)$. Then

$$\int_{S^2_\kappa} P_k(\kappa\cdot\alpha) Y_l^m(\kappa) d\kappa = \frac{4\pi}{2l+1} Y_l^m(\alpha) \delta_{k,l}.$$

Substitute this addition formula into (7.3), the formula (7.1) follows. □

Lemma 7.1 is the basis for calculating the Fourier transform of $\sqrt{\mu}\varphi_{n,l,m}$.

Lemma 7.2. *Let $\varphi_{n,l,m}$ be the functions defined in* (2.1)*, then for $n, l \in \mathbb{N}$, $|m| \leq l$, we have*

$$\widehat{\sqrt{\mu}\varphi_{n,l,m}}(\xi) = (-i)^l (2\pi)^{\frac{3}{4}} \Big(\frac{1}{\sqrt{2} n! \Gamma(n+l+\frac{3}{2})}\Big)^{\frac{1}{2}} \Big(\frac{|\xi|}{\sqrt{2}}\Big)^{2n+l} e^{-\frac{|\xi|^2}{2}} Y_l^m\Big(\frac{\xi}{|\xi|}\Big). \tag{7.4}$$

Proof. Define $H(\xi) = \Big(\frac{|\xi|}{\sqrt{2}}\Big)^{2n+l} e^{-\frac{|\xi|^2}{2}} Y_l^m\Big(\frac{\xi}{|\xi|}\Big)$, and by Lemma 7.1 with $r = |v||\xi|$, we can compute the inverse Fourier transform of H,

$$\begin{aligned}
\mathcal{F}^{-1}(H)(v) &= \Big(\frac{1}{2\pi}\Big)^3 \int_{\mathbb{R}^3} e^{iv\cdot\xi}\Big(\frac{|\xi|}{\sqrt{2}}\Big)^{2n+l} e^{-\frac{|\xi|^2}{2}} Y_l^m\Big(\frac{\xi}{|\xi|}\Big) d\xi \\
&= \frac{1}{4\pi^3}\int_0^\infty \Big(\frac{\rho}{\sqrt{2}}\Big)^{2n+l+2} e^{-\frac{\rho^2}{2}} \bigg(\int_{S^2_\kappa} e^{i|v|\rho\kappa\cdot\alpha} Y_l^m(\kappa)d\kappa\bigg) d\rho \\
&= i^l\Big(\frac{1}{2\pi}\Big)^{\frac{3}{2}} Y_l^m\Big(\frac{v}{|v|}\Big)\Big(\frac{2\sqrt{2}}{|v|}\Big)^{\frac{1}{2}} \int_0^\infty \Big(\frac{\rho}{\sqrt{2}}\Big)^{2n+l+\frac{3}{2}} e^{-\frac{\rho^2}{2}} J_{l+\frac{1}{2}}(|v|\rho)d\rho.
\end{aligned}$$

By using the standard formula, see (6.2.15) in [8],

$$L_n^{(l+\frac{1}{2})}(x) = \frac{e^x x^{-\frac{l+\frac{1}{2}}{2}}}{n!}\int_0^{+\infty} t^{n+\frac{l+\frac{1}{2}}{2}} J_{l+\frac{1}{2}}(2\sqrt{xt})e^{-t}dt,$$

we have

$$L_n^{(l+\frac{1}{2})}\Big(\frac{|v|^2}{2}\Big) = \sqrt{2}\frac{e^{\frac{|v|^2}{2}}\big(\frac{|v|}{\sqrt{2}}\big)^{-(l+\frac{1}{2})}}{n!}\int_0^\infty \Big(\frac{\rho}{\sqrt{2}}\Big)^{2n+l+\frac{3}{2}} J_{l+\frac{1}{2}}(|v|\rho)e^{-\frac{\rho^2}{2}}d\rho.$$

Therefore,

$$\mathcal{F}^{-1}(H)(v) = \Big(\frac{1}{2\pi}\Big)^{\frac{3}{2}} n! i^l e^{-\frac{|v|^2}{2}}\Big(\frac{|v|}{\sqrt{2}}\Big)^l L_n^{(l+\frac{1}{2})}\Big(\frac{|v|^2}{2}\Big) Y_l^m\Big(\frac{v}{|v|}\Big).$$

Recall the expression of (2.1), one can verify

$$\sqrt{\mu}\varphi_{n,l,m}(v) = (-i)^l(2\pi)^{\frac{3}{4}}\Big(\frac{1}{\sqrt{2}n!\Gamma(n+l+\frac{3}{2})}\Big)^{\frac{1}{2}}\mathcal{F}^{-1}(H)(v).$$

Henceforth, (7.4) yields. □

7.3. **Spherical Harmonics.** The following results with respect to the spherical harmonics is significant. For $l, \tilde{l} \in \mathbb{N}, |m| \le l, |\tilde{m}| \le \tilde{l}$,

$$Y_l^m Y_{\tilde{l}}^{\tilde{m}} = \sum_{l'}\sum_{m'}\bigg(\int_{\mathbb{S}^2} Y_l^m(\kappa)Y_{\tilde{l}}^{\tilde{m}}(\kappa)Y_{l'}^{-m'}(\kappa)d\kappa\bigg)Y_{l'}^{m'} \tag{7.5}$$

where $|m'| \le l'$ and $\tilde{l} - l \le l' \le \tilde{l} + l$. More explicitly, in order to have a non-vanishing integral, the parameters m', l' satisfy

$$m' = m + \tilde{m},\ \ l' = l + \tilde{l} - 2j \text{ with } 0 \le j \le \min(l, \tilde{l}). \tag{7.6}$$

For more details, See (86) in Chap. 3 in [9], [17].

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