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On $p$-almost direct products and residual properties of pure braid groups of nonorientable surfaces

Paolo Bellingeri and Sylvain Gervais

Abstract

We prove that the $n$th pure braid group of a nonorientable surface (closed or with boundary, but different from $\mathbb{R}P^2$) is residually 2-finite. Consequently, this group is residually nilpotent. The key ingredient in the closed case is the notion of $p$-almost direct product, which is a generalization of the notion of almost direct product. We prove therefore also some results on lower central series and augmentation ideals of $p$-almost direct products.

1 Introduction

Let $M$ be a surface (orientable or not, possibly with boundary) and $F_n(M) = \{(x_1, \ldots, x_n) \in M^n / x_i \neq x_j \text{ for } i \neq j\}$ its $n$th configuration space. The fundamental group $\pi_1(F_n(M))$ is called the $n$th pure braid group of $M$ and shall be denoted by $P_n(M)$.

The mapping class group $\Gamma(M)$ of $M$ is the group of isotopy classes of homeomorphisms $h : M \rightarrow M$ which are identity on the boundary. Let $X_n = \{z_1, \ldots, z_n\}$ a set of $n$ distinguished points in the interior of $M$; the pure mapping class group $P\Gamma(M, X_n)$ relatively to $X_n$ is the group of the isotopy classes of homeomorphisms $h : M \rightarrow M$ satisfying $h(z_i) = z_i$ for all $i$: since this group does not depend on the choice of the set $X_n$ but only on its cardinality we can write $P_n\Gamma(M)$ instead of $P\Gamma(M, X_n)$. Forgetting the marked points, we get a morphism $P_n\Gamma(M) \rightarrow \Gamma(M)$ whose kernel is known to be isomorphic to $P_n(M)$ when $M$ is not a sphere, a torus, a projective plane or a Klein bottle (see [Sc,GJ]).

Now, recall that if $\mathcal{P}$ is a group-theoretic property, then a group $G$ is said to be residually $\mathcal{P}$ if, for all $g \in G$, $g \neq 1$, there exists a group homomorphism $\varphi : G \rightarrow H$ such that $H$ is $\mathcal{P}$ and $\varphi(g) \neq 1$. We are interested in this paper to the following properties: nilpotence, being free and being a finite $p$-group for a prime number $p$ (mostly $p = 2$). Recall that, if for subgroups $H$ and $K$ of $G$, $[H,K]$ is the subgroup generated by $\{[h,k] / (h,k) \in H \times K\}$ where $[h,k] = h^{-1}k^{-1}hk$, the lower central series of $G$, $(\Gamma_k G)_{k \geq 1}$, is defined inductively by $\Gamma_1 G = G$ and $\Gamma_{k+1} G = [G, \Gamma_k G]$. It is well known that $G$ is residually nilpotent if, and only if, $\bigcap_{k=1}^{+\infty} \Gamma_k G = \{1\}$. From the lower central series of $G$ one can define another filtration $D_1(G) \supseteq D_2(G) \supseteq \ldots$ setting $D_1(G) = G$, and for $i \geq 2$, defining $D_i(G) = \{x \in G \mid \exists n \in \mathbb{N}^*, x^n \in \Gamma_i(G)\}$. After Garoufalidis and Levine [GLE], this filtration is called rational lower central series of $G$ and a group $G$ is residually torsion-free nilpotent if, and only if, $\bigcap_{i=1}^{+\infty} D_i(G) = \{1\}$.

When $M$ is an orientable surface of positive genus (possibly with boundary) or a disc with holes, it is proved in [BGG] and [BB] that $P_n(M)$ is residually torsion-free nilpotent for all $n \geq 1$. The fact that a
group is residually torsion-free nilpotent has several important consequences, notably that the group is bi-orderable [MR] and residually p-finite [Gr]. The goal of this article is to study the nonorientable case and, more precisely, to prove the following:

**Theorem 1** The n\textsuperscript{th} pure braid group of a nonorientable surface different from \( \mathbb{R}P^2 \) is residually 2-finite.

In the case of \( P_n(\mathbb{R}P^2) \) we give some partial results at the end of Section 4. Since a finite 2-group is nilpotent, a residually 2-finite group is residually nilpotent. Thus, we have

**Corollary 1** The n\textsuperscript{th} pure braid group of a nonorientable surface different from \( \mathbb{R}P^2 \) is residually nilpotent.

Remark that in [Go] it was shown that the n\textsuperscript{th} pure braid group of a nonorientable surface is not bi-orderable and therefore it is not residually torsion-free nilpotent. Let us notice also that if pure braid groups of nonorientable surfaces with boundary are residually p for a prime \( p \neq 2 \) therefore pure braid groups of nonorientable closed surfaces are also residually \( p \) (Remark 3); however since the technique proposed in the nonorientable case applies only for \( p = 2 \), the question if pure braid groups of nonorientable surfaces are residually \( p \) for \( p \neq 2 \) is still open (recall that there are groups residually \( p \) for infinitely many primes \( p \) which are not residually torsion-free nilpotent, see [HI]).

Remark that one can prove that finite type invariants separate classical braids using the fact that the pure braid group \( P_n \) is residually nilpotent without torsion (see [Pa]). Moreover using above residual properties it is possible to construct algebraically a universal finite type invariant over \( \mathbb{Z} \) on the classical braid group \( B_n \) (see [Pa]). Similar constructions were afterwards proposed for braids on orientable surfaces (see [BF, GP]): in a further paper we will explore the relevance of Theorem 1 in the realm of finite type invariants over \( \mathbb{Z}/2\mathbb{Z} \) for braids on non orientable surfaces.

From now on, \( M = N_{g,b} \) is a nonorientable surface of genus \( g \) with \( b \) boundary components, simply denoted by \( N_g \) when \( b = 0 \). We will see \( N_g \) as a sphere \( S^2 \) with \( g \) open discs removed and \( g \) Möbius strips glued on each circle (see figure 2 where each crossed disc represents a Möbius strip). The surface \( N_{g,b} \) is obtained from \( N_g \) by removing \( b \) open discs. The mapping class groups \( \Gamma(N_{g,b}) \) and pure mapping class group \( P_n \Gamma(N_{g,b}) \) will be denoted respectively \( \Gamma_{g,b} \) and \( \Gamma_{g,b}^n \).

The paper is organized as follows. In Section 2, we prove Theorem 1 for surfaces with boundary: following what the authors did in the orientable case (see [BGG]), we embed \( P_n(N_{g,b}) \) in a Torelli group. The difference here is that we must consider mod 2 Torelli groups. In Section 3 we introduce the notion of \( p \)-almost direct product, which generalizes the notion of almost direct product (see Definition 1) and we prove some results on lower central series and augmentations ideals of \( p \)-almost direct products (Theorems 4 and 5) that can be compared with similar results on almost direct products (Theorem 3.1 in [FR] and Theorem 3.1 in [Pa]).

In Section 4, the existence of a split exact sequence

\[
1 \longrightarrow P_{n-1}(N_{g,1}) \longrightarrow P_n(N_g) \longrightarrow \pi_1(N_g) \longrightarrow 1
\]

and results from Section 2 and 3 are used to prove Theorem 1 in the closed case (Theorem 7). The method is similar to the one developed for orientable surface in [BB]: the difference will be that the semi-direct product \( P_{n-1}(N_{g,1}) \times \pi_1(N_g) \) is a 2-almost-direct product (and not an almost-direct product as in the case of closed oriented surfaces).

For proving the main result of the paper, we will also need a group presentation for \( P_n(N_{g,b}) \) when \( b \geq 1 \). Although generators of this group seem to be known, we could not find a group presentation in the literature. Thus, we give one in the Appendix (Theorem A).
2 The case of non-empty boundary

In this section, \( N = N_{g,b} \) is a nonorientable surface of genus \( g \geq 1 \) with boundary (i.e. \( b \geq 1 \)). In this case, one has \( P_n(N) = \text{Ker}(\Gamma^n_{g,b} \rightarrow \Gamma_{g,b}) \) for all \( n \geq 1 \).

2.1 Notations

We will follow notations from [PS]. A simple closed curve in \( \mathbb{N} \) is an embedding \( \alpha : S^1 \rightarrow N \setminus \partial N \). Such a curve is said two-sided (resp. one-sided) if it admits a regular neighborhood homeomorphic to an annulus (resp. a Möbius strip). We shall consider the following elements in \( \Gamma_{g,b} \):

- If \( \alpha \) is a two-sided simple closed curve in \( N \), \( \tau_\alpha \) is a Dehn twist along \( \alpha \).
- Let \( \mu \) and \( \alpha \) be two simple closed curves such that \( \mu \) is one-sided, \( \alpha \) is two-sided and \( |\alpha \cap \mu| = 1 \). A regular neighborhood \( K \) (resp. \( M \)) of \( \alpha \cup \mu \) (resp. \( \mu \)) is diffeomorphic to a Klein bottle with one hole (resp. a Möbius strip). Pushing \( M \) once along \( \alpha \), we get a diffeomorphism of \( K \) fixing the boundary: it can be extended via the identity to \( N \). Such a diffeomorphism is called a cross-cap slide, and denoted by \( Y_{\mu,\alpha} \).
- Consider a one-sided simple closed curve \( \mu \) containing exactly one marked point \( \epsilon_i \). Sliding \( \epsilon_i \) once along \( \mu \), we get a diffeomorphism \( S_\mu \) of \( N \) which is identity outside a regular neighborhood of \( \mu \). Such a diffeomorphism will be called puncture slide along \( \mu \).

2.2 Blowup homomorphism

In this section, we recall the construction of the Blowup homomorphism \( \eta^n_{g,b} : \Gamma^n_{g,b} \rightarrow \Gamma_{g+n,b} \) given in [Sz1], [Sz2] and [PS].

Let \( U = \{ z \in \mathbb{C} : |z| \leq 1 \} \) and, for \( i = 1, \ldots, n \), fix an embedding \( e_i : U \rightarrow N \) such that \( e_i(0) = \epsilon_i \), \( e_i(U) \cap e_j(U) = \emptyset \) if \( i \neq j \) and \( e_i(U) \cap \partial N = \emptyset \) for all \( i \). If we remove the interior of each \( e_i(U) \) (thus getting the surface \( N_{g+b+n} \)) and identify, for each \( z \in \partial U \), \( e_i(z) \) with \( e_i(-z) \), we get a nonorientable surface of genus \( g+n \) with \( b \) boundary components, that is to say a surface homeomorphic to \( N_{g+n,b} \). Let us denote by \( \gamma_i = e_i(S^1) \) the boundary of \( e_i(U) \), and by \( \mu_i \) its image in \( N_{g+n,b} \): it is a one-sided simple closed curve.

Now, let \( h \) be an element of \( \Gamma^n_{g,b} \). It can be represented by a homeomorphism \( N_{g,b} \rightarrow N_{g,b} \), still denoted \( h \), such that

1. \( h(e_i(z)) = e_i(z) \) if \( h \) preserves local orientation at \( e_i(\gamma_i) \);
2. \( h(e_i(z)) = e_i(-z) \) if \( h \) reverses local orientation at \( e_i(\gamma_i) \).

Such a homeomorphism \( h \) commutes with the identification leading to \( N_{g+n,b} \) and thus induces an element \( \eta(h) \in \Gamma_{g+n,b} \). It is proved in [Sz2] that the map \( \eta^n_{g,b} = \eta : \Gamma^n_{g,b} \rightarrow \Gamma_{g+n,b} \) which sends \( h \) to \( \eta(h) \) is well defined for \( n = 1 \), but the proof also works for \( n > 1 \). This homomorphism is called blowup homomorphism.

Proposition 1 The blowup homomorphism \( \eta^n_{g,b} : \Gamma^n_{g,b} \rightarrow \Gamma_{g+n,b} \) is injective if \( (g+n,b) \neq (2,0) \).
Remark 1 This result is proved in [Sz1] for \((g, b) = (0, 1)\), but the proof can be adapted in our case as follows.

Proof. Suppose that \(h : N_{g,b} \rightarrow N_{g,b}\) is a homeomorphism satisfying \(h(z_i) = z_i\) for all \(i\) and \(\eta(h) : N_{g+n,b} \rightarrow N_{g+n,b}\) is isotopic to identity. Then \(h\) is isotopic to a map equal to identity on \(c_i(U)\) for all \(i\) (otherwise, \(\mu_i\) is isotopic to \(\mu_i^{-1}\) since \(\eta(h)(\mu_i)\) is isotopic to \(\mu_i\)) and its restriction to \(N_{g,b+n}\) is an element of the kernel of the natural map \(\Gamma_{g,b+n} \rightarrow \Gamma_{g+n,b}\) induced by gluing a Möbius strip on \(n\) boundary components. However, this kernel is generated by the Dehn twists along the curves \(\gamma_i\) (see [St, theorem 3.6]). Now, any \(\gamma_i\) bounds a disc with one marked point in \(N_{g,b}\); the corresponding Dehn twist is trivial in \(\Gamma_{g,b}\) and therefore \(h\) is isotopic to identity. \(\square\)

2.3 Embedding \(P_n(N_{g,b})\) in \(\Gamma_{g+n+2(b-1),1}\)

Gluing a one-holed torus onto \(b-1\) boundary components of \(N_{g,b}\), we get \(N_{g,b}\) as a subsurface of \(N_{g+2(b-1),1}\). This inclusion induces a homomorphism \(\lambda_{g,b}^n : \Gamma_{g,b} \rightarrow \Gamma_{g+2(b-1),1}\) which is injective (see [St]). Thus, the composed map \(\lambda_{g,b}^n \circ \eta_{g,b}^n : \Gamma_{g,b} \rightarrow \Gamma_{g+n+2(b-1),1}\) is also injective.

Recall that the mod \(p\) Torelli group \(I_p(N_{g,1})\) is the subgroup of \(\Gamma_{g,1}\) defined as the kernel of the action of \(\Gamma_{g,1}\) on \(H_1(N_{g,1}; \mathbb{Z}/p\mathbb{Z})\). In the following we will consider in particular the case of the mod 2 Torelli group \(I_2(N_{g,1})\).

Proposition 2 If \(b \geq 1\), \(\lambda_{g,b}^n(P_n(N_{g,b}))\) is a subgroup of the Torelli subgroup \(I_2(N_{g+n+2(b-1),1})\).

Proof. The image of the generators (see figures 2, 6 and theorem A) \((B_{i,j})_{1 \leq i < j \leq n}, (\rho_{k,l})_{1 \leq k \leq n, 1 \leq l \leq g}\) of \(P_n(N_{g,b})\) under \(\lambda_{g,b}^n\) are respectively (see figure 1):

- Dehn twist along curves \(\beta_{i,j}\), which bound a subsurface homeomorphic to \(N_{2,1}\);
- crosscap slides \(Y_{\mu_k,\alpha_{k,l}}\);
- the product \(\tau_{u,t} \tau_{\delta_i}^{-1}\) of Dehn twists along the bounding curves \(\xi_{u,t}\) and \(\delta_t\).
According to [Sz2], all these elements are in the mod 2 Torelli subgroup $I_2(N_{g+n+2(b-1),1})$. \hfill \qed

**Remark 2** The embedding provided in Proposition 2 does not hold for $I_p(N_{g+n+2(b-1),1})$, when $p \neq 2$: for example, the cross slide $Y_{\mu_k,\alpha_k,1}$ is not in the mod $p$ Torelli subgroup since it sends $\mu_k$ to $\mu_k^{-1}$.

**2.4 Conclusion of the proof**

We shall use the following result, which is a straightforward consequence of a similar result for mod $p$ Torelli groups of orientable surfaces due to L. Paris [P]:

**Theorem 2** Let $g \geq 1$. The mod $p$ Torelli group $I_p(N_{g,1})$ is residually $p$-finite.

**Proof.** We use Dehn-Nielsen-Baer Theorem (see for instance Theorem 5.15.3 of [CVZ]) which states that $\Gamma_{g,1}$ embeds in $\text{Aut}(\pi_1(N_{g,1}))$. Since $\pi_1(N_{g,1})$ is free we can apply Theorem 1.4 in [P] which claims that if $G$ is a free group, its mod $p$ Torelli group (i.e. the kernel of the canonical map from $\text{Aut}(G)$ to $\text{GL}(H_1(G,\mathbb{F}_p))$) is residually $p$-finite. Therefore $I_p(N_{g,1})$ is residually $p$-finite. \hfill \qed

**Theorem 3** Let $g \geq 1$, $b > 0$, $n \geq 1$. $P_n(N_{g,b})$ is residually 2-finite.

**Proof.** The group $P_n(N_{g,b})$ is a subgroup of $I_2(N_{g+n+2(b-1),1})$ by Proposition 2 and by injectivity of the map $\lambda_{g,b}^n$. Then by Theorem 2 it follows that $P_n(N_{g,b})$ is residually 2-finite. \hfill \qed

**3 $p$-almost direct products**

**3.1 On residually $p$-finite groups**

Let $p$ be a prime number and $G$ a group. If $H$ is a subgroup of $G$, we denote by $H^p$ the subgroup generated by $\{h^p \mid h \in H\}$. Following [P], we define the lower $\mathbb{F}_p$-linear central filtration $(\gamma_n^p G)_{n \in \mathbb{N}}$ of $G$ by $\gamma_1^p G = G$ and, for $n \geq 1$, $\gamma_{n+1}^p G$ is the subgroup of $G$ generated by $[G, \gamma_n^p G] \cup (\gamma_n^p G)^p$. Note that the subgroups $\gamma_n^p G$ are characteristic in $G$ and that the quotient group $G/\gamma_n^p G$ is nothing but the first homology group $H_1(G;\mathbb{F}_p)$. The followings are proved in [P]:

- for $m, n \geq 1$, $[\gamma_n^p G, \gamma_m^p G] \subset \gamma_{m+n}^p G$;
- a finitely generated group $G$ is a finite $p$-group if, and only if, there exists some $N \geq 1$ such that $\gamma_N^p G = \{1\}$;
- a finitely generated group $G$ is residually $p$-finite if, and only if, $\bigcap_{n=1}^{+\infty} \gamma_n^p G = \{1\}$;

and clearly, if $f : G \rightarrow G'$ is a group homomorphism, then $f(\gamma_n^p G) \subset \gamma_n^p G'$ for all $n \geq 1$.

**Definition 1** Let $1 \longrightarrow A \longrightarrow B \overset{\lambda}{\longrightarrow} C \longrightarrow 1$ be a split exact sequence.

- If the action of $C$ induced on $H_1(A;\mathbb{Z})$ is trivial (i.e. the action is trivial on $A^{Ab} = A/[A,A]$), we say that $B$ is a almost direct product of $A$ and $C$.
- If the action of $C$ induced on $H_1(A;\mathbb{F}_p)$ is trivial (i.e. the action is trivial on $A/\gamma_2^p A$), we say that $B$ is a $p$-almost direct product of $A$ and $C$. 

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Let us remark that, as in the case of almost direct products (Proposition 6.3 of [BGoGu]), the fact to be a $p$-almost direct product does not depend on the choice of the section.

**Proposition 3** Let $1 \rightarrow A \xrightarrow{\Lambda} B \xrightarrow{\Lambda} C \xrightarrow{1}$ be a split exact sequence of groups. Let $\sigma, \sigma'$ be sections for $\lambda$, and suppose that the induced action of $C$ on $A$ via $\sigma$ on $H_1(A; \mathbb{F}_p)$ is trivial. Then the same is true for the section $\sigma'$.

**Proof.** Let $a \in A$ and $c \in C$. By hypothesis, $\sigma(c)\Lambda(\sigma(c))^{-1} \equiv c \mod \gamma^p_2A$. Let $\sigma'$ be another section for $\lambda$. Then $\lambda \circ \sigma'(c) = \lambda \circ \sigma(c)$, and so $\sigma'(c)(\sigma(c))^{-1} \in \text{Ker}(\lambda)$. Thus there exists $a' \in A$ such that $\sigma'(c) = a' \sigma(c)$, and hence

$$\sigma'(c)\Lambda(\sigma(c))^{-1} \equiv a' \sigma(c)\Lambda(\sigma(c))^{-1} a' = a \equiv a \mod \gamma^p_2A.$$ 

Thus the induced action of $C$ on $H_1(A; \mathbb{F}_p)$ via $\sigma'$ is also trivial. \qed

The first goal of this section is to prove the following Theorem (see Theorem 3.1 in [FR] for an analogous result for almost direct products).

**Theorem 4** Let $1 \rightarrow A \xrightarrow{\Lambda} B \xrightarrow{\Lambda} C \xrightarrow{1}$ be a split exact sequence where $B$ is a $p$-almost direct product of $A$ and $C$. Then, for all $n \geq 1$, one has a split exact sequence

$$1 \rightarrow \gamma^p_nA \xrightarrow{\lambda_n} \gamma^p_nB \xrightarrow{\Lambda} \gamma^p_nC \xrightarrow{1}$$

where $\lambda_n$ and $\sigma_n$ are restrictions of $\lambda$ and $\sigma$.

We shall need the following preliminary result.

**Lemma 1** Under the hypotheses of Theorem 4, one has, for all $m, n \geq 1$

$$[\gamma^p_m C', \gamma^p_n A] \subset \gamma^p_{m+n} A$$

where $C'$ denotes $\sigma(C)$.

**Proof.** First, we prove by induction on $n$ that $[C', \gamma^p_n A] \subset \gamma^p_{n+1} A$ for all $n \geq 1$. The case $n = 1$ corresponds to the hypotheses: the action of $C$ on $H_1(A; \mathbb{F}_p) = A/\gamma^p_2 A$ is trivial if, and only if, $[C', A] \subset \gamma^p_2 A$. Thus, suppose that $[C', \gamma^p_n A] \subset \gamma^p_{n+1} A$ for some $n \geq 1$ and let us prove that $[C', \gamma^p_{n+1} A] \subset \gamma^p_{n+2} A$. In view of the definition of $\gamma^p_{n+1} A$, we have to prove that $[C', A, \gamma^p_n A] \subset \gamma^p_{n+2} A$ and $[C', (\gamma^p_n A)^p] \subset \gamma^p_{n+2} A$. For the first case, we use a classical result (see [MKS], theorem 5.2) which says

$$[C', A, \gamma^p_n A] = [\gamma^p_n A, [C', A]] [A, [\gamma^p_n A, C']]$$

We have just seen that $[C', A] \subset \gamma^p_2 A$ thus $[\gamma^p_n A, [C', A]] \subset [\gamma^p_n A, \gamma^p_2 A] \subset \gamma^p_{n+2} A$. Then, the induction hypotheses says that $[\gamma^p_n A, C'] \subset \gamma^p_{n+1} A$ thus $[A, [\gamma^p_n A, C']] \subset [A, \gamma^p_{n+1} A] \subset \gamma^p_{n+2} A$. The second case works as follows: for $c \in C'$ and $x \in \gamma^p_n A$, one has, using the fact that $[u, uv] = [u, w][u, v][u, v]$ (see [MKS])

$$[c, x^p] = [c, x][c, x^{p-1}][c, x^{p-1}][c, x^{p-1}][c, x^{p-1}][c, x^{p-1}][c, x^{p-1}] \cdots [c, x] \cdot [c, x^{p-1}][c, x][c, x^2][c, x][c, x^3][c, x] \cdots [c, x^{p-1}][c, x].$$

Since $c \in C'$ and $x \in \gamma^p_n A$, one has $[c, x^p] \in [C', \gamma^p_n A] \subset \gamma^p_{n+1} A$ for all $i$, $1 \leq i \leq p - 1$, which leads to $[c, x^p] \in (\gamma^p_{n+1} A)^p \subset \gamma^p_{n+2} A$ and $[[c, x^i], x] \in [\gamma^p_{n+1} A, A] \subset \gamma^p_{n+2} A$. 

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Now, we suppose that $[\gamma_p^m C', \gamma_p^m A] \subset \gamma_p^{m+n} A$ for some $m \geq 1$ and all $n \geq 1$ and prove that $[\gamma_p^{m+1} C', \gamma_p^m A] \subset \gamma_p^{m+n+1} A$. As above, there are two cases which work on the same way:

(i) \([C', \gamma_p^m C', \gamma_p^m A] \subset [\gamma_p^m A, C'] \subset [\gamma_p^{m+1} A, \gamma_p^m C'] \subset \gamma_p^{m+n+1} A\).

(ii) For $c \in \gamma_p^m C'$ and $x \in \gamma_p^m A$, one has

$$[c^p, x] = [c, [x, c^{p-1}]] = [c, c^{p-1}][c, c^{p-1}].$$

which is an element of $\gamma_p^{m+n+1} A$ by induction hypotheses. 

\[\square\]

**Proof of Theorem 4.** Restrictions of $\lambda$ and $\sigma$ give rise to morphisms $\lambda_n : \gamma^n_p B \rightarrow \gamma^n_p C$ and $\sigma_n : \gamma^n_p C \rightarrow \gamma^n_p B$ satisfying $\lambda_n \circ \sigma_n = \text{Id}_{\gamma_n^p C}$, $\sigma_n$ is onto and $\text{Ker}(\lambda_n) = A \cap \gamma^n_p B$. Thus, we need to prove that $A \cap \gamma^n_p B = \gamma^n_p A$ for all $n \geq 1$. Clearly one has $\gamma^n_p A \subset A \cap \gamma^n_p B$. In order to prove the converse inclusion, we follow the method developed in [FR] for almost semi-direct product and define $\tau : B \rightarrow B$ by $\tau(b) = (\sigma \lambda(b))^{-1} b$. This map has the following properties:

(i) since $\lambda \sigma = \text{Id} c$, $\tau(B) \subset A$;

(ii) for $x \in B$, $\tau(x) = x$ if, and only if, $x \in A$;

(iii) for $(b_1, b_2) \in B^2$, $\tau(b_1 b_2) = [\sigma \lambda(b_1), \tau(b_1)^{-1}] \tau(b_1) \tau(b_2)$;

(iv) for $b \in B$, setting $a = \tau(b)$ and $c = \sigma \lambda(b)$, we get $b = ca$ with $c \in C' = \sigma (C)$ and $a \in A$, this decomposition being unique.

We claim that $\tau(\gamma^n_p B) \subset \gamma^n_p A$ for all $n \geq 1$. From this, we conclude easily the proof: if $x \in A \cap \gamma^n_p B$, then $x = \tau(x) \in \gamma^n_p A$.

One has $\tau(\gamma^n_1 B) \subset \gamma^n_1 A$. Suppose inductively that $\tau(\gamma^n_m B) \subset \gamma^n_p A$ for some $n \geq 1$ and let us prove that $\tau(\gamma^n_{m+1} B) \subset \gamma^n_{m+1} A$. Suppose first that $x$ is an element of $\gamma^n_p B$. Then using $(iii)$ we get:

$$\tau(x^p) = [\sigma \lambda(x), \tau(x^{p-1})^{-1}] \tau(x^{p-1}) \tau(x)$$

$$\vdots$$

$$= [\sigma \lambda(x), \tau(x^{p-1})^{-1}] [\sigma \lambda(x), \tau(x^{p-2})^{-1}] \cdots [\sigma \lambda(x), \tau(x)^{-1}] \tau(x)^p.$$
We sketch the proof which is almost verbatim the same as the proof of Theorem 3.1 in [Pa]. Let $R$ we get that $3.2$ Augmentation ideals

corollary 2 Let $1 \rightarrow A' \rightarrow B \xrightarrow{\lambda} C \rightarrow 1$ be a split exact sequence such that $B$ is a $p$-almost direct product of $A$ and $C$. If $A$ and $C$ are residually $p$-finite, then $B$ is residually $p$-finite.

3.2 Augmentation ideals

Given a group $G$ and $\mathbb{K} = \mathbb{Z}$ or $\mathbb{F}_2$ we will denote by $\mathbb{K}[G]$ the group ring of $G$ over $\mathbb{K}$ and by $\mathbb{K}[G]$ the augmentation ideal of $G$. The group ring $\mathbb{K}[G]$ is filtered by the powers $\mathbb{K}[G]^j$ of $\mathbb{K}[G]$ and we can define the associated graded algebra $gr(\mathbb{K}[G]) = \oplus \mathbb{K}[G]^j / \mathbb{K}[G]^{j+1}$.

The following theorem provides a decomposition formula for the augmentation ideal of a $2$-almost direct product (see Theorem 3.1 in [Pa] for an analogous in the case of almost direct products).

Let $A \ltimes C$ be a semi-direct product between two groups $A$ and $C$. It is a classical result that the map $a \otimes c \mapsto ac$ induces a $\mathbb{K}$-isomorphism from $\mathbb{K}[A] \otimes \mathbb{K}[C]$ to $\mathbb{K}[A \ltimes C]$. Identifying these two $\mathbb{K}$-modules, we have the following:

**Theorem 5** If $A \ltimes C$ is a $2$-almost direct product then :

$$\mathbb{F}_2[A \ltimes C] = \mathbb{F}_2[A] \mathbb{F}_2[C]^h$$ for all $k$.

**Proof.** We sketch the proof which is almost verbatim the same as the proof of Theorem 3.1 in [Pa]. Let $R_k = \sum_{i+h=k} \mathbb{F}_2[A]^i \mathbb{F}_2[C]^h$; $R_k$ is a descending filtration on $\mathbb{F}_2[A] \mathbb{F}_2[C]$, and with the above identification, we get that $R_k \subset \mathbb{F}_2[A \ltimes C]^k$. To verify the other inclusion we have to check that $\prod_{j=1}^k (a_j c_j - 1) \in R_k$ for every $a_1, \ldots, a_k$ in $A$ and $c_1, \ldots, c_k$ in $C$. Actually it is enough to verify that $e = \prod_{j=1}^k (e_j - 1) \in R_k$ either $e_j \in A$ or $e_j \in C$ (see Theorem 3.1 in [Pa] for a proof of this fact): we call $e$ a special element. We associate to a special element $e$ an element in $\{0, 1\}^k$: let $\text{type}(e) = (\delta(e_1), \ldots, \delta(e_k))$ where $\delta(e_j) = 0$ if $e_j \in A$ and $\delta(e_j) = 1$ if $e_j \in C$. We will say that the special element $e$ is standard if $\text{type}(e) = (0, \ldots, 0, 1, \ldots, 1)$.

In this case $e \in \mathbb{F}_2[A]^i \mathbb{F}_2[C]^h \subset R_k$ and we are done. We claim that we can reduce all special elements to linear combinations of standard elements. If $e$ is not standard, then it must be of the form

$$e = \prod_{i=1}^r (a_i - 1) \prod_{j=1}^s (c_i - 1)(c - 1) \prod_{l=1}^t (e_i - 1)$$

Thus we have $e \in R_k$ for some $k$.
where \(a_1, \ldots, a_r, a \in A, c_1, \ldots, c_s, c \in A, \hat{c} = \prod_{i=1}^t (e_i - 1)\) is special and \(r + s + t + 2 = k\). Therefore

\[
\text{type}(e) = \left(0, \ldots, 0, 1, \ldots, 1, 0, \delta(e_1), \ldots, \delta(e_t)\right).
\]

Now we can use the assumption that \(A \times C\) is a 2–almost direct product to claim that one has commutation relations in \(\mathbb{Z}[A \times C]\) expressing the difference \((c - 1)(a - 1) - (a - 1)(c - 1)\) as a linear combination of terms of the form

\[
(a' - 1)(a'' - 1)c \quad \text{with } a', a'' \in A
\]

for any \(a \in A\) and \(c \in C\). In fact,

\[
(c - 1)(a - 1) - (a - 1)(c - 1) = ca - ac = (cac^{-1}a^{-1} - 1)ac = (f - 1)ac
\]

where \(f = [c^{-1}, a^{-1}] \in [C, A] \subset \gamma^2_2(A)\) by lemma 1. We can decompose \(f\) as \(f = h_1k_1 \cdots h_mk_m\) where, for \(j = 1, \ldots, m, h_j\) belongs to \([A, A]\) and \(k_j = (k_j')^2\) for some \(k_j' \in A\). One knows (see for instance [Ch] p. 194) that for \(j = 1, \ldots, m\) \((h_j - 1)\) is a linear combination of terms of the form

\[
(h_j' - 1)(h_j'' - 1)\alpha_j \quad \text{with } h_j', h_j'', \alpha_j \in A.
\]

On the other hand for \(j = 1, \ldots, m\) we have also that

\[
(k_j - 1) = (k_j' - 1)(k_j'' - 1) \quad \text{with } k_j' \in A \quad \text{since the coefficients are } \mathbb{F}_2.
\]

Then, recalling that \((hk - 1) = (h - 1)k + (k - 1)\) for any \(h, k \in A\), we can conclude that \(f - 1\) can be rewritten as a linear combination of terms of the form

\[
(f' - 1)(f'' - 1)\alpha \quad \text{with } f', f'', \alpha \in A
\]

and that \((c - 1)(a - 1) - (a - 1)(c - 1)\) is a linear combination of terms of the form

\[
(f' - 1)(f'' - 1)ac \quad \text{with } f', f'', \alpha \in A.
\]

Rewriting \((f'' - 1)\alpha\) as \((f''\alpha - 1) - (\alpha - 1)\) we obtain that the difference \((c - 1)(a - 1) - (a - 1)(c - 1)\) can be seen as a linear combination of terms of the form

\[
(a' - 1)(a'' - 1)c \quad \text{with } a', a'' \in A.
\]

Therefore \(e\) can be rewritten as a sum whose first term is the special element

\[
e' = \prod_{i=1}^r (a_i - 1) \prod_{j=1}^s (c_j - 1)(a - 1)(c - 1) \prod_{l=1}^t (e_l - 1)
\]

and whose second term is a linear combination of elements of the form \(e''c\) where

\[
e'' = \prod_{i=1}^r (a_i - 1) \prod_{j=1}^s (c_j - 1)(a' - 1)(a'' - 1) \prod_{l=1}^t (c_e c^{-1} - 1)c.
\]

Using the lexicographic order from the left, one has \(\text{type}(e) > \text{type}(e')\) and \(\text{type}(e) > \text{type}(e'')\).

By induction on the lexicographic order we infer that \(e'\) and \(e''\) belong to \(R_k\): since \(R_k \cdot c \subset R_k\) for any \(c \in C\) it follows that \(e\) belongs to \(R_k\) and we are done. \(\square\)
4 The closed case

4.1 A presentation of $P_n(N_g)$ and induced identities

We recall a group presentation of $P_n(N_g)$ given in [GG3]: the geometric interpretation of generators is provided in Figure 2.

**Theorem 6 ([GG3])** For $g \geq 2$ and $n \geq 1$, $P_n(N_g)$ has the following presentation:

**generators:** $(B_{i,j})_{1 \leq i < j \leq n}$ and $(\rho_{k,l})_{1 \leq k \leq n}$.

**relations:**

(a) for all $1 \leq i < j \leq n$ and $1 \leq r < s \leq n$,

\[
B_{r,s}B_{i,j}B_{r,s}^{-1} = \begin{cases} 
B_{i,j} & \text{if } i < r < s < j \text{ or } r < s < i < j \\
B_{i,j}^{-1}B_{r,s}^{-1}B_{i,j}B_{r,s} & \text{if } r < i = s < j \\
B_{i,j}^{-1}B_{r,s}^{-1}B_{i,j} & \text{if } i = r < s < j \\
B_{i,j}^{-1}B_{r,s}^{-1}B_{i,j}B_{r,s}^{-1}B_{r,s} & \text{if } r < i < s < j 
\end{cases} \]

(b) for all $1 \leq i < j \leq n$ and $1 \leq k, l \leq g$,

\[
\rho_{i,k}\rho_{j,l}\rho_{i,k}^{-1} = \begin{cases} 
\rho_{i,k} & \text{if } k < l \\
\rho_{j,l}^{-1}\rho_{i,k} & \text{if } k = l \\
\rho_{i,k}^{-1}B_{i,j}^{-1}\rho_{j,l}B_{i,j}^{-1}\rho_{i,k}^{-1} & \text{if } k > l 
\end{cases} \]

(c) for all $1 \leq i < n$, $\rho_{i,1}^2 \cdots \rho_{i,g}^2 = T_i$ where $T_i = B_{1,i} \cdots B_{i-1,i}B_{i,i+1} \cdots B_{i,n}$

(d) for all $1 \leq i < j < n$, $1 \leq k \leq n$, $k \neq j$ and $1 \leq l \leq g$,

\[
\rho_{k,i}B_{i,j}\rho_{k,l}^{-1} = \begin{cases} 
B_{i,j} & \text{if } i < k \text{ or } j < k \\
\rho_{i,k}^{-1}B_{i,j}^{-1}\rho_{j,l} & \text{if } k = i \\
\rho_{i,k}^{-1}B_{i,j}^{-1}\rho_{j,l}B_{i,j}^{-1}\rho_{i,k}^{-1}B_{k,j}^{-1}B_{k,j} & \text{if } i < k < j 
\end{cases} \]

Figure 2: Generators of $P_n(N_g)$

For $1 \leq k \leq g$, let us consider the element $a_k$ in $P_n(N_g)$ given by $a_k = \rho_{k,g-1}\rho_{k,g}$ and set $U = a_n \cdots a_2$.

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**generators:** $(B_{i,j})_{1 \leq i < j \leq n}$ and $(\rho_{k,l})_{1 \leq k \leq n}$.

**relations:**

(a) for all $1 \leq i < j \leq n$ and $1 \leq r < s \leq n$,

\[
B_{r,s}B_{i,j}B_{r,s}^{-1} = \begin{cases} 
B_{i,j} & \text{if } i < r < s < j \text{ or } r < s < i < j \\
B_{i,j}^{-1}B_{r,s}^{-1}B_{i,j}B_{r,s} & \text{if } r < i = s < j \\
B_{i,j}^{-1}B_{r,s}^{-1}B_{i,j} & \text{if } i = r < s < j \\
B_{i,j}^{-1}B_{r,s}^{-1}B_{i,j}B_{r,s}^{-1}B_{r,s} & \text{if } r < i < s < j 
\end{cases} \]

(b) for all $1 \leq i < j \leq n$ and $1 \leq k, l \leq g$,

\[
\rho_{i,k}\rho_{j,l}\rho_{i,k}^{-1} = \begin{cases} 
\rho_{i,k} & \text{if } k < l \\
\rho_{j,l}^{-1}\rho_{i,k} & \text{if } k = l \\
\rho_{i,k}^{-1}B_{i,j}^{-1}\rho_{j,l}B_{i,j}^{-1}\rho_{i,k}^{-1} & \text{if } k > l 
\end{cases} \]

(c) for all $1 \leq i < n$, $\rho_{i,1}^2 \cdots \rho_{i,g}^2 = T_i$ where $T_i = B_{1,i} \cdots B_{i-1,i}B_{i,i+1} \cdots B_{i,n}$

(d) for all $1 \leq i < j < n$, $1 \leq k \leq n$, $k \neq j$ and $1 \leq l \leq g$,

\[
\rho_{k,i}B_{i,j}\rho_{k,l}^{-1} = \begin{cases} 
B_{i,j} & \text{if } i < k \text{ or } j < k \\
\rho_{i,k}^{-1}B_{i,j}^{-1}\rho_{j,l} & \text{if } k = i \\
\rho_{i,k}^{-1}B_{i,j}^{-1}\rho_{j,l}B_{i,j}^{-1}\rho_{i,k}^{-1}B_{k,j}^{-1}B_{k,j} & \text{if } i < k < j 
\end{cases} \]

Figure 2: Generators of $P_n(N_g)$

For $1 \leq k \leq g$, let us consider the element $a_k$ in $P_n(N_g)$ given by $a_k = \rho_{k,g-1}\rho_{k,g}$ and set $U = a_n \cdots a_2$.

**Lemma 2** The following relations holds in $P_n(N_g)$:
```
Proof. Some of these identities can easily be verified drawing corresponding braids. This is the case for example for the first, the fourth and the eighth ones (see figure 3, 4 and 5). Let us give an algebraic proof for the others.
(3) By relation (b₁), $$\rho_{1,g}^{-1}$$ commutes with $$\rho_{j,g}$$ for $$2 \leq j \leq n$$. Thus, using relation (e), we get:

$$\begin{align*}
\rho_{1,g}^{-1} U \rho_{1,g}^{-1} &= \rho_{1,g}^{-1} a_{n} \cdots a_{2} \rho_{1,g}^{-1} \\
&= \rho_{1,g}^{-1} (\rho_{n,g-1} \rho_{n,g}) \cdots (\rho_{2,g-1} \rho_{2,g}) \rho_{1,g}^{-1} \\
&= (B_{1,n}^{-1} \rho_{n,g-1} \rho_{n,g}) \cdots (B_{1,2}^{-1} \rho_{2,g-1} \rho_{2,g}) \\
&= B_{1,n}^{-1} B_{1,n-1}^{-1} \cdots B_{1,2}^{-1} (\rho_{n,g-1} \rho_{n,g}) \cdots (\rho_{2,g-1} \rho_{2,g}) \text{ by (d₁)} \\
&= T_{1}^{-1} U.
\end{align*}$$

(5) Let $$j$$ and $$k$$ be integers such that $$1 \leq j < k \leq n$$. By (d₁), $$a_{1}, \ldots, a_{j-1}$$ commute with $$B_{j,k}$$. Then, one has

$$\begin{align*}
a_{j} B_{j,k} a_{j}^{-1} &= a_{j} B_{j,k} a_{j}^{-1} \\
&= (\rho_{j,g-1} \rho_{j,g} B_{j,k} \rho_{j,g} \rho_{j,g}^{-1} B_{j,k} \rho_{j,g} \rho_{j,g}^{-1}) \text{ by (d₂)} \\
&= \rho_{j,g-1} \rho_{k,g}^{-1} B_{j,k} \rho_{k,g} \rho_{j,g}^{-1} \text{ by (b₁)} \\
&= \rho_{k,g}^{-1} (\rho_{k,g-1} B_{j,k} \rho_{k,g}^{-1} \rho_{k,g}) \text{ by (d₂)} \\
&= a_{k}^{-1} B_{j,k} a_{k}
\end{align*}$$

and we get

$$\begin{align*}
a_{n} \cdots a_{1} B_{j,k} a_{j}^{-1} \cdots a_{1}^{-1} &= a_{n} \cdots a_{k+1} a_{k} a_{k-1} \cdots a_{j+1} a_{j}^{-1} B_{j,k} a_{k} a_{k-1}^{-1} \cdots a_{1}^{-1} \cdots a_{k+1}^{-1} \cdots a_{n}^{-1} \\
&= a_{n} \cdots a_{k+1} B_{j,k} a_{k}^{-1} \cdots a_{1}^{-1} \text{ by (g)} \\
&= B_{j,k} \text{ by (d₁)}.
\end{align*}$$

(6) By (d₁), $$a_{1} = \rho_{1,g-1} \rho_{1,g}$$ commutes with $$B_{i,j}$$ for $$2 \leq i < j \leq n$$. Thus, relation (i) is a direct consequence of (h).

(7) A direct consequence of (h) since $$T_{1} = B_{1,2} \cdots B_{1,n}$$.

$$\square$$

4.2 The pure braid group $$P_{n}(N_{g})$$ is residually 2-finite

Following [GG1], one has, for $$g \geq 2$$, a split exact sequence

$$\begin{align*}
1 \longrightarrow P_{n-1}(N_{g,1}) \overset{\mu}{\longrightarrow} P_{n}(N_{g}) \overset{\lambda}{\longrightarrow} P_{1}(N_{g}) = \pi_{1}(N_{g}) \longrightarrow 1
\end{align*}$$

(1)
where $\lambda$ is induced by the map which forgets all strands except the first one, and $\mu$ is defined by capping the boundary component by a disc with one marked point (the first strand in $P_n(N_g)$). According to the definition of $\mu$ and to Theorem A, $\text{Im}(\mu)$ is generated by $\{\rho_{i,k}, \ 2 \leq i \leq n, \ 1 \leq k \leq g\} \cup \{B_{i,j}, \ 2 \leq i < j \leq n\}$.

The section given in [GG1] is geometric, i.e. it is induced by a crossed section at the level of fibrations. In order to study the action of $\pi_1(N_g)$ on $P_{n-1}(N_{g,1})$, we need an algebraic one. Recall that $\pi_1(N_g)$ has a group presentation with generators $p_1, \ldots, p_g$ and the single relation $p_1^2 \cdots p_g^2 = 1$. We define the set map $\sigma : \pi_1(N_g) \rightarrow P_n(N_g)$ by setting

$$
\sigma(p_i) = \begin{cases} 
\rho_{1,i} & \text{for } 1 \leq i \leq g - 3, \\
\rho_{1,g-2}^{-1}U^{-1} & \text{for } i = g - 2, \\
U \rho_{1,g-1} & \text{for } i = g - 1, \\
\rho_{1,g}T_1^{-1} & \text{for } i = g.
\end{cases}
$$

**Proposition 4** The map $\sigma$ is a well defined homomorphism satisfying $\lambda \circ \sigma = \text{Id}_{\pi_1(N_g)}$.

**Proof.** Since $\lambda(p_{1,i}) = p_i$ for all $1 \leq i \leq g$ and $\lambda(U) = \lambda(T_1) = 1$, one has clearly $\lambda \sigma = \text{Id}_{\pi_1(N_g)}$ if $\sigma$ is a group homomorphism. Thus, we have just to prove that $\sigma(p_1)^2 \cdots \sigma(p_g)^2 = 1$:

$$
\sigma(p_1)^2 \cdots \sigma(p_g)^2 = (\rho_{1,1}^2 \cdots \rho_{1,g-3}^2)(\rho_{1,g-2}^{-2}U^{-1})^2(\rho_{1,g-1}^2)^2(\rho_{1,g}T_1^{-1})^2
$$

$\lambda \sigma(1) = \lambda(1) = 1$ by (c).

So, the exact sequence (1) splits. In order to apply Theorem 4, we have to prove that the action of $\pi_1(N_g)$ on $P_{n-1}(N_{g,1})$ is trivial on $H_1(P_{n-1}(N_{g,1}); F_2)$. This is the claim of the following proposition.

**Proposition 5** For all $x \in \text{Im}(\sigma)$ and $a \in \text{Im}(\mu)$, one has $[x^{-1}, a^{-1}] = xax^{-1}a^{-1} \in \gamma_2^2(\text{Im}(\mu))$.

**Proof.** It is enough to prove the result for $a \in \{B_{j,k}, \ 2 \leq j < k \leq n\} \cup \{B_{j,l}, \ 2 \leq j \leq n \text{ and } 1 \leq l \leq g\}$ and $x \in \{\sigma(p_1), \ldots, \sigma(p_g)\}$, respectively sets of generators of $\text{Im}(\mu)$ and $\text{Im}(\sigma)$. Suppose first that $2 \leq j < k \leq n$. One has:

- $[\sigma(p_1)^{-1}, B_{j,k}^{-1}] = [\rho_{1,1}^{-1}, B_{j,k}^{-1}] = 1$ for $1 \leq i \leq g - 3$ by (d1);
- $[\sigma(p_{g-2})^{-1}, B_{j,k}^{-1}] = [U \rho_{1,g-2}^{-1}, B_{j,k}^{-1}] = 1$ by (d1) and (i);
- $[\sigma(p_{g-1})^{-1}, B_{j,k}^{-1}] = [\rho_{1,g-1}^{-1}U^{-1}, B_{j,k}^{-1}] = 1$ by (d1) and (i);
- $[\sigma(p_g)^{-1}, B_{j,k}^{-1}] = [T_1 \rho_{1,g}^{-1}, B_{j,k}^{-1}] = 1$ by (d1) and (k).

Now, let $j$ and $l$ be integers such that $2 \leq j \leq n$ and $1 \leq l \leq g$ and let us first prove that $[\rho_{1,i}^{-1}, \rho_{j,l}^{-1}] \in \gamma_2^2(\text{Im}(\mu))$ for all $i$, $1 \leq i \leq n$:
• this is clear for $i < l$ by (b); 
• for $i = l$, the relation (b2) gives $[\rho_{1,l}^{-1}, \rho_{j,l}^{-1}] = \rho_{j,l}^{-1} B_{i,j}^{-1} \rho_{j,l}$. But 
\[ B_{i,j}^{-1} = B_{2,j} \cdots B_{j-1,j} B_{j+1,j} \cdots B_{n,j} \rho_{j,g}^{-2} \cdots \rho_{j,l}^{-2} \] (relation (c)) 
is an element of $\gamma_2^2(\text{Im}(\mu))$ by (e), thus we get $[\rho_{1,l}^{-1}, \rho_{j,l}^{-1}] \in \gamma_2^2(\text{Im}(\mu))$.

• If $l < i$ then $[\rho_{1,i}^{-1}, \rho_{j,i}^{-1}] = [B_{1,j} \rho_{j,i}^{-1} B_{1,j} \rho_{j,i}, \rho_{j,i}^{-1}]$ by (b3) so $[\rho_{1,i}^{-1}, \rho_{j,i}^{-1}] \in \gamma_2(\text{Im}(\mu))$ since $\rho_{j,i}, \rho_{j,i}$ and $B_{1,j}$ are elements of $\text{Im}(\mu)$.

From this, we deduce the following facts.

1. $[\sigma(p_l)^{-1}, \rho_{j,l}^{-1}] \in \gamma_2^2(\text{Im}(\mu))$ for $i \leq g - 3$ since $\sigma(p_l) = \rho_{1,i}$.

2. $[\sigma(p_{g-2})^{-1}, \rho_{j,l}^{-1}] = [U \rho_{1,g-2}^{-1}, \rho_{j,l}^{-1}] = \rho_{1,g-2} [U, \rho_{j,k}^{-1}] \rho_{1,g-2}^{-1} \rho_{j,k}^{-1}$. But $U$ and $\rho_{j,k}^{-1}$ are elements of $\text{Im}(\mu)$ thus $[U, \rho_{j,k}^{-1}] \in T_2(\text{Im}(\mu)) \subset \gamma_2^2(\text{Im}(\mu))$. Consequently, $\rho_{1,g-2} [U, \rho_{j,k}^{-1}] \rho_{1,g-2}^{-1} \in \gamma_2^2(\text{Im}(\mu))$ since $\gamma_2^2(\text{Im}(\mu))$ is a characteristic subgroup of $\text{Im}(\mu)$ and $\text{Im}(\mu)$ is normal in $P_n(N_g)$. Thus, we get $[\sigma(p_{g-2})^{-1}, \rho_{j,l}^{-1}] \in \gamma_2^2(\text{Im}(\mu))$. In the same way, one has 
\[ [\sigma(p_{g-1})^{-1}, \rho_{j,l}^{-1}] = [\rho_{1,g-1}^{-1} U^{-1}, \rho_{j,l}^{-1}] = U [\rho_{1,g-1}^{-1}, \rho_{j,l}^{-1}] U^{-1} [U, \rho_{j,l}^{-1}] \in \gamma_2^2(\text{Im}(\mu)). \]

4. At last,
\[ [\sigma(p_g)^{-1}, \rho_{j,l}^{-1}] = [T_1 \rho_{1,g}^{-1}, \rho_{j,l}^{-1}] = \rho_{1,g} [T_1, \rho_{j,l}^{-1}] \rho_{1,g}^{-1} \rho_{j,l}^{-1} \in \gamma_2^2(\text{Im}(\mu)) \]
since $T_1, \rho_{j,l} \in \text{Im}(\mu)$ and $[\rho_{1,g}, \rho_{j,l}^{-1}] \in \gamma_2^2(\text{Im}(\mu))$.

We are now ready to prove the main result of this section.

**Theorem 7** For all $g \geq 2$ and $n \geq 1$, the pure braid group $P_n(N_g)$ is residually 2-finite.

**Proof.** Proposition 4 says that the sequence

\[ 1 \longrightarrow P_{n-1}(N_{g,1}) \longrightarrow P_n(N_g) \longrightarrow \pi_1(N_g) \longrightarrow 1 \]
splits. Now $P_{n-1}(N_{g,1})$ is residually 2-finite (Theorem 3). It is proved in [B1] and [B2] that $\pi_1(N_g)$ is residually free for $g \geq 4$, so it is residually 2-finite. This result is proved in [LM] (lemma 8.9) for $g = 3$. When $g = 2$, $\pi_1(N_2)$ has presentation $(a, b | aba^{-1} = b^{-1})$ so is a 2-almost direct product of $\mathbb{Z}$ by $\mathbb{Z}$. Since $\mathbb{Z}$ is residually 2-finite, $\pi_1(N_2)$ is residually 2-finite by corollary 2. So, using Proposition 5 and Corollary 2, we can conclude that $P_n(N_g)$ is residually 2-finite.

**Remark 3** It follows from the proof of Theorem 7 that, when $g \geq 2$, if $P_n(N_{g,1})$ is residually $p$-finite for some $p \neq 2$ then the pure braid group $P_n(N_g)$ is also residually $p$-finite.
4.3 The case $P_n(\mathbb{RP}^2)$

The main reason to exclude $N_1 = \mathbb{RP}^2$ in Theorem 7 is that the exact sequence (1) doesn’t exist in this case, but forgetting at most $n - 2$ strands we get the following exact sequence ($1 \leq m \leq n - 2$; see [VB])

$$1 \to P_{m}(N_{1,n-m}) \to P_n(\mathbb{RP}^2) \to P_{n-m}(N_g) \to 1. $$

This sequence splits if, and only if $n = 3$ and $m = 1$ (see [GG2]). Thus, what we know is the following:

- $P_1(\mathbb{RP}^2) = \pi_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$: $P_1(\mathbb{RP}^2)$ is a 2-group.
- $P_2(\mathbb{RP}^2) = Q_8$, the quaternion group (see [VB]): $P_2(\mathbb{RP}^2)$ is a 2-group.
- One has the exact sequence

$$1 \to P_1(N_{1,2}) \to P_3(\mathbb{RP}^2) \to P_2(\mathbb{RP}^2) \to 1$$

where $P_1(N_{1,2}) = \pi_1(N_{1,2})$ is a free group of rank 2, thus is residually 2-finite. Since $P_2(\mathbb{RP}^2)$ is 2-finite, we can conclude that $P_3(\mathbb{RP}^2)$ is residually 2-finite using lemma 1.5 of [Gr].

Appendix: a group presentation for $P_n(N_{g,b})$

Here we apply classical method (see [B, GG3]) to give a presentation of the $n^{th}$ pure braid group of a nonorientable surface with boundary. Since $b \geq 1$, we’ll see $N_{g,b}$ as a disc $D^2$ with $g + b - 1$ open discs removed and $g$ Möbius strips glued on $g$ boundary components so obtained (see figure 6).

![Figure 6: Generators $x_{k,t}$ for $P_n(N_{g,b})$, $b \geq 1$](image)

**Theorem A** For $g \geq 1$, $b \geq 1$ and $n \geq 1$, $P_n(N_{g,b})$ has the following presentation:

**Generators:** $(B_{i,j})_{1 \leq i < j \leq n}$, $(\rho_{k,l})_{1 \leq k \leq n}$ and $(x_{u,t})_{1 \leq u \leq n}$,

$1 \leq t \leq g$

$1 \leq t \leq b-1$
relations: (a) for all \( 1 \leq i < j \leq n \) and \( 1 \leq r < s \leq n \),

\[
B_{r,s}B_{i,j}B_{r,s}^{-1} = \begin{cases} 
B_{i,j} & \text{if } i < r < s < j \text{ or } r < s < i < j \\
B_{j,i}B_{i,j}B_{r,j}B_{i,j} & \text{if } r < i = s < j \\
B_{j,i}B_{r,j}B_{i,j} & \text{if } i = r < s < j \\
B_{s,j}B_{r,j}B_{i,j}B_{r,j}B_{s,j} & \text{if } r < i < s < j \\
\end{cases}
\]

(b) for all \( 1 \leq i < j \leq n \) and \( 1 \leq k, l \leq g \),

\[
\rho_{i,k}\rho_{j,l}\rho_{i,k}^{-1} = \begin{cases} 
\rho_{j,l} & \text{if } k < l \\
\rho_{j,k}B_{i,j}^{-1}\rho_{j,k}^2 & \text{if } k = l \\
\rho_{j,k}B_{i,j}^{-1}\rho_{j,k}B_{i,j}\rho_{j,k}^{-1}B_{i,j}\rho_{j,k} & \text{if } k > l \\
\end{cases}
\]

(d) for all \( 1 \leq i < j \leq n \), \( 1 \leq k \leq n \), \( k \neq j \) and \( 1 \leq l \leq g \),

\[
\rho_{k,l}B_{i,j}\rho_{k,l}^{-1} = \begin{cases} 
B_{i,j} & \text{if } k < i \text{ or } j < k \\
\rho_{j,i}B_{i,j}^{-1}\rho_{j,i} & \text{if } k = i \\
\rho_{j,i}B_{i,j}^{-1}\rho_{j,i}B_{i,j}\rho_{j,i}^{-1}B_{i,j}\rho_{j,i} & \text{if } i < k < j \\
\end{cases}
\]

(l) for all \( 1 \leq i < j \leq n \), \( 1 \leq u \leq n \), \( u \neq j \) and \( 1 \leq t \leq b - 1 \),

\[
x_{u,t}B_{i,j}x_{u,t}^{-1} = \begin{cases} 
x_{i,j} & \text{if } u < i \text{ or } j < u \\
x_{j,i}B_{u,j}x_{i,j} & \text{if } u = i \\
x_{j,i}B_{u,j}x_{i,j}B_{a,j}^{-1}B_{a,j}x_{i,j}^{-1}B_{a,j} & \text{if } i < u < j \\
\end{cases}
\]

(m) for all \( 1 \leq k \leq n \), \( k \neq u \), \( 1 \leq l \leq g \) and \( 1 \leq t \leq b - 1 \),

\[
x_{u,t}\rho_{k,l}x_{u,t}^{-1} = \begin{cases} 
\rho_{k,l} & \text{if } k < u \\
x_{k,t}\rho_{k,l}B_{u,k}\rho_{k,l}B_{a,k}x_{k,t}^{-1}B_{a,k} & \text{if } u < k \\
\end{cases}
\]

(n) for all \( 1 \leq i < j \leq n \) and \( 1 \leq i \leq s, t \leq b - 1 \),

\[
x_{i,t}x_{s,t}x_{i,t}^{-1} = \begin{cases} 
x_{s,t} & \text{if } t < s \\
x_{s,t}B_{i,s}x_{s,t}B_{i,s}^{-1}x_{s,t} & \text{if } t = s \\
x_{s,t}B_{i,s}x_{s,t}B_{i,s}x_{s,t}B_{i,s}^{-1}B_{i,s} & \text{if } s < t \\
\end{cases}
\]

Proof. \(1 \rightarrow \pi_1(N_{g,b} \setminus \{z_1, \ldots, z_n\}, z_{n+1}) \overset{\alpha}{\longrightarrow} P_{n+1}(N_{g,b}) \overset{\beta}{\longrightarrow} P_n(N_{g,b}) \rightarrow 1.\)

The presentation is correct for \( n = 1 \): \( P_1(N_{g,b}) = \pi_1(N_{g,b}) \) is free on the \( \rho_{l,i} \)'s and \( x_{1,i} \)'s for \( 1 \leq l \leq g \) and \( 1 \leq t \leq b - 1 \). Suppose inductively that \( P_n(N_{g,b}) \) has the given presentation. Then, observe that \( \{B_{i,n+1}/1 \leq i \leq n\} \cup \{\rho_{n+1,l}/1 \leq l \leq g\} \cup \{x_{n+1,t}/1 \leq t \leq b - 1\} \) is a free generators set of \( \text{Im}(\alpha) \) and \( (B_{i,j})_{1 \leq i < j \leq n},(\rho_{k,l})_{1 \leq k \leq n},(x_{u,t})_{1 \leq u \leq n} \) are coset representative for the considered generators of \( P_n(N_{g,b}) \). There are three types of relations for \( P_{n+1}(N_{g,b}) \). The first one comes from the relations in \( \text{Im}(\alpha) \): none are here, since this group is free. The second type comes from the relations in \( P_n(N_{g,b}) \): they lift to the same relations in \( P_{n+1}(N_{g,b}) \). Finally, the third type arrives by studying the action of \( P_n(N_{g,b}) \) on \( \text{Im}(\alpha) \) by conjugation. We leave to the reader to verify that this action corresponds to the given relations.

Remark 4 What precedes proves that \( P_{n+1}(N_{g,b}) \) is a semidirect product of the free group.
\( \pi_1(N_{g,b} \setminus \{z_1, \ldots, z_n\}, z_{n+1}) \) by \( P_n(N_{g,b}) \). Therefore, by recurrence, we get that \( P_{n+1}(N_{g,b}) \) is an iterated semidirect product of (finitely generated) free groups.

References


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