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COVERING TECHNIQUES IN AUSLANDER-REITEN THEORY

CLAUDIA CHAIO, PATRICK LE MEUR, AND SONIA TREPODE

Abstract. Given a finite dimensional algebra over a perfect field the text introduces covering functors over the mesh category of any modulated Auslander-Reiten component of the algebra. This is applied to study the composition of irreducible morphisms between indecomposable modules in relation with the powers of the radical of the module category.

Introduction

Let A be a finite-dimensional algebra over a field $\Bbbk$. The representation theory of A deals with the category $\operatorname{mod} A$ of finitely generated (right) A-modules. In particular it aims at describing the indecomposable modules up to isomorphism and the morphisms between them. For this purpose the Auslander-Reiten theory gives useful tools such as irreducible morphisms and almost split sequences. These two particular concepts have been applied to study singularities of algebraic varieties and Cohen-Macaulay modules over commutative rings.

Let $\operatorname{ind} A$ be the full subcategory of $\operatorname{mod} A$ containing one representative of each isomorphism class of indecomposable A-modules. Given $X, Y \in \operatorname{ind} A$, a morphism $f\colon X \to Y$ is called *irreducible* if it lies in $\operatorname{rad}\backslash\operatorname{rad}^2$. Here rad denotes the radical of the module category, that is, the ideal in $\operatorname{mod} A$ generated by the non-isomorphisms between indecomposable modules. The powers rad^ℓ of the radical are recursively defined by $\operatorname{rad}^{\ell+1} = \operatorname{rad}^\ell \cdot \operatorname{rad} = \operatorname{rad} \cdot \operatorname{rad}^\ell$. The Auslander-Reiten theory encodes part of the information of $\operatorname{mod} A$ in the Auslander-Reiten quiver $\Gamma(\operatorname{mod} A)$. This concentrates much of the combinatorial information on the irreducible morphisms and almost split sequences. However it does not give a complete information on the composition of two (or more) irreducible morphisms. For example the composition of n irreducible morphisms obviously lies in rad^n but it may lie in rad^{n+1}. It is proved in [IT84b, Thm. 13.3] that if these irreducible morphisms form a sectional path then their composition lies in $\operatorname{rad}^n\backslash\operatorname{rad}^{n+1}$. This result was made more precise for finite-dimensional algebras over algebraically closed fields in a study [CLMT11] of the degrees of irreducible morphisms (in the sense of [Liu92]) and their relationship to the representation type of the algebra. The results in [CLMT11] are based on well-behaved functors introduced first in [Rie80, BG82] for (selfinjective) algebras of finite representation type. This text presents general constructions of well-behaved functors with application to composition of irreducible morphisms.

Let Γ be a connected component of the Auslander-Reiten quiver of A (or, an Auslander-Reiten component, for short). Let $\operatorname{ind}\Gamma$ be the full subcategory of $\operatorname{ind} A$

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with set of objects the modules $X \in \operatorname{ind} A$ lying in Γ. Beyond the combinatorial structure on Γ, the mesh-category $\Bbbk(\Gamma)$ is a first approximation of $\operatorname{ind}\Gamma$ taking into account the composition of irreducible morphisms. Actually Igusa and Todorov have shown that Γ comes equipped with a $\Bbbk$-modulation ([IT84a]), which is called *standard* here and which includes the division algebra $\kappa_X = \operatorname{End}_A(X)/\operatorname{rad}(X,X)$ and the $\kappa_X - \kappa_Y$-bimodule $\operatorname{irr}(X,Y) = \operatorname{rad}(X,Y)/\operatorname{rad}^2(X,Y)$ for every $X, Y \in \Gamma$. The category $\Bbbk(\Gamma)$ may be defined by generators and relations (see Section 1 for details). Its objects are the modules $X \in \Gamma$, the generators are the classes of morphisms $u \in \kappa_X$ (as morphisms in $\Bbbk(\Gamma)(X,X)$) and $u \in \operatorname{irr}(X,Y)$ (as morphisms in $\Bbbk(\Gamma)(X,Y)$), for every $X, Y \in \Gamma$, and the ideal of relations is the mesh ideal.

When $\Bbbk$ is a perfect field this text introduces a covering functor of $\operatorname{ind}\Gamma$ in order to get information about the composition of irreducible morphisms in Γ.

The Auslander-Reiten component is called *standard* if there exists an isomorphism of categories $\Bbbk(\Gamma) \simeq \operatorname{ind}\Gamma$. Not all Auslander-Reiten components are standard and in many cases there even exist no functor $\Bbbk(\Gamma) \to \operatorname{ind}\Gamma$. For instance if Γ has oriented cycles then such a functor is likely not to exist. This may be bypassed replacing the mesh category $\Bbbk(\Gamma)$ by that of a suitable translation quiver $\widetilde{\Gamma}$ with a $\Bbbk$-modulation such that there exists a covering $\pi\colon \widetilde{\Gamma} \to \Gamma$. It appears that the composition of irreducible morphisms in $\operatorname{ind}\Gamma$ may be studied using $\Bbbk(\widetilde{\Gamma})$ provided that there exists a so-called well-behaved functor $\Bbbk(\widetilde{\Gamma}) \to \operatorname{ind}\Gamma$. These functors were first considered by Riedtmann [Rie80] in her study of the shapes of the Auslander-Reiten quivers of selfinjective algebras of finite representation type over algebraically closed fields and next by Bongartz and Gabriel [BG82] for algebras of finite representation type over algebraically closed fields. The present article considers well-behaved functors for modulated translation quivers when $\Bbbk$ is a perfect field. Let $(\kappa_x, M(x,y))_{x,y}$ be the $\Bbbk$-modulation of $\widetilde{\Gamma}$ induced by the standard modulation of Γ, that is, $\kappa_x = \kappa_{\pi x}$ and $M(x,y) = \operatorname{irr}(\pi x, \pi y)$ for every $x, y \in \widetilde{\Gamma}$. Then a functor $F\colon \Bbbk(\widetilde{\Gamma}) \to \operatorname{ind}\Gamma$ is well-behaved if it induces isomorphisms $\kappa_x \simeq \kappa_{\pi x}$ and $M(x,y) \simeq \operatorname{irr}(\pi x, \pi y)$, for every $x, y \in \Gamma$. The construction of F relies on three fundamental facts. Firstly, if one tries to construct such an F then it is quite natural to proceed by induction. The translation quiver $\widetilde{\Gamma}$ is called *with length* if any two paths in Γ having the same source and the same target have the same length. As mentionned above an inductive construction is likely not to work if $\widetilde{\Gamma}$ has oriented cycles and actually simple examples show that this construction fails if $\widetilde{\Gamma}$ is not with length. Note that $\widetilde{\Gamma}$ is with length when $\widetilde{\Gamma}$ is the universal cover of [BG82]. Secondly, if $x \in \widetilde{\Gamma}$ then the ring homomorphism $\kappa_x \hookrightarrow \Bbbk(\widetilde{\Gamma})(x,x) \xrightarrow{F} \operatorname{End}_A(\pi x)$ is a section of the quotient homomorphism $\operatorname{End}_A(X) \twoheadrightarrow \kappa_x$. In view of the Wedderburn-Malcev theorem this section is most likely to exist in the framework of algebras over perfect fields. Finally, given an irreducible morphism $f\colon X \to Y$ with $X, Y \in \Gamma$ then there exist $x, y \in \widetilde{\Gamma}$ and $u \in \Bbbk(\widetilde{\Gamma})(x,y)$ such that $f - Fu \in \operatorname{rad}^2$. In view of studying the composition of irreducible morphisms in $\operatorname{ind}\Gamma$ one may wish to have an equality $f = Fu$. This would permit to *lift* the study into $\Bbbk(\widetilde{\Gamma})$ where the composition of morphisms is better understood because of the mesh ideal. Keeping in mind these comments the main result of this text is the following.

Theorem A. *Let A be a finite-dimensional algebra over a perfect field $\Bbbk$. Let Γ be an Auslander-Reiten component of A. Let $\pi\colon \widetilde{\Gamma} \to \Gamma$ be a covering of translation quivers where $\widetilde{\Gamma}$ is with length. There exists a well-behaved functor $F\colon \Bbbk(\widetilde{\Gamma}) \to \operatorname{ind}\Gamma$.*

The study of the composition of irreducible mophisms in $\operatorname{ind}\Gamma$ using such a covering functor F is made possible by the following lifting (or, covering) property of F which is the second main result of the text. No assumption is made on length.

Theorem B. *Let A be a finite-dimensional algebra over a perfect field $\Bbbk$. Let Γ be an Auslander-Reiten component of A. Let $\pi\colon \widetilde{\Gamma} \to \Gamma$ be a covering of translation quivers. Let $F\colon \Bbbk(\widetilde{\Gamma}) \to \operatorname{ind}\Gamma$ be a well-behaved functor, $x, y \in \widetilde{\Gamma}$ and let $n \geqslant 0$.*

(a) The two following maps induced by F are bijective

$$\bigoplus_{Fz=Fy} \mathfrak{R}^n\Bbbk(\widetilde{\Gamma})(x,z)/\mathfrak{R}^{n+1}\Bbbk(\widetilde{\Gamma})(x,z) \quad \to \quad \operatorname{rad}^n(Fx,Fy)/\operatorname{rad}^{n+1}(Fx,Fy)$$

$$\bigoplus_{Fz=Fy} \mathfrak{R}^n\Bbbk(\widetilde{\Gamma})(z,x)/\mathfrak{R}^{n+1}\Bbbk(\widetilde{\Gamma})(z,x) \quad \to \quad \operatorname{rad}^n(Fy,Fx)/\operatorname{rad}^{n+1}(Fy,Fx) \quad .$$

(b) The two following maps induced by F are injective

$$\bigoplus_{Fz=Fy} \Bbbk(\widetilde{\Gamma})(x,z) \to \operatorname{Hom}_A(Fx,Fy) \quad \text{and} \quad \bigoplus_{Fz=Fy} \Bbbk(\widetilde{\Gamma})(z,x) \to \operatorname{Hom}_A(Fy,Fx).$$

(c) Γ is generalized standard if and only if F is a covering functor, that is, the two maps of (b) are bijective (see [BG82, 3.1]*).*

Here $\mathfrak{R}\Bbbk(\widetilde{\Gamma})$ is the ideal in $\Bbbk(\widetilde{\Gamma})$ generated by the morphisms in $M(x,y)$, for every arrow $x \to y$ in $\widetilde{\Gamma}$. Call it the *radical* of $\Bbbk(\widetilde{\Gamma})$ by abuse of terminology. Define its powers $\mathfrak{R}^n\Bbbk(\widetilde{\Gamma})$ like for the radical of $\operatorname{mod} A$. Here is an interpretation of Theorem B. Both $\Bbbk(\widetilde{\Gamma})$ and $\operatorname{ind}\Gamma$ are filtered by the powers of their respective radicals. The above theorem asserts that F induces a covering functor $\operatorname{gr}\Bbbk(\widetilde{\Gamma}) \to \operatorname{gr}\operatorname{ind}\Gamma$ (in the sense of [BG82]) between the associated graded categories.

This text is therefore organised as follows. Section 1 is a reminder on Auslander-Reiten theory, modulated translation quivers and their mesh-categories, and coverings of translation quivers; it also introduces strongly irreducible morphisms. Section 2 proves the above theorems. Section 3 gives an application to the composition of irreducible morphisms.

In the sequel $\Bbbk$ denotes a perfect field. Hence the tensor product over $\Bbbk$ of two finite-dimensional division algebras is semi-simple. Also if R is a finite-dimensional $\Bbbk$-algebra and $J \subseteq R$ is a two-sided ideal such that R/J is a division $\Bbbk$-algebra, then the natural surjection $R \twoheadrightarrow R/J$ admits a section $R/J \hookrightarrow R$ as a $\Bbbk$-algebra. The composition of mappings $f\colon X \to Y$ and $g\colon Y \to Z$ is denoted by $fg\colon X \to Z$.

1. Preliminaries

1.1. **Reminder on Auslander-Reiten theory.** Following [ARS97, V.7, p.178], denote by rad the *radical* of $\operatorname{mod} A$. This is the ideal of $\operatorname{mod} A$ generated by the non invertible morphisms between indecomposable modules. Its powers are defined recursively by $\operatorname{rad}^0 = \operatorname{mod} A$ and $\operatorname{rad}^{n+1} = \operatorname{rad}\cdot\operatorname{rad}^n$ $(= \operatorname{rad}^n\cdot\operatorname{rad})$ for every integer $n \geqslant 0$. For $X \in \operatorname{mod} A$ indecomposable, the division algebra $\operatorname{End}_A(X)/\operatorname{rad}(X,X)$ is denoted by κ_X and, given $u \in \operatorname{End}_A(X)$, its residue class in κ_X is denoted by $\overline{u}$.

A morphism f in $\operatorname{mod} A$ is *irreducible* ([ARS97, V.5, p. 166]) if it is neither a section nor a retraction and for all decompositions $f = uv$, then u is a section or else v is a retraction. Given $X, Y \in \operatorname{mod} A$ indecomposable, a morphism $X \to Y$ is irreducible if and only if it belongs to rad and not to rad^2, and the quotient

$\mathrm{rad}(X,Y)/\mathrm{rad}^2(X,Y)$ is called the *space of irreducible morphisms* from X to Y and denoted by $\mathrm{irr}(X,Y)$. Given $f \in \mathrm{rad}(X,Y)$, its image in $\mathrm{irr}(X,Y)$ is denoted by $\overline{f}$. Hence, $\mathrm{irr}(X,Y)$ is a $\kappa_X - \kappa_Y$-bimodule, or, a left $\kappa_X \otimes_{\Bbbk} \kappa_Y^{\mathrm{op}}$-module, such that $\overline{u} \cdot \overline{f} \cdot \overline{v} = \overline{ufv}$ for all $u \in \mathrm{End}_A(X)$, $f \in \mathrm{rad}(X,Y)$ and $v \in \mathrm{End}_A(Y)$.

A *minimal left almost split* morphism ([ARS97, V.1, 137-138]) is a morphism $f\colon X \to Y$ in $\mathrm{mod}\, A$ that is not a section, such that any morphism with domain X and which is not a section factors through f and such that, for all $u \in \mathrm{End}_A(X)$, if $uf = f$ then u is invertible. Such a morphism features the following properties ([ARS97, V.1, VII.1 Prop. 1.3 p. 230]): X is indecomposable and, given any direct sum decomposition $Y = \oplus_{i=1}^r X_i^{n_i}$ where $X_1, \dots, X_r$ are indecomposable and pairwise non isomorphic, if f is denoted coordinatewise as $[f_{i,j} \ ; \ {\scriptstyle 1 \leqslant i \leqslant r \atop 1 \leqslant j \leqslant n_i}]$, then, for all $i \in \{1, \dots, r\}$, the family $\{\overline{f_{i,j}}\}_j$ is a basis of $\mathrm{irr}(X, X_i)$ over κ_{X_i}. These properties characterise minimal left almost split morphisms. Any indecomposable A-module X is the domain of such a morphism $X \to Y$ which happens to be the quotient morphism $X \to X/\mathrm{soc}(X)$ when X is injective and, in general, any irreducible morphism $X \to Z$ is equal to the composite morphism $X \to Y \to Z$ for some retraction $Y \to Z$. Minimal right almost split morphisms are defined dually and feature dual properties.

An *almost split sequence* ([ARS97, V.1, p.144]) is an exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathrm{mod}\, A$ such that $X \to Y$ is minimal left almost split and $Y \to Z$ is minimal right almost split. Any indecomposable and non-projective $Z \in \mathrm{mod}\, A$ (or, non-injective $X \in \mathrm{mod}\, A$) is the end term (or, the first term, respectively) of such a (unique up to isomorphism) sequence ([ARS97, V.1, Thm. 1.15 p. 145]), in such a case, X is called the *Auslander-Reiten translate* of Z and denoted by $\tau_A Z$.

The *Auslander-Reiten* quiver of $\mathrm{mod}\, A$ ([ARS97, VII.1 p. 225]) is the pair $(\Gamma(\mathrm{mod}\, A), \tau_A)$ where $\Gamma(\mathrm{mod}\, A)$ is the quiver with vertices the objects of $\mathrm{ind}\, A$ and such that for all $X, Y \in \mathrm{ind}\, A$ there is an arrow $X \to Y$ if and only if there exists an irreducible morphism $X \to Y$. Its connected components are called *Auslander-Reiten components*. These quivers are particular instances of *translation quivers* ([ARS97, VII.4] p. 248). Those are the pairs (Γ, τ) where Γ is a locally finite quiver with two distinguished sets of vertices called *projectives* and *injectives*, respectively, and τ, the *translation*, is a bijective mapping from non-projective vertices to non-injective vertices and such that such that for all vectices x, y with x non-projective, there is an arrow $y \to x$ if and only if there is an arrow $\tau x \to y$.

The *valuation* (a, b) of an arrow $X \to Y$ in $\Gamma(\mathrm{mod}\, A)$ is defined such that a is the maximal integer such that there exists a minimal right almost split morphism $X^a \oplus Z \to Y$ and b is the maximal integer such that there exists a minimal left almost split morphism $X \to Y^b \oplus Z$. Equivalently, $(a, b) = (\dim_{\kappa_X} \mathrm{irr}(X,Y), \dim_{\kappa_Y^{\mathrm{op}}} \mathrm{irr}(X,Y))$.

1.2. **Strongly irreducible morphisms.** The following definition is given over arbitrary fields. However, it is relevant mostly when $\Bbbk$ is a perfect field, see below.

Definition. Let $\Bbbk$ be any field. Let A be a finite dimensional $\Bbbk$-algebra. Let $X, X_1, \dots, X_r \in \mathrm{ind}\, A$ be such that $X_1, \dots, X_r \in \mathrm{ind}\, A$ are pairwise non isomorphic and let $n_1, \dots, n_r$ be positive integers. A *strongly irreducible* morphism $X \to \oplus_{i=1}^r X_i^{n_i}$ is an irreducible morphism $[f_{i,j};\ 1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant n_i]$ with the following properties in the left $\kappa_X \otimes_{\Bbbk} \kappa_{X_i}^{\mathrm{op}}$-module $\mathrm{irr}(X, X_i)$, for all $i \in \{1, \dots, r\}$.

(a) $\sum_j \kappa_X \cdot \overline{f_{i,j}} \cdot \kappa_{X_i} = \oplus_j \kappa_X \cdot \overline{f_{i,j}} \cdot \kappa_{X_i}$.

(b) $\sum_j \kappa_X \cdot \overline{f_{i,j}} \cdot \kappa_{X_i}$ is a direct summand of $\mathrm{irr}(X, X_i)$ as a left $\kappa_X \otimes_{\Bbbk} \kappa_{X_i}^{\mathrm{op}}$-module.

Note that, f is strongly irreducible if and only if, for all i, so is $X \to X_i^{n_i}$. As mentioned previously, this definition is mostly relevant when $\Bbbk$ is a perfect field because of the following standard result.

Lemma. *Let $\Bbbk$ be a perfect field.*

(1) Let E, F be finite dimensional division $\Bbbk$-algebras. Then, $E \otimes_\Bbbk F$ is semi-simple.

(2) Let A be a finite dimensional $\Bbbk$-algebra. Let $X, Y \in \operatorname{ind} A$. Then, $\kappa_X \otimes_\Bbbk \kappa_Y^{\mathrm{op}}$ is semi-simple.

Proof. (1) follows from [Pie82, 10.7 Corollary of p. 188] and [Pie82, 10.7 Corollary b p. 192]). (2) follows from (1). □

Here are some comparison elements between strongly irreducible morphisms and irreducible morphisms, $\Bbbk$ here need not be perfect. Keep the notation used in the previous definition. First, condition (b) holds automatically when $\Bbbk$ is perfect. Next, f is irreducible if and only if so is each $f_{i,j}$ and the following assertions are true in the right κ_{X_i}-vector space $\operatorname{irr}(X, X_i)$, for all $i \in \{1, \dots, r\}$.

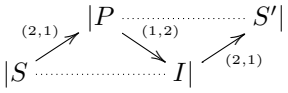
(a') $\sum_j \overline{f_{i,j}} \cdot \kappa_{X_i} = \oplus_j \overline{f_{i,j}} \cdot \kappa_{X_i}$.

(b') $\sum_j \overline{f_{i,j}} \cdot \kappa_{X_i}$ is a direct summand of $\operatorname{irr}(X, X_i)$.

Note that condition (b') is independent of f because κ_{X_i} is a division algebra. Clearly, conditions (a) and (b) in the above definition are stronger than conditions (a') and (b'). In particular, there are irreducible morphisms which are not strongly irreducible. Here is a example (see [ARS97, VII.2, p. 235]). Let $\Bbbk = \mathbb{R}$ and

$$A = \{ \left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right) \mid a \in \mathbb{C},\, b \in \mathbb{C},\, c \in \mathbb{R} \} \,.$$

Then, $\Gamma(\operatorname{mod} A)$ has the following shape, as a valued quiver,



where S, P, I, S' are simple projective, projective non simple, injective non simple and simple injective, respectively. Consider an almost split sequence

$$0 \to S \xrightarrow{f} P \xrightarrow{g} I \to 0 \,.$$

Then, there exists an automorphism $u \colon P \to P$ fitting in an almost split sequence

$$0 \to P \xrightarrow{[g,\ ug]} I^2 \to S' \to 0 \,.$$

The valuation of the arrow $P \to I$ of $\Gamma(\operatorname{mod} A)$ is $(1, 2)$. Therefore $\dim_{\kappa_P} \operatorname{irr}(P, I) = 1$. Consequently, $\operatorname{irr}(P, I)$ is generated by a single element as a left κ_P-vector space, and hence also as a left $\kappa_P \otimes_\mathbb{R} \kappa_I^{\mathrm{op}}$-module. Accordingly, the left $\kappa_P \otimes_\mathbb{R} \kappa_I^{\mathrm{op}}$-module $\operatorname{irr}(P, I)$ is simple. Thus, it equals both $\kappa_P \cdot \overline{g} \cdot \kappa_I$ and $\kappa_P \cdot \overline{ug} \cdot \kappa_I$. In particular, $g \colon P \to I$ is strongly irreducible; and $[g\ ug]$ is not strongly irreducible despite being irreducible, because the sum $\kappa_P \cdot \overline{g} \cdot \kappa_I + \kappa_P \cdot \overline{ug} \cdot \kappa_I$ is not direct.

From now on, assume that $\Bbbk$ is perfect. Here are hints on how to construct strongly irreducible morphisms and how to obtain irreducible morphisms from them. Consider an arrow $X \to Y$ in $\Gamma(\operatorname{mod} A)$. Since $\kappa_X \otimes_\Bbbk \kappa_Y^{\mathrm{op}}$ is a semi-simple algebra, there is a direct sum decomposition $\operatorname{irr}(X, Y) = \oplus_{j=1}^n S_j$, where $S_1, \dots, S_n$ are simple left $\kappa_X \otimes_\Bbbk \kappa_Y^{\mathrm{op}}$-modules. For each j, let $f_j \in \operatorname{rad}(X, Y)$ be such that $\overline{f_j}$

is a generator of the simple module S_j. Then, the following morphism is strongly irreducible

$$[f_j;\, 1 \leqslant j \leqslant n] : X \to Y^n \,. \tag{1}$$

Any strongly irreducible morphism $X \to Y^m$ ($m \geqslant 1$) is equal to the composition of a morphism such as (1), for adequate choices, with a retraction $Y^n \to Y^m$. Now, for every $j \in \{1, \dots, n\}$, consider S_j as a right κ_Y-vector space; then, using a basis of κ_X over $\Bbbk$ it follows that there exists a family $\{u_{j,k}\}_k$ of automorphisms of X with at most $\dim_\Bbbk \kappa_X$ terms and such that

$$S_j = \oplus_k \overline{u_{j,k} f_j} \cdot \kappa_Y \,.$$

Accordingly, there is a direct sum decomposition of right κ_Y-vector spaces

$$\operatorname{irr}(X, Y) = \oplus_{j,k} \overline{u_{j,k} f_j} \cdot \kappa_Y \,.$$

Hence, this sum has $\dim_{\kappa_Y} \operatorname{irr}(X, Y)$ terms and the following morphism is irreducible,

$$[u_{j,k} f_j;\, j, k] \colon X \to Y^{\dim_{\kappa_Y} \operatorname{irr}(X,Y)} \,. \tag{2}$$

In particular, if the valuation of the arrow $X \to Y$ of $\Gamma(\operatorname{mod} A)$ has the shape $(a, 1)$, then all the direct sums in (1) and (2) have exactly one term. Actually, these considerations show the equivalence of the following assertions.

(i) The irreducible morphism (2) is strongly irreducible.
(ii) $\operatorname{irr}(X, Y)$ is the direct sum of $\dim_{\kappa_Y} \operatorname{irr}(X, Y)$ simple $\kappa_X \otimes_\Bbbk \kappa_Y^{\mathrm{op}}$-modules.
(iii) For all $j \in \{1, \dots, n\}$, the $\Bbbk$-vector spaces $\kappa_X \cdot \overline{f_j} \cdot \kappa_Y$ and $\overline{f_j} \cdot \kappa_Y$ are equal.
(iv) $\kappa_X \cdot \overline{f_j}$ is contained in $\overline{f_j} \cdot \kappa_Y$, for all $j \in \{1, \dots, n\}$.

Going back to the general situation of an irreducible morphism $f \colon X \to \oplus_{i=1}^r X_i^{n_i}$, it would be worth characterising when it is strongly irreducible. In view of the previous considerations, each of the following conditions is sufficient.

(1) $n_1 = \cdots = n_r = 1$.
(2) $\kappa_X \cdot \overline{f_{i,j}}$ is contained in $\overline{f_{i,j}} \cdot \kappa_{X_i}$, for all i, j; indeed, in such case, the sum $\sum_{i,j} \kappa_X \cdot \overline{f_{i,j}} \cdot \kappa_{X_i}$ is direct because the morphism $X \to \oplus_i X_i^{n_i}$ is irreducible and, for every i, j, the $\Bbbk$-vector spaces $\kappa_X \cdot \overline{f_{i,j}} \cdot \kappa_{X_i}$ and $\overline{f_{i,j}} \cdot \kappa_{X_i}$ are equal.
(3) $\kappa_X = \Bbbk$; indeed, this assertion implies (2).
(4) $\Bbbk$ is algebraically closed; indeed, this assertion implies (3).
(5) For each $i \in \{1, \dots, r\}$, the arrow $X \to X_i$ of $\Gamma(\operatorname{mod} A)$ has valuation $(a_i, 1)$; indeed, this entails (1).

Recall ([Liu92, 1.7]) that if an arrow $X \to Y$ of $\Gamma(\operatorname{mod} A)$ has finite left degree, then it has valuation $(1, b)$ or $(a, 1)$. In the former case, if $b \geqslant 2$, then there exists an irreducible morphism $X \to Y^b$ which is necessarily not strongly irreducible.

1.3. **Factorisation through minimal almost split morphisms.** The reader is referred to [ARS97] for basics on Auslander-Reiten theory. In the sequel the factorisation property of minimal almost split morphisms is used as follows.

Lemma. *Let $u \colon X \to Y$ be a left minimal almost split morphism, $Z \in \operatorname{mod} A$ and $v \in \operatorname{rad}^{n+1}(X, Z)$ for some $n \geqslant 0$. There exists $w \in \operatorname{rad}^n(Y, Z)$ such that $v = uw$.*

Proof. Since $v \in \operatorname{rad}^{n+1}$, there exist an integer $N \geqslant 1$, indecomposable modules $X_1, \dots, X_N$ and morphisms $v_i' \in \operatorname{rad}(X, X_i)$ and $v_i'' \in \operatorname{rad}^n(X_i, Z)$, for every $i \in \{1, \dots, N\}$, such that $v = \sum_{i=1}^N v_i' v_i''$. For every $i \in \{1, \dots, N\}$, there exists $w_i \in$

$\mathrm{Hom}_A(Y, X_i)$ such that $v'_i = uw_i$. Let $w = \sum_{i=1}^{N} w_i v''_i$. Then, $w \in \mathrm{rad}^n(Y, Z)$ and $v = uw$. □

1.4. **Modulated translation quivers and their mesh-categories.** Let Γ be a translation quiver. For a non-projective vertex x, the subquiver of Γ formed by the arrows starting in τx and the arrows arriving in x is called the *mesh* starting in τx.

A *($\Bbbk$)-modulation* on Γ is the following data

(i) a division $\Bbbk$-algebra κ_x for every vertex $x \in \Gamma$,
(ii) a non-zero $\kappa_x - \kappa_y$ bimodule $M(x, y)$ for every arrow $x \to y$ in Γ,
(iii) a $\Bbbk$-algebra isomorphism $\tau_* \colon \kappa_x \xrightarrow{\sim} \kappa_{\tau x}$ for every vertex $x \in \Gamma$,
(iv) a non-degenerate $\kappa_y - \kappa_y$-linear map $\sigma_* \colon M(y, x) \otimes_{\kappa_x} M(\tau x, y) \to \kappa_y$ (the left κ_x-module structure on $M(\tau x, y)$ is defined using its structure of left $\kappa_{\tau x}$-module and $\tau_* \colon \kappa_x \to \kappa_{\tau x}$).

With such a structure, Γ is called a *modulated* translation quiver. If A is a finite-dimensional algebra over a field $\Bbbk$ then the Auslander-Reiten quiver $\Gamma(\mathrm{mod}\, A)$ has a $\Bbbk$-modulation as follows ([IT84a, 2.4, 2.5]), in the rest of the text this modulation is called the *standard* modulation of $\Gamma(\mathrm{mod}\, A)$; the restriction of the standard modulation of $\Gamma(\mathrm{mod}\, A)$ to any Auslander-Reiten component is also called the standard modulation of that component. For every non-projective $X \in \mathrm{ind}\, A$ fix an almost split sequence $0 \to \tau_A X \to E \to X \to 0$ in $\mathrm{mod}\, A$. Then

- $\kappa_X = \mathrm{End}_A(X)/\mathrm{rad}(X, X)$ for every $X \in \mathrm{ind}\, A$,
- $M(X, Y) = \mathrm{irr}(X, Y)$ for every arrow $X \to Y$ in $\Gamma(\mathrm{mod}\, A)$,
- for every $X \in \mathrm{ind}\, A$ and every morphism $u \colon X \to X$ defining the residue class $\overline{u} \in \kappa_X$, let $\tau_* \overline{u} \colon \tau_A X \to \tau_A X$ be the residue class $\overline{v}$ where $v \colon \tau_A X \to \tau_A X$ is a morphism fitting into a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tau_A X & \longrightarrow & E & \longrightarrow & X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle v} & & \downarrow & & \downarrow{\scriptstyle u} & & \\ 0 & \longrightarrow & \tau_A X & \longrightarrow & E & \longrightarrow & X & \longrightarrow & 0\,, \end{array}$$

- let $X, Y \in \mathrm{ind}\, A$ with X non-projective and assume that there is an arrow $Y \to X$ in $\Gamma(\mathrm{mod}\, A)$. Let $\overline{u} \in M(\tau_A X, Y)$ and $\overline{v} \in M(Y, X)$ be residue classes of morphisms $u \colon \tau_A X \to Y$ and $v \colon Y \to X$, respectively. Then define $\sigma_*(\overline{v} \otimes \overline{u})$ as the composition $\overline{v'u'}$ where u', v' are morphisms fitting into a commutative diagram

$$\begin{array}{ccccccccc} & & & & & & Y & & \\ & & & & & {\scriptstyle v'}\swarrow & \downarrow{\scriptstyle v} & & \\ 0 & \longrightarrow & \tau_A X & \longrightarrow & E & \longrightarrow & X & \longrightarrow & 0 \\ & & & {\scriptstyle u}\searrow & \swarrow{\scriptstyle u'} & & & & \\ & & & & Y & & & & \end{array}\,.$$

This construction does not depend on the initial choice of the almost split sequences up to an isomorphism of modulated translation quivers ([IT84a, 2.5]). In the sequel $\Gamma(\mathrm{mod}\, A)$ is considered as a modulated translation quiver as above.

If Γ is a modulated translation quiver, its *mesh-category* $\Bbbk(\Gamma)$ is defined as follows ([IT84a, 1.7]). Let S be the semi-simple category whose object set is the set of vertices in Γ and such that $S(x, y) = \kappa_x$ if $x = y$ and $S(x, y) = 0$ otherwise. The collection $\{M(x, y)\}_{x \to y \text{ in } \Gamma}$ naturally defines an $S - S$-bimodule denoted by M.

The *path-category* is the tensor category $T_S(M)$ also denoted by $\Bbbk\Gamma$. The *mesh-ideal* is the ideal in $\Bbbk\Gamma$ generated by a collection of morphisms $\gamma_x\colon \tau x \to x$ indexed by the non-projective vertices $x \in \Gamma$. Given a non-projective vertex $x \in \Gamma$, a morphism $\gamma_x\colon \tau x \to x$ in $\Bbbk\Gamma$ is defined as follows. For every arrow $y \to x$ ending in x, fix a basis $(u_1, \dots, u_d)$ of the κ_y-vector space $M(y, x)$. Let $(u_1^*, \dots, u_d^*)$ be the associated dual basis of the κ_y-vector space $M(\tau x, y)$ under the pairing σ_* (that is, $\sigma_*(u_i \otimes u_j^*)$ is 1 if $i = j$ and 0 otherwise). Then $\gamma_x = \sum_{y \to x \text{ in } \Gamma} \sum_i u_i^* u_i \in \Bbbk\Gamma(\tau x, x)$. This morphism does not depend on the choice of the basis $(u_1, \dots, u_d)$. The mesh-category is then defined as the quotient category of $\Bbbk\Gamma$ by the mesh-ideal.

Let Γ be a component of $\Gamma(\operatorname{mod} A)$ endowed with its standard modulation. As proved in [IT84a, Sect. 2], the mesh-category $\Bbbk(\Gamma)$ does not depend on the choice of the almost split sequences used to define the modulation up to an isomorphism of $\Bbbk$-linear categories. The following lemma explains how to recover the mesh-relations γ_X and the pairing σ_* starting from a different choice of almost split sequences.

Lemma. *In the previous setting, let $X \in \Gamma$ be non-projective and let*

$$0 \to \tau_A X \xrightarrow{\alpha} \bigoplus_{i=1}^{r} X_i^{n_i} \xrightarrow{\beta} X \to 0$$

be the almost split sequence ending in X that is used in the definition of the modulation on Γ, where $X_1, \dots, X_r \in \Gamma$ are pairwise distinct. Let

$$0 \to \tau_A X \xrightarrow{f=\left[f_{i,j}\,;\ \substack{1\leqslant i\leqslant r\\ 1\leqslant j\leqslant n_i}\right]} \bigoplus_i X_i^{n_i} \xrightarrow{g=\left[g_{i,j}\,;\ \substack{1\leqslant i\leqslant r\\ 1\leqslant j\leqslant n_i}\right]^t} X \to 0$$

be another almost split sequence where the $f_{i,j}\colon \tau_A X \to X_i$ and the $g_{i,j}\colon X_i \to X$ are the components of f and g, respectively. Then there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \tau_A X & \xrightarrow{\alpha} & \bigoplus_i X_i^{n_i} & \xrightarrow{\beta} & X & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \| & & \\
0 & \longrightarrow & \tau_A X & \xrightarrow{f} & \bigoplus_i X_i^{n_i} & \xrightarrow{g} & X & \longrightarrow & 0
\end{array}$$

where u, v are isomorphisms. With this data:

(a) $(\overline{g_{i,1}}, \dots, \overline{g_{i,n_i}})$ *is a basis of the left κ_{X_i}-module* $\operatorname{irr}(X_i, X)$ *and the corresponding dual basis of the right κ_{X_i}-module* $\operatorname{irr}(\tau_A X, X_i)$ *under the pairing σ_* is $(\overline{uf_{i,1}}, \dots, \overline{uf_{i,n_i}})$, for every $i \in \{1, \dots, r\}$,*

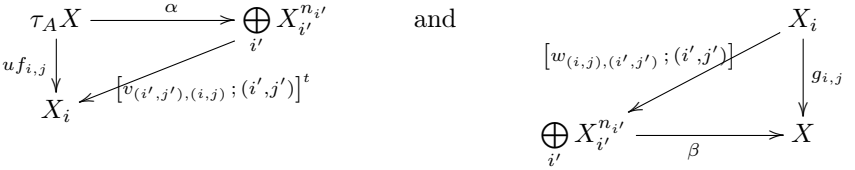
(b) $\gamma_X = \sum_{i,j} \overline{uf_{i,j}} \otimes \overline{g_{i,j}}$.

Proof. (b) follows directly from (a) and from the definition of γ_X. It therefore suffices to prove (a). The existence of u and v are direct consequences of the basic properties of almost split sequences; also, for every i, the given n_i-tuples are indeed bases because they arise from a left (or right) minimal almost split morphism (see 1.1). Let $w\colon \oplus_i X_i^{n_i} \to \oplus_i X_i^{n_i}$ be v^{-1}. For every $i, i' \in \{1, \dots, r\}$, and every $1 \leqslant j \leqslant n_i$, and every $1 \leqslant j' \leqslant n_{i'}$, let $v_{(i,j),(i',j')}\colon X_i \to X_{i'}$ and $w_{(i,j),(i',j')}\colon X_i \to X_{i'}$ be the respective components of v and w, from the j-th component X_i of $X_i^{n_i}$ to the j'-th component $X_{i'}$ of $X_{i'}^{n_{i'}}$. Then, the equality

$wv = \mathrm{Id}$ reads

$$(3)\quad \forall (i,j),(i'',j'') \qquad \sum_{(i',j')} w_{(i,j),(i',j')} v_{(i',j'),(i'',j'')} = \begin{cases} \mathrm{Id}_{X_i} & \text{if } (i,j)=(i'',j'') \\ 0 & \text{otherwise.} \end{cases}$$

Now, consider the equalities $uf = \alpha v$ in $\mathrm{Hom}_A(\tau_A X, \oplus_i X_i^{n_i})$ and $g = (wvg =)w\beta$ in $\mathrm{Hom}_A(\oplus_i X^{n_i}, X)$. Given $i \in \{1,\dots,r\}$ and $j \in \{1,\dots,n_i\}$, the composition with the (i,j)-th canonical section $X_i \to X_i^{n_i} \to \oplus_{i'} X_{i'}^{n_{i'}}$ and with the (i,j)-th canonical retraction $\oplus_{i'} X_{i'}^{n_{i'}} \to X_i^{n_i} \to X_i$, respectively yield that the following diagrams are commutative,



Thus $\sigma_*\left(\overline{g_{i,j}} \otimes \overline{uf_{i'',j''}}\right) = \sum_{i',j'} w_{(i,j),(i',j')} v_{(i',j'),(i'',j'')}$ for every (i,j) and (i'',j''). This and (3) show (a). □

1.5. **Radical in mesh-categories.** Let $\widetilde{\Gamma}$ be a modulated translation quiver. Recall that the radical $\mathfrak{R}\Bbbk(\widetilde{\Gamma})$ of $\Bbbk(\widetilde{\Gamma})$ was defined in the introduction. For every arrow $x \to y$ in $\widetilde{\Gamma}$ the natural map $M(x,y) \to \Bbbk(\widetilde{\Gamma})(x,y)$ is one-to-one. And the $\kappa_x - \kappa_y$-bimodule $\mathfrak{R}\Bbbk(\widetilde{\Gamma})(x,y)$ decomposes as $M(x,y) \oplus \mathfrak{R}^2\Bbbk(\widetilde{\Gamma})(x,y)$. The description of the ideal $\mathfrak{R}^\ell\Bbbk(\widetilde{\Gamma})$ is easier when $\widetilde{\Gamma}$ is with length as shows the following proposition. It is central in this text and used without further reference. The proof is a small variation of [Cha10, 2.1] where the description was first proved in the case $\kappa_x = \Bbbk$ for every vertex $x \in \widetilde{\Gamma}$.

Proposition. *Let $\widetilde{\Gamma}$ be a translation quiver with length and $x, y \in \widetilde{\Gamma}$. If there is a path of length ℓ from x to y in $\widetilde{\Gamma}$, then:*
(a) $\Bbbk(\widetilde{\Gamma})(x,y) = \mathfrak{R}\Bbbk(\widetilde{\Gamma})(x,y) = \mathfrak{R}^2\Bbbk(\widetilde{\Gamma})(x,y) = \cdots = \mathfrak{R}^\ell\Bbbk(\widetilde{\Gamma})(x,y)$.
(b) $\mathfrak{R}^i\Bbbk(\widetilde{\Gamma})(x,y) = 0$ *if* $i > \ell$.

1.6. **Coverings of translation quivers.** See [BG82, 1.3] for more details. A *covering of translation quivers* is a quiver morphism $p\colon \widetilde{\Gamma} \to \Gamma$ such that

(a) $\Gamma, \widetilde{\Gamma}$ are translation quivers, Γ is connected,
(b) a vertex $x \in \widetilde{\Gamma}$ is projective (or injective, respectively) if and only if so is px,
(c) p commutes with the translations in Γ and $\widetilde{\Gamma}$ (where these are defined),
(d) for every vertex $x \in \widetilde{\Gamma}$ the map $\alpha \mapsto p(\alpha)$ induces a bijection from the set of arrows in $\widetilde{\Gamma}$ starting in x (or ending in x) to the set of arrows in Γ starting in px (or ending in px, respectively).

Let $\pi\colon \widetilde{\Gamma} \to \Gamma$ be a covering of translation quivers. If Γ is modulated by division $\Bbbk$-algebras κ_x and bimodules $M(x,y)$ for every vertex x and every arrow $x \to y$, then $\widetilde{\Gamma}$ is modulated by the division $\Bbbk$-algebra $\kappa_x := \kappa_{\pi x}$ at the vertex $x \in \widetilde{\Gamma}$ and by the bimodule $M(\pi x, \pi y)$ for every arrow $x \to y$ in $\widetilde{\Gamma}$. In this text, this modulation on $\widetilde{\Gamma}$ is called *induced* by the modulation of Γ. When $\widetilde{\Gamma}$ is with length, it has a *length function,* that is, a map $x \mapsto \ell(x)$ defined on vertices such that $\ell(y) = \ell(x) + 1$ for every arrow $x \to y$. See [BG82, 1.6] for the construction of such

a function in the particular case where $\widetilde{\Gamma}$ is the universal cover of Γ ([BG82, 1.2,1.3]). Note that quivers have no parallel arrows here hence the notion of universal cover coincides with that of *generic cover* used in [CLMT11]). In the rest of the text, whenever Γ is an Auslander-Reiten component of A and $\widetilde{\Gamma}$ is its universal cover, the modulation of $\widetilde{\Gamma}$ induced by the standard modulation of Γ is referred to as the *universal* modulation.

From now on, A is a finite-dimensional $\Bbbk$-algebra. Its Auslander-Reiten components and their coverings are modulated as in 1.4 and above, respectively. To avoid possible confusions, upper-case letters $(X, Y, \ldots)$ stand for vertices in Auslander-Reiten components and lower-case letters $(x, y, \ldots)$ stand for vertices in other translation quivers. But the same notation (κ, M) is used for all $\Bbbk$-modulations.

2. Well-behaved functors

Recall the convention set in the introduction: whenever $f\colon X \to Y$ and $g\colon Y \to Z$ are two mappings, their composition is denoted by $fg\colon X \to Z$. Let Γ be an Auslander-Reiten component of A and $\pi\colon \widetilde{\Gamma} \to \Gamma$ be a covering of translation quivers such that $\widetilde{\Gamma}$ is connected. This section introduces the notion of well-behaved functors $F\colon \Bbbk(\widetilde{\Gamma}) \to \operatorname{ind}\Gamma$ where $\widetilde{\Gamma}$ is equipped with the modulation induced from the standard one restricted to Γ (see 1.4). These functors are proved to exist when $\widetilde{\Gamma}$ is with length. Such objects were first introduced over algebraically closed fields for Auslander-Reiten quivers of algebras of finite representation type ([Rie80, Sect. 1] and [BG82, Sect. 2]). The section also proves the lifting properties of well-behaved functors (see [CLMT11, Sect. 2] when $\Bbbk$ is algebraically closed).

Until the end of 2.5 the following convention is used implicitely. If $f\colon X \to \oplus_{i=1}^{r} X_i^{n_i}$ is an irreducible morphism it is assumed that $X \in \Gamma$, that $X_1, \ldots, X_r \in \Gamma$ are pairwise distinct, and $n_1, \ldots, n_r \geqslant 1$. Then f is written $f = [f_{i,j}\,;\, i, j]$.

2.1. **Sections of residue fields and of spaces of irreducible morphisms.** Let $X \in \Gamma$ be a vertex. The following theorem plays a central role in this article.

Theorem (Wedderburn, Malcev (see [Pie82, 11.6 Corollary on p. 211])). *Let $\Bbbk$ be a perfect field. Let E be a finite dimensional $\Bbbk$-algebra. Then, there exists a section $E/\mathrm{rad}E \to E$ of the $\Bbbk$-algebra canonical surjection $E \to E/\mathrm{rad}E$.*

For a given section $\kappa_X = \mathrm{End}_A(X)/\mathrm{rad}(X,X) \to \mathrm{End}_A(X)$ of the $\Bbbk$-algebra surjection $\mathrm{End}_A(X) \to \mathrm{End}_A(X)/\mathrm{rad}(X,X)$, the image is denoted by $\Bbbk_X$. Then $\Bbbk_X \subseteq \mathrm{End}_A(X)$ is a subalgebra such that $\mathrm{End}_A(X) = \Bbbk_X \oplus \mathrm{rad}(X,X)$ as a $\Bbbk$-vector space and the surjection $\mathrm{End}_A(X) \twoheadrightarrow \kappa_X$ restricts to a $\Bbbk$-algebra isomorphism $\Bbbk_X \xrightarrow{\sim} \kappa_X$. For short, $\Bbbk_X$ is called a *section of* κ_X.

Let $X \to Y$ be an arrow in Γ. Then $X \not\simeq Y$ and $\mathrm{Hom}_A(X,Y) = \mathrm{rad}(X,Y)$. Suppose given sections $\Bbbk_X \subseteq \mathrm{End}_A(X)$ and $\Bbbk_Y \subseteq \mathrm{End}_A(Y)$ of κ_X and κ_Y, respectively. Then $\mathrm{irr}(X,Y)$ is a $\Bbbk_X - \Bbbk_Y$-bimodule using the isomorphisms $\Bbbk_X \xrightarrow{\sim} \kappa_X$ and $\Bbbk_Y \xrightarrow{\sim} \kappa_Y$. By a $\Bbbk_X - \Bbbk_Y$-*linear section* (of $\mathrm{irr}(X,Y)$) is meant a section $\mathrm{irr}(X,Y) \to \mathrm{rad}(X,Y)$ of the canonical surjection $\mathrm{rad}(X,Y) \twoheadrightarrow \mathrm{irr}(X,Y)$ in the category of $\Bbbk_X - \Bbbk_Y$-bimodules. Such a section always exists because the $\Bbbk$-algebra $\Bbbk_X \otimes_{\Bbbk} \Bbbk_Y^{op}$ is semisimple. Note that the datum of a linear section depends on the choice of the algebra sections $\Bbbk_X$ and $\Bbbk_Y$.

2.2. **Well-behaved functors.** The following definition extends to the case of perfect fields the already known definition of well-behaved functors when the base field is algebraically closed ([BG82, 3.1] and [Rie80, 2.2]). Keep the setting established at the beginning of this section. A $\Bbbk$-linear functor $F\colon \Bbbk(\widetilde{\Gamma}) \to \operatorname{ind}\Gamma$ is called *well-behaved* if

(a) $Fx = \pi x$ for every vertex $x \in \widetilde{\Gamma}$,
(b) for every vertex $x \in \widetilde{\Gamma}$, the $\Bbbk$-algebra map from $\kappa_{\pi x} = \kappa_x$ to $\operatorname{End}_A(\pi x)$ defined by $u \mapsto F(u)$ is a section of the natural surjection $\operatorname{End}_A(\pi x) \twoheadrightarrow \kappa_{\pi x}$. Its image is denoted by $\Bbbk_x$,
(c) for every arrow $x \to y$ in $\widetilde{\Gamma}$, the following $\Bbbk$-linear composite map is a $\Bbbk_x - \Bbbk_y$-linear section in the obvious way

$$\operatorname{irr}(\pi x, \pi y) = M(x,y) \xrightarrow{\mathrm{Id}} \Bbbk(\widetilde{\Gamma})(x,y) \xrightarrow{F} \operatorname{Hom}_A(\pi x, \pi y)\,.$$

Note that if F is as in the definition then distinct vertices $x, x' \in \widetilde{\Gamma}$ such that $\pi x = \pi x'$ may give rise to different sections $\Bbbk_x$ and $\Bbbk_{x'}$ in $\operatorname{End}_A(\pi x)$. The data of a section $\Bbbk_x \subseteq \operatorname{End}_A(\pi x)$ of $\kappa_{\pi x}$, for every vertex $x \in \widetilde{\Gamma}$, and that of a $\Bbbk_x - \Bbbk_y$-linear section $M(x,y) \to \operatorname{Hom}_A(\pi x, \pi y)$, for every arrow $x \to y$ in $\widetilde{\Gamma}$ determine a unique $\Bbbk$-linear functor $\Bbbk\widetilde{\Gamma} \to \operatorname{ind}\Gamma$. It induces a $\Bbbk$-linear functor $F\colon \Bbbk(\widetilde{\Gamma}) \to \operatorname{ind}\Gamma$ if and only if it vanishes on $\gamma_x = \gamma_{\pi x}$ for every non-projective vertex $x \in \widetilde{\Gamma}$. In such a case, F is well-behaved. Moreover, any well-behaved functor arises in this way.

2.3. **Local sections on almost split sequences.** The existence of well-behaved functors is based on the following technical lemma. It aims at constructing sections that are compatible with the standard modulation of Γ, in some sense.

Lemma. *Let $X \in \Gamma$ be a non-injective vertex and let*

$$0 \to X \xrightarrow{f} \bigoplus_{i=1}^{r} X_i^{n_i} \xrightarrow{g} \tau_A^{-1}X \to 0$$

be an almost split sequence. Let $\Bbbk_X \subseteq \operatorname{End}_A(X)$ and $\Bbbk_{X_i} \subseteq \operatorname{End}_A(X_i)$ (for every $i \in \{1,\dots,r\}$) be sections of κ_X and κ_{X_i} respectively, and let $\operatorname{irr}(X, X_i) \hookrightarrow \operatorname{rad}(X, X_i)$ be a $\Bbbk_X - \Bbbk_{X_i}$-linear section which maps $\overline{f_{i,1}}, \dots, \overline{f_{i,n_i}}$ to $f_{i,1}, \dots, f_{i,n_i}$ respectively, for every $i \in \{1,\dots,r\}$. There exists a section $\Bbbk_{\tau_A^{-1}X} \hookrightarrow \operatorname{End}_A(\tau_A^{-1}X)$ of $\kappa_{\tau_A^{-1}X}$ and a $\Bbbk_{X_i} - \Bbbk_{\tau_A^{-1}X}$-linear section $\operatorname{irr}(X_i, \tau_A^{-1}X) \to \operatorname{rad}(X_i, \tau_A^{-1}X)$ (for every $i \in \{1,\dots,r\}$) such that

(a) it maps $\overline{g_{i,1}}, \dots, \overline{g_{i,n_i}}$ to $g_{i,1}, \dots, g_{i,n_i}$ respectively, for every $i \in \{1,\dots,r\}$,
(b) the induced map $\oplus_{i=1}^{r} \operatorname{irr}(X, X_i) \otimes_{\kappa_{X_i}} \operatorname{irr}(X_i, \tau_A^{-1}X) \to \operatorname{Hom}_A(X, \tau_A^{-1}X)$ vanishes on $\gamma_{\tau_A^{-1}X}$.

Proof. Note that if such sections do exist then $(g_{i,1}, \dots, g_{i,n_i})$ must be a basis over $\Bbbk_{X_i}$ of the image of $\operatorname{irr}(X_i, \tau_A^{-1}X) \to \operatorname{rad}(X_i, \tau_A^{-1}X)$ because g is a right minimal almost split morphism. In particular, the section $\Bbbk_{\tau_A^{-1}X} \subseteq \operatorname{End}_A(\tau_A^{-1}X)$ must be such that the left $\Bbbk_{X_i}$-submodule of $\operatorname{rad}(X_i, \tau_A^{-1}X)$ generated by $g_{i,1}, \dots, g_{i,n_i}$ is a $\Bbbk_{X_i} - \Bbbk_{\tau_A^{-1}X}$-submodule of $\operatorname{rad}(X_i, \tau_A^{-1}X)$, for every $i \in \{1,\dots,r\}$. The proof therefore proceeds as follows: 1) define a section $\Bbbk_{\tau_A^{-1}X} \subseteq \operatorname{End}_A(\tau_A^{-1}X)$ satisfying this last condition, 2) define sections $\operatorname{irr}(X_i, \tau_A^{-1}X) \to \operatorname{rad}(X_i, \tau_A^{-1}X)$ so that (a) is satisfied, and 3) prove (b).

1) Let $\varphi \in \mathrm{End}_A(\tau_A^{-1}X)$ and define a new representative $\varphi_1 \in \mathrm{End}_A(\tau_A^{-1}X)$ of $\overline{\varphi} \in \kappa_{\tau_A^{-1}X}$ as follows. Let $0 \to X \xrightarrow{\alpha} \oplus_i X_i^{n_i} \xrightarrow{\beta} \tau_A^{-1}X \to 0$ be the almost split sequence used in the definition of the standard modulation of Γ. So there exists an isomorphism of exact sequences

$$\begin{array}{ccccccccc} (4) & 0 & \longrightarrow & X & \xrightarrow{\alpha} & \bigoplus_i X_i^{n_i} & \xrightarrow{\beta} & \tau_A^{-1}X & \longrightarrow 0 \\ & & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \| & \\ & 0 & \longrightarrow & X & \xrightarrow[f]{} & \bigoplus_i X_i^{n_i} & \xrightarrow[g]{} & \tau_A^{-1}X & \longrightarrow 0 \end{array} \quad .$$

There also exists a commutative diagram

$$\begin{array}{ccccccccc} (5) & 0 & \longrightarrow & X & \xrightarrow{f} & \bigoplus_i X_i^{n_i} & \xrightarrow{g} & \tau_A^{-1}X & \longrightarrow 0 \\ & & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \varphi} & \\ & 0 & \longrightarrow & X & \xrightarrow[f]{} & \bigoplus_i X_i^{n_i} & \xrightarrow[g]{} & \tau_A^{-1}X & \longrightarrow 0 \end{array}$$

for some morphisms θ and ψ. The pair (θ, ψ) in the above diagram is not unique; however for any different pair (θ', ψ') such that the diagram (5) still commutes after replacing θ and ψ by θ' and ψ', respectively, then there exists $h\colon \oplus_i X^{n_i} \to X$ such that $\theta - \theta' = hf$; given that $(\psi - \psi')f = f(\theta - \theta')$ and that f is a monomorphism, it follows that $\psi - \psi' = fh$; note that none of the X_i is isomorphic to X because f is irreducible; therefore h lies in rad, and hence $\psi - \psi'$ lies in rad^2. The diagrams (4) and (5) yield the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{\alpha} & \bigoplus_i X_i^{n_i} & \xrightarrow{\beta} & \tau_A^{-1}X & \longrightarrow 0 \\ & & \downarrow{\scriptstyle u\psi u^{-1}} & & \downarrow{\scriptstyle v\theta v^{-1}} & & \downarrow{\scriptstyle \varphi} & \\ 0 & \longrightarrow & X & \xrightarrow[\alpha]{} & \bigoplus_i X_i^{n_i} & \xrightarrow[\beta]{} & \tau_A^{-1}X & \longrightarrow 0 \end{array} \quad .$$

Therefore, $\tau_*(\overline{\varphi}) = \overline{u\psi u^{-1}} \in \kappa_X$. The previous discussion on the uniqueness of the pair (θ, ψ) shows that $\overline{\psi}$ is uniquely determined by φ even though ψ is not. Now let $\psi_1 \in \Bbbk_X$ be the representative of ψ, that is, $\overline{\psi_1} = \overline{\psi}$. Since the section $\mathrm{irr}(X, X_i) \to \mathrm{rad}(X, X_i)$ is $\Bbbk_X - \Bbbk_{X_i}$-linear and maps $\overline{f_{i,1}}, \ldots, \overline{f_{i,n_i}}$ to $f_{i,1}, \ldots, f_{i,n_i}$, respectively, and since $(\overline{f_{i,1}}, \ldots, \overline{f_{i,n_i}})$ is a basis of the right $\Bbbk_{X_i}$-module $\mathrm{irr}(X, X_i)$, there is a unique matrix $\eta_i \in M_{n_i}(\Bbbk_{X_i})$, considered as an endomorphism of $X_i^{n_i}$, making the following square commute for $i \in \{1, \ldots, r\}$

$$\begin{array}{ccc} X & \xrightarrow{[f_{i,1}, \ldots, f_{i,n_i}]} & X_i^{n_i} \\ {\scriptstyle \psi_1}\downarrow & & \downarrow{\scriptstyle \eta_i} \\ X & \xrightarrow[[f_{i,1}, \ldots, f_{i,n_i}]]{} & X_i^{n_i} \end{array} \quad .$$

Therefore there exists a unique $\varphi_1 \in \mathrm{End}_A(\tau_A^{-1}X)$ making the following diagram commute where $\eta\colon \oplus_i X_i^{n_i} \to \oplus_i X_i^{n_i}$ is the morphism defined by $\{\eta_i\colon X_i^{n_i} \to X_i^{n_i}\}_i$.

$$\begin{array}{ccccccccc} (6) & 0 & \longrightarrow & X & \xrightarrow{f} & \bigoplus_i X_i^{n_i} & \xrightarrow{g} & \tau_A^{-1}X & \longrightarrow 0 \\ & & & \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \varphi_1} & \\ & 0 & \longrightarrow & X & \xrightarrow[f]{} & \bigoplus_i X_i^{n_i} & \xrightarrow[g]{} & \tau_A^{-1}X & \longrightarrow 0 \end{array} \quad .$$

Hence the following diagram commutes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{\alpha} & \bigoplus_i X_i^{n_i} & \xrightarrow{\beta} & \tau_A^{-1}X & \longrightarrow & 0 \\ & & {\scriptstyle u\psi_1 u^{-1}}\downarrow & & \downarrow{\scriptstyle v\eta v^{-1}} & & \downarrow{\scriptstyle \varphi_1} & & \\ 0 & \longrightarrow & X & \xrightarrow[\alpha]{} & \bigoplus_i X_i^{n_i} & \xrightarrow[\beta]{} & \tau_A^{-1}X & \longrightarrow & 0 \end{array} .$$

This entails $\overline{\varphi_1} = \tau_*^{-1}(\overline{u\psi_1 u^{-1}})$. But $\overline{\psi_1} = \overline{\psi}$ so that $\overline{u\psi_1 u^{-1}} = \overline{u\psi u^{-1}}$. Thus $\overline{\varphi_1} = \tau_*^{-1}(\overline{u\psi u^{-1}}) = \overline{\varphi}$. This construction therefore yields a well-defined map

$$\begin{array}{rccl} s\colon & \mathrm{End}_A(\tau_A^{-1}X) & \to & \mathrm{End}_A(\tau_A^{-1}X) \\ & \varphi & \mapsto & \varphi_1 \ \text{(such that } \overline{\varphi_1} = \overline{\varphi}). \end{array}$$

Since $\tau_*\colon \kappa_{\tau_A^{-1}X} \to \kappa_X$ is a $\Bbbk$-algebra isomorphism and $\eta_1, \dots, \eta_r$ are determined by ψ_1 (and the fixed $\Bbbk$-algebra sections), the map s is a $\Bbbk$-algebra homomorphism. Moreover if $\varphi \in \mathrm{rad}(\tau_A^{-1}X, \tau_A^{-1}X)$ then $\overline{\varphi} = 0$ and the representative ψ_1 in $\Bbbk_X$ of $\tau_*(\overline{\varphi}) = 0$ is 0; therefore, $\eta_1, \dots, \eta_r = 0$ and $\varphi_1 = 0$; in other words s vanishes on $\mathrm{rad}(\tau_A^{-1}X, \tau_A^{-1}X)$. Hence, s induces a $\Bbbk$-algebra homomorphism

$$\overline{s}\colon \kappa_{\tau_A^{-1}X} \to \mathrm{End}_A(\tau_A^{-1}X)$$

such that $\overline{s}(\overline{\varphi}) = \varphi$ for all $\varphi \in \mathrm{End}_A(\tau_A^{-1}X)$. Denote by $\Bbbk_{\tau_A^{-1}X}$ the image of $\overline{s}$. Thus, $\Bbbk_{\tau_A^{-1}X} \subseteq \mathrm{End}_A(\tau_A^{-1}X)$ is a $\Bbbk$-algebra section of $\kappa_{\tau_A^{-1}X}$.

2) In order to prove (a) there only remains to define a $\Bbbk_{X_i} - \Bbbk_{\tau_A^{-1}X}$-linear section $\mathrm{irr}(X_i, \tau_A^{-1}X) \to \mathrm{rad}(X_i, \tau_A^{-1}X)$ which maps $\overline{g_{i,1}}, \dots, \overline{g_{i,n_i}}$ to $g_{i,1}, \dots, g_{i,n_i}$, respectively, for every $i \in \{1, \dots, r\}$. Let $i \in \{1, \dots, r\}$. Since g is right minimal almost split, then $\{\overline{g_{i,j}}\}_{1 \leqslant j \leqslant n_i}$ is a basis of $\mathrm{irr}(X_i, \tau_A^{-1}X)$ as a left κ_{X_i}-vector space, and hence as a left $\Bbbk_{X_i}$-vector space. Denote by s_i the unique $\Bbbk_{X_i}$-linear mapping

$$s_i\colon \mathrm{irr}(X_i, \tau_A^{-1}X) \to \mathrm{rad}(X_i, \tau_A^{-1}X)$$

which maps $\overline{g_{i,j}}$ to $g_{i,j}$ for every $j \in \{1, \dots, n_i\}$. In view of the assumption on the section $\Bbbk_{X_i} \subseteq \mathrm{End}_A(X_i)$ of κ_{X_i}, the construction of s_i entails that the identity mapping of $\mathrm{irr}(X_i, \tau_A^{-1}X)$ is equal to the composite morphism

$$\mathrm{irr}(X_i, \tau_A^{-1}X) \xrightarrow{s_i} \mathrm{rad}(X_i, \tau_A^{-1}X) \twoheadrightarrow \mathrm{irr}(X_i, \tau_A^{-1}X)\,,$$

where the right hand-side arrow is the natural surjection. In order to prove that s_i is a $\Bbbk_{X_i} - \Bbbk_{\tau_A^{-1}X}$-linear section, there only remains to prove that s_i is $\Bbbk_{\tau_A^{-1}X}$-linear. Let $\varphi \in \Bbbk_{\tau_A^{-1}X}$. Using the above notation in the construction of $\Bbbk_{\tau_A^{-1}X}$, one has $\varphi = \varphi_1$. It follows from (6) that

$$[g_{i,1}, \dots, g_{i,n_i}]^t \cdot \varphi = \eta_i \cdot [g_{i,1}, \dots, g_{i,n_i 1}]^t\,.$$

Writing the endomorphism $\eta_i\colon X_i^{n_i} \to X_i^{n_i}$ coordinatewise as $[\eta_{i,j,j'}\,;\, 1 \leqslant j, j' \leqslant n_i]$, where $\eta_{i,j,j'} \in \Bbbk_{X_i}$ for all j, j', it follows that

$$(\forall j \in \{1, \dots, n_i\})\ g_{i,j}\varphi = \sum_{j'=1}^{n_i} \eta_{i,j,j'} g_{i,j'}\,.$$

Applying s_i which is $\Bbbk_{X_i}$-linear and satisfies $s_i(\overline{g_{i,j}}) = g_{i,j}$ for all j entails that

$$(\forall j \in \{1, \dots, n_i\})\ s_i(\overline{g_{i,j}\varphi}) = g_{i,j}\varphi = s_i(\overline{g_{i,j}})\varphi\,.$$

Accordingly, s_i is $\Bbbk_{\tau_A^{-1}X}$-linear, which finishes the proof of (a).

3) There only remains to prove (b). According to the lemma in 1.4, the commutative diagram (4) entails that $\gamma_{\tau_A^{-1}X} = \sum_{i=1}^{r}\sum_{j=1}^{n_i} \overline{u f_{i,j}} \otimes \overline{g_{i,j}}$. Let $u' \in \Bbbk_X$ be the representative of $\overline{u} \in \kappa_X$. The image of $\gamma_{\tau_A^{-1}X}$ under the map $\oplus_{i=1}^{r}\mathrm{irr}(X, X_i) \otimes_{\kappa_{X_i}} \mathrm{irr}(X_i, \tau_A^{-1}X) \to \mathrm{Hom}_A(X, \tau_A^{-1}X)$ induced by all the considered sections is therefore $\sum_{i,j} u'\, f_{i,j}\, g_{i,j} = u' \sum_{i,j} f_{i,j}\, g_{i,j} = u' f g = 0$. □

Dual considerations yield the following dual version of the preceding lemma.

Lemma. *Let $X \in \Gamma$ be a non-injective vertex and let*

$$0 \to X \xrightarrow{f} \bigoplus_{i=1}^{r} X_i^{n_i} \xrightarrow{g} \tau_A^{-1}X \to 0$$

be an almost split sequence. Let $\Bbbk_{\tau_A^{-1}X} \subseteq \mathrm{End}_A(X)$ and $\Bbbk_{X_i} \subseteq \mathrm{End}_A(X_i)$ (for $i \in \{1, \dots, r\}$) be sections of $\kappa_{\tau_A^{-1}X}$ and κ_{X_i} respectively, and let $\mathrm{irr}(X_i, \tau_A^{-1}X) \hookrightarrow \mathrm{rad}(X_i, \tau_A^{-1}X)$ be a $\Bbbk_{X_i}$–$\Bbbk_{\tau_A^{-1}X}$-linear section under which $\overline{g_{i,1}}, \dots, \overline{g_{i,n_i}}$ are mapped to $g_{i,1}, \dots, g_{i,n_i}$ respectively, for every $i \in \{1, \dots, r\}$. There exists a section $\Bbbk_X \hookrightarrow \mathrm{End}_A(X)$ of κ_X and a $\Bbbk_X - \Bbbk_{X_i}$-linear section $\mathrm{irr}(X, X_i) \to \mathrm{rad}(X, X_i)$ (for every $i \in \{1, \dots, r\}$) such that

(a) it maps $\overline{f_{i,1}}, \dots, \overline{f_{i,n_i}}$ to $f_{i,1}, \dots, f_{i,n_i}$ respectively, for every $i \in \{1, \dots, r\}$,
(b) the induced map $\oplus_{i=1}^{r}\mathrm{irr}(X, X_i) \otimes_{\kappa_{X_i}} \mathrm{irr}(X_i, \tau_A^{-1}X) \to \mathrm{Hom}_A(X, \tau_A^{-1}X)$ vanishes on $\gamma_{\tau_A^{-1}X}$.

2.4. **Inductive construction of well-behaved functors.** In view of proving the existence of well-behaved functors $\Bbbk(\widetilde{\Gamma}) \to \mathrm{ind}\,\Gamma$ it is necessary to consider $\Bbbk$-linear functors $F\colon \Bbbk\chi \to \mathrm{ind}\,\Gamma$ where χ is a full and convex subquiver of $\widetilde{\Gamma}$. Here the notation $\Bbbk\chi$ stands for the full subcategory of $\Bbbk\widetilde{\Gamma}$ with object set being the set of vertices in χ. Following 2.2, such a functor $F\colon \Bbbk\chi \to \mathrm{ind}\,\Gamma$ is called well-behaved if: (a) $Fx = \pi x$ for every vertex $x \in \chi$; (b) the $\Bbbk$-algebra homomorphism $\kappa_x \hookrightarrow \Bbbk\chi(x, x) \to \mathrm{End}_A(\pi x)$ is a section of the natural surjection $\mathrm{End}_A(\pi x) \twoheadrightarrow \kappa_{\pi x} = \kappa_x$, for every vertex $x \in \chi$ (as above, the image of the section is denoted by $\Bbbk_x$); (c) the $\Bbbk$-linear composite map $\mathrm{irr}(\pi x, \pi y) = M(x, y) \hookrightarrow \Bbbk\chi(x, y) \to \mathrm{Hom}_A(\pi x, \pi y)$ is a $\Bbbk_x - \Bbbk_y$-linear section, for every arrow $x \to y$ in χ; and (d) it vanishes on $\gamma_x = \gamma_{\pi x}$ for every non-projective vertex $x \in \chi$ such that $\tau x \in \chi$.

Lemma. *Assume that $\widetilde{\Gamma}$ is with length. Let ℓ be a length function on the vertices in $\widetilde{\Gamma}$. Let $F\colon \Bbbk\chi \to \mathrm{ind}\,\Gamma$ be a well-behaved functor where $\chi \subseteq \widetilde{\Gamma}$ is a full and convex subquiver distinct from $\widetilde{\Gamma}$ and satisfying the following two conditions*

(i) either there is no arrow $x \to x_1$ in $\widetilde{\Gamma}$ such that $x_1 \in \chi$ and $x \notin \chi$, or else there is an upper bound on the integers $\ell(x)$ where x runs through the vertices in $\widetilde{\Gamma}\backslash\chi$ such that there exists an arrow $x \to x_1$ in $\widetilde{\Gamma}$ satisfying $x_1 \in \chi$,
(ii) either there is no arrow $x_1 \to x$ in $\widetilde{\Gamma}$ such that $x_1 \in \chi$ and $x \notin \chi$, or else there is a lower bound on the integers $\ell(x)$ where x runs through the vertices in $\widetilde{\Gamma}\backslash\chi$ such that there exists an arrow $x_1 \to x$ in $\widetilde{\Gamma}$ satisfying $x_1 \in \chi$.

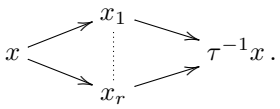
Then there exists at least one arrow such as in (i) or (ii) and for every arrow $x \to x_1$ (or, $x_1 \to x$) in $\widetilde{\Gamma}$ such that $x \notin \chi$ and $x_1 \in \chi$, with $\ell(x)$ maximal (or, minimal, respectively) for these properties, there exists a well-behaved functor

$F'\colon \Bbbk\chi' \to \operatorname{ind}\Gamma$ *which extends* F*, where* χ' *denotes the full subquiver of* $\widetilde{\Gamma}$ *with vertices* x *and the vertices in* χ*.*

Proof. There exists an arrow $x \to x_1$ or $x_1 \to x$ in $\widetilde{\Gamma}$ such that $x \notin \chi$ and $x_1 \in \chi$ because $\chi \subsetneq \widetilde{\Gamma}$. Assume the former (the latter is dealt with similarly) and choose x so that $\ell(x)$ is maximal ((i)). Let χ' be the full subquiver of $\widetilde{\Gamma}$ with vertices x and those of χ. Then χ' is convex in $\widetilde{\Gamma}$ because $\ell(x)$ is maximal. Let $x \to x_1, \dots, x \to x_r$ be the arrows in $\widetilde{\Gamma}$ starting in x and ending in some vertex in χ. Note that if x is non-injective and $\tau^{-1}x \in \chi$ then these are all the arrows in $\widetilde{\Gamma}$ starting in x because $\ell(x)$ is maximal. In order to extend $F\colon \Bbbk\chi \to \operatorname{ind}\Gamma$ to a functor $F'\colon \Bbbk\chi' \to \operatorname{ind}\Gamma$ distinguish two cases according to whether x is non-injective and $\tau^{-1}x \in \chi$, or not. For every $i \in \{1, \dots, r\}$ let $\Bbbk_i \subseteq \operatorname{End}_A(\pi x_i)$ be the section $F(\kappa_{x_i})$ of $\kappa_{\pi x_i}$.

Assume first that either x is injective, or else x is non-injective and $\tau^{-1}x \notin \chi$. Fix any $\Bbbk$-algebra section $\Bbbk_x \subseteq \operatorname{End}_A(\pi x)$ of $\kappa_{\pi x}$. Let $i \in \{1, \dots, r\}$. The $\Bbbk$-algebra isomorphisms $\kappa_{x_i} \to \Bbbk_i$ (induced by F) and $\kappa_x \to \Bbbk_x$ allow one to consider $M(x, x_i)$ as a $\Bbbk_x - \Bbbk_i$-bimodule. For this structure, the quotient map $\operatorname{Hom}_A(\pi x, \pi x_i) = \operatorname{rad}(\pi x, \pi x_i) \overset{p_i}{\twoheadrightarrow} \operatorname{irr}(\pi x, \pi x_i)$ is $\Bbbk_x - \Bbbk_i$-linear. On the other hand the $\Bbbk$-algebra $\Bbbk_x \otimes_{\Bbbk} \Bbbk_i^{op}$ is semi-simple. Hence there exists a $\Bbbk_x - \Bbbk_i$-linear section $M(x, x_i) = \operatorname{irr}(\pi x, \pi x_i) \xrightarrow{s_i} \operatorname{Hom}_A(\pi x, \pi x_i)$ of p_i. The sections $\Bbbk_x \subseteq \operatorname{End}_A(\pi x)$ and s_i ($i \in \{1, \dots, r\}$) extend $F\colon \Bbbk\chi \to \operatorname{ind}\Gamma$ to a $\Bbbk$-linear functor $F'\colon \Bbbk\chi' \to \operatorname{ind}\Gamma$ satisfying the conditions (a), (b) and (c) in the definition of well-behaved functors. Moreover, condition (d) is satisfied for F' because it is so for F and because, if $y \in \widetilde{\Gamma}$ is non-projective and $y, \tau y \in \chi'$ then $y, \tau y \in \chi$. This proves the lemma when either x is injective or else x is non-injective and $\tau^{-1}x \notin \chi$.

Assume now that x is non-injective and $\tau^{-1}x \in \chi$. The mesh in $\widetilde{\Gamma}$ starting in x has the form



For simplicity let $X = \pi x$ and $X_i = \pi x_i$ for every $i \in \{1, \dots, r\}$ so that $\pi\tau^{-1}x = \tau_A^{-1}X$. Let $n_1, \dots, n_r \geqslant 1$ be the integers such that there is an almost split sequence $0 \to X \to \oplus_{i=1}^r X_i^{n_i} \to \tau_A^{-1}X \to 0$. Since $F\colon \Bbbk\chi \to \operatorname{ind}\Gamma$ is well-behaved, it induces a $\Bbbk$-algebra section $\Bbbk_{\tau_A^{-1}X} \subseteq \operatorname{End}_A(\tau_A^{-1}X)$ of $\kappa_{\tau^{-1}x} = \kappa_{\tau_A^{-1}X}$. It also induces a $\Bbbk_i - \Bbbk_{\tau_A^{-1}X}$-linear section $M(x_i, \tau^{-1}x) = \operatorname{irr}(X_i, \tau_A^{-1}X) \to \operatorname{rad}(X_i, \tau_A^{-1}X)$, for every $i \in \{1, \dots, r\}$. Let $i \in \{1, \dots, r\}$, let $\{g_{i,j}\colon X_i \to \tau_A^{-1}X\}_{1 \leqslant j \leqslant n_i}$ be the image under this $\Bbbk_i - \Bbbk_{\tau_A^{-1}X}$-linear section of a basis of the $\Bbbk_i$-vector space $M(x_i, \tau^{-1}x)$. Thus, the morphism $\bigoplus_{i=1}^r X_i^{n_i} \xrightarrow{g := [g_{i,j}\,;\, i,j]^t} \tau_A^{-1}X$ is right minimal almost split. Let $f = [f_{i,j}\,;\, i,j] : X \to \oplus_{i=1}^r X_i^{n_i}$ be its kernel. Then 2.3 yields a $\Bbbk$-algebra section $\Bbbk_X \subseteq \operatorname{End}_A(\tau_A^{-1}X)$ of κ_X together with $\Bbbk_X - \Bbbk_i$-linear sections $M(x, x_i) = \operatorname{irr}(X, X_i) \to \operatorname{rad}(X, X_i)$, for $1 \leqslant i \leqslant r$, such that the induced map

$$\bigoplus_{i=1}^r M(x, x_i) \otimes_{\kappa_{x_i}} M(x_i, \tau^{-1}x) \longrightarrow \operatorname{Hom}_A(X, \tau_A^{-1}X)$$

vanishes on $\gamma_{\tau_A^{-1}X}$. These new sections clearly extend $F\colon \Bbbk\chi \to \operatorname{ind}\Gamma$ to a well-behaved functor $F'\colon \Bbbk\chi' \to \operatorname{ind}\Gamma$. Like in the previous case $F'\colon \Bbbk\chi' \to \operatorname{ind}\Gamma$ is

well-behaved. Finally χ' satisfies both conditions (i) and (ii) in the statement of the lemma since $\chi'\backslash\chi$ consists of one vertex. □

2.5. **Existence of well-behaved functors.** For a strongly irreducible morphism $f\colon X \to \oplus_{i=1}^{r} X_i^{n_i}$, the $\kappa_X - \kappa_{X_i}$-bimodule $\kappa_X \cdot \overline{f_{i,j}} \cdot \kappa_{X_i}$ is projective and given by an idempotent of $\kappa_X \otimes \kappa_{X_i}^{\mathrm{op}}$, for all i, j, hence, the following are equivalent

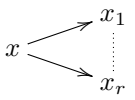
- for all i, j, the annihilator of $\overline{f_{i,j}}$ over $\kappa_X \otimes_{\Bbbk} \kappa_{X_i}^{\mathrm{op}}$ is trivial,
- for all i, the family $\{\overline{f_{i,j}}\}_j$ of $\mathrm{irr}(X, X_i)$ is free over $\kappa_X \otimes_{\Bbbk} \kappa_{X_i}^{\mathrm{op}}$.

The following result implies Theorem A.

Proposition. *Let A be a finite-dimensional algebra over a perfect field $\Bbbk$. Let Γ be an Auslander-Reiten component of A. Let $\pi\colon \widetilde{\Gamma} \to \Gamma$ be a covering of translation quivers where $\widetilde{\Gamma}$ is connected and with length. Endow Γ with its standard $\Bbbk$-modulation and $\widetilde{\Gamma}$ with the induced $\Bbbk$-modulation.*

(1) There exists a well-behaved functor $F\colon \Bbbk(\widetilde{\Gamma}) \to \operatorname{ind}\Gamma$.

(2) Let $X \in \Gamma$ and let $f\colon X \to \oplus_{i=1}^{r} X_i^{n_i}$ be a strongly irreducible morphism. Assume that the annihilator of $\overline{f_{i,j}}$ over $\kappa_X \otimes_{\Bbbk} \kappa_{X_i}^{\mathrm{op}}$ is zero for all i, j. Let $x \in \pi^{-1}(X)$ and let



be the full subquiver of $\widetilde{\Gamma}$ such that $\pi x_i = X_i$. Then there exists a well-behaved functor $F\colon \Bbbk(\widetilde{\Gamma}) \to \operatorname{ind}\Gamma$ such that F maps $\overline{f_{i,j}} \in \Bbbk(\widetilde{\Gamma})(x, x_i)$ to $f_{i,j}$ for every $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, n_i\}$.

Proof. It is possible to assume that A is connected and not semi-simple. Both (1) and (2) are proved with the same inductive construction yet with a specific initialisation step for each. In case (1), let $r = 1$, let $X \to X_1$ be any arrow in Γ and let $x \to x_1$ be any arrow in $\widetilde{\Gamma}$ with image under π equal to $X \to X_1$. In both cases, let χ_0 be the full subquiver of $\widetilde{\Gamma}$ with vertices $x, x_1, \dots, x_r$.

Let Σ be the set of pairs $(\chi, F\colon \Bbbk\chi \to \operatorname{ind}\Gamma)$ where $\chi \subseteq \widetilde{\Gamma}$ is a full and convex subquiver containing χ_0 and which satisfies conditions (i) and (ii) in 2.4, and $F\colon \Bbbk\chi \to \operatorname{ind}\Gamma$ is a well-behaved functor with the following additional condition in case (2): It maps $\overline{f_{i,j}} \in \Bbbk(\widetilde{\Gamma})(x, x_i)$ to $f_{i,j}$ for all i, j. This set is ordered: $(\chi, F) \leqslant (\chi', F')$ if and only if $\chi \subseteq \chi'$ and F' restricts to F.

Let ℓ be a length function on $\widetilde{\Gamma}$. Denote $\ell(x)$ by ℓ_0. Denote by $\widetilde{\Gamma}_{\geqslant \ell_0}$ the full subquiver of $\widetilde{\Gamma}$ with set of vertices being $\{x' \in \widetilde{\Gamma} \mid \ell(x') \geqslant \ell_0\}$. Consider the subsets Σ_1, Σ_2 and Σ_3 of Σ defined as follows.

- Σ_1 consists of those $(\chi, F) \in \Sigma$ such that the restriction of ℓ to χ is bounded below by ℓ_0 and bounded above by $\ell_0 + 1$.
- Σ_2 consists of those $(\chi, F) \in \Sigma$ such that $\chi \subseteq \widetilde{\Gamma}_{\geqslant \ell_0}$ and χ is stable under predecessors in $\widetilde{\Gamma}_{\geqslant \ell_0}$, that is, any oriented path in $\widetilde{\Gamma}_{\geqslant \ell_0}$ with endpoint in χ is contained in χ.
- Σ_3 consists of the those $(\chi, F) \in \Sigma$ such that $\widetilde{\Gamma}_{\geqslant \ell_0} \subseteq \chi$ and χ is stable under successors in $\widetilde{\Gamma}$.

The set Σ_1 is not empty. Indeed, let $F\colon \Bbbk\chi_0 \to \operatorname{ind}\Gamma$ be as follows. In case (1), let $\Bbbk_X$ and $\Bbbk_{X_1}$ be any sections of κ_X and κ_{X_1} in $\mathrm{End}_A(X)$ and $\mathrm{End}_A(X_1)$,

respectively, let $\operatorname{irr}(X, X_1) \to \operatorname{rad}(X, X_1)$ be any section of the $\Bbbk_X - \Bbbk_{X_1}$-linear canonical surjection $\operatorname{rad}(X, X_1) \to \operatorname{irr}(X, X_1)$ and let $F\colon \Bbbk\chi_0 \to \operatorname{ind}\widetilde{\Gamma}$ be given by the sections $\Bbbk_X$, $\Bbbk_{X_1}$ and $\operatorname{irr}(X, X_1) \to \operatorname{rad}(X, X_1)$. In this case, (χ_0, F) lies in Σ_1. In case (2), fix $\Bbbk$-algebra sections $\Bbbk_X \subseteq \operatorname{End}_A(X)$ of κ_X, and $\Bbbk_{X_i} \subseteq \operatorname{End}_A(X_i)$ of κ_{X_i}, for every $i \in \{1, \dots, r\}$. Consider an index $i \in \{1, \dots, r\}$. By assumption on f, there exists a direct sum decomposition $\operatorname{irr}(X, X_i) = (\oplus_j \Bbbk_X \cdot \overline{f_{i,j}} \cdot \Bbbk_{X_i}) \oplus M_i$ as $\Bbbk_X \otimes_\Bbbk \Bbbk_{X_i}^{\mathrm{op}}$-modules. Let $\operatorname{irr}(X, X_i) \to \operatorname{rad}(X, X_i)$ be the following $\Bbbk_X \otimes_\Bbbk \Bbbk_{X_i}^{\mathrm{op}}$-linear section of the natural surjection $\operatorname{rad}(X, X_i) \twoheadrightarrow \operatorname{irr}(X, X_i)$ (recall that the algebra $\Bbbk_X \otimes_\Bbbk \Bbbk_{X_i}^{\mathrm{op}}$ is semi-simple),

- for all j, the composite morphism $\Bbbk_X \cdot \overline{f_{i,j}} \cdot \Bbbk_{X_i} \hookrightarrow \operatorname{irr}(X, X_i) \to \operatorname{rad}(X, X_i)$ is the $\Bbbk_X \otimes_\Bbbk \Bbbk_{X_i}^{\mathrm{op}}$-linear section of the composite morphism $\operatorname{rad}(X, X_i) \twoheadrightarrow \operatorname{irr}(X, X_i) \twoheadrightarrow \Bbbk_X \cdot \overline{f_{i,j}} \cdot \Bbbk_{X_i}$ which maps $\overline{f_{i,j}}$ onto $f_{i,j}$, recall that the annihilator of $\overline{f_{i,j}}$ over $\Bbbk_X \otimes_\Bbbk \Bbbk_{X_i}^{\mathrm{op}}$ is trivial,
- the composite morphism $M_i \hookrightarrow \operatorname{irr}(X, X_i) \to \operatorname{rad}(X, X_i)$ is any $\Bbbk_X \otimes_\Bbbk \Bbbk_{X_i}^{\mathrm{op}}$-linear section of the composite morphism $\operatorname{rad}(X, X_i) \twoheadrightarrow \operatorname{irr}(X, X_i) \twoheadrightarrow M_i$.

These sections define a well-behaved functor $F\colon \Bbbk\chi_0 \to \operatorname{ind}\Gamma$. Then $(\chi_0, F) \in \Sigma_1$.

Therefore, by definition, Σ_1 is totally inductive. Hence it has at least one maximal element, say, (χ_1, F_1), by the Kuratowski-Zorn Lemma. Apply 2.4 to it: since (χ_1, F_1) is maximal in Σ_1, it follows that χ_1 is stable under predecessors in $\widetilde{\Gamma}_{\geqslant \ell_0}$. Thus, Σ_2 contains (χ_1, F_1), and hence it is not empty.

Since Σ_2 is not empty, then it is totally inductive by construction. Consider a maximal element (χ_2, F_2) in Σ_2. In this case, χ_2 equals $\widetilde{\Gamma}_{\geqslant \ell_0}$. Indeed, apply 2.4 to it: since (χ_2, F_2) is maximal in Σ_2, it follows that there is no arrow in $\widetilde{\Gamma}$ connecting a vertex in χ_2 to a vertex in $\widetilde{\Gamma}_{\geqslant \ell_0} \backslash \chi_2$. Hence, should there exist $x' \in \widetilde{\Gamma}_{\geqslant \ell_0} \backslash \chi_2$, then $\chi_2 \cup \{x'\}$ would be a full subquiver of $\widetilde{\Gamma}$ and any $\Bbbk$-algebra section of $\kappa_{\pi x'}$ in $\operatorname{End}_A(\pi x')$ would define an extension of F_2 to a well-behaved functor $\chi_2 \cup \{x'\} \to \operatorname{ind} A$, a contradiction to (χ_2, F_2) being maximal in Σ_2. Thus, $\chi_2 = \widetilde{\Gamma}_{\geqslant \ell_0}$. In particular, Σ_3 is not empty.

Therefore, by definition, Σ_3 is totally inductive. Consider a maximal element (χ_3, F_3) in Σ_3. By absurd, assume that the inclusion $\chi_3 \subseteq \widetilde{\Gamma}$ is strict. Then, there exists $x' \in \widetilde{\Gamma} \backslash \chi_3$ with $\ell(x')$ maximal because $\widetilde{\Gamma}_{\geqslant \ell_0} \subseteq \chi_3$. There is no arrow $x_1 \to x'$ such that $x_1 \in \chi_3$ because χ_3 is stable under successors. Should there exist no arrow $x' \to x_1$ in $\widetilde{\Gamma}$ such that $x_1 \in \chi_3$, then $\chi_3 \cup \{x'\}$ would be a full subquiver of $\widetilde{\Gamma}$ containing $\widetilde{\Gamma}_{\geqslant \ell_0}$ and stable under successors, and any $\Bbbk$-algebra section of $\kappa_{\pi x'}$ in $\operatorname{End}_A(\pi x')$ would define an extension of F_3 to a well-behaved functor $\chi_3 \cup \{x'\} \to \operatorname{ind} A$, a contradiction to (χ_3, F_3) being maximal in Σ_3. There hence exists an arrow $x' \to x_1$ such that $x_1 \in \chi_3$. Now, apply 2.4 to (χ_3, F_3): by assumption on $\ell(x')$, there exists a well-behaved functor $F_3'\colon \chi_3' \to \operatorname{ind} A$ which extends F_3, where χ_3' is the full subquiver of $\widetilde{\Gamma}$ with vertices x' and those of χ_3. The assumption on $\ell(x')$ entails that $(\chi_3', F_3') \in \Sigma_3$, a contradiction to (χ_3, F_3) being maximal in Σ_3. Thus, $\chi_3 = \widetilde{\Gamma}$. □

The authors acknowledge the referee for pointing out the missing hypothesis on the annihilators in a previous version of this text. It would be interesting to determine when such annihilators are indeed trivial. For instance, this is the case when $\kappa_X = \Bbbk$ because, then, $\kappa_X \otimes_\Bbbk \kappa_{X_i}^{\mathrm{op}}$ is a division algebra for all i.

2.6. **Covering property of well-behaved functors.** Theorem B is an adaptation of [CLMT11, Thm. B] to perfect fields.

Proof of Theorem B. The proof uses a specific left minimal almost split morphism and a specific almost split sequence that arise from F and which are now introduced. Let $X = \pi x$. For every arrow in $\widetilde{\Gamma}$ with source x (say, with target x'), fix one basis over $\kappa_{\pi x'}$ of $\mathrm{irr}(\pi x, \pi x')$. Putting these bases together (for all the arrows in $\widetilde{\Gamma}$ with source x) yields a sequence of morphisms in $\Bbbk(\widetilde{\Gamma})$ with domain x. Say, the sequence is $(\alpha_i)_{i=1,\dots,r}$ where the codomain of α_i is denoted by x_i (there may be repetitions in the sequence of codomains). Set $X_i = \pi x_i$ and $a_i = F(\alpha_i)$ for every i. In particular $a_i \colon X \to X_i$ is an irreducible morphism and $\overline{a_i} = \alpha_i$ if $\overline{a_i}$ is considered as lying in $\Bbbk(\widetilde{\Gamma})(x, x_i)$. By construction, $[a_1, \dots, a_r] \colon X \to \oplus_{i=1}^r X_i$ is a left minimal almost split morphism. If x is non-injective then a completes into an almost split sequence $0 \to X \xrightarrow{a} \oplus_{i=1}^r X_i \xrightarrow{b} \tau_A^{-1} X \to 0$ as follows. For every $X' \in \{X_1, \dots, X_r\}$ the family $\{\overline{a_i}\}_{i\, s.t.\, X'=X_i}$ is a basis of $\mathrm{irr}(X, X')$ over $\kappa_{X'}$; let $\{\beta_i\}_{i\, s.t.\, X'=X_i}$ be the corresponding dual basis of $\mathrm{irr}(X', \tau_A^{-1}X)$ over $\kappa_{X'}$ (for the standard $\Bbbk$-modulation of Γ); For every $i \in \{1, \dots, r\}$ such that $X' = X_i$ set $b_i \colon X_i \to \tau_A^{-1}X$ to be the image of $\beta_i \in \Bbbk(\widetilde{\Gamma})(x_i, \tau^{-1}x)$ under F (hence, if one considers $\overline{b_i}$ as lying in $\Bbbk(\widetilde{\Gamma})(x_i, \tau^{-1}x)$ then $\overline{b_i} = \beta_i$). By construction $\gamma_{\tau^{-1}x} = \sum_{i=1}^r \overline{a_i} \otimes \overline{b_i}$. Therefore $\sum_{i=1}^r a_i b_i = 0$ because F is well-behaved and maps each $\overline{a_i}$ and $\overline{b_i}$ to a_i and b_i, respectively. Since moreover a is left minimal almost split, the morphism $b = [b_1, \dots, b_r]^t$ is right minimal almost split. Thus (a, b) forms the announced almost split sequence.

(a) The two maps are dual to each other so only the first one is taken care of.

The surjectivity for every x is proved by induction on $n \geqslant 0$. If $n = 0$ it follows from: $\mathrm{rad}^0(Fx, Fy)/\mathrm{rad}(Fx, Fy)$ is κ_{Fx} or 0 according to whether $Fx = Fy$ or $Fx \neq Fy$, and $\mathcal{R}^0\Bbbk(\widetilde{\Gamma})(x, z)/\mathcal{R}\Bbbk(\widetilde{\Gamma})(x, z)$ is κ_x or 0 according to whether $x = z$ or $x \neq z$. If $n \geqslant 1$, if the surjectivity is already proved for indices smaller that n, and if $f \in \mathrm{rad}^n(Fx, Fy)$ is given, then there exists $(u_i)_i \in \oplus_{i=1}^r \mathrm{rad}^{n-1}(Fx_i, Fy)$ such that $f = \sum_i a_i u_i$ (1.3); for every i, there exists $(\beta_{i,z})_z \in \oplus_{Fz=Fy} \mathcal{R}^{n-1}\Bbbk(\widetilde{\Gamma})(x_i, z)$ such that $u_i = \sum_z F(\beta_{i,z}) \bmod \mathrm{rad}^n$ (induction hypothesis); thus $f - \sum_z F\left(\sum_i \alpha_i \beta_{i,z}\right) \in \mathrm{rad}^{n+1}$. This proves the surjectivity at index n.

The injectivity for every x is also proved by induction on $n \geqslant 0$. If $n = 0$ it follows from: $\Bbbk(\widetilde{\Gamma})(x, z) = \mathcal{R}\Bbbk(\widetilde{\Gamma})(x, z)$ if $x \neq z$, and $\Bbbk(\widetilde{\Gamma})(x, x) = \kappa_x = \kappa_{Fx}$, and F induces a section $\kappa_x \to \mathrm{End}_A(Fx)$ of the canonical surjection $\mathrm{End}_A(Fx) \to \kappa_{Fx}$. Let $n \geqslant 1$. Assume the injectivity for indices smaller that n. Let $(\phi_z) \in \oplus_{Fz=Fy} \mathfrak{R}^n\Bbbk(\widetilde{\Gamma})(x, z)$ be such that $\sum_z F(\phi_z) \in \mathrm{rad}^{n+1}(Fx, Fy)$. Using the surjectivity and 1.3 yields $(\psi_z)_z \in \oplus_{Fz=Fy} \mathfrak{R}^{n+1}\Bbbk(\widetilde{\Gamma})(x, z)$ together with $(u_i)_i \in \oplus_{i=1}^r \mathrm{rad}^{n+1}(Fx_i, Fy)$ such that $\sum_z F(\phi_z - \psi_z) = \sum_i a_i u_i$. On the other hand, $n \geqslant 1$ and $\{\alpha_j\}_{j \in \{1,\dots,r\}}$ contains a basis of $M(x, x_i)$, which is part of the induced modulation of $\widetilde{\Gamma}$, over κ_{x_i} for every $i \in \{1, \dots, r\}$. The construction of $\Bbbk(\widetilde{\Gamma})$ therefore yields $(\theta_{i,z})_i \in \oplus_{i=1}^r \Bbbk(\widetilde{\Gamma})(x_i, z)$ such that $\phi_z - \psi_z = \sum_i \alpha_i \theta_{i,z}$, for every z. Putting these morphisms together and using that $a_i = F(\alpha_i)$ for every i yields $\sum_i a_i \left(\sum_z F(\theta_{z,i}) - u_i\right) = 0$. Now distinguish two cases according to whether x is injective or not. If x is injective then $\sum_z F(\theta_{i,z}) = u_i$ which, following the induction hypothesis, implies that $\theta_{i,z} \in \mathfrak{R}^n\Bbbk(\widetilde{\Gamma})$ for every i and every z. Thus $\phi_z = \psi_z + \sum_i \alpha_i \theta_{i,z} \in \mathfrak{R}^{n+1}\Bbbk(\widetilde{\Gamma})$ for every z. If x is not injective there exists $v \in \mathrm{Hom}_A(\tau_A^{-1}Fx, Fy)$ such that $\sum_z F(\theta_{i,z}) - u_i =$

b_iv for every i. Using the surjectivity yields $(\chi_z)_z \in \oplus_{Fz=Fy}\Bbbk(\widetilde{\Gamma})(\tau^{-1}x,z)$ such that $v = \sum_z F(\chi_z) \bmod \mathrm{rad}^{n-1}$. In particular $b_iv = \sum_z F(\beta_i\chi_z) \bmod \mathrm{rad}^n$ for every i. Hence $\sum_z F(\theta_{i,z} - \beta_i\chi_z) = u_i \bmod \mathrm{rad}^n$. Therefore $\theta_{i,z} - \beta_i\chi_z \in \mathfrak{R}^n\Bbbk(\widetilde{\Gamma})$ for every i and every z (by induction and because $u_i \in \mathrm{rad}^{n+1}$). Since moreover $\sum_i \alpha_i\beta_i = 0$, $\psi_z \in \mathfrak{R}^{n+1}\Bbbk(\widetilde{\Gamma})$ and $\phi_z = \psi_z + \sum_i \alpha_i\theta_{i,z}$ for every z, it follows that $\phi_z \in \mathfrak{R}^{n+1}\Bbbk(\widetilde{\Gamma})$.

(b) follows from (a) and from 1.5 (part (b)).

(c) follows from (a), (b) and the fact that Γ is generalised standard, that is, $\bigcap_{n\geqslant 0}\mathrm{rad}^n(X,Y) = 0$ for every $X, Y \in \Gamma$. □

3. Application to compositions of irreducible morphisms

Recall the convention set in the introduction: whenever $f\colon X \to Y$ and $g\colon Y \to Z$ are two mappings, their composition is denoted by $fg\colon X \to Z$. The following equivalence was proved in [CLMT11, Prop. 5.1] when $\Bbbk$ is algebraically closed and under the additional assumption that the valuation of the involved arrows are trivial. This last assumption is dropped here.

Proposition. *Let $X_1, \dots, X_{n+1} \in \operatorname{ind} A$. The following conditions are equivalent*

(a) there exist irreducible morphisms $X_1 \xrightarrow{h_1} \cdots \xrightarrow{h_n} X_{n+1}$ such that $h_1 \cdots h_n \in \mathrm{rad}^{n+1}\backslash\{0\}$,

(b) there exist irreducible morphisms $f_i\colon X_i \to X_{i+1}$ and morphisms $\varepsilon_i\colon X_i \to X_{i+1}$, for every i, such that $f_1 \cdots f_n = 0$, such that $\varepsilon_1 \cdots \varepsilon_n \neq 0$ and such that, for every i, either $\varepsilon_i \in \mathrm{rad}^2$ or else $\varepsilon_i = f_i$.

Proof. The implication $(b) \Rightarrow (a)$ was proved in [CT10, Thm. 2.7] (the proof there works for artin algebras and the standard hypothesis made there plays no role for this implication). Assume (a). Let Γ be the component of $\Gamma(\operatorname{mod} A)$ containing $X_1, \dots, X_{n+1}$, let $\pi\colon \tilde{\Gamma} \to \Gamma$ be the universal covering and $F\colon \Bbbk(\widetilde{\Gamma}) \to \operatorname{ind}\Gamma$ be a well-behaved functor (2.5). Let $x_1 \in \pi^{-1}(X_1)$. There is a unique path $\gamma\colon x_1 \to x_2 \to \cdots \to x_{n+1}$ in $\widetilde{\Gamma}$ which image under π is $X_1 \to X_2 \to \cdots \to X_{n+1}$. Let $h_i\colon X_i \to X_{i+1}$ $(1 \leqslant i \leqslant n)$ be irreducible morphisms such that $h_1 \cdots h_n \in \mathrm{rad}^{n+1}\backslash\{0\}$ and consider $\overline{h_i} \in \mathrm{irr}(X_i, X_{i+1})$ as lying in $\Bbbk(\widetilde{\Gamma})(x_i, x_{i+1})$. Let $h_i' = h_i - F(\overline{h_i})$ for $1 \leqslant i \leqslant n$. Then $h_i' \in \mathrm{rad}^2$ because F is well-behaved. Therefore $F(\overline{h_1} \cdots \overline{h_n}) \in \mathrm{rad}^{n+1}$. Since $\mathfrak{R}^{n+1}\Bbbk(\widetilde{\Gamma})(x_1, x_{n+1}) = 0$ (the path γ has length n, 1.5), it follows that $\overline{h_1} \cdots \overline{h_n} = 0$ (2.6). This and $h_1 \cdots h_n \neq 0$ imply that $F(\overline{h_1} \cdots \overline{h_n}) - h_1 \cdots h_n \neq 0$ that is, the sum of the morphisms

$$F(\overline{h_1}) \cdots F(\overline{h_{i_1-1}}) h_{i_1}' F(\overline{h_{i_1+1}}) \cdots F(\overline{h_{i_t-1}}) h_{i_t}' F(\overline{h_{i_t+1}}) \cdots F(\overline{h_n}),$$

for $t \in \{1, \dots, n\}$ and $1 \leqslant i_1 < \cdots < i_t \leqslant n$, is non-zero. Hence there exists $t \in \{1, \dots, n\}$ and $1 \leqslant i_1 < \cdots < i_t \leqslant n$ such that the corresponding term in the above sum is non-zero. Define $f_j := F(\overline{h_j})$, and $\varepsilon_j := F(\overline{h_j})$ if $j \notin \{i_1, \dots, i_t\}$ or $\varepsilon_j := h_j'$ if $j \in \{i_1, \dots, i_t\}$. Then $\{f_i, \varepsilon_i\}_{i=1,\dots,n}$ fits the requirements of (b). □

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