Approximation result for a Mumford-Shah functional adapted to a segmentation problem

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Abstract

We consider a segmentation problem of image close to a binary one. After introducing a Mumford Shah energy like, we perform an approximation of this model. We prove a Γ-convergence result and show why this approximation is suitable for numerics.

1 Introduction

This paper is motivated by a 3D-image segmentation process with thin tubular structures. We have introduced a model which is derived from Mumford and Shah's one in [7]. Mathematical analysis of such model is difficult. So, in order to make numerical experiments, we perform an approximation in the sense of Γ-convergence. More precisely, an image is modeled as an application g defined on Ω with values on $[0,1]$ where Ω is an open, bounded and regular subset of $\mathbb{R}^n$. We assume that g is close to a binary function. More precisely, its histogram has two distincts modes that we assume to be 0 and 1. We prove that the model

$$\begin{array}{ll}(\mathcal{P}): & \text{Min}\left\{E(p)\colon p\in SBV(\Omega), p(x)\in\{0;1\}\ \text{ a.e. } x\in\Omega\right\}, \\ & \text{where } E(p)=\frac{1}{2}\int_\Omega (p-g)^2\mathrm{d}x+\beta\mathcal{H}^{n-1}(S_p).\end{array} \tag{1.1}$$

where S_p is the jump set of the binary function p, may be approximated by

$$\begin{array}{ll}(\mathcal{P}_\varepsilon): & \text{Min}\left\{E_\varepsilon(p)\colon p\in W^{1,2}(\Omega), 0\le p(x)\le 1 \text{ a.e. } x\in\Omega\right\} \\ & \text{where } E_\varepsilon(p)=\frac{1}{2}\int_\Omega (p-g)^2\mathrm{d}x+\beta\int_\Omega\left(9\varepsilon|\nabla p|^2+\frac{p^2(1-p)^2}{\varepsilon}\right)\mathrm{d}x\end{array} \tag{1.2}$$

in the sense of the Γ-convergence. In the application which motivated this paper, we set $n=3$ but we prove this result for any $n\ge 1$.
In a companion paper [4] we perform a tuning for the parameters of this approximated model in order to detect thin structures of tubes in 3D images and we give there a complete bibliography of the subject. In the present paper we introduce the theoretical framework (section 2) then we prove the Γ-convergence of the model (section 3) and give additional properties related to numerical aspects (section 4).

2 Functional framework and Γ-convergence

2.1 Functional spaces

We work in this paper with spaces of functions with bounded variation that we recall thereafter, for more details see [5], [1].

Definition 2.1. *Let Ω be an open subset $\mathbb{R}^n$, and $u \in L^1(\Omega)$. We define $BV(\Omega)$, the space of functions with bounded variations:*

$$BV(\Omega) = \{u \in L^1(\Omega) \colon TV(u) < +\infty\},$$

where

$$TV(u) = \sup \left\{ \int_\Omega u(x) div(\xi)(x) dx \colon \xi \in C_c^1(\Omega), \|\xi\|_\infty \leq 1 \right\}.$$

The space $BV(\Omega)$, with the norm $\|u\|_{BV(\Omega)} = \|u\|_{L^1} + TV(u)$, is a Banach space. The derivative, in a distributional sense, of an element $u \in BV(\Omega)$ is a Radon measure, denoted Du, and $TV(u) = \int_\Omega |Du|$ is the total variation of u. We introduce the following notation for a measure μ and a set B, $\mu \llcorner B$ is the measure such that:

$$\mu \llcorner B(A) = \mu(A \cap B).$$

We recall useful properties on the structure of the gradient of BV functions in next theorem.

Theorem 2.1. *Let $u \in BV(\Omega)$ then we have the following decomposition:*

$$Du = \nabla u \cdot \mathcal{L}_n + (u^+ - u^-)\nu_u \mathcal{H}^{n-1} \llcorner S_u + C_u,$$

where

- $\nabla u = \frac{Du}{\mathcal{L}_n}$,
- *S_u is a countable union of hypersurfaces,*
- *$\nu_u \colon S_u \to \mathbf{S}^{n-1}$ is a measurable normal unitary vector of S_u,*
- *u^-, u^+ are the lower and upper approximated limits of u,*
- $C_u \perp \mathcal{H}^{n-1}$.

Definition 2.2. *For any $u \in BV(\Omega)$, we set:*

- *$\nabla u \cdot \mathcal{L}_n$ as the regular part of Du,*
- *S_u as the jump set of u,*
- *$(u^+ - u^-)\nu_u \mathcal{H}^{n-1} \llcorner S_u$ as the jump part of Du,*

- C_u *as the Cantor part of* Du.

Definition 2.3. *The set of BV-functions whose derivative have a Cantor part equal to zero is called the special set of functions with bounded variation, denoted* $SBV(\Omega)$.

Theorem 2.2. *Let* $u = \mathbf{1}_A$ *an indicator function such that* $u \in SBV(\Omega)$ *then we have the following decomposition:*

$$Du = \nu_u \mathcal{H}^{n-1} \llcorner S_u.$$

The space SBV has a useful slicing property.

Definition 2.4. *Let* $\nu \in \mathbf{S}^{n-1}$ *(unit sphere of* $\mathbb{R}^n$*) and* Ω *be an open subset of* $\mathbb{R}^n$. *Let* Π_ν *be the orthogonal plane to* ν*:*

$$\Pi_\nu = \{x \in \mathbb{R}^n : \langle \nu, x \rangle = 0\},$$

where $\langle \cdot, \cdot \rangle$ *stands for the scalar* $\mathbb{R}^n$*-product. For any* $x \in \Pi_\nu$ *we define*

$$\Omega_x = \{t \in \mathbb{R} : x + t\nu \in \Omega\}$$

and

$$\Omega_\nu = \{x \in \Pi_\nu : \Omega_x \neq \emptyset\},$$

see figure 2.1.

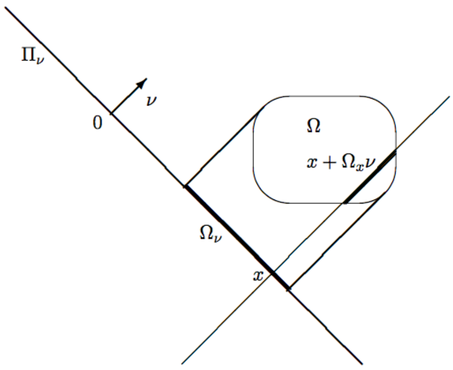

Figure 2.1: Slicing the open Ω

As in [2], we introduce a projection on the open set Ω_x (see figure 2.1). For any function u defined on $\Omega \subset \mathbb{R}^n$ and any $x \in \Omega_\nu$, we set

$$\begin{array}{rcl} u_x : \Omega_x & \to & \mathbb{R} \\ t & \to & u(x + t\nu) \end{array}$$

Theorem 2.3. *Let $u \in L^\infty(\Omega)$ be a function such that, for all $\nu \in \mathbf{S}^{n-1}$,*

i) $u_x \in SBV(\Omega_x)$ for $\mathcal{H}^{n-1}$ a.e. $x \in \Omega_\nu$,

ii) $\int_{\Omega_\nu} \left\{ \int_{\Omega_x} |\nabla u_x| dt + \mathcal{H}^0(S_{u_x}) \right\} d\mathcal{H}^{n-1}(x) < +\infty$ where S_v is the jump set of the function v (see 2.2);

then, $u \in SBV(\Omega)$ and $\mathcal{H}^{n-1}(S_u) < +\infty$.
Conversely, let $u \in SBV(\Omega) \cap L^\infty$ be such that $\mathcal{H}^{n-1}(S_u) < +\infty$. Then i) and ii) are satisfied. Moreover, we have

iii) $\langle \nabla u(x + t\nu), \nu \rangle = \nabla u_x(t)$, for a.e. $t \in \Omega_x$ and $\mathcal{H}^{n-1}$-a.e. $x \in \Omega_\nu$,

iv) there exists a measurable function $\nu_u : S_u \to \mathbf{S}^{n-1}$ depending on S_u, such that

$$\int_{S_u} \langle \nu_u, \nu \rangle d\mathcal{H}^{n-1}(x) = \int_{\Omega_\nu} \mathcal{H}^0(S_{u_x}) d\mathcal{H}^{n-1}(x).$$

2.2 Γ-convergence theory

Usually, a standard compactness argument is sufficient to prove the existence of minimizing solutions. In our case, we use a specific notion of convergence which will be specified here, since compactness is not ensured for usual topologies. As before, for more details see [3].

Definition 2.5. *Let (X, d) a metric space, $(F_n)_{n\in\mathbb{N}}$ a sequence of functions defined on X with values in $\mathbb{R}\cup\{+\infty\}$ and $F : X \to \mathbb{R}\cup\{+\infty\}$. The sequence $(F_n)_{n\in\mathbb{N}}$ Γ-converges to F on $x \in X$ if the following assertions are true:*

i) for all $(x_n)_{n\in\mathbb{N}}$ which converges to x:

$$F(x) \leq \liminf_{n\to\infty} F_n(x_n),$$

ii) there exists $(y_n)_{n\in\mathbb{N}}$ which converges to x such that:

$$F(x) \geq \limsup_{n\to\infty} F_n(y_n),$$

This concept is the convergence of the approximate minimum. More precisely, if for any n, p_n is a minimum of F_n and $(p_n)_n$ converges to p, then according to this concept, p is a minimum of F.
We shall use the following result.

Proposition 2.1. *Let X a topological space, $(f_n)_{n\in\mathbb{N}}$ a sequence of functions defined on X with values in $\mathbb{R} \cup \{+\infty\}$ which Γ-converges to f and h a continuous function defined on X with values in $\mathbb{R} \cup \{+\infty\}$. Then the sequence $(f_n + h)_{n\in\mathbb{N}}$ Γ-converges to $f + h$.*

3 A Γ-convergence result

We now prove that the family of functionals $(E_\varepsilon)_{\varepsilon>0}$ defined by (1.2) Γ-converges to E defined by (1.1). For more simplicity in the proof we assume that $\beta = 1$ in this section, its value doesn't change the arguments of the proof. We denote $W_b^{1,2}(\Omega)$ the following space

$$W_b^{1,2}(\Omega) = \left\{p \in W^{1,2}(\Omega)\colon 0 \leq p(x) \leq 1 \text{ a.e. } x \in \Omega\right\}.$$

Let $\mathbb{B}(\Omega; [0,1])$ be the space of measurable functions defined on Ω which take their values in $[0,1]$. This space is endowed with the almost everywhere convergence topology. It's a Fréchet space (see [2]).

Let F be the penalization term of the functional E:

$$\begin{aligned} F\colon \mathbb{B}(\Omega;[0,1]) &\to \mathbb{R}\cup\{+\infty\} &(3.1)\\ p &\mapsto \begin{cases} \mathcal{H}^{n-1}(S_p) & \text{if } p \in SBV(\Omega),\\ +\infty & \text{otherwise}\end{cases} &(3.2)\end{aligned}$$

and F_ε be its approximation:

$$\begin{aligned} F_\varepsilon\colon \mathbb{B}(\Omega;[0,1]) &\to \mathbb{R}\cup\{+\infty\}\\ p &\mapsto \begin{cases} \int_\Omega \left(9\varepsilon|\nabla p|^2 + \frac{p^2(1-p)^2}{\varepsilon}\right) \mathrm{d}x, & \text{if } p \in W_b^{1,2}(\Omega),\\ +\infty & \text{otherwise.}\end{cases}\end{aligned}$$

Let H be the fitting data term:

$$\begin{aligned} H\colon \mathbb{B}(\Omega;[0,1]) &\to \mathbb{R}\cup\{+\infty\}\\ p &\to \frac{1}{2}\int_\Omega (p-g)^2 \mathrm{d}x.\end{aligned}$$

Let $(p_k)_{k\in\mathbb{N}} \subset \mathbb{B}(\Omega;[0,1])$ be a a.e. converging sequence to some p. According Lebesgue theorem and the fact that $\|p_k\|_\infty \leq 1$, the sequence $(H(p_k))_{k\in\mathbb{N}}$ converges to $H(p)$ and H is continuous for the almost everywhere convergence topology. So, it is sufficient to prove the Γ-convergence of F_ε to F. Indeed, proposition 2.1 with $X = \mathbb{B}(\Omega;[0;1])$, implies the Γ-convergence of $E_\varepsilon = F_\varepsilon + H$ to $E = F + H$.
We introduce the following notation:

$$F_\varepsilon(p,U) = \int_U \left(9\varepsilon|\nabla p|^2 + \frac{p^2(1-p)^2}{\varepsilon}\right) \mathrm{d}x, \qquad F(p,U) = \mathcal{H}^{n-1}(S_p \cap U)$$

and

$$F_-(p,U) = \inf\left\{\liminf_{k\to\infty} F_{\varepsilon_k}(p_k,U) | p_k \overset{a.e.}{\to} p, (p_k)_{k\in\mathbb{N}} \subset W_b^{1,2}(U)\right\}. \qquad (3.3)$$

The functional F_- is also called the *lower*-Γ limit of F.
We shall prove the following theorem.

Theorem 3.1. *Let $(\varepsilon_k)_{k\in\mathbb{N}}$ be a sequence which converges to 0^+ and $p \in SBV(\Omega)$ be a function taking its values in $\{0,1\}$. Then*

i)
$$F(p,\Omega) \leq F_-(p,\Omega) \tag{3.4}$$
holds;

ii) there exists a sequence $(p_k)_{k\in\mathbb{N}} \subset W_b^{1,2}(\Omega)$ which converges a.e. to p such that:
$$F(p,\Omega) \geq \limsup_{k\to\infty} F_{\varepsilon_k}(p_k,\Omega). \tag{3.5}$$

According to this theorem, we deduce that $F(p,\Omega) = F_-(p,\Omega)$ for $p \in SBV(\Omega)$ taking its values in $\{0,1\}$. The proof is detailed thereafter.

3.1 The inequality for the *lower* Γ-limit (3.4)

We now prove the first inequality (3.4). We shall need the following properties of this functional:

a) $F_-(p,\cdot)$ is supperaddditive on open sets, that is
$$A \cap B = \emptyset \qquad \Rightarrow \qquad F_-(p, A\cup B) \geq F_-(p,A) + F_-(p,B),$$

b) $F_-(p,\cdot)$ is non decreasing:
$$A \subset B \qquad \Rightarrow \qquad F_-(p,A) \leq F_-(p,B).$$

By a diagonal extraction, there exists a sequence $(p_k)_{k\in\mathbb{N}} \subset W_b^{1,2}(\Omega)$ such that:
$$\begin{cases} p_k \overset{a.e.}{\to} p, \\ F_{\varepsilon_k}(p_k,\Omega) \to F_-(p,\Omega) \end{cases} \tag{3.6}$$

We assume that $F_-(p,\Omega) < +\infty$ otherwise (3.4) is ensured. We have to prove:
$$F(p,\Omega) \leq \liminf_{k\to\infty} F_{\varepsilon_k}(p_k,\Omega)$$

As in [2], we perform the proof in two steps: the first step deals with dimension 1. The second generalizes it for dimension $n \geq 2$.

3.1.1 The one-dimensional case

Assume $\Omega = I$ is an open interval. To avoid confusion, when $n = 1$, we denote the approximating functionals by F_ε^1 et their lower Γ-limit, defined in (3.3), by F_-^1.

Lemma 3.1. *Let $B_\eta(x) \subset \mathbb{R}$ be an open ball and $q \in \mathbb{B}(B_\eta(x))$ be a function taking its values in $\{0;1\}$. Assuming*

$$\forall \rho \in]0, \eta[, \qquad q \notin W^{1,2}(B_\rho(x)),$$

then we have:

$$\forall \rho \in]0, \eta[, \qquad F^1_-(q, B_\rho(x)) \geq 1.$$

In particular, if $I \subset \mathbb{R}$ is an open and bounded interval such that $q \in \mathbb{B}(I)$ takes its values in $\{0;1\}$ and $F^1_-(q, I) < +\infty$ then $q \in SBV(I)$. Moreover, there exists $J \subset I$ with cardinal less than $F^1_-(q, I)$ such that

i) p is constant on each connected component of $I \setminus J$,

ii) $\mathcal{H}^0(S_q \cap I) \leq F^1_-(q, I)$.

Proof. We can assume that $F^1_-(q, B_\rho(x)) < +\infty$ for any $\rho \in]0; \eta[$. As before, there exists a sequence $(q_k)_{k\in\mathbb{N}} \subset W^{1,2}_b(B_\rho(x))$ such that:

$$\begin{cases} q_k \overset{a.e.}{\to} q, \\ F_{\varepsilon_k}(q_k, B_\rho(x)) \to F^1_-(q, B_\rho(x)) \end{cases} \tag{3.7}$$

Since q takes its values a.e. in $\{0, 1\}$ and $q \notin W^{1,2}(B_{\rho'}(x))$ for any $\rho' \in]0, \rho[$, there exists two sequences $(y^1_k)_{k\in\mathbb{N}}$ and $(y^2_k)_{k\in\mathbb{N}}$ such that:

$$y^1_k \to x,\ q_k(y^1_k) \to 1 \text{ and } y^2_k \to x,\ q_k(y^2_k) \to 0.$$

However, we have:

$$F^1_{\varepsilon_k}(q_k, B_\rho(x)) \geq \int_{x-\rho}^{x+\rho} \left(9\varepsilon_k |\nabla q_k(y)|^2 + \frac{q^2_k(1 - q_k(y))^2}{\varepsilon_k} \right) dy.$$

We introduce:

$$A := 3\sqrt{\varepsilon_k} |\nabla q_k(y)|, \qquad B := \frac{q_k(1 - q_k(y))}{\sqrt{\varepsilon_k}}.$$

With the inequality $A^2 + B^2 \geq 2AB$, we get:

$$F^1_{\varepsilon_k}(q_k, B_\rho(x)) \geq 6 \int_{x-\rho}^{x+\rho} |\nabla q_k(y)| q_k(y)(1 - q_k(y))\, dy.$$

As $[y^1_k, y^2_k] \subset B_\rho(x)$, we obtain:

$$F^1_{\varepsilon_k}(q_k, B_\rho(x)) \geq 6 \int_{y^1_k}^{y^2_k} |\nabla q_k(y)| q_k(y)(1 - q_k(y))\, dy.$$

Since $q_k \in W^{1,2}(I)$, we may use the change of variable $t = q_k(s)$. This yields:

$$\begin{aligned} F^1_{\varepsilon_k}(q_k, B_\rho(x)) &\geq 6\int_{q_k(y_k^1)}^{q_k(y_k^2)} t(1-t)\mathrm{d}t, \\ F^1_{\varepsilon_k}(q_k, B_\rho(x)) &\geq 6\left|\frac{q_k^2(y_k^1) - q_k^2(y_k^2)}{2} - \frac{q_k^3(y_k^1) - q_k^3(y_k^2)}{3}\right| \end{aligned}$$

We know that $q_k(y_k^1) \to 1$ and $q_k(y_k^2) \to 0$, so that we deduce:

$$\frac{q_k^2(y_k^1) - q_k^2(y_k^2)}{2} - \frac{q_k^3(y_k^1) - q_k^3(y_k^2)}{3} \to \frac{1}{6}.$$

It comes,

$$\liminf F^1_{\varepsilon_k}(q_k, B_\rho(x)) \geq 1.$$

and we can conclude

$$F^1_-(q, B_\rho(x)) \geq 1. \tag{3.8}$$

We set

$$J = \left\{ x \in I \colon \forall \rho > 0,\ q \notin W^{1,2}(B_\rho(x)) \right\}.$$

The functional $F^1_-(q,\cdot)$ is super-additive and non-decreasing (see (3.1)). With inequality (3.8) we get

$$F^1_-(q, I) \geq \mathcal{H}^0(J).$$

According to the fact that q takes its values a.e. in $\{0;1\}$, then q is constant on the connected components of $I \setminus J$ and $S_q \cap I = J$. Therefore, we may conclude

$$F^1_-(q, I) \geq \mathcal{H}^0(S_q \cap I).$$

□

3.1.2 Generalization to dimensions $n \geq 2$

We now extend the proof to any higher dimension n by a slicing argument. Recall that we want to prove $F_-(p,\Omega) \geq F(p,\Omega)$.
Let $(p_k)_{k\in\mathbb{N}} \subset W_b^{1,2}(\Omega)$ as in (3.6). Let A be an arbitrary open subset of Ω. Assume that $F_-(p, A) < +\infty$ and $F_{\varepsilon_k}(p_k, A) \to F_-(p, A)$. Let $\nu \in \mathbf{S}^{n-1}$ be an unit vector, we have:

$$\begin{aligned} F_{\varepsilon_k}(p_k, A) &= \int_A \left(9\varepsilon_k |\nabla p_k(x)|^2 + \frac{p_k^2(x)(1-p_k(x))^2}{\varepsilon_k} \right) \mathrm{d}x \\ &\geq \int_A \left(9\varepsilon_k |\langle \nabla p_k(x), \nu\rangle|^2 + \frac{p_k^2(x)(1-p_k(x))^2}{\varepsilon_k} \right) \mathrm{d}x \end{aligned}$$

The Lebesgue measure of $\mathbb{R}^n$ projected on Π_ν is the Hausdorff measure with dimension $n-1$ (see for example [5]). Applying Fubini theorem, we get:

$$\begin{aligned} F_{\varepsilon_k}(p_k, A) &\geq \int_{A_\nu} \int_{A_x} \left(9\varepsilon_k |\langle \nabla p_k(x+t\nu), \nu \rangle|^2 + \frac{p_k(x+t\nu)^2(1-p_k(x+t\nu))^2}{\varepsilon_k} \right) \mathrm{d}t \; \mathrm{d}\mathcal{H}^{n-1}(x) \\ &\geq \int_{A_\nu} F^1_{\varepsilon_k}((p_k)_x, A_x) \; \mathrm{d}\mathcal{H}^{n-1}(x). \end{aligned}$$

Fatou lemma yields:

$$F_-(p, A) \geq \int_{A_\nu} \liminf F^1_{\varepsilon_k}((p_k)_x, A_x) \; \mathrm{d}\mathcal{H}^{n-1}(x).$$

As $(p_k)_x \in SBV(A_x)$ a.e. $x \in A_\nu$, with lemma 3.1, we have:

$$F_-(p, A) \geq \int_{A_\nu} \mathcal{H}^0(S_{(p)_x} \cap A_x) \; \mathrm{d}\mathcal{H}^{n-1}(x).$$

Using theorem 2.3 gives:

$$\int_{A_\nu} \mathcal{H}^0(S_{(p)_x} \cap A_x) \; \mathrm{d}\mathcal{H}^{n-1}(x) = \int_{S_p \cap A} |\langle \nu_p, \nu \rangle| \; \mathrm{d}\mathcal{H}^{n-1}(x),$$

so that

$$F_-(p, A) \geq \int_{S_p \cap A} |\langle \nu_p, \nu \rangle| \; \mathrm{d}\mathcal{H}^{n-1}(x). \tag{3.9}$$

As p belongs to $SBV(\Omega)$, according to theorem 2.1, the set S_p is a countable union of $\mathcal{C}^1$-hypersurfaces. Then, we obtain:

$$\mathcal{H}^{n-1}(S_p) = \sup_{\mathcal{A}, \mathcal{V}} \left\{ \sum_{i=1}^{\infty} \int_{S_p \cap A_i} |\langle \nu_p, \nu_i \rangle| \; \mathrm{d}\mathcal{H}^{n-1}(x) \right\}$$

where the supremum is taken over all the families $\mathcal{A} = (A_i)_{i=1\cdots\infty}$ of open and pairwise disjoint subsets of Ω and $\mathcal{V} = (\nu_i)_{i=1\cdots\infty} \subset \mathbf{S}^{n-1}$. Applying the inequality (3.9) to each term, we have:

$$\mathcal{H}^{n-1}(S_p) \leq \sup_{\mathcal{A}} \left\{ \sum_{i=1}^{\infty} F_-(p, A_i) \right\}.$$

where the supremum is only taken over all the families $\mathcal{A}$.

Since $F_-(p, \cdot)$ is super-additive and non decreasing (see (3.1)), then we may conclude that

$$\mathcal{H}^{n-1}(S_p) \leq F_-(p, \Omega).$$

$\square$

3.2 The inequality for the *higher* Γ-limit (3.5)

To prove the second part (3.5), we construct a sequence of functions $(p_k)_{k\in\mathbb{N}}$ converging to p, so that the higher bound of the energy is lower than $F(p,\Omega)$. We will use the following lemma due to Modica [6].

Lemma 3.2. *Let A be an open and bounded subset of Ω with a non empty Lipschitz boundary ∂A. Let*

$$h(x) = \begin{cases} +dist(x,\partial A) \text{ if } x \in A, \\ -dist(x,\partial A) \text{ otherwise} \end{cases}$$

and for all $t > 0$, $S_t = \{x \in A \colon h(x) = t\}$.
Then h is a Lipschitz function, and we have:

$$|Dh(x)| = 1 \text{ a.e. } x \in \Omega \text{ and } \lim_{t\to 0} \mathcal{H}^{n-1}(S_t \cap \Omega) = \mathcal{H}^{n-1}(\partial A \cap \Omega)\ .$$

We may now prove the second part (3.5) of theorem 3.

Proof. Set:

$$h(x) = \begin{cases} +\text{dist}(x,\{p=0\}) \text{ if } p(x) = 1, \\ -\text{dist}(x,\{p=1\}) \text{ if } p(x) = 0, \end{cases}$$

$$\chi_0(t) = \begin{cases} 1 \text{ si } t \geq 0, \\ 0 \text{ si } t < 0. \end{cases}$$

It is clear that $p = \chi_0 \circ h$. We construct p_k as $p_k = \chi_k \circ h$. In this case, we have:

$$F_{\varepsilon_k}(p_k) = \int_\Omega \left(9\varepsilon_k |\nabla \chi_k(h(x))|^2 |\nabla h(x)|^2 + \frac{\chi_k^2(h(x))(1-\chi_k(h(x)))^2}{\varepsilon_k} \right) \mathrm{d}x.$$

The function p belongs to $SBV(\Omega)$. According to theorem 2.1 and lemma 3.2, we have $|\nabla h(x)| = 1$ for almost every $x \in \Omega$. Then, we may apply coarea formula (see for example [5]) to obtain:

$$\begin{aligned} F_{\varepsilon_k}(p_k) &= \int_{t=-\infty}^{+\infty} \varphi_{\varepsilon,k}(t) \mathcal{H}^{n-1}(\{h=t\})\ \mathrm{d}t, \\ \text{where} \qquad & \varphi_{\varepsilon,k}(t) = 9\varepsilon_k |\nabla \chi_k(t)|^2 + \frac{\chi_k^2(t)(1-\chi_k(t))^2}{\varepsilon_k}. \end{aligned}$$

The minimization of this energy is a 1D-problem. The Euler-lagrange equation associated to this problem is:

$$(9\varepsilon_k(\chi_k')^2)' = \left(\frac{\chi_k^2(1-\chi_k)^2}{\varepsilon_k} \right)'$$

Thus, there exists $c_k > 0$ such that:

$$9\varepsilon_k(\chi_k^{'})^2 = c_k + \frac{\chi_k^2(1-\chi_k)^2}{\varepsilon_k} \tag{3.10}$$

Let $\overline{\chi_k}$ a solution of the equation (3.10) such that:

$$\begin{cases} \overline{\chi_k}(t) = 0 & \text{if } t \leq 0, \\ 3\sqrt{\varepsilon_k}\overline{\chi_k}^{'} = \sqrt{c_k + \frac{\overline{\chi_k}^2(1-\overline{\chi_k})^2}{\varepsilon_k}} & \text{if } t \in [0, \eta_k], \\ \overline{\chi_k}(t) = 1 & \text{if } t \geq \eta_k. \end{cases}$$

As $\overline{\chi_k}^{'} \geq \sqrt{\frac{c_k}{9\varepsilon_k}}$, we have $\eta_k \leq \sqrt{\frac{c_k}{9\varepsilon_k}}$. If $\frac{c_k}{\varepsilon_k} \to 0$ then $\overline{\chi_k} \circ h$ converges to p almost everywhere.
Set $A = 3\sqrt{\varepsilon_k}\overline{\chi_k}^{'}$ and $B = \frac{\overline{\chi_k}(1-\overline{\chi_k})}{\sqrt{\varepsilon_k}}$. We have $A^2 = c_k + B^2$ and

$$\begin{aligned} A^2 + B^2 &= 2AB + c_k + 2B^2 - 2AB, \\ &= 2AB + c_k + 2B(B - A), \\ &= 2AB + c_k + 2B(B - \sqrt{c_k + B^2}), \\ &= 2AB + c_k - 2c_k\frac{B}{B + \sqrt{c_k + B^2}}. \end{aligned}$$

Replacing it in $F_{\varepsilon_k}(p_k)$ leads to:

$$\begin{aligned} F_{\varepsilon_k}(p_k) &= \int_{t=0}^{\eta_k} (A^2 + B^2)\mathcal{H}^{n-1}(\{h = t\}) \text{ dt} \\ &= \int_{t=0}^{\eta_k} (2AB + c_k - 2c_k\frac{B}{B + \sqrt{c_k + B^2}})\mathcal{H}^{n-1}(\{h = t\}) \text{ dt} \\ &\leq \int_{t=0}^{\eta_k} (2AB + c_k)\mathcal{H}^{n-1}(\{h = t\}) \text{ dt}. \end{aligned}$$

Replacing A and B by its values, we get:

$$F_{\varepsilon_k}(p_k) \leq \int_{t=0}^{\eta_k} 6\overline{\chi_k}^{'}\overline{\chi_k}(1 - \overline{\chi_k})\mathcal{H}^{n-1}(\{h = t\}) \text{ dt} + \int_{t=0}^{\eta_k} c_k\mathcal{H}^{n-1}(\{h = t\}) \text{ dt}.$$

As $p \in SBV$. So, $t \to \mathcal{H}^{n-1}(\{h = t\})$ is continuous at $t = 0$ and converges to $\mathcal{H}^{n-1}(S_p)$ with theorem 2.1. Thus, the second term converges to 0. Applying Fatou lemma gives

$$\begin{aligned} \limsup F_{\varepsilon_k}(p_k) &\leq \mathcal{H}^{n-1}(S_p) \limsup \int_{t=0}^{\eta_k} \frac{3}{4}\overline{\chi_k}^{'}(1 - \chi_k^2) \text{ dt} \\ &\leq \mathcal{H}^{n-1}(S_p) \int_0^1 6s(1 - s) \text{ ds} \\ &\leq \mathcal{H}^{n-1}(S_p). \end{aligned}$$

$\square$

4 Convergence of solutions

In this section we prove an existence result for the minimizing problem of E_ε, with $\varepsilon > 0$ fixed; we set p_ε such solution. Then, we show the convergence a.e. of an extracted subsequence $(p_{\varepsilon_k})_{\varepsilon_k \in \mathbb{N}}$, with $\varepsilon_k \to 0^+$, to a binary function $p \in SBV(\Omega)$. To demonstrate that, we use the Γ-convergence result proved in section 3.

4.1 Lower semi-continuity of E_ε

For $\varepsilon > 0$ fixed, we can find a minimizer of E_ε on $W_b^{1,2}(\Omega)$. Contrary to E, the functional E_ε is lower semi-continuous for the usual topology.

Theorem 4.1. *Let $\varepsilon > 0$, the problem $(\mathcal{P}_\varepsilon)$ (1.2) admits solutions. More precisely, if $(p_k)_{k\in\mathbb{N}} \subset W_b^{1,2}(\Omega)$ is a minimizing sequence of E_ε, then there exists a sub-sequence $(p_{k_l})_{l\in\mathbb{N}}$ converging a.e. to $p \in W^{1,2}(\Omega)$ such that*

$$E_\varepsilon(p) = \lim_{l\to\infty} E_\varepsilon(p_{k_l}) = \min\left\{E_\varepsilon(q) \colon q \in W_b^{1,2}(\Omega)\right\}.$$

Proof. Let $(p_k)_{k\in\mathbb{N}} \subset W_b^{1,2}(\Omega)$ be a minimizing sequence of E_ε. In particular, the sequence $(E_\varepsilon(p_k))_{k\in\mathbb{N}}$ is bounded. Since

$$9\varepsilon\beta\|\nabla p_k\|^2_{L^2(\Omega)} \le E_\varepsilon(p_k)$$

and

$$\|p_k\|^2_{L^2(\Omega)} \le \|p_k\|_{L^\infty(\Omega)}\mathcal{L}^n(\Omega) \le \mathcal{L}^n(\Omega),$$

then this sequence is bounded in $W^{1,2}(\Omega)$. So, there exists a subsequence $(p_{k_l})_{l\in\mathbb{N}}$ which weakly converges to $p \in W^{1,2}(\Omega)$. The functional $q \to \int_\Omega |\nabla q|^2$ is lower semi-continuous for the weak topology of $W^{1,2}(\Omega)$. According to Fatou lemma, the functionals

- $q \to \int_\Omega (q-g)^2$
- $q \to \int_\Omega q^2(1-q)^2$

are lower semi-continuous for the topology of the a.e. convergence. Then, we may conclude

$$E_\varepsilon(p) \le \liminf_{l\to\infty} E_\varepsilon(p_{k_l}).$$

Since $(p_k)_{k\in\mathbb{N}}$ is a minimizing sequence then

$$E_\varepsilon(p) = \lim_{l\to\infty} E_\varepsilon(p_{k_l}) = \min\left\{E_\varepsilon(q) \colon q \in W_b^{1,2}(\Omega)\right\}.$$

□

4.2 Compactness result of $(\mathcal{P}_\varepsilon)_{\varepsilon>0}$

According to 4.1, there exists p_ε solution of $(\mathcal{P}_\varepsilon)$. For numerical applications, we need the convergence of the solutions $(p_{\varepsilon_k})_{k\in\mathbb{N}}$ for any $\varepsilon_k \to 0^+$ to a solution of $(\mathcal{P})$. In this section, we prove that this family is compact in the following sense.

Theorem 4.2. *Let $(\varepsilon_k)_{k\in\mathbb{N}}$ be a sequence converging to 0^+ and let $(p_k)_{k\in\mathbb{N}} \subset W_b^{1,2}(\Omega)$ be such that for any $k \in \mathbb{N}$, p_k is a minimizer of E_{ε_k}. Then, there exists a subsequence $(p_{k_l})_{l\in\mathbb{N}}$ which converges a.e. to $p \in \mathbb{B}(\Omega)$. Moreover, we have:*

i) $p(x) \in \{0; 1\}$ a.e. $x \in \Omega$,

ii) $p \in SBV(\Omega)$ and $\mathcal{H}^{n-1}(S_p) < +\infty$,

iii) p is a solution of $(\mathcal{P})$.

This result also prove that the space $SBV(\Omega; \{0; 1\})$ is "optimal" for the previous Γ-convergence result: it is the largest domain for which this result is satisfied.

Proof. Let $q \in SBV(\Omega)$ an arbitrary function such that q takes its values in $\{0; 1\}$ and $\mathcal{H}^{n-1}(S_q) < \infty$. According theorem 3, there exists a sequence $(q_k)_{k\in\mathbb{N}}$ such that $(E_{\varepsilon_k}(q_k))_{k\in\mathbb{N}}$ converges to $E(q)$. In particular, the sequence $(E_{\varepsilon_k}(q_k))_{k\in\mathbb{N}}$ is bounded, we denote M its higher bound. Since p_k is a minimizer of E_{ε_k} for any k then $(E_{\varepsilon_k}(p_k))_{k\in\mathbb{N}}$ is bounded by M too.
We set $c_k = 3p_k^2 - 2p_k^3$. Note that $c_k \in W^{1,2}(\Omega)$ and that the chain rule gives $\nabla c_k = 6\nabla p_k p_k(1 - p_k)$, so we have

$$\begin{aligned}
\int_\Omega |\nabla c_k| &= 6\int_\Omega |\nabla p_k| p_k(1-p_k)\mathrm{d}x,\\
&\leq \int_\Omega 9\varepsilon_k |\nabla p_k|^2 \mathrm{d}x + \int_\Omega \frac{p_k^2(1-p_k)^2}{\varepsilon_k}\mathrm{d}x,\\
&\leq \frac{M}{\beta}.
\end{aligned}$$

Thus $(c_k)_{k\in\mathbb{N}}$ is bounded in $W^{1,1}(\Omega)$. So, there exists a subsequence $(c_{k_l})_{l\in\mathbb{N}}$ which converges a.e. to $c \in \mathbb{B}(\Omega)$. According that $x \to 3x^2 - 2x^3$ is a bicontinuous isomorphism from $[0, 1]$ to itself, there exists $p \in \mathbb{B}(\Omega)$ such that $(p_{k_l})_{l\in\mathbb{N}}$ converges a.e. to p. We have

$$\forall k \in \mathbb{N}, \qquad \int_\Omega \frac{p_k^2(1-p_k)^2}{\varepsilon_k}\mathrm{d}x \leq \frac{M}{\beta}.$$

As $(\varepsilon_k)_{k\in\mathbb{N}}$ converges to 0^+, Fatou lemma yields

$$\int_\Omega p^2(1-p)^2 \mathrm{d}x = 0,$$

so that $p(x) \in \{0;1\}$ a.e. $x \in \Omega$.
Let us prove ii. As

$$\forall k \in \mathbb{N}, \qquad E_{\varepsilon_k}(p_k) \leq M.$$

then $F_-(p,\Omega) \leq M < +\infty$.
Let $\nu \in \mathbb{S}^{n-1}$ be a fixed vector, and $(r_k)_{k\in\mathbb{N}} \subset W_b^{1,2}(\Omega)$ such that

$$\begin{cases} r_k \stackrel{a.e.}{\to} p, \\ F_{\varepsilon_k}(r_k,\Omega) \to F_-(p,\Omega). \end{cases}$$

Since $r_k \in W^{1,2}(\Omega)$, then $(r_k)_x$ (defined by 2.1) belongs to $W^{1,2}(\Omega_x)$ and we have:

$$\langle \nabla r_k(x+t\nu), \nu\rangle = \nabla (r_k)_x.$$

As

$$\int_\Omega \left(9\varepsilon_k |\langle \nabla r_k, \nu\rangle|^2 + \frac{r_k^2(1-r_k)^2}{\varepsilon_k} \right) \mathrm{d}x \leq F_{\varepsilon_k}(r_k,\Omega).$$

Fubini theorem gives

$$\int_{\Omega_\nu} \int_{\Omega_x} \left[9\varepsilon_k |\langle \nabla r_k(x+t\nu), \nu\rangle|^2 + \frac{r_k^2(1-r_k)^2}{\varepsilon_k} \right] \mathrm{d}t \; \mathrm{d}\mathcal{H}^{n-1}(x) \leq F_{\varepsilon_k}(r_k,\Omega).$$

and

$$\int_{\Omega_\nu} F^1_{\epsilon_k}((r_k)_x, \Omega_x) \mathrm{d}\mathcal{H}^{n-1}(x) \leq F_{\varepsilon_k}(r_k,\Omega).$$

We apply Fatou lemma to get

$$\begin{aligned} \int_{\Omega_\nu} \liminf F^1_{\epsilon_k}((r_k)_x, \Omega_x) \mathrm{d}\mathcal{H}^{n-1}(x) &\leq F_-(p,\Omega), \\ \int_{\Omega_\nu} F^1_-((p_x), \Omega_x) \mathrm{d}\mathcal{H}^{n-1}(x) &\leq F_-(p,\Omega). \end{aligned}$$

We deduce that $F^1_-((p_x),\Omega_x) < +\infty$ is finite $\mathcal{H}^{n-1}$-a.e. $x \in \Omega_\nu$. Otherwise, $((r_k)_x)_{k\in\mathbb{N}}$ converges to p_x a.e. on Ω_x and p_x takes its values on $\{0;1\}$ $\mathcal{H}^{n-1}$-a.e. $x \in \Omega_\nu$. According to lemma 3.1, we have

$$p_x \in SBV(\Omega_x)$$

for $\mathcal{H}^{n-1}$-a.e. $x \in \Omega_\nu$. Moreover, we have proved that $\mathcal{H}^0(S_{p_x} \cap \Omega_x) \leq F^1_-(p_x,\Omega_x)$ and $\nabla p_x = 0$, so we have

$$\int_{\Omega_\nu} \left\{ \int_{\Omega_x} |\nabla p_x| \mathrm{dt} + \mathcal{H}^0(S_{p_x}) \right\} \mathrm{d}\mathcal{H}^{n-1}(x) < +\infty.$$

Now, the conditions of theorem 2.3 are satisfied. We can conclude that $p \in SBV(\Omega)$ and $\mathcal{H}^{n-1}(S_p) < +\infty$.
Let us prove iii.

□

5 Conclusion

We have shown in section 3 that $(E_\varepsilon)_\varepsilon$ Γ-converges to E (theorem 3.1) in SBV and this is the best we can have (theorem 4.2). Furthermore, we have shown that this approximation is suitable for numerical experiments since the solutions of $(\mathcal{P}_\varepsilon)$ are approximations of $(\mathcal{P})$ (theorem 4.1).
Practically, we replace the problem

$$(\mathcal{P})\colon \mathrm{Min}\,\{E(p)\colon p \in SBV(\Omega),\ \text{and } p \text{ takes its values in } \{0,1\}\}$$

by the problem

$$(\mathcal{P}_\varepsilon)\colon \mathrm{Min}\left\{E_\varepsilon(p)\colon p \in W_b^{1,2}(\Omega)\right\}$$

with ε fixed and small. In the companion paper [4], we study this approximate problem with the specifics constraints for thin tubes.

References

[1] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of Bounded Variation and Free Discontinuity Problems.* Oxford Science Publications, 2000.

[2] L. Ambrosio and V. M. Tortorelli. Approximation of functionals depending on jumps by elliptic functionals via gamma-convergence. *Communications on Pure and Applied Mathematics*, XLIII:999–1036, 1990.

[3] H. Attouch, G. Buttazzo, and G. Michaille. *Variational Analysis in Sobolev and BV spaces.* SIAM, 2005.

[4] M. Bergounioux and D. Vicente. A mumford-shah geometrical model for the detection of thin structures. 2013.

[5] L. C. Evans and R. F. Gariepy. *Measure Theory and Fine Properties of Functions.* Studies in Advanced Mathematics, 1992.

[6] L. Modica and S. Mortola. The gradient theory of phase transitions and the minimal interface criterion. *Archive for Rational Mechanics and Analysis*, 98:123–142, 1987.

[7] D. Mumford and J. Shah. Optimal approximations by piecewise smooth functions and associated variationnal problems. *Communications on Pure and Applied Mathematics*, XLVII(5):577–685, 1989.