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On isometries of Product of normed linear spaces.

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Abstract. We give a condition on norms under which two vector normed spaces $X$ and $Y$ are isometrically isomorphic if and only if $X \times \mathbb{R}$ and $Y \times \mathbb{R}$ are isometrically isomorphic. We also prove that this result fail for arbitrary norms even if $X = Y = \mathbb{R}^2$ by building a generic counterexamples.

Keyword, phrase: Normed vector space and isometries.

1 Introduction

We are interested in this paper in the following question. Let $X$ and $Y$ be vector spaces and let $N_X$ and $N_Y$ be two norms on $(X \times \mathbb{R}, N_X)$ and $(Y \times \mathbb{R}, N_Y)$ respectively. The norm $N_X$ on $X$ (and in a similar way $N_Y$) denotes $N_X(x,0)$ for all $x \in X$.

Problem. It is true that $(X \times \mathbb{R}, N_X)$ and $(Y \times \mathbb{R}, N_Y)$ are isometrically isomorphic if and only if $(X, N_X)$ and $(Y, N_Y)$ are isometrically isomorphic?

We begin by showing that in the general case the answer to this question is no for arbitrary norms $N_X$ and $N_Y$, even when $X$ and $Y$ are two dimensional vector spaces, by constructing a generic counterexamples (See Theorem 1). We prove then in Theorem 2 that the result is true for all norms $(N_X, N_Y)$ satisfying the following property $(P)$.

Definition 1 Let $X$ and $Y$ be two vector spaces. Let $N_X$ and $N_Y$ be two norms on $X \times \mathbb{R}$ and $X \times \mathbb{R}$ respectively. We say the the pair $(N_X, N_Y)$ satisfy the property $(P)$ if for all $x \in X$ and all $y \in Y$:

$$N_X(x,0) = N_Y(y,0) \Rightarrow N_X(x, \lambda) = N_Y(y, \lambda), \forall \lambda \in \mathbb{R}.$$ 

In all the article we identify $X$ with $X \times \{0\}$ and the norm $N_X$ on $X$ denotes $N_X(x,0)$ for all $x \in X$.

Exemples 1 Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed vector spaces. Let $p \in [1, +\infty]$ and

$$N_{X,p}(x,t) := (\|x\|_X^p + |t|^p)^{\frac{1}{p}},$$

$$N_{X,\infty}(x,t) := \max(\|x\|_X, |t|),$$

for all $(x,t) \in X \times \mathbb{R}$. In a similar way we define $N_{Y,p}$ and $N_{Y,\infty}$. Then the pairs $(N_{X,p}, N_{Y,p})$ and $(N_{X,\infty}, N_{Y,\infty})$ satisfies the property $(P)$. 

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We give in the following proposition a more general examples.

**Proposition 1** Let \( N_{X2} \) be any norm on \( \mathbb{R}^2 \) such that \( N_X(x, t) := N_{X2}(|x|_X, |t|) \) for all \((x, t) \in X \times \mathbb{R}\) defined a norm on \( X \times \mathbb{R} \) (Similarly we define \( N_Y \) on \( Y \times \mathbb{R}\).) Then \((N_X, N_Y)\) satisfy the property \((P)\).

**Proof.** Let \( x \in X \) and \( y \in Y \) be such that \( N_X(x, 0) = N_{Y}(y, 0) \). Then \( N_{X2}(|x|_X, 0) = N_{X2}(|y|_X, 0) \) and so \( |x|_X N_{X2}(1, 0) = |y|_Y N_{X2}(1, 0) \), which implies that \( |x|_X = |y|_Y \). It follows that \( N_{X2}(|x|_X, |\lambda|) = N_{X2}(|y|_Y, |\lambda|) \) for all \( \lambda \in \mathbb{R} \).

The problem mentioned above was motivated at the first time in [1] by questions connected to the Banach-Stone theorem, and solved positively only for the particular norms \( N_{X, p} \) and \( N_{Y, p} \) when \( p \in [1, +\infty] \setminus \{2\} \). The technique used in [1] did not include the case \( p=2 \). The property \((P)\) bear is more general and allowed to include varied norms. We give in section 4 other simple examples of applications of Theorem 2.

## 2 A generic counterexample.

**Theorem 1** Let \( X = Y = \mathbb{R}^2 \). For each norm \( \| \cdot \|_X \) on \( X \) there exists a norm \( \| \cdot \|_Y \) on \( Y \), a norm \( N_X \) on \( X \times \mathbb{R} \) and a norm \( N_Y \) on \( Y \times \mathbb{R} \) such that:

1. \((X, \| \cdot \|_X)\) is not isometrically isomorphic to \((Y, \| \cdot \|_Y)\).
2. \((X \times \mathbb{R}, N_X)\) is isometrically isomorphic to \((Y \times \mathbb{R}, N_Y)\).
3. the restriction of \( N_X \) to \( X \) coincide with \( \| \cdot \|_X \) and the restriction of \( N_Y \) to \( Y \) coincide with \( \| \cdot \|_Y \).

**Proof.** Let \( p \in [1, +\infty] \). Let us define \( N_X \) and \( N_Y \) as follow:

\[
N_X(x_1, x_2, t) := \left( |(x_1, x_2)|^p_X + |t|^p \right)^{\frac{1}{p}}, \quad \forall (x_1, x_2, t) \in X \times \mathbb{R}
\]

and

\[
N_Y(y_1, y_2, s) := \left( |y_2|^p + \frac{|(y_1, s)|_Y^p}{a^p} \right)^{\frac{1}{p}}, \quad \forall (y_1, y_2, s) \in Y \times \mathbb{R}.
\]

Where \( a = \|(1, 0)\|_X \). Let us define the norm \( \| \cdot \|_{Y, p} \) on \( Y \) as follows \( \| (y_1, y_2) \|_{Y, p} := \left( |y_1|^p + |y_2|^p \right)^{\frac{1}{p}} \) for all \((y_1, y_2) \in Y \). Clearly,

\[
N_X(x_1, x_2, 0) = \| (x_1, x_2) \|_X, \quad \forall (x_1, x_2) \in X
\]

and

\[
N_Y(y_1, y_2, 0) = \left( |y_1|^p + |y_2|^p \right)^{\frac{1}{p}} := \| (y_1, y_2) \|_{Y, p}, \quad \forall (y_1, y_2) \in Y.
\]

Since \( \frac{|(y_1, 0)|_Y^p}{a^p} = |y_1| \cdot \frac{\| (1, 0) \|_Y^p}{a^p} = |y_1| \). On the other hand, the following map is an isometric isomorphism:

\[
\Theta : (X \times \mathbb{R}, N_X) \rightarrow (Y \times \mathbb{R}, N_Y)
\]

\[
(x_1, x_2, t) \mapsto (ax_1, t, ax_2).
\]

Now, there exist cases:

**Case 1**: If every point of the sphere \( S_X \) of \( X \) is an extreme point, we choose \( p = 1 \) and so \( S_Y \) has no non extreme point since in this case \( \| (y_1, y_2) \|_{Y, 1} = |y_1| + |y_2| \) (For example \( \frac{1}{2}, \frac{1}{2} \)) is not extreme for \( \| \cdot \|_{Y, 1} \). Consequently \( X \) and \( Y \) cannot be isometrically isomorphic.

**Case 2**: If there exists some point of the sphere \( S_X \) which is not extreme point then we choose \( p = 2 \) and so every points of \( S_Y \) is an extreme point since \( \| (y_1, y_2) \|_{Y, 2} = |y_1|^2 + |y_2|^2 \) is the euclidean norm. Also \( X \) and \( Y \) cannot be isometrically isomorphic.■

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3 Isometries between product spaces.

Theorem 2 Let X and Y be a vector spaces. Suppose that $(N_X, N_Y)$ satisfy the property (P). Then $(X \times \mathbb{R}, N_X)$ and $(Y \times \mathbb{R}, N_Y)$ are isometrically isomorphic if and only if $(X, N_X)$ and $(Y, N_Y)$ are isometrically isomorphic.

The proof of the above theorem is given in section 3.2 after some lemmas.

3.1 Notations and lemmas.

We need some notations and lemmas. Let $\Theta : (X \times \mathbb{R}, N_X) \to (Y \times \mathbb{R}, N_Y)$ be an isomorphism isometric. We set $(a, u) = \Theta^{-1}(0, 1)$ and $(b, v) = \Theta(0, 1)$. Let us define the linear continuous map $\chi_X$ as follow:

$$\chi_X : X \times \mathbb{R} \rightarrow \mathbb{R}$$

$$(x, t) \mapsto t$$

We define analogously the map $\chi_Y$ by

$$\chi_Y : Y \times \mathbb{R} \rightarrow \mathbb{R}$$

$$(y, t) \mapsto t$$

We obtain the following linear map on $Y \times \{0\}$:

$$\chi_X \circ \Theta^{-1} : Y \times \{0\} \rightarrow \mathbb{R}$$

Analogously we have also the linear map on $X \times \{0\}$:

$$\chi_Y \circ \Theta : X \times \{0\} \rightarrow \mathbb{R}$$

Let us set $X_0 := Ker(\chi_Y \circ \Theta)$ and $Y_0 := Ker(\chi_X \circ \Theta^{-1})$.

Remark 1 The linear spaces $X_0$ and $Y_0$ are not necessarily closed since $\chi_X$ and $\chi_Y$ are not necessarily continuous.

Lemma 1 $X_0$ and $Y_0$ are isometrically isomorphic. More precisely, the map

$$\Theta : (X_0, N_X) \rightarrow (Y_0, N_Y)$$

$$(z, 0) \mapsto \Theta(z, 0)$$

(1)

is an isomorphism isometric.

Proof. Since $\Theta$ is an isomorphism isometric, it suffices to show that the restriction of $\Theta$ to $X_0$ is onto. Indeed, let $(y, 0) \in Y_0$. Clearly, $(z, 0) := \Theta^{-1}(y, 0) \in X_0$ since $\chi_Y \circ \Theta(\Theta^{-1}(y, 0)) = \chi_Y(0, 0) = 0$ and we have $(y, 0) = \Theta(z, 0)$.

Lemma 2 We have only two cases.

Case1: $u \neq 0$. In this case, we have $X \times \{0\} = X_0$.

Case2: $u = 0$. In this case we have $\Theta^{-1}(0, 1) = (a, 0)$ and $X \times \{0\} = X_0 \oplus \mathbb{R}(a, 0)$.

Similarly we have,

Case1: $v \neq 0$. In this case, we have $Y \times \{0\} = Y_0$.

Case2: $v = 0$. In this case we have $\Theta(0, 1) = (b, 0)$ and $Y \times \{0\} = Y_0 \oplus \mathbb{R}(b, 0)$.
Proof. For all \( x \in X \) there exists \((y_x, \lambda_x) \in Y \times \mathbb{R}\) such that

\[
(x, 0) = \Theta^{-1}(y_x, \lambda_x) = \Theta^{-1}(y_x, 0) + \lambda \Theta^{-1}(0, 1) = \Theta^{-1}(y_x, 0) + \lambda \Theta(a, u) = \Theta^{-1}(y_x, 0) + (\lambda \Theta(a, u))
\]  \hfill (2)

Since \( \Theta^{-1}(y_x, 0) \in X_0 \subset X \times \{0\} \) and also \((x, 0) \in X \times \{0\}\), then from the above equation we obtain that \((\lambda \Theta(a, u)) \in X \times \{0\}\) which implies that \(\lambda \Theta(a, u) = 0\). So we have:

Case 1: \( u \neq 0 \). In this case, \( X \times \{0\} = X_0 \). Indeed, if \( u \neq 0 \) then \( \lambda_x = 0 \) and so \((x, 0) = \Theta^{-1}(y_x, 0) \in X_0\), for all \( x \in X \) i.e \( X \times \{0\} \subset X_0 \). On the other hand we know that \( X_0 \subset X \times \{0\} \).

Case 2: \( u = 0 \). In this case we have \( \Theta^{-1}(0, 1) = (a, 0) \) and so \( X = X_0 \oplus \mathbb{R}(a, 0) \). Indeed. We have \( X_0 \cap \mathbb{R}(a, 0) = \{0, 0\} \), since if \( \alpha \) is a real number such that \( \alpha(a, 0) \in X_0 \) then \( 0 = \chi_Y \Theta(\alpha(a, 0)) = \alpha \chi_Y (0, 1) = \alpha \). In other words from (2), for all \( x \in X \), there exist \((y_x, \lambda_x) \in Y \times \mathbb{R}\) such that

\[
(x, 0) = \Theta^{-1}(y_x, 0) + \lambda \Theta(a, u).
\]

whit \( \Theta^{-1}(y_x, 0) \in X_0 \). Thus \( X \times \{0\} \subset X_0 \oplus \mathbb{R}(a, 0) \subset X \times \{0\} \) and so \( X \times \{0\} = X_0 \oplus \mathbb{R}(a, 0) \).

In a similar way we obtain the second part of the lemma.

Lemma 3 We have, \( u = 0 \) if and only if \( v = 0 \).

Proof. Suppose that \( v = 0 \). Then for all \((x, t) \in X \times \mathbb{R}\), we have \( \Theta(x, t) = \Theta(x, 0) + \Theta(0, t) = \Theta(x, 0) + t \Theta(0, 1) = \Theta(x, 0) + (t, 0) \). Now, we are going to prove that \( u = 0 \).

Suppose that the contrary hold, that is \( u \neq 0 \). Then \( X \times \{0\} = X_0 \) (See the case 1. in Lemma 2). So \( \Theta(x, 0) \in \Theta(X \times \{0\}) = \Theta(X_0) = Y_0 \), since \( \Theta \) is an isomorphism isometric from \( X_0 \) onto \( Y_0 \) (See the formula (1)). Now since \( Y_0 \subset Y \times \{0\} \), then \( \Theta(x, 0) + (t, 0) \in Y \times \{0\} \). In other words, \( \Theta(x, t) \in X \times \{0\} \) for all \((x, t) \in X \times \mathbb{R}\). So \( \Theta(X \times \mathbb{R}) \subset Y \times \{0\} \). But \( \Theta \) is an isomorphism between \( X \times \mathbb{R} \) and \( Y \times \mathbb{R} \). This implies that \( Y \times \{0\} = Y \times \mathbb{R} \) which is impossible. Thus \( u = 0 \). In a similar way we obtain the converse.

3.2 Proof of Theorem 2 and some corollaries.

We give now the proof of the main result.

Proof of Theorem 2. For the “if” part, let \( T : (X, N_X) \to (Y, N_Y) \) be an isomorphism isometric. Let us define \( \Theta : (X \times \mathbb{R}, N_{X}) \to (Y \times \mathbb{R}, N_Y) \) by \( \Theta(x, \lambda) = (T(x), \lambda) \). Then, clearly \( \Theta \) is an isomorphism and by the property \((P)\) it is also isometric. We prove now the “only if” part. By combining Lemma 2 and Lemma 3 we have that:

Case 1: If \( u \neq 0 \) and \( v \neq 0 \), then \( X \times \{0\} = X_0 \) and \( Y \times \{0\} = Y_0 \). So by Lemma 1 we conclude that \( X \times \{0\} \) and \( Y \times \{0\} \) are isometrically isomorphic for the norms \( N_X \) and \( N_Y \). So \( (X, N_X) \) and \( (Y, N_Y) \) are isometrically isomorphic.

Case 2: If \( u = 0 \) and \( v = 0 \), using Lemma 2 we have that \( \Theta^{-1}(0, 1) = (a, 0) \) and \( X \times \{0\} = X_0 \oplus \mathbb{R}(a, 0) \) and \( \Theta(0, 1) = (b, 0) \) and \( Y \times \{0\} = Y_0 \oplus \mathbb{R}(b, 0) \). Now we prove that the map

\[
\psi : X \times \{0\} = X_0 \oplus \mathbb{R}(a, 0) \to Y \times \{0\} = Y_0 \oplus \mathbb{R}(b, 0)
\]

\[
(z, 0) + \lambda(a, 0) \mapsto \Theta(z, 0) + \lambda(b, 0)
\]

is an isomorphism isometric. Indeed, the fact that \( \psi \) is linear and onto map is clear by using Lemma 1. Let us prove that \( \psi \) is isometric for the norms \( N_X \) and \( N_Y \). Since \( (z, 0) \in X_0 \), by (1) there exist \((y, 0) \in Y_0 \) such that \( \Theta(z, 0) = (y, 0) \). Since

\[
(z, 0) + \lambda(a, 0) = \Theta^{-1}(\Theta(z, 0)) + \lambda \Theta^{-1}(0, 1)
\]

\[
= \Theta^{-1}(\Theta(z, 0) + (0, \lambda))
\]

\[
= \Theta^{-1}((y, \lambda))
\]
then, using the fact that $\Theta^{-1}$ is isometric we have

$$N_X((z,0) + \lambda(a,0)) = N_X(\Theta^{-1}(y,\lambda)) = N_Y(y,\lambda).$$

(3)

On the other hand we known that $(b,0) = \Theta(0,1)$ so $\Theta(z,0) + \lambda(b,0) = \Theta(z,0) + \lambda(0,1) = \Theta(z,\lambda)$. Thus, using the fact that $\Theta$ is isometric we have.

$$N_Y(\psi((z,0) + \lambda(a,0))) = N_Y(\Theta(z,0) + \lambda(b,0)) = N_Y(\Theta(\psi(z),\lambda)) = N_X(z,\lambda).$$

(4)

But $N_X(z,0) = N_Y(y,0)$ since $\Theta(z,0) = (y,0)$ and $\Theta$ is isometric. Since $(N_X, N_Y)$ satisfy the property $(P)$ then $N_X(z,\lambda) = N_Y(y,\lambda)$. Thus, using the formulas (3) and (4) we obtain that $\psi$ is isometric.

Remark 2 By induction, we can easily extend the above theorem to $X \times \mathbb{R}^n$ $(n \in \mathbb{N}^*)$ if we assume that $(N_X, N_Y)$ is a pair of norms satisfying the following property $(P^n)$: for all $x \in X$ all $y \in Y$, all $i \in \{1, 2, ..., n\}$ and all $(s_1, s_2, ..., s_i)$; $(s'_1, s'_2, ..., s'_i) \in \mathbb{R}^i$ : if $N_X(x, s_1, s_2, ..., s_i, 0, ..., 0) = N_Y(y, s'_1, s'_2, ..., s'_i, 0, ..., 0)$ then $N_X(x, s_1, s_2, ..., s_i, 0, ..., 0) = N_Y(y, s'_1, s'_2, ..., s'_i, \lambda, 0, ..., 0), \forall \lambda \in \mathbb{R}$.

\textbf{Example 2} Let $p \in [1, +\infty]$, and

$$N_{X,p}(x, s_1, ..., s_n) = (\|x\|_X^p + \sum_{k=1}^n |s_k|^p)^{\frac{1}{p}},$$

$$N_{X,\infty}(x, s_1, ..., s_n) = \max(\|x\|_X, |s_1|, ..., |s_n|)$$

for all $(x, s_1, ..., s_n) \in X \times \mathbb{R}^n$. In a similar way we define $N_{Y,p}$ and $N_{Y,\infty}$. Then the pairs $(N_{X,p}, N_{Y,p})$ and $(N_{X,\infty}, N_{Y,\infty})$ satisfies the property $(P^n)$.

\textbf{Corollary 1} Let $X$ and $Y$ be a vector spaces. Let $n \in \mathbb{N}^*$ and suppose that $(N_X, N_Y)$ satisfy $(P^n)$. Then $(X \times \mathbb{R}^n, N_X)$ and $(Y \times \mathbb{R}^n, N_Y)$ are isometrically isomorphic if and only if $(X, N_X)$ and $(Y, N_Y)$ are isometrically isomorphic.

As a remark we have the following corollary for inner product spaces. Note that a non complete inner product space has no orthonormal basis in general (See [5]). The symbol $\cong$ means “isometrically isomorphic”.

\textbf{Corollary 2} Let $(H, \|\cdot\|_H)$ and $(L, \|\cdot\|_L)$ be two inner product space (not necessary complete). Then $(H, \|\cdot\|_H) \cong (L, \|\cdot\|_L)$ if and only if for all finite dimensional subspaces $E \subset H$ and $F \subset L$ such that $\dim(E) = \dim(F)$ we have that $(E^\perp, \|\cdot\|_H) \cong (F^\perp, \|\cdot\|_L)$. Where $E^\perp$ and $F^\perp$ denotes the orthogonal of $E$ and $F$ respectively.

\textbf{Proof.} Let $E \subset H$ and $F \subset L$ such that $\dim(E) = \dim(F) = n$ for $n \in \mathbb{N}$. By the classical projection theorem on a complete vector subspace of an inner product space, we have $H = E^\perp \oplus E$ and $L = F^\perp \oplus F$. On the other hand it is clear that $(H, \|\cdot\|_H) \cong (E^\perp \times \mathbb{R}^n, N_{E^\perp,2})$ and $(L, \|\cdot\|_L) \cong (F^\perp \times \mathbb{R}^n, N_{F^\perp,2})$, where $N_{E^\perp,2}$ and $N_{F^\perp,2}$ are defined as in the Example 2 with $p = 2$. Since $(N_{E^\perp,2}, N_{F^\perp,2})$ satisfy $(P^n)$ then from Corollary 1 we obtain $(E^\perp, \|\cdot\|_H) \cong (F^\perp, \|\cdot\|_L)$. The converse is clear.
4 Applications.

We give in this section two applications of Theorem 2. We denote by \( (C^1[0,1],N_{C^1[0,1]}) \) the space of continuously differentiable functions on \([0,1]\) endowed with the norm \( N_{C^1[0,1]}(f) := N_{\mathbb{R}^2}(\|f\|_{\infty},\|f(0)\|) \), where \( N_{\mathbb{R}^2} \) denotes any norm satisfying Proposition 1. Let \((X,\|\cdot\|_X)\) be a Banach space. We denote by \( N_X \) the norm defined on \( X \times \mathbb{R} \) by \( N_X(x,t) := N_{\mathbb{R}^2}(\|x\times x,\|t\|) \) for all \((x,t) \in X \times \mathbb{R}\). Finally, we denote by \((C[0,1],\|\cdot\|_{\infty})\) the space of continuous functions on \([0,1]\) endowed with the supremum norm.

**Proposition 2** We have \((X \times \mathbb{R},N_X) \cong (C^1[0,1],N_{C^1[0,1]}),\) if and only if \((X,\|\cdot\|_X) \cong (C[0,1],\|\cdot\|_{\infty})\).

**Proof.** Let us define the norm \( N_{C[0,1]} \) on \([0,1] \times \mathbb{R} \) by \( N_{C[0,1]}(g,t) := N_{\mathbb{R}^2}(\|g\|_{\infty},\|t\|) \) for all \((g,t) \in [0,1] \times \mathbb{R} \). Let us consider the map

\[
\chi : (C^1[0,1],N_{C^1[0,1]}) \rightarrow ([0,1] \times \mathbb{R},N_{C[0,1]}),
\]

\[
f \mapsto (f',f(0))
\]

Clearly, \( \chi \) is an isomorphism. So we have \((X \times \mathbb{R},N_X) \cong (C[0,1] \times \mathbb{R},N_{C[0,1]}).

Since \((N_X,N_{C[0,1]})\) satisfy the property \((P)\) by Proposition 1 then using Theorem 2 we obtain that \((X,\|\cdot\|_X) \cong (C[0,1],\|\cdot\|_{\infty}),\) since \( N_X(\|x\times x,0\|) = \|x\times x\|_{\mathbb{R}^2}(1,0) \) and \( N_{C[0,1]}(g,0) = \|g\|_{\infty} N_{\mathbb{R}^2}(1,0) \).

Let us recall some notions. Let \( K \) and \( C \) be convex subsets of vector spaces. A function \( T : K \rightarrow C \) is said to be affine if for all \( x,y \in K \) and \( 0 \leq \lambda \leq 1, T(\lambda x + (1-\lambda)y) = \lambda T(x) + (1-\lambda)T(y). \) The set of all continuous real-valued affine functions on a convex subset \( K \) of a topological vector space will be denoted by \( Aff(K). \) Clearly, all translates of continuous linear functionals are elements of \( Aff(K), \) but the converse is not true in general (see [4] page 22). However, we do have the following relationship.

**Proposition 3** ([4], Proposition 4.5) Assume that \( K \) is a compact convex subset of a separated locally convex space \( X \) then

\[
\left\{ a \in Aff(K) : a = r + x^*_a \right\} \text{ for some } x^*_a \in X^* \text{ and some } r \in R \}
\]

is dense in \( Aff(K),\|\cdot\|_X \) where \( \|\cdot\|_X \) denotes the norm of uniform convergence.

But in the particular case when \( X \) is a Banach space and \( K = (B_{X^*},w^*) \) is the unit ball of the dual space \( X^* \) endowed with the weak star topology, the well known result due to Banach and Dieudonné states that:

**Theorem 3** (Banach-Dieudonné). The space \( Aff(0(B_{X^*}),\|\cdot\|_{\infty}) \) is isometrically identified to \( (X,\|\cdot\|). \) In other words, \( Aff(0(B_{X^*})) = \{ \tilde{z} \in B_{X^*} : z \in X \}. \) Where \( Aff(0(B_{X^*})) \) denotes the space of all affine weak star continuous functions that vanish at 0 and \( \tilde{z} : p \mapsto p(z) \) for all \( p \in X^* \) and \( \tilde{z} \in B_{X^*} \) denotes the restriction of \( \tilde{z} \) to \( B_{X^*}. \)

Now, let \( X \) and \( Y \) be two Banach spaces and let us endowed the space \( Aff(B_{X^*}) \) (and in a similar way the space \( Aff(B_{Y^*}) \)) with the norm \( N(f) := N_{\mathbb{R}^2}(\|f - f(0)\|_{\infty},\|f(0)\|) \) for all \( f \in Aff(B_{X^*}), \) where \( N_{\mathbb{R}^2} \) denotes any norm on \( \mathbb{R}^2 \) satisfying Proposition 1. We obtain the following version of the Banach-Stone theorem for affine functions (For more information about the Banach-Stone theorem see [2] and [3]).

**Proposition 4** Let \( X \) and \( Y \) be two Banach spaces. Then the following assertions are equivalent.

1. \( (Aff(B_{X^*}),N) \) and \( (Aff(B_{Y^*}),N) \) are isometrically isomorphic.
2. \( (X,\|\cdot\|_X) \) and \( (Y,\|\cdot\|_Y) \) are isometrically isomorphic.
Proof. Let $\tilde{N}$ be the norm on $\text{Aff}_0(B_{X^*}) \times \mathbb{R}$ defined by $\tilde{N}(f_0, t) := N_{\mathbb{R}^2}(\|f_0\|_\infty, |t|)$ for all $(f_0, t) \in \text{Aff}_0(B_{X^*}) \times \mathbb{R}$. Let us consider the map,

$$
\chi : (\text{Aff}(B_{X^*}), N) \to (\text{Aff}_0(B_{X^*}) \times \mathbb{R}, \tilde{N})
$$

$$
f \mapsto (f - f(0), f(0))
$$

Clearly, $\chi$ is an isometric isomorphism. Thus using Theorem 1 we have that $(\text{Aff}(B_{X^*}), N)$ and $(\text{Aff}(B_{X^*}), N)$ are isometrically isomorphic if and only if $(\text{Aff}_0(B_{Y^*}), \|\cdot\|_\infty)$ and $(\text{Aff}_0(B_{Y^*}), \|\cdot\|_\infty)$ are isometrically isomorphic, which is equivalent by Theorem 3 to the fact that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are isometrically isomorphic.\[\square\]

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References


