Angular statistics of Lagrangian trajectories in turbulence
Wouter Bos, Benjamin Kadoch, Kai Schneider

To cite this version:

HAL Id: hal-01085070
https://hal.archives-ouvertes.fr/hal-01085070
Submitted on 20 Nov 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Angular multiscale statistics of Lagrangian trajectories in turbulence

Wouter J.T. Bos\textsuperscript{1}, Benjamin Kadoch\textsuperscript{2} and Kai Schneider\textsuperscript{3}
\textsuperscript{1} LMFA, CNRS UMR 5509, Ecole Centrale de Lyon, Universit\'e de Lyon, Ecully, France
\textsuperscript{2} IUSTI, CNRS UMR 7343, Aix-Marseille University, Marseille, France
\textsuperscript{3} M2P2, CNRS UMR 7340 & CMI, Aix-Marseille University, Marseille, France

The angle between subsequent particle displacement increments is evaluated as a function of the timelag in isotropic turbulence. It is shown that the evolution of this angle contains two well-defined power-laws, reflecting the multi-scale dynamics of high-Reynolds number turbulence. The probability density function of the directional change is shown to be self-similar and well approximated by an analytically derived model assuming Gaussianity and independence of the velocity and the Lagrangian acceleration.

PACS numbers: 47.27.Jv, 47.27.Gs

Advances in experimental devices and numerical simulations over the last two decades have opened the way to a Lagrangian characterization of turbulent flows [1–3]. The structural description of the statistical dynamics of turbulence has thereby shifted from the investigation of spatial correlations of instantaneous velocity fields to the study of temporal correlations along fluid particle trajectories. In the Lagrangian reference frame, the spatio-temporal complexity of turbulence manifests itself through the spiraling chaotic motion of fluid particles, changing direction at every timescale. This directional change of Lagrangian tracers, as a function of the timelag between two observations, is the subject of the present work. Instantaneous measures of the curvature in turbulence have been investigated in the past five years for academic turbulent flows, both in three [4, 5] and in two space dimensions [6, 7]. Curvature is dominated by the small-scale structures and contains only little information on the multiscale dynamics of turbulent flows. Multi-scale dynamics can be measured by Lagrangian structure functions [1, 3], but those do not contain any direct information on the curvature of the trajectories.

The related measure which represents the coarse grained curvature over a time interval was only recently introduced by Burov \textit{et al.} [8]. More precisely, in this last work the directional change of a particle was introduced, and the characteristics of this new measure were investigated in various types of random walks. In the present work, we will show how this measure can characterize the time-correlation of the direction of a fluid particle in a turbulent flow. In particular we will show how the multi-scale character of a turbulent flow can be revealed by considering the timelag dependence of the directional change.

We define the Lagrangian spatial increment as

$$\delta X(x_0, t, \tau) = X(x_0, t) - X(x_0, t - \tau)$$

where \(X(x_0, t)\) is the position of a fluid particle at time \(t\), passing through point \(x_0\) at the reference time \(t = t_0\) and advected by a velocity field \(\mathbf{u}\), i.e. \(dX/dt = \mathbf{u}\). The cosine of the angle \(\Theta(t, \tau)\) between subsequent particle increments, introduced in [8], is

$$\cos(\Theta(t, \tau)) = \frac{\delta X(x_0, t, \tau) \cdot \delta X(x_0, t + \tau, \tau)}{|\delta X(x_0, t, \tau)| \cdot |\delta X(x_0, t + \tau, \tau)|}.$$  \hspace{1cm} (2)

The angle is illustrated in Figure 1 (top). Rather than considering its instantaneous evolution, its averaged absolute value is of particular interest in an isotropic random velocity field. The ensemble average will be denoted in the following by

$$\theta(\tau) \equiv \langle |\Theta(t, \tau)| \rangle.$$  \hspace{1cm} (3)

We omitted the time-dependence since we will consider statistically stationary flow in the following. For short
time lags, $\theta(\tau)$ should be close to zero, whereas for times long compared to the correlation time associated with the spiraling motion $\theta(\tau)$ should tend to $\pi/2$ by symmetry.

For short times the instantaneous angle $\Theta(\tau,t)$ is related to the curvature $\kappa$ (see Figure 1 (top)) by the relation

$$\kappa(t) = \lim_{\tau \to 0} \frac{|\Theta(t,\tau)|}{2\pi \|u(t)\|},$$

with $u$ being the velocity. How the angle varies in between the short and long-time limits is the main subject of the present work and we will show that the dependence of $\theta(\tau)$ on the timelag contains the signature of the multi-scale dynamics of a turbulent flow.

The database used to investigate the behaviour of $\theta(\tau)$ is described in [9, 10]. The simulation was carried out using standard pseudo-spectral techniques, following 8.10^6 fluid particles in a statistically stationary isotropic turbulent flow during 5.8 integral timescales in a periodic cube of dimension $2\pi$. The resolution is 1024^3 gridpoints. The Kolmogorov time-scale $\tau_K = (\nu/\epsilon)^{1/2} = 0.036$, where $\epsilon = 0.31$ is the mean dissipation rate and $\nu = 4.10^{-4}$ the kinematic viscosity. The Lagrangian integral timescale is of the order of the Eulerian integral timescale. The Taylor-scale Reynolds number is $R_\lambda = 225$.

Figure 2 shows $\theta(\tau)$ in double-logarithmic representation. The angle increases monotonously from zero to $\pi/2$, and this latter value is approached for values of $\tau$ of the order of the Lagrangian integral timescale. Two power-laws can be identified in this graph, with a cross-over around twice the Kolmogorov timescale. The origin of these power-laws will now be elucidated.

For our phenomenological explanation, we consider high-Reynolds-number isotropic turbulence, containing flow structures on a wide range of different scales. We consider short timelags $\tau \ll T$, where $T$ is the Lagrangian integral timescale of the flow. In this limit, the angle $\Theta(\tau)$ can be approximated using a Taylor-expansion by,

$$\frac{l_\parallel}{l_\perp} \approx |\tan(\Theta/2)| \approx |\Theta/2|.$$
a relation which can be refined to take into account intermittency corrections [12–14]. Omitting these corrections, expressions (10) and (11) yield,

$$\theta(\tau) \sim \frac{\tau}{T} R_{\lambda}^{1/2} \quad \text{for } \tau \ll \tau_K. \quad (12)$$

At timescales larger than \( \tau_K \), but smaller than \( T \), i.e., in the inertial interval, the above approximations to obtain (9) are still valid. However, the subsequent approximation, that \( \sigma_\alpha(\tau) \) is independent of \( \tau \) is not valid anymore. Indeed, the perpendicular acceleration fluctuates rapidly in time, on a time-scale of the order of \( \tau_K \). Coarse-graining the acceleration over an interval \( \tau > \tau_K \), the influence of the more rapidly fluctuating scales is filtered out. Indeed, even if their contribution to the rms acceleration is dominant, if the coarse-graining is performed before considering the norm, positive and negative contributions will cancel each other. The remaining variance will be predominantly caused by scales with a time-scale larger than, or comparable to \( \tau \). Following classical Kolmogorov phenomenology [15, 16], the acceleration induced by inertial range structures with typical timescale \( \tau \) will be of the order

$$\sigma_\alpha(\tau) \sim (\epsilon/\tau)^{1/2}. \quad (13)$$

This estimate is obtained by neglecting the viscous contribution to the acceleration, a reasonable assumption even near the dissipation range scales [13], and realizing that the acceleration is due to pressure forces, which satisfy, at inertial range scales to a good approximation Kolmogorov-scaling [17, 18]. The scale of such eddies is proportional to

$$l(\tau) \sim \tau^{3/2} \epsilon^{1/2}. \quad (14)$$

The reciprocal dependence of the acceleration variance on \( \tau \) in expression (13) illustrates that the smallest scales are most efficient in accelerating the fluid particles. After the influence of the scales smaller than \( \tau \) is removed by the coarse graining, the remaining dominant contribution is caused by the smallest scales still present, i.e., with timescale \( \tau \). It is therefore those scales, with correlation-time \( \tau \) which will deviate particles from their trajectory over a lengthscale of the order of the correlation-length of the structures. This phenomenological picture is illustrated in Fig. 1, bottom, where it can be understood intuitively that scales of the size \( l \ll l(\tau) \) are too small to efficiently contribute to a perpendicular displacement averaged over a time-interval \( \tau \).

Combining (13) and (8) we obtain in the inertial range

$$\theta(\tau) \sim \tau^{1/2} \frac{\epsilon^{1/2}}{\sigma_u} \sim \left(\frac{\tau}{T}\right)^{1/2} \quad \text{for } \tau_K \ll \tau \ll T. \quad (15)$$

Again, this scaling is observable in Figure 2, even though the power-law is less well present than in the dissipation range. This is better appreciated by considering the compensated angle, \( \bar{\theta}(\tau) \equiv \theta(\tau)/\sigma_u(\epsilon \tau)^{1/2} \), plotted in the inset of the figure. The slow emergence of inertial ranges with the Reynolds number in Lagrangian statistics is fairly common [19] and is was recently even argued that they might be non-existent [20]. In the present case, the emergence of a plateau is undeniable. This might be because the inertial range scaling of the mean-angle \( \bar{\theta}(\tau) \) is not directly related to the Lagrangian structure functions. Indeed, the scaling is induced by considering the coarse-grained Lagrangian acceleration, a quantity of which the scaling is related to that of the Eulerian pressure gradient.

The above arguments and results considered the average value \( \theta(\tau) \) only. Further information, in particular on higher order moments, is contained in the probability density function (PDF) of the instantaneous angle and its evolution with \( \tau \). Those PDFs of the angle \( \Theta(t, \tau) \) and its cosine are shown in Figure 3. It is observed that the \( P_{\Theta}(\theta) \) for small \( \tau \) consists of a peak near zero, whereas for long-times a symmetric distribution between 0 and \( \pi \) is obtained. This latter distribution corresponds to the distribution between two randomly chosen vectors in three-dimensions. Its distribution is given by

FIG. 3: PDFs of (a) \( \Theta \) and (b) \( \cos(\Theta) \) for different timelags. Solid black lines indicate the long time asymptotic form of the PDFs.
The measure we investigated in the foregoing allows a different angle of attack on the simultaneous characterisation of the multi-scale character of turbulence and the scale-dependent curvature of Lagrangian fluid particle trajectories. In this light an interesting perspective is to clarify the link between the current work and the results obtained using the recently introduced longitudinal Lagrangian structure functions [21].

Acknowledgments. The authors are indebted to Oliver Kamps and Michael Wilczek who provided us the DNS data used in the present investigation.


