Combinatorics, Words and Symbolic Dynamics

Eda Cesaratto, Brigitte Vallée

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Combinatorics, Words and Symbolic Dynamics

Edited by
Valérie Berté and Michel Rigo

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1

Pseudo-randomness of a random Kronecker sequence. An instance of dynamical analysis

Eda Cesaratto and Brigitte Vallée

1.1 Introduction.

A measure of randomness on the unit interval $\mathcal{I} := [0, 1]$ tests how a sequence $\mathcal{X} \subset \mathcal{I}$ differs from a “truly random” sequence. See books (Drmota and Tichy, 1997; Kuipers and Niederreiter, 1974) for a general discussion on the subject. Such a measure describes the difference between the behaviour of the truncated sequence $\mathcal{X}_{(T)}$ formed with the first T terms of the sequence and a “truly random” sequence formed with T elements of $\mathcal{I}$, and explains what happens when the truncation T becomes large.

Here, we focus on the case of the *Kronecker sequence* $\mathcal{K}(\alpha)$, formed of the fractional parts of the multiples of a real α . The three-distance theorem states that there are at most three possible distinct distances between geometric consecutive points in the truncated sequence $\mathcal{K}_{(T)}(\alpha)$. The length and the exact number of these distinct distances depend on the continued fraction expansion of the real α . Such a sequence $\mathcal{K}(\alpha)$ is thus very particular, and is surely *not* pseudo-random at all. On the other hand, it can be precisely studied since its main parameters are expressed as a function of the continued fraction expansion of the real α . This explains why there are many existing works which deal with various randomness measures of the Kronecker sequence, notably the discrepancy. However, they all adopt an “individual” point of view: for which reals α , and for which integers T , the discrepancy of the sequence $\mathcal{K}(\alpha)$ is minimal, maximal?

Our points of view.

Here, we adopt different points of view, which appear to be new:

(1) We focus on the special case when the truncation T gives rise to the two-distance phenomenon: the computations are easier, but already show very interesting phenomena. In particular, we introduce two different types of truncation, which depend on the *position* $\mu \in [0, 1]$ of the truncation: the *boundary* positions which correspond to $\mu = 0$ or $\mu = 1$, and the *generic* positions, with $\mu \in]0, 1[$ and we describe how they lead to different probabilistic behaviours.

(2) We study *five different* parameters of interest: each of the two distances (the small distance and the large distance), the covered space by the “large” intervals, and, finally, two randomness measures, the discrepancy and the Arnold measure.

(3) We adopt a *probabilistic* point of view: we choose the “input” α at random, and we wish to study the randomness of a *random* sequence $\mathcal{K}(\alpha)$. We estimate in particular the mean values of our main parameters, when the truncation T tends to ∞ . This is different from the existing works which adopt an “individual” point of view.

(4) We consider the usual case when α is a *random real* number, but we also focus on the study of the rational model, where α is a *random rational* of the unit interval. This case is always forgotten in the literature of the subject, except in the paper (Cesaratto et al., 2006). However, it appears in a natural way in the general framework of pseudo-random generators, where it is useful to study modular arithmetic progressions of the form $\{k \mapsto k \cdot u \pmod{\nu}, \quad k \leq T\}$. Of course, the study is interesting only if $T < \nu$, and it is useful to relate the truncation T and the denominator ν so that they tend both to ∞ .

(5) It is already known that the *size* of the digits¹ in the continued fraction expansion of α plays an important rôle in the behaviour of the Kronecker sequence $\mathcal{K}(\alpha)$. This is why we deal with the *constrained* models where the input α (rational or real) has all its digits in its continued fraction expansion bounded by some integer M . This leads to the other two models: the *constrained real* model and the *constrained rational* model.

Finally, we study five parameters, four probabilistic models (real versus rational, unconstrained versus constrained), and two types of truncations. We then obtain forty statements...

Main results.

Our results exhibit the following common features about the mean values of our five parameters:

(a) A strong *parallelism* between the *real* and in the *rational* models. For instance, when the truncation T has a boundary position, the three mean values for covered space, discrepancy and Arnold measure, tend to finite limits, and the limits in the real model and in the rational model coincide.

(b) A precise *transition* between the *constrained model*, where all the digits in the continued fraction expansion are bounded by M and the *unconstrained model*, where there is no constraint on the digit size (this is the case when $M = \infty$). We then study three cases $M < \infty$, $M = \infty$, and $M \rightarrow \infty$.

(c) A different behaviour, according to the *position* $\mu \in [0, 1]$ of the truncation T . There are two main cases: the *boundary* positions where μ equals 0 or 1, and the *generic* positions where $\mu \in]0, 1[$.

¹ They are more usually called *partial quotients*, but, in the paper, they will be called digits.

(d) A characterisation of the *worst position* of the truncations T which leads to the *worst mean pseudo-randomness* behaviour, for the discrepancy or the Arnold measure.

Methodology.

This work strongly uses the dynamical analysis methodology, developed by Vallée (Flajolet and Vallée, 1998; Baladi and Vallée, 2005; Vallée, 2006), which combines tools imported from dynamics, such as transfer operators, with various tools of analytic combinatorics: generating functions, Dirichlet series, Landau theorem, Tauberian theorem.

Computing the successive digits of the continued fraction expansion of a real number α of the unit interval is performed by the iteration of the *Gauss map* $S: \mathcal{I} \rightarrow \mathcal{I}$. This map associates with a non-zero $x \in \mathcal{I}$ the number $S(x) = 1/x - m(x)$ with $m(x) := \lfloor 1/x \rfloor$ and satisfies furthermore $S(0) = 0$. With the trajectory of α under the iteration of S , it is easy to compute the sequence of the digits in the continued fraction expansion, namely the sequence $(m(\alpha), m(S(\alpha)), m(S^2(\alpha)), \dots, m(S^k(\alpha)) \dots)$. The dynamical system $(\mathcal{I}, S)$ is often called the *Euclidean dynamical system*, since the execution of the Euclid algorithm on the pair (u, v) of integers with $u \leq v$ is closely related to the trajectory of the rational u/v under S .

When this process is viewed as a dynamical system, it is then possible to use all the tools of the dynamical systems theory, and, amongst them, a central object: the transfer operator. The transfer operator $\mathbf{H}_s$ associated with the Euclidean dynamical system is omnipresent in all the studies which are related to the continued fraction expansions. Here, it plays the rôle of a generating operator, since it generates itself generating functions, that are here of Dirichlet type. In this context, the asymptotic analysis would be closely related to the dominant eigenvalue of the transfer operator. Then, as it is already the case in analytic combinatorics, there are two main steps: The combinatorial step deals with the transfer operator, and views it as a generating operator, which generates itself the generating functions of interest (that are of Dirichlet type, and depend on a complex parameter s). The analytic step deals with the geometry of the dynamical system, translates it into spectral properties of the transfer operator, then into analytical properties for the generating functions. The dominant eigenvalue of the operator plays there the same rôle as the dominant singularity in “classical” analytic combinatorics (see (Vallée, 2006) for a general survey of the methodology).

Plan of the paper

Section 1.2 introduces the five parameters of interest which describe the pseudo-randomness of the Kronecker sequence $\mathcal{K}(\alpha)$ and provides their expressions as a function of the main characteristics of the continued fraction expansion of the real

α . Then Section 1.2.6 describes the general framework for the analysis. We introduce two notions: the position for truncations, the balance for costs, which leads to a classification of our parameters: the distances are unbalanced, whereas the other three parameters (covered space, discrepancy and Arnold measure) are balanced. The probabilistic models are described in Section 1.3. Section 1.4 contains the (forty) statements which describe the results. Then, Section 1.5 explains the general methodology, based on dynamical analysis: it introduces the Euclidean dynamical system, defined via the Gauss map, and explains how various transfer operators related to this dynamical system are viewed as generating operators which generate themselves the main objects of interest. It describes the main steps of the proofs.

The next two sections are devoted to outline the proofs and may be skipped by the reader who is not interested by technical details. Section 1.6, focusses on balanced parameters, and introduces a further classification between costs (ordinary versus extremal) whereas Section 1.7 deals with the unbalanced case. The reader who is interested by the complete proofs may read the long version of the paper (Cesaratto and Vallée, 2014). Finally, Section 1.8 summarizes basic facts on functional analysis and Section 1.9 describes some open problems.

1.2 Five parameters for Kronecker sequences.

In this section, we introduce the parameters of interest which describe the behaviour of a Kronecker sequence. First, we recall the two-distance theorem, and introduce, together with the two distances themselves, three measures of randomness (covered space, discrepancy and Arnold measure). One considers a sequence $\mathcal{X}$ of the unit interval $\mathcal{I} := [0, 1]$, and, for an integer T , the truncated sequence $\mathcal{X}_{\langle T \rangle}$ formed by the first T elements of the sequence $\mathcal{X}$. After re-ordering the sequence $\mathcal{X}_{\langle T \rangle}$, one obtains an increasing sequence $\{y_i : i \in \llbracket 1, T \rrbracket\}$, and the distance $y_{i+1} - y_i$ between consecutive elements is denoted by δ_i , whereas the last distance $\delta_{\langle T \rangle}$ is defined as $\delta_{\langle T \rangle} := 1 + y_1 - y_{\langle T \rangle}$.

An instance of this general framework is described in Figure 1.1. It involves the Kronecker sequence defined in the next section.

1.2.1 Kronecker sequences and the two-distance phenomenon.

The *Kronecker sequence* $\mathcal{K}(\alpha)$ associates with a real α of the unit interval $\mathcal{I} := [0, 1]$ the fractional parts of the multiples of α ,

$$\mathcal{K}(\alpha) := \{\{n\alpha\}; n \in \mathbb{N}\} . \quad (1.1)$$

Here, $\{t\}$ denotes the fractional part of t , namely $\{t\} = t - \lfloor t \rfloor$, where $\lfloor t \rfloor$ denotes the integer part. Remark that, for a rational α of the form $\alpha = u/v$ with coprime integers (u, v) , the sequence is completely defined for $n < v$.

Figure 1.1 Description of the truncated sequence $\mathcal{K}_{(T)}(\alpha)$ of the Kronecker sequence $\mathcal{K}(\alpha)$ defined in (1.1), with $\alpha = 7/17$, and truncation $T = 4$

The truncated sequence $\mathcal{K}_{(T)}(\alpha)$ is a very well studied object. Remark that the set of “useful” truncations T is the set $\mathbb{N}$ of integers (for an irrational α) and the set $\llbracket 1, v-1 \rrbracket$ for a rational $\alpha = u/v$. The truncated sequence $\mathcal{K}_{(T)}(\alpha)$ is described with the three main parameters which describe the continued fraction expansion of the real α , namely

- (a) the digits m_k ,
- (b) the denominators q_k of the approximants p_k/q_k of α , also named continuants,
- (c) the differences $\theta_k := q_{k-1}\eta_{k-1} = |q_{k-1}\alpha - p_{k-1}|$.

For an irrational α , these sequences are defined for any index $k \geq 1$; for a rational $\alpha := u/v$ with coprime integers (u, v) , these sequences are defined for $k \in \llbracket 1, P(\alpha) \rrbracket$, where $P(\alpha)$, called the depth of α is the smallest index k for which q_k equals v .

Section 1.5.1 recalls the main facts about the three sequences. Here, Figure 1.2 gives an example of these parameters, for the rational number $\alpha = 7/17$.

$$\frac{7}{17} = \frac{1}{2 + \frac{1}{2 + \frac{1}{3}}}$$

$m_1 = 2$	$m_2 = 2$	$m_3 = 3$
$q_1 = 2$	$q_2 = 5$	$q_3 = 17$
$\theta_1 = 7/17$	$\theta_2 = 3/17$	$\theta_3 = 1/17$

Figure 1.2 The continued fraction expansion of the rational $\alpha = 7/17$. The depth equals 3, and the three sequences (m_k, q_k, θ_k) are defined for $k \in \llbracket 1, 3 \rrbracket$

The truncated sequence $\mathcal{K}_{(T)}(\alpha)$ satisfies a crucial property which explains its interest: This is the three-distance property, which states that, for any truncation T , the truncated sequence $\mathcal{K}_{(T)}(\alpha)$ possesses only two or three distinct distances between geometric consecutive points. With the previous definition of δ_i , this means that the distances δ_i take only three possible values. Both the characterisation of pairs (T, α) for which there exist only two distances, and the determination of the distances themselves depend on the sequences $(m_k), (q_k), (\theta_k)$. The three-distance theorem was conjectured and proved in (Surányi, 1958; Sós, 1958; S.Świerczkowski, 1959).

Figure 1.3 Two instances of a truncated Kronecker sequence $\mathcal{K}_{(T)}(\alpha)$ relative to $\alpha = 7/17$, with two truncations $T = 5$ and $T = 6$. The distances are denoted by δ_i and the vectors of these distances are $\delta = (1/17)(4, 3, 4, 3, 3)$ for $T = 5$, (on the left), and $\delta = (1/17)(1, 3, 3, 4, 3, 3)$ for $T = 6$ (on the right)

Theorem A. *Let α be a real of the unit interval. Consider the Kronecker sequence $\mathcal{K}(\alpha)$ and a truncation T for the sequence. Then, the truncated Kronecker sequence $\mathcal{K}_{(T)}(\alpha)$ has the three-distance property:*

There are at most three distinct values for the distances between geometrically consecutive points. There are only two distinct values when the truncation T is written as

$$T = m \cdot q_k + q_{k-1}, \quad \text{with } k \geq 1, \quad m \in \llbracket 1, m_{k+1} \rrbracket, \quad (1.2)$$

where the two sequences (q_k) and (m_k) are defined by the continued fraction expansion of the real α . Such an truncation T is called a two-distance truncation (with respect to α .)

For any fixed α , the disjoint intervals $[q_{k-1} + q_k, q_{k+1} + q_k[$ (for $k \geq 1$) provide a covering of the set of truncations. Then, any truncation T belongs to a unique interval $[q_{k-1} + q_k, q_{k+1} + q_k[$, and the index k is called the index of T . A truncation with index k is written as

$$T = m \cdot q_k + q_{k-1} + r \quad \text{with } m \in \llbracket 1, m_{k+1} \rrbracket \text{ and } r \in \llbracket 0, q_k - 1 \rrbracket,$$

and a two-distance truncation T is associated with a “remainder” $r = 0$.

Example 1.2.1 Figure 1.3 gives two examples of a truncated Kronecker sequence. With the values of the sequence (q_k) relative to the rational $\alpha = 7/17$ given in Figure 1.2, we see that the integer $T = 5$ is a two-distance truncation, whereas $T = 6$ is a three-distance truncation.

1.2.2 Two-distance truncations and positions.

In Eq. (1.2), the digit m may vary in the whole interval $\llbracket 1, m_{k+1} \rrbracket$, and it is well known that the digit m_{k+1} has an infinite mean value. This is why the digit m may play an important rôle.

Boundary cases. There are two particular cases for the digit m , namely, the two *boundary* cases, where $m = 1$ or $m = m_{k+1}$. In this last case, due to Relation (1.12), the digit m_{k+1} plays the same rôle as $m = 0$ (up to a translation on index $k := k + 1$) and does *not* actually appear. This is why these two two-distance truncations are particular.

Generic cases. It would be also important to consider the other possible values for m in the interval $\llbracket 1, m_{k+1} \rrbracket$. Our model focusses on the *position* of the digit m with respect to the *maximal possible digit* m_{k+1} , as we explain now.

We first consider the linear function $[0, 1] \rightarrow [1, m_{k+1}]$ which associates with a real $\mu \in [0, 1]$ the (real) number $1 + \mu(m_{k+1} - 1)$ of the interval $\llbracket 1, m_{k+1} \rrbracket$. However, we are interested in integer values, and it proves convenient to deal with the closest² integer $\lfloor \cdot \rfloor$. We thus associate with a real $\mu \in [0, 1]$, called the *position*, the integer $m \in \llbracket 1, m_{k+1} \rrbracket$ equal to $m = \lfloor 1 + \mu(m_{k+1} - 1) \rfloor$. When now the position μ is fixed and the index k varies, this defines the truncation sequence at position μ , namely

$$T_k^{(\mu)} := \lfloor 1 + \mu(m_{k+1} - 1) \rfloor q_k + q_{k-1} . \quad (1.3)$$

Example 1.2.2 Figure 1.4 displays the sequence $T_k^{(\mu)}$ in two cases. When $\alpha = 7/17$, the depth equals 3 (see Figure 1.3) and the only possible values for the index k are $k = 1, 2$. When α is the irrational quadratic number $\alpha = [1, 3, 1, 3, \dots]$, the definition of the two-distance truncation $T_k^{(\mu)}$ depends on the parity of the index k .

Figure 1.4 The truncation sequence $T_k^{(\mu)}$ in two cases: on the left, for the rational $\alpha = 7/17 = [2, 2, 3]$ – on the right, for the irrational quadratic $\alpha = [1, 3, 1, 3, \dots]$

1.2.3 The two distances and the covered space.

The two *distances* $\Gamma_{\langle T \rangle}$ and the covered space $V_{\langle T \rangle}$ are simple parameters that give a first description of the “randomness” of the sequence $\mathcal{K}_{\langle T \rangle}(\alpha)$.

Theorem B. *Let α be a real of the unit interval, with its three sequences (q_k) , (θ_k) and (m_k) , and consider a two-distance truncation T described in (1.2). Then, the*

² Here $\lfloor a + 1/2 \rfloor = a + 1$ for naturals a .

smallest distance $\hat{\Gamma}_{\langle T \rangle}$ and the largest distance $\tilde{\Gamma}_{\langle T \rangle}$ are respectively equal to

$$\hat{\Gamma}_{\langle T \rangle} = \theta_{k+1}, \quad \tilde{\Gamma}_{\langle T \rangle} = \theta_k - (m-1)\theta_{k+1}.$$

There are $T - q_k$ small distances and q_k large distances and the space covered with all the “large” intervals, called the covered space, and denoted by $V_{\langle T \rangle}$ equals

$$V_{\langle T \rangle} = q_k(\theta_k - (m-1)\theta_{k+1}).$$

Example 1.2.3 For $\alpha = 7/17$ and $T = 5$ of $\alpha = 7/17$, the two distances (the small one and the large one) and the covered space satisfy

$$\hat{\Gamma}_{\langle T \rangle}(\alpha) = \frac{3}{17}, \quad \tilde{\Gamma}_{\langle T \rangle}(\alpha) = \frac{4}{17}, \quad V_{\langle T \rangle}(\alpha) = \frac{12}{17}.$$

When α is fixed, the asymptotic behaviour of the sequences $\log q_k(\alpha)$ or $\log \theta_k(\alpha)$ (for $k \rightarrow \infty$) is deeply studied, via ergodic theorems. This leads to convergence results which hold almost everywhere,

$$\frac{1}{k} \log q_k(\alpha) \rightarrow_{a.e} \frac{\mathcal{E}}{2}, \quad \frac{1}{k} \log \theta_k(\alpha) \rightarrow_{a.e} -\frac{\mathcal{E}}{2} \quad (1.4)$$

and involve the entropy $\mathcal{E}$ of the Euclidean dynamical system (see Section 1.5.2), equal to $\pi^2/(6 \log 2)$. These sequences are also studied for rationals α , when the index k is a linear function of the depth $P(\alpha)$ [see (Daireaux and Vallée, 2004; Lhote and Vallée, 2008)]. In suitable probabilistic models, there are also asymptotic normal laws for the parameters $\log q_k(\alpha)$, $\log \theta_k(\alpha)$, easily obtained via dynamical analysis.

Here, the expressions of distances $\Gamma_{\langle T \rangle}$ involve θ_k and θ_{k+1} , and we may expect, with (1.4), an exponential behaviour for the mean value $\mathbb{E}[\theta_k]$. This is actually true, and proven as a corollary of Theorem 1.4.4. The “surprise” comes from the value of the exponential rate, which is not $\exp[-\mathcal{E}/2]$. This rate is expressed with the dominant eigenvalue $\lambda(s)$ of the transfer operator $\mathbf{H}_s$, defined in (1.7), and is equal to $\lambda(3/2)$. The dominant eigenvalue $\lambda(s)$ plays an important rôle in many studies on Euclidean algorithms (see for instance (Vallée, 2006)), and the value $\lambda(2) \sim 0.1994$ is omnipresent in this context, together with the value $-\lambda'(1)$, equal to the entropy $\mathcal{E}$, that has just been mentioned. This is the first time that the constant $\lambda(3/2) \sim 0.3964$ appears in a natural way in the Euclidean context.

To the best of our knowledge, the covered space was not previously studied. We show that, on the average, the space covered by the large distances is always close to the space covered by the small distances. Their means are sometimes equal, for particular truncations T . The difference between these two mean covered spaces is always less than $1/(4 \log 2) \sim 0.311$ [see Theorems 1.4.1 and 1.4.2].

The distances $\Gamma_{\langle T \rangle}$ and the covered space $V_{\langle T \rangle}$ give a first description of the “randomness” of the sequence $\mathcal{H}_{\langle T \rangle}(\alpha)$. There exist more refined measures of randomness, namely the discrepancy and the Arnold measure, that are described in the next sections. The discrepancy compares the ordered sequence (y_j) to the fixed regular sequence j/T , whereas the Arnold constant directly deals with the distances δ_i .

1.2.4 The discrepancy.

The discrepancy is a measure of how closely the truncated sequence $\mathcal{X}_{\langle T \rangle}$ approximates the uniform distribution on $\mathcal{I}$. For a general study of discrepancy, see the two books (Kuipers and Niederreiter, 1974) and (Drmota and Tichy, 1997). For a finite set $\mathcal{Y}$, we denote by $|\mathcal{Y}|$ its cardinality.

A sequence $\mathcal{X}$ of the unit interval $\mathcal{I}$ is said to be uniformly distributed if, for any interval $\mathcal{J} \subset \mathcal{I}$,

$$\lim_{T \rightarrow \infty} \frac{1}{T} |\mathcal{X}_{\langle T \rangle} \cap \mathcal{J}| = \lambda(\mathcal{J}),$$

the length of the interval $\mathcal{J}$. The *discrepancies* $D_{\langle T \rangle}(\mathcal{X}), \Delta_{\langle T \rangle}(\mathcal{X})$, given by

$$D_{\langle T \rangle}(\mathcal{X}) := \sup_{\mathcal{J} \subset \mathcal{I}} \left| \frac{1}{T} |\mathcal{X}_{\langle T \rangle} \cap \mathcal{J}| - \lambda(\mathcal{J}) \right|, \quad \Delta_{\langle T \rangle}(\mathcal{X}) := T D_{\langle T \rangle}(\mathcal{X}),$$

(where the supremum is taken over all the intervals $\mathcal{J} \subset \mathcal{I}$) estimate the speed of convergence towards the uniform distribution. As it is explained in (van Ravenstein, 1985), the discrepancy can be expressed with the “signed” distances between the ordered sequence (y_j) and the reference sequence (j/T) , namely

$$D_{\langle T \rangle}^+(\mathcal{X}) = \sup_{j \in \llbracket 1, T \rrbracket} \left(\frac{j}{T} - y_j \right), \quad D_{\langle T \rangle}^-(\mathcal{X}) = \sup_{j \in \llbracket 1, T \rrbracket} \left(y_j - \frac{j-1}{T} \right)$$

so that the relation $D_{\langle T \rangle}(\mathcal{X}) = D_{\langle T \rangle}^+(\mathcal{X}) + D_{\langle T \rangle}^-(\mathcal{X})$ holds. In conclusion, the notion of discrepancy is mainly based on the comparison between the ordered sequence y_j with the reference sequence j/T .

There exists a closed formula due to (van Ravenstein, 1985) which expresses the discrepancy of the Kronecker sequence as a function of sequences m_k, q_k, θ_k :

Theorem C. [Discrepancy for a Kronecker sequence] *Let α be a real of the unit interval, with its three sequences (q_k) , (θ_k) and (m_k) , and consider a two-distance truncation T described in (1.2). Then the discrepancy $\Delta_{\langle T \rangle}(\alpha)$ of the truncated sequence $\mathcal{X}_{\langle T \rangle}(\alpha)$ satisfies*

$$\Delta_{\langle T \rangle}(\alpha) := T \cdot D_{\langle T \rangle}(\alpha) = 1 + (mq_k + q_{k-1} - 1)(\theta_k - m\theta_{k+1}).$$

Example 1.2.4 In the left example of Figure 1.3, relative to $\alpha = 7/17$ and $T = 5$, one has $\Delta_{\langle T \rangle}(\alpha) = 21/17$.

The discrepancy of Kronecker sequences is very-well studied. Works of Weyl, Hardy and other authors proved that $(1/T)\Delta_{\langle T \rangle}(\alpha)$ tends to zero and asked the question of the speed of convergence to 0. The following is known:

(a) There exists a lower bound C_1 that holds for any $\Delta_{\langle T \rangle}(\alpha)$, for any pair (T, α) .

(b) There exists an absolute constant $C_2 = (66 \log 4)^{-1}$ such that, for every irrational α , there exists an infinite sequence of truncations T for which $\Delta_{\langle T \rangle}(\alpha) \geq C_2 \log T$ [see (Schmidt, 1972)].

(c) The discrepancy $\Delta_{\langle T \rangle}(\alpha)$ is of order $O(\log T)$ if and only if the sequence of digits which appear in the continued fraction expansion of the real α admits bounded averages [see (Behnke, 1922, 1924)].

We adopt here a probabilistic point of view, and we consider (only) two-distance truncations with a given position (defined in Section 1.2.2). Then, the discrepancy $\Delta_{\langle T \rangle}(\alpha)$ depends on α , in a direct way via α , and in an indirect way, via $T(\alpha)$. Our results often provide a quantitative probabilistic version of the previous assertions.

(a') When T is at a boundary position ($\mu = 0, 1$), and for a random real α , we prove that the mean discrepancy tends to a finite limit for $T \rightarrow \infty$ [see Theorem 1.4.1].

(b') The mean value of the discrepancy becomes infinite as soon as T is a truncation with a generic position [see Theorem 1.4.3]. We recover a “logarithmic” behaviour for the discrepancy, when α is a random rational, with an estimate on the average value of the constant C_2 . We also exhibit the position of truncation where the mean discrepancy is maximal.

(c') In the constrained model, we consider reals α which have all their digits bounded by M , and we limit ourselves to two-distance truncations. Then, we do not exhibit a logarithmic behaviour of order $\Theta(\log T)$ for the mean discrepancy that remains of constant order, with a constant which is now logarithmic with respect to M . We will return to the differences between our results described in (c') and Assertion (c) in the conclusion [Section 1.9].

1.2.5 Another measure of randomness: the Arnold measure.

There exists another measure of randomness, recently proposed in (Arnold, 2003, 2004) and much less studied. V. I. Arnold considered the normalized mean-value of the square of the distances δ_i 's between successive elements y_j of the truncated sequence $\mathcal{X}_{\langle T \rangle}$,

$$A_{\langle T \rangle}(\mathcal{X}) = \frac{1}{T} \sum_{i=1}^T \left(\frac{\delta_i}{\frac{1}{T}} \right)^2 = T \sum_{i=1}^T \delta_i^2,$$

and he proposed this constant as a measure of the randomness of the sequence $\mathcal{X}$. In the case where only K distinct distances δ_i appear, with a respective number of occurrences equal to T_i , so that

$$T = T_1 + T_2 + \dots + T_K, \quad 1 = T_1 \delta_1 + T_2 \delta_2 + \dots + T_K \delta_K,$$

$$A_{\langle T \rangle} = (T_1 + T_2 + \dots + T_K) \cdot (T_1 \delta_1^2 + T_2 \delta_2^2 + \dots + T_K \delta_K^2) = 1 + \sum_{1 \leq i < j \leq K} T_i T_j (\delta_i - \delta_j)^2.$$

The minimum possible value of $A_{\langle T \rangle}$ equals 1: it is reached when the sequence

gives rise to a regular T -gon. More generally, the value of A is close to 1 when the geometric distances between consecutive elements are close to each other. The maximum value of $A_{\langle T \rangle}$ equals T : it is obtained in the degenerate case when the sequence $\mathcal{X}_{\langle T \rangle}$ has only one value, since, in this case $A_{\langle T \rangle} = T \cdot 1 = T$. More generally, the value of $A_{\langle T \rangle}$ is close to T when all the geometric distances between consecutive elements are small except one which is then close to 1.

On the other hand, a random choice of T independent uniformly distributed points on the unit torus leads to what Arnold calls the “freedom-liking” value $A_{\langle T \rangle}^*$. Defining two integrals whose domain is the portion $\mathcal{P}$ of the hyperplane of $\mathbb{R}^T$ defined by $x_1 \geq 0, x_2 \geq 0, \dots, x_{\langle T \rangle} \geq 0, x_1 + \dots + x_{\langle T \rangle} = 1$,

$$I_1 := \int_{\mathcal{P}} (x_1^2 + \dots + x_{\langle T \rangle}^2) dx_1 \dots dx_{\langle T \rangle}, \quad I_2 := \int_{\mathcal{P}} dx_1 \dots dx_{\langle T \rangle}$$

$$\text{one obtains } A_{\langle T \rangle}^* = T \cdot \frac{I_1}{I_2} = \frac{2T}{T+1}, \quad A_{\langle T \rangle}^* \rightarrow 2 \quad \text{for } T \rightarrow \infty.$$

From these observations, it can be inferred that, for a given sequence of the unit interval, the value of A is a measure of randomness: if A is “much smaller” than A^* , this means “mutual repulsion”, while if A is “much larger” than A^* , this means “mutual attraction”. On the opposite side, from these two extremal types of non-randomness, the fact that A is “close” to A^* can be considered as a sign of randomness.

There exists a closed formula, given in (Cesaratto et al., 2006) which expresses the Arnold measure of the Kronecker sequence as a function of sequences m_k, q_k, θ_k :

Theorem D. [Arnold measure for a Kronecker sequence] *Let α be a real of the unit interval, with its three sequences (q_k) , (θ_k) and (m_k) , and consider a two-distance truncation T described in (1.2).*

The Arnold measure $A_{\langle T \rangle}(\alpha)$ of the truncated Kronecker sequence $\mathcal{X}_{\langle T \rangle}(\alpha)$ equals

$$A_T(\alpha) = (mq_k + q_{k-1}) [((m-1)q_k + q_{k-1})\theta_{k+1}^2 + q_k(\theta_k - (m-1)\theta_{k+1})^2] .$$

Example 1.2.5 In the left example of Figure 1.3, relative to $\alpha = 7/17$ and $T = 5$, one has $A_{\langle T \rangle}(\alpha) = 295/289$.

The study of the Arnold measure is just beginning now, with the proposal of Arnold himself in Problem 2003-2 of (Arnold, 2004, 2003), and the work of Cesaratto, Plagne and Vallée (Cesaratto et al., 2006). As we already said, we consider truncation $T = T(\alpha)$ with a given position, and the Arnold measure $A_{\langle T \rangle}(\alpha)$ depends on α , in a direct way via α , and in an indirect way, via $T(\alpha)$. In the paper (Cesaratto et al., 2006), the authors study the mean Arnold measure for one of the boundary truncations. They show that this mean value tends to a finite limit in the random model.

We first make this result more precise, as we prove that this limit is the same in the

other probabilistic models (rational case, or constrained case) [see Theorem 1.4.1]. Then, we study a more general situation, and we prove the following:

For boundary truncations, the small mean values displayed in Theorem 1.4.1 may be considered as a “sign of mutual attraction”. Generic truncations give rise to large mean values, which is interpreted by Arnold as a “sign of mutual repulsion”. Moreover, for such truncations, we exhibit the same features as for the discrepancy: When α is a random real, the mean value of the Arnold measure becomes infinite [see Theorem 1.4.3]. We recover a “logarithmic” behaviour for the Arnold measure, when α is a random rational, or when α has all his digits bounded by M (the so-called constrained model). We also exhibit the position of truncation where the mean Arnold measure is maximal. This is not the same position as for the discrepancy.

1.2.6 Main principles for our study.

The expressions of all the parameters of interest –distances, covered space, discrepancy and Arnold measure– that have been previously described are gathered in the table of Figure 1.5 which also focusses on boundary truncations.

$\widehat{\Gamma}_{\langle T \rangle}$	θ_{k+1}	
$\tilde{\Gamma}_{\langle T \rangle}$	$\theta_k - (m-1)\theta_{k+1}$	
$V_{\langle T \rangle}$	$q_k(\theta_k - (m-1)\theta_{k+1})$	
$\Delta_{\langle T \rangle}$	$1 + (mq_k + q_{k-1})(\theta_k - m\theta_{k+1})$	
$A_{\langle T \rangle}$	$(mq_k + q_{k-1}) [((m-1)q_k + q_{k-1})\theta_{k+1}^2 + q_k(\theta_k - (m-1)\theta_{k+1})^2]$	
$m = 1 \text{ [} \mu = 0 \text{]}$		$m = m_{k+1} \text{ [} \mu = 1 \text{]}$
$\widehat{\Gamma}_{\langle T \rangle}$	θ_{k+1}	θ_k
$\tilde{\Gamma}_{\langle T \rangle}$	θ_k	$\theta_k + \theta_{k+1}$
$V_{\langle T \rangle}$	$q_k \theta_k$	$q_{k-1}(\theta_k + \theta_{k+1})$
$\Delta_{\langle T \rangle}$	$1 + (q_k + q_{k-1})(\theta_k - \theta_{k+1})$	$1 + q_k \theta_{k+1}$
$A_{\langle T \rangle}$	$(q_k + q_{k-1}) [q_{k-1} \theta_{k+1}^2 + q_k \theta_k^2]$	$q_k [(q_k - q_{k-1})\theta_k^2 + q_{k-1}(\theta_k + \theta_{k+1})^2]$

Figure 1.5 The expression of the five parameters for a general truncation (above) and for a boundary truncation (below).

We recall here the main objectives of our work. Most of the existing works deal with a fixed real number α , study the pseudo-randomness of the truncated sequence $\mathcal{H}_{\langle T \rangle}(\alpha)$ and mainly deal with the discrepancy measure. They ask the question: For a given α , for which T , the discrepancy is maximal? minimal?

We limit ourselves to the two-distance framework, and ask (and answer) the same kind of questions. However, we deal with

- (i) various parameters X for pseudo-randomness (not only the discrepancy),
- (ii) various subsets $\mathcal{A} \subset [0, 1]$ which gather particular real numbers α ,

(iii) particular subsets $\mathcal{T}$ of truncations T , defined by a given position μ .

We study the asymptotics of the mean value $\mathbb{E}_{\mathcal{A}}[X_{\langle T \rangle}]$ for truncations T of the family $\mathcal{T}$, when $T \rightarrow \infty$. We are particularly interested in the triples $(\mathcal{A}, X, \mathcal{T})$ which give rise to a mean of logarithmic order.

1.2.7 Position of the truncation and decomposition of costs.

All the five parameters are written as a sum of *elementary costs* that all involve the three sequences m_k, q_k, θ_k under the form

$$R_k := m^e q_{k-1}^a q_k^b \theta_k^c \theta_{k+1}^d \quad \text{with } m \in \llbracket 1, m_{k+1} \rrbracket.$$

Here, the integers a, b, c, d, e belong to the interval $\llbracket 0, 3 \rrbracket$. Truncations $T = T(\alpha)$ with a given position μ define, when the index k varies, the *truncation sequence* $T_k^{(\mu)}$ at position μ [see Eq. (1.3)], and, for each parameter X of interest, this defines the sequence at position μ ,

$$X_k^{(\mu)} := X_{\langle T \rangle} \quad \text{when } T = T_k^{(\mu)}.$$

Such a sequence $X_k^{(\mu)}$ involves elementary costs

$$R_k^{(\mu)} := \lfloor \mu(m_{k+1} - 1) + 1 \rfloor^e q_{k-1}^a q_k^b \theta_k^c \theta_{k+1}^d. \quad (1.5)$$

Section 1.6.3 explains why it is sometimes convenient to decompose such a cost [see Eq. (1.28)]. This leads to more general elementary costs

$$R_k^{(\tau)} := \tau(m_{k+1}) q_{k-1}^a q_k^b \theta_k^c \theta_{k+1}^d, \quad (1.6)$$

whose expression involves a function τ (called a *digit-cost*) which always be of polynomial growth.

1.2.8 Balanced and unbalanced costs.

We wish to study the mean values of such costs defined in Eq. (1.5); the study is not a priori easy, because the random variables $m_{k+1}, q_j, \theta_\ell$ are correlated. We are in particular interested in the expectation $\mathbb{E}[R_k]$ of the cost $R_k := R_k^{(\tau)}$ defined in (1.6) when α is a real uniformly distributed with respect to the Lebesgue measure; however, we may begin with the study of the expectation $\mathbb{E}[\log R_k]$. This is much easier because $\log R_k$ is now a sum of costs: with the results recalled in (1.4), the estimate $\mathbb{E}[\log m_{k+1}] = O(1)$, and the polynomial growth of the digit-cost τ , we obtain

$$\begin{aligned} \mathbb{E}[\log R_k^{(\tau)}] &= O(\mathbb{E}[\log m_{k+1}]) + |(a+b) - (c+d)| k \frac{\mathcal{O}}{2} + O(1) \\ &= |(a+b) - (c+d)| k \frac{\mathcal{O}}{2} + O(1). \end{aligned}$$

This proves that the expectation $\mathbb{E}[\log R_k]$ may have two distinct behaviours, according as the difference $(a+b) - (c+d)$ between exponents of q_k 's and exponents

of θ_k 's be zero or not. If the difference is zero, then $\mathbb{E}[|\log R_k|]$ is of constant order, whereas it is linear with respect to k if the difference is not zero. This is why it is natural to consider two types of costs, and the following definition.

Definition 1.2.6 A cost of the form $R_k := \tau(m_{k+1}) q_{k-1}^a q_k^b \theta_k^c \theta_{k+1}^d$ is said to be *balanced* if $a + b = c + d$ and *unbalanced* otherwise. The *imbalance* is the difference $v := (c + d) - (a + b)$. For a balanced cost, the *balance* f is the common value $f = a + b = c + d$. A balanced cost with balance f only depends on the triple (a, d, τ) ,

$$R_k = \tau(m_{k+1}) (q_k \theta_k)^f \left(\frac{q_{k-1}}{q_k} \right)^a \left(\frac{\theta_{k+1}}{\theta_k} \right)^d.$$

Balanced costs. The three parameters (covered space, discrepancy, Arnold measure) involve balanced costs, with various balances f and they are themselves called balanced parameters: One has $f = 1$ for the covered space or the discrepancy, and $f = 2$ for the Arnold measure. We also deal with the constant cost $R = 1$ of balance $f = 0$. These balanced parameters are studied in our first three theorems [Theorems 1.4.1, 1.4.2, 1.4.3].

Unbalanced costs. We may expect an exponential behaviour for general costs of the form $q_k^b \theta_k^c$ when the imbalance $v := b - c$ is not zero. Such costs are studied in Theorem 1.4.4. The distances $\Gamma_{\langle T \rangle}$ are particular³ cases of such unbalanced parameters, with $c = 1, b = 0$. They are studied as a corollary of Theorem 1.4.4.

Resemblances and differences. This explains the difference between the behaviours of unbalanced costs and balanced costs. Furthermore, the factor digit m_{k+1} is important for balanced costs, whereas it does not actually intervene for unbalanced costs, where the exponential behaviour annihilates the polynomial factor m^e . However, the principles of the analysis, based on dynamical analysis, are similar for the two types of costs (balanced or unbalanced), and described in Section 1.5. Then, Section 1.6, performs the analysis for balanced costs whereas Section 1.7 analyses unbalanced costs.

1.3 Probabilistic models.

Section 1.2 has introduced five parameters of interest for describing the truncated Kronecker sequence $\mathcal{K}_{\langle T \rangle}(\alpha)$, namely the two distances $\Gamma_{\langle T \rangle}$, the covered space $V_{\langle T \rangle}$, the discrepancy $\Delta_{\langle T \rangle}$ and the Arnold measure $A_{\langle T \rangle}$.

We denote by $X_{\langle T \rangle}$ any parameter of this family; when it is associated with the sequence of truncations $T = T_k^{(\mu)}$, at position μ , it gives rise to the sequence $X_k^{(\mu)}$.

³ There is a possible occurrence of a factor m^e with $e \in \{0, 1\}$, which does not modify the exponential behaviour

When μ is fixed, the sequence $X_k^{(\mu)}$ clearly depends on the real α , since the Kronecker sequence is defined through this real α . But, the sequence of truncations is itself defined via the continued fraction expansion of α and it also depends on α .

There are two main situations, as we now explain: For real models, the index k is not *a priori* related with α and it may be equal to any integer, whereas, in rational models, the index k will be a function of the depth of the rational α .

1.3.1 Real probabilistic models.

We choose a position μ for the truncation T . This defines, for each real α and for each index k , a *unique* value $X_k^{(\mu)}$. We study the asymptotics of the mean value of $X_k^{(\mu)}$, when the index $k \rightarrow \infty$.

Unconstrained real model. The real α belongs to the unit interval $\mathcal{I}$, which is endowed with the Lebesgue measure ν and, for each position μ , we are interested in the asymptotics of the mean value $\mathbb{E}[X_k^{(\mu)}]$ when the index k of truncation tends to ∞ .

Constrained real model. The constrained model is parametrized by the upper bound M on the digits of the continued fraction expansion. More precisely, the set $\mathcal{I}^{[M]}$ gathers all the reals α of the interval $\mathcal{I}$, whose digits in the continued fraction expansion are at most M . The set $\mathcal{I}^{[M]}$ is a Cantor set, with zero Lebesgue measure, and we endow it with the Hausdorff measure ν_M corresponding to its Hausdorff dimension σ_M that we now introduce.

Roughly speaking, given a subset $E \subset \mathcal{I}$ and a real number σ , its σ -dimensional Hausdorff measure is defined in terms of δ -covers, i.e. denumerable covers of E by intervals of length at most δ . We consider all the δ -covers of E and minimize the sum of the σ -th powers of the interval lengths. The limit when δ tends to zero of this minimum is the σ -dimensional measure of the set E . The Hausdorff dimension of E is the value σ for which the σ -dimensional Hausdorff measure changes from infinite to zero.

The Hausdorff dimension σ_M of $\mathcal{I}^{[M]}$ and the measure ν_M are well studied. In our framework, they defined with the transfer operator $\mathbf{H}_{M,s}$, defined in (1.8), and recalled in Section 1.8.3. The dimension σ_M is the solution of the equation $\lambda_M(s) = 1$ which involves the dominant eigenvalue $\lambda_M(s)$ of the operator $\mathbf{H}_{M,s}$, and the measure ν_M is the dominant eigenmeasure of the adjoint $\mathbf{H}_{M,s}^*$ of the operator $\mathbf{H}_{M,s}$.

For each position μ of the truncation, we are interested in the asymptotics of the mean value $\mathbb{E}^{[M]}[X_k^{(\mu)}]$ when the index k tends to ∞ .

1.3.2 Rational probabilistic models for balanced costs.

Here, the set of interest is

$$\Omega = \mathbb{Q} \cap \mathcal{I} = \left\{ \frac{u}{v} \in \mathbb{Q}; \quad 1 \leq u \leq v, \quad \gcd(u, v) = 1 \right\}.$$

We denote by $P(\alpha)$ the depth of the rational α . Now, the index k of the truncation $T = T(\alpha)$ must satisfy the inequality $k < P(\alpha)$. It proves convenient to choose the index k not too close to the ends of the interval $\llbracket 1, P(\alpha) \rrbracket$, which leads to the following definition, already used in (Cesaratto et al., 2006).

Definition 1.3.1 A function $K : \Omega \rightarrow \mathbb{N}$ is an admissible function of the depth if there exist a function $F : \mathbb{N} \rightarrow \mathbb{N}$, an integer p_0 , and two real numbers $\beta^- > 0$ and $\beta^+ < 1$ such that

- (i) K factorizes as $K = F(P)$,
- (ii) For any integer $p \geq p_0$, one has $F(p) \in [\beta^- p, \beta^+ p]$

We fix an admissible function $K = F(P)$ of the depth. Then, when α is fixed, the index k of the truncation is fixed and equal to $F(P(\alpha))$. If the position μ is fixed, this defines, for each $\alpha \in \Omega$, a *unique* truncation $T_K^{(\mu)}$ and a *unique* value $X_K^{(\mu)}$.

Unconstrained rational model. The rational model is parametrized by an integer $N > 0$ which is an upper bound on the denominator v , and the set of interest is the subset Ω_N of Ω formed of rationals u/v whose denominator v is at most N ,

$$\Omega_N = \left\{ \frac{u}{v} \in \Omega; \quad v \leq N \right\} = \left\{ \frac{u}{v} \in \mathbb{Q}; \quad 1 \leq u \leq v \leq N, \quad \gcd(u, v) = 1 \right\}.$$

It is equipped with the uniform probability. We study the asymptotics of the mean value $\mathbb{E}_N[X_K^{(\mu)}]$ when α is a random rational of Ω_N and $N \rightarrow \infty$.

Constrained rational model. Here, the set of interest is parametrized by a pair (M, N) of integers, where N is the upper bound for the denominator v and M is the upper bound for the digits m_k which appear in the continued fraction. Then, for any pair (N, M) of integers with $N \geq 1, M \geq 2$, the set of interest is

$$\Omega_N^{[M]} = \left\{ \frac{u}{v} \in \Omega; \quad v \leq N; \quad m_k(u/v) \leq M, \quad \forall k \in \llbracket 1, P(u/v) \rrbracket \right\}.$$

It is equipped with the uniform probability. As previously, we study the asymptotics of the mean value $\mathbb{E}_N^{[M]}[X_K^{(\mu)}]$ when M is fixed, α is a random rational of $\Omega_N^{[M]}$ and $N \rightarrow \infty$.

1.3.3 Rational probabilistic models for unbalanced costs.

Here, we expect results which involve the index k of the truncation. In the rational model, it then proves convenient to adopt a more restrictive definition for the index k , and only consider particular admissible functions of the depth. For each real⁴ $\delta \in]0, 1[$, we consider the *unique* particular admissible function, which is called the δ -fraction of the depth and is defined as

$$K_\delta : \Omega \rightarrow \mathbb{N}, \quad K_\delta(x) = \lfloor \delta P(x) \rfloor.$$

This notion has been already introduced in previous works on the subject (Daireaux and Vallée, 2004; Lhote and Vallée, 2008).

⁴ We will see that we further restrict ourselves to a rational δ .

1.3.4 Transition from the constrained to the unconstrained models.

When the constraint M becomes large, it is natural to expect that the results in the constrained case have “a limit” which coincides with the result in the unconstrained case. It is also natural to study the constrained problem in the rational model where the digit-bound $M = M(N)$ is a function of the denominator-bound N of rationals. Since the equality $M \leq N$ holds, we recover the unconstrained case when $M(N) = N$. The set $\Omega_N^{[M(N)]}$ has been precisely studied in the paper (Cesaratto and Vallée, 2011) for a general function $M = M(N)$.

1.4 Statements of the main results.

All along our analysis, the *transfer operator* $\mathbf{H}_s$ of the Euclidean dynamical system plays a fundamental rôle. This operator depends of a complex parameter s and can be viewed as an extension of the Riemann zeta function $s \mapsto \zeta(2s)$. It acts on the functional space $\mathcal{C}^1(\mathcal{J})$, and transforms a function $g \in \mathcal{C}^1(\mathcal{J})$ into a function

$$\mathbf{H}_s[g](x) := \sum_{m \geq 1} \frac{1}{(m+x)^{2s}} g\left(\frac{1}{m+x}\right). \quad (1.7)$$

This operator is precisely defined in Section 1.5.2. Some of its extensions are introduced in Section 1.5.5, and Section 1.8 summarizes the main properties that are useful here.

This operator $\mathbf{H}_s$ is used here as a *generating* operator for the characteristics of the continued fraction expansion, namely the digits m_k , the continuants q_k, θ_k . In the constrained framework, the *constrained transfer operator* $\mathbf{H}_{M,s}$,

$$\mathbf{H}_{M,s}[g](x) := \sum_{m \leq M} \frac{1}{(m+x)^{2s}} \cdot g\left(\frac{1}{m+x}\right) \quad (1.8)$$

replaces the operator $\mathbf{H}_s$.

These transfer operators (constrained or unconstrained) play a crucial rôle in all the proofs. However, they do not explicitly appear in the statements of the first three theorems, with three exceptions: the factor $1/\log 2$ (equal to the value of the Gauss density at zero), the subdominant factor ρ (related to the subdominant spectral radius of the operator $\mathbf{H}_1$) and finally the exponent γ , (equal to the width of a vertical strip around $\Re s = 1$ where the operator $(I - \mathbf{H}_s)^{-1}$ has a polynomial growth for large $|\Im s|$). The situation differs in Theorem 1.4.4, where the dominant eigenvalue $\lambda_M(s)$ of the operator $\mathbf{H}_{M,s}$ plays an explicit rôle, for (possibly infinite) integers M .

1.4.1 Balanced parameters for a boundary truncation.

Our first result deals with balanced parameters and truncations at a boundary position, namely $\mu = 0$ [case $-$] or $\mu = 1$ [case $+$]. This is the simplest case: here, the

mean values of the three parameters (covered space, discrepancy and Arnold measure) share the same behaviour in the four probabilistic models (real versus rational, unconstrained versus constrained): they admit a finite limit. The limit is the same in the real and the rational frameworks, and only depends on the constraint M . Moreover, there is a clear transition to the unconstrained model. Finally, there are explicit values for the limits in the unconstrained model.

Theorem 1.4.1 [Covered space, discrepancy and Arnold measure for boundary truncations T]. *Consider a two-distance truncation T at a boundary position $\mu = 0$ (case $-$) or $\mu = 1$ (case $+$). In the rational model, consider an admissible function K of the depth. Then, the following holds:*

(a) *There exist $\rho < 1$, $\gamma > 0$, and, for each constraint M (possibly infinite), and each parameter $X \in \{V, \Delta, A\}$, there exist two constants $x_M^\pm$, for which the associated sequences $X_k^\pm$ satisfy*

$$\begin{aligned} \text{[Real model]} \quad & \text{for } k \rightarrow \infty, \quad \mathbb{E}^{[M]}[X_k^\pm] = x_M^\pm + O(\rho^k), \\ \text{[Rational model]} \quad & \text{for } N \rightarrow \infty, \quad \mathbb{E}_N^{[M]}[X_K^\pm] = x_M^\pm + O(N^{-\gamma}). \end{aligned}$$

The hidden constants are uniform with respect to M .

(b) *For $M = \infty$, the numerical values of the limits $x_\infty^\pm$ are displayed in Figure 1.6.*

(c) *When $M \rightarrow \infty$, there is a transition to the unconstrained model, and*

$$x_M^\pm = x_\infty^\pm + O\left(\frac{1}{M}\right).$$

	Covered space V	Discrepancy Δ	Arnold measure A
x_∞^-	$\frac{1}{2} + \frac{1}{4\log 2} \sim 0.861$	$1 + \frac{1}{2\log 2} \sim 1.721$	$\frac{2}{3} + \frac{1}{3\log 2} \sim 1.147$
x_∞^+	$\frac{1}{2}$	$1 + \frac{1}{4\log 2} \sim 1.360$,	$\frac{2}{3} + \frac{1}{4\log 2} \sim 1.027$.

Figure 1.6 Values of the limits $s_\infty^\pm$, $d_\infty^\pm$, $a_\infty^\pm$ of the mean values of V , Δ , A in the unconstrained models.

1.4.2 Covered space for a generic truncation.

Our second result deals with the covered space in the case when the truncation is “generic”, and defined by its position $\mu \in]0, 1[$. The situation is similar as what happens when the truncation is at a boundary position (Theorem 1.4.1). For any constraint M (possibly infinite), there is a finite limit for the mean values in the rational model, equal to the limit of the real model. And the limits $s_M(\mu)$ of the constrained models tend to the limit $s_\infty(\mu)$ of the unconstrained model.

Theorem 1.4.2 [Covered space for a generic truncation T]. *Consider, for $\mu \in]0, 1[$, a two-distance truncation at position μ . In the rational model, consider an admissible function K of the depth. Then, the following holds for the sequence $V_k^{(\mu)}$:*

(a) *There exist $\rho < 1$, $\gamma > 0$, and, for each constraint M (possibly infinite), and each $\mu \in]0, 1[$, there exists a constant $s_M(\mu)$, for which one has*

$$\begin{aligned} \text{[Real model]} \quad & \text{for } k \rightarrow \infty, \quad \mathbb{E}^{[M]}[V_k^{(\mu)}] = s_M(\mu) + O(\rho^k), \\ \text{[Rational model]} \quad & \text{for } N \rightarrow \infty, \quad \mathbb{E}_N^{[M]}[V_K^\pm] = s_M(\mu) + O(N^{-\gamma}). \end{aligned}$$

The hidden constants are uniform with respect to M and μ . (b) When $\mu \rightarrow 0^+$ or $\mu \rightarrow 1^-$, there is a transition to the boundary cases

$$s_M(\mu) \xrightarrow{\mu \rightarrow 0^+} s_M^-, \quad s_M(\mu) \xrightarrow{\mu \rightarrow 1^-} s_M^+.$$

(c) *When $M \rightarrow \infty$, there is a transition to the unconstrained models,*

$$s_M(\mu) = s_\infty(\mu) + O\left(\frac{1}{M}\right).$$

The hidden constant is uniform with respect to μ .

1.4.3 Discrepancy and Arnold measure for a generic truncation.

The situation deeply differs from the two previous results (Theorems 1.4.1 and 1.4.2). Previously, the parameter has a similar behaviour in the four probabilistic models, and its mean values admit finite limits. The behaviour of the discrepancy and Arnold measure is now completely different from what happens when the truncation is at a boundary position (Theorem 1.4.1). Their mean values are infinite in the unconstrained real model. Their mean values in the other three models are all of logarithmic order, with respect to N in the rational unconstrained model, or with respect to M in the two constrained models.

The constant which appears in the dominant logarithmic term is of the same type for discrepancy and Arnold measure: it depends on the position μ as a simple polynomial which equals 0 at $\mu = 0$ and $\mu = 1$. It is of degree two and equals $\mu(1 - \mu)$ for the discrepancy; it is of degree three and equals $\mu(1 - \mu)^2$ for the Arnold measure. This entails an interesting result: in the three probabilistic models where the mean values are finite, the asymptotic mean value of the discrepancy is maximal at a position $\mu = 1/2$ whereas the asymptotic mean value of the Arnold measure is maximal at a position $\mu = 1/3$.

Theorem 1.4.3 [Discrepancy and Arnold measure for a generic truncation T]. *Consider, for $\mu \in]0, 1[$, a two-distance truncation at position μ . In the rational model, consider an admissible function K of the depth.*

Then, there exist $\rho < 1$, $\gamma > 0$, $\theta > 0$ such that, for any $\mu \in]0, 1[$, for each parameter $X \in \{\Delta, A\}$, the following holds for the sequence $X_k^{(\mu)}$:

(a) [Unconstrained real model] *The mean values $\mathbb{E}[X_k^{(\mu)}]$ are infinite.*

(b) [Unconstrained rational model] *There exist constants $a(\mu), d(\mu)$ for which*

$$\begin{aligned}\mathbb{E}_N[\Delta_K^{(\mu)}] &= \mu(1-\mu)\log_2 N + d(\mu) + O(N^{-\gamma}), \\ \mathbb{E}_N[A_K^{(\mu)}] &= \mu(1-\mu)^2\log_2 N + a(\mu) + O(N^{-\gamma}).\end{aligned}$$

The hidden constants are uniform with respect to μ .

(c) [Constrained models]. *For any finite constraint M , there exist constants $x_M(\mu)$ for which*

$$\begin{aligned}\text{[Real model]} \quad & \text{for } k \rightarrow \infty, \quad \mathbb{E}^{[M]}[X_k^{(\mu)}] = x_M(\mu) + O(p^k), \\ \text{[Rational model]} \quad & \text{for } N \rightarrow \infty, \quad \mathbb{E}_N^{[M]}[X_K^{(\mu)}] = x_M(\mu) + O(N^{-\theta/\log M}).\end{aligned}$$

The hidden constants are uniform with respect to M .

(d) [Transition to the unconstrained model]. *There exist constants $\widehat{d}(\mu), \widehat{a}(\mu)$ for which one has, when $M \rightarrow \infty$,*

$$d_M(\mu) - \mu(1-\mu)\log_2 M \rightarrow \widehat{d}(\mu), \quad a_M(\mu) - \mu(1-\mu)\log_2 M \rightarrow \widehat{a}(\mu)$$

with a speed of convergence of order $O(\log^2 M/M)$.

In the constrained rational model, it is possible to choose the digit bound M as a function of the denominator bound N . There are two interesting choices:

Choice $M = N$. We recover the main behaviour of the parameters in the unconstrained rational model, with a remainder term of order $O(1)$. This is why the constants $d(\mu)$ and $\widehat{d}(\mu)$ [or $a(\mu)$ and $\widehat{a}(\mu)$] are not (a priori) equal for $\mu \in]0, 1[$.

Choice $M = \Theta(\log N)$. Indeed, in the paper (Cesaratto and Vallée, 2011), it is proven that there is a threshold phenomenon on the cardinality of the subset $\Omega_N^{[M]}$, depending on the relative order of the digit-bound M with respect to the length $n := \log N$ of the denominator. In particular, for $M = an$

$$\frac{|\Omega_N^{[an]}|}{|\Omega_N|} \rightarrow \exp\left[-\frac{12}{a\pi^2}\right] \quad \text{for } N \rightarrow \infty.$$

On this subset $\Omega_N^{[an]}$, the mean value of the discrepancy or the Arnold measure at a generic position are of order $\Theta(\log \log N)$.

1.4.4 Distances and products $q_k^b \theta_k^c$.

We now wish to study the two distances, the small distance $\widehat{\Gamma}_{\langle T \rangle}$ and the large distance $\widetilde{\Gamma}_{\langle T \rangle}$, in the four probabilistic models. The expression of distances mainly involves the random variables θ_k ; more precisely, for any⁵ truncation T of index k , the distances $\Gamma_{\langle T \rangle}$ belong to the interval $[\theta_{k+1}, \theta_k]$.

⁵ In the three-distance framework, this result holds for each of the three distances.

A more general study. It may be of independent interest to perform a probabilistic study of products of the general form $q_k^b \theta_k^c$, related to any pair (b, c) of real numbers.

We first remark that the case $b = c$ leads to a (ordinary) balanced cost, and this type of cost was already studied in the previous section. When the imbalance $v := c - b$ was close to zero, the study of the variable $q_k^b \theta_k^c$ was already performed in (Lhote and Vallée, 2008; Vallée, 1997). This was a first step to prove that the variables $\log q_k$ or $\log \theta_k$ asymptotically follow a gaussian law, both in the real and rational models.

We are here mainly interested in other situations, with two main motivations. First, the study of distances introduces the pair $(b = 0, c = 1)$. Second, the approximation of a number α by its k -th convergent p_k/q_k leads to the pair $(b = -1, c = 1)$. In the real case, the mean value is of exponential type, with a ratio which involves the value $\lambda(2) \sim 0.1994$ of the dominant eigenvalue of the transfer operator. This last value⁶ $\lambda(2)$ (discovered in 1994...) plays a central role in the analysis of the Gauss Algorithm (Daudé et al., 1997), and its occurrence in this approximation context was remarked for the first time in (Flajolet and Vallée, 1998).

Products $q_k^b \theta_k^c$. The following Theorem 1.4.4 proves that the probabilistic behaviour of this product mainly depends on the imbalance equal here to $v := c - b$. It will be directly applied to the study of distances, where v equals 1.

Here, the spectral objects of the transfer operator $\mathbf{H}_{M,s}$ explicitly appear in the statements of the result, via the dominant eigenvalue $\lambda_M(s)$, the Hausdorff dimension σ_M solution of the equation $\lambda_M(s) = 1$, or a “perturbation” $\sigma_M(v, \delta)$ of σ_M . In the real models, the mean value $\mathbb{E}^{[M]}[q_k^b \theta_k^c]$ has an exponential behaviour with respect to the index k (decreasing or increasing, according to the sign of v), and the exponential rate depends on the constraint M and the imbalance v : it is equal to $\lambda_M(\sigma_M + v/2)$.

In the rational models, we consider truncations whose index is defined by the δ -fraction of the depth, and already described in Section 1.3.3. In this case, the mean value $\mathbb{E}_N^{[M]}[q_k^b \theta_k^c]$ has a polynomial behaviour with respect to the denominator N (decreasing or increasing, according to the sign of v), and the exponent of N depends on the imbalance v , the constraint M and the fraction δ : it is equal to the difference $2(\sigma_M(v, \delta) - \sigma_M)$ where the function $\sigma_M(v, \delta)$ is defined in (1.9).

Theorem 1.4.4 [Parameters $q_k^b \theta_k^c$] *Consider an integer $M \geq 2$ possibly infinite, and denote by $\lambda_M(s)$ the dominant eigenvalue of the (possibly) constrained transfer operator $\mathbf{H}_{M,s}$ defined in (1.8). Denote by σ_M the Hausdorff dimension of the set $\mathcal{J}^{[M]}$, by t_M the real in $\mathbb{R}$ defined by $t_M = -\infty$ for $M < \infty$ and $t_M = -1$ for $M = \infty$. The following holds and strongly involves the imbalance $v := c - b$.*

[Real model]

(i) *If $v \leq t_M$, the mean value $\mathbb{E}^{[M]}[q_k^b \theta_k^c]$ is infinite for any integer k .*

⁶ Flajolet called it the Vallée constant...

(ii) If $\mathfrak{v} > t_M$, the mean value $\mathbb{E}^{[M]}[q_k^b \theta_k^c]$ is finite, and satisfies

$$\mathbb{E}^{[M]}[q_k^b \theta_k^c] = D_M(b, c) \lambda_M^k(\sigma_M + \mathfrak{v}/2) \left[1 + O\left(\rho(b, c)^k\right) \right] \quad [k \rightarrow \infty],$$

for some positive constants $D_M(b, c) > 0$, and $\rho(b, c) < 1$. According to the sign of $\mathfrak{v}$, the following holds for the mean value $\mathbb{E}^{[M]}[q_k^b \theta_k^c]$ when $k \rightarrow \infty$,

- (a) For $\mathfrak{v} > 0$, the mean value tends to 0,
- (b) For $t_M < \mathfrak{v} < 0$, the mean value tends to $+\infty$.
- (c) For $\mathfrak{v} = 0$, the mean value tends to the constant $D_M(b, b)$.

[Rational model] For any $\delta \in [0, 1]$, and any integer $M \in \llbracket 2, \infty \rrbracket$, there is a unique solution, denoted by $\sigma_M(\mathfrak{v}, \delta)$, of the equation

$$\lambda_M^{1-\delta}(\sigma) \cdot \lambda_M^\delta(\sigma + \mathfrak{v}/2) = 1, \quad (1.9)$$

with $\sigma_M(0, \delta) = \sigma_M$, $\sigma_M(\mathfrak{v}, \delta) < \sigma_M$ (for $\mathfrak{v} > 0$), $\sigma_M(\mathfrak{v}, \delta) > \sigma_M$ (for $\mathfrak{v} < 0$).

(i) For any triple (δ, b, c) , with a rational $\delta \in]0, 1[$, the mean value of the product $q_k^b \theta_k^c$ on $\Omega_N^{[M]}$, when the index $k = \lfloor \delta P \rfloor$ is the δ -fraction of the depth P , satisfies

$$\mathbb{E}_N^{[M]}[q_k^b \theta_k^c] = D_M(\delta, b, c) N^{2(\sigma_M(\mathfrak{v}, \delta) - \sigma_M)} (1 + \varepsilon(N)),$$

for some positive constant $D_M(\delta, b, c)$ and $\varepsilon_M(N) = o(1)$.

(ii) According to the sign of $\mathfrak{v}$, the following holds for the mean value $\mathbb{E}_N^{[M]}[q_k^b \theta_k^c]$, when $N \rightarrow \infty$,

- (a) For $\mathfrak{v} > 0$, the mean value tends to 0,
- (b) For $\mathfrak{v} < 0$, the mean value tends to $+\infty$,
- (c) For $\mathfrak{v} = 0$, and any $\delta \in]0, 1[$, the mean value tends to $D_M(b, b)$, the constant of the real case.

(iii) When M is large and $\mathfrak{v}$ is near 0, the error term $\varepsilon(N)$ is of order $O(N^{-\gamma})$ for some positive γ , the quantity γ and the hidden constants in the O -term being uniform with respect to M , b and c .

Application to distances. As we already said, the study of distances is obtained with the choice $\mathfrak{v} := c - b = 1$. Then, for any two-distance truncation T whose index k tends to ∞ , the mean value $\mathbb{E}^{[M]}[\Gamma_{\langle T \rangle}]$ is exponentially decreasing, with a rate

$$\lambda_M \left(\sigma_M + \frac{1}{2} \right).$$

When $M = \infty$, one has $\sigma_\infty = 1$, and the exponential rate is $\lambda(3/2) \sim 0.3964$.

In the rational models, for any two-distance truncation, whose index is the δ -fraction of the depth, with $\delta \in \mathbb{Q} \cap]0, 1[$, the mean value $\mathbb{E}_N^{[M]}[\Gamma_{\langle T \rangle}]$ is polynomially decreasing with respect to the denominator N , and the exponent of N equals $2(\sigma_M(\delta) - \sigma_M)$, where $\sigma_M(\delta)$ is solution of the equation

$$\lambda_M^{1-\delta}(\sigma) \cdot \lambda_M^\delta(\sigma + 1/2) = 1.$$

1.5 Dynamical Analysis.

We now describe in an informal way the dynamical analysis methodology which mixes analysis of algorithms and dynamical systems theory. The random variables of interest are generated by transfer operators (plain or extended) of the Euclidean dynamical system, and their asymptotic probabilistic behaviour is dictated by the dominant spectral objects of these transfer operators. The analysis in the real model only needs a precise knowledge of operators when their complex parameter s is close to the real axis, whereas the study in the rational model also needs their precise behaviour when the parameter s is far from the real axis.

1.5.1 Continued fraction expansion.

The *Gauss map* $S : \mathcal{I} \rightarrow \mathcal{I}$ is defined by

$$S(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor = \left\{ \frac{1}{x} \right\} \quad \text{for } x \neq 0, \quad S(0) = 0,$$

where $\lfloor \cdot \rfloor$ denotes the integer part, and $\{ \cdot \}$ denotes the fractional part. The pair $(\mathcal{I}, S)$ defines a dynamical system, which is called in the sequel the *Euclidean dynamical system*. The restriction of S to the interval $\mathcal{I}_m := [1/(m+1), 1/m]$ is the mapping $S_{[m]} : \mathcal{I}_m \rightarrow \mathcal{I}$ defined by $S_{[m]}(x) = (1/x) - m$ whose inverse mapping $h_{[m]} : \mathcal{I} \rightarrow \mathcal{I}_m$ is defined by $h_{[m]}(x) = 1/(m+x)$.

The trajectory $(x, S(x), S^2(x), \dots, S^k(x), \dots)$ of the real x reaches 0 if and only if x is rational. For a rational x , the first index k for which $S^k(x) = 0$ is called the *depth* of x and is denoted by $P(x)$. The sequence of the *digits* is defined as

$$(m_1(x), m_2(x), \dots, m_k(x), \dots) \quad \text{where} \quad m(x) := \left\lfloor \frac{1}{x} \right\rfloor, \quad m_{k+1}(x) = m(S^k(x)),$$

and x admits a *continued fraction expansion* (CFE) of the form

$$x = \frac{1}{m_1 + \frac{1}{m_2 + \frac{1}{\ddots + \frac{1}{m_k + \frac{1}{\ddots}}}}} = [m_1, m_2, \dots, m_k, \dots].$$

The continued fraction expansion is finite if and only if x is rational. In any case (rational or irrational), a truncation of the continued fraction expansion at depth $k \leq P(x)$ produces two continued fraction expansions: the *beginning part* $[m_1, m_2, \dots, m_k]$ and the *ending part* $[m_{k+1}, m_{k+2}, \dots, m_{k+\ell}, \dots]$. The beginning part defines the linear fractional transformation (LFT),

$$g_k := h_{[m_1]} \circ h_{[m_2]} \circ \dots \circ h_{[m_k]}, \quad \text{with} \quad g_k(y) = \frac{p_{k-1}y + p_k}{q_{k-1}y + q_k}$$

together with the rational $p_k/q_k = g_k(0)$, which is often called the k -th approximant of x . The ending part defines the real $x_k := S^k(x) = [m_{k+1}, m_{k+2}, \dots]$ via the equality

$$x = g_k(x_k), \quad \text{or} \quad x_k = \frac{\theta_{k+1}(x)}{\theta_k(x)} \quad \text{with} \quad \theta_k(x) := |q_{k-1}x - p_{k-1}|.$$

The beginning continuant q_k and the distance θ_k (also called here ending continuant) are expressed with the derivative of g_k ,

$$\frac{1}{q_k^2} = |g'_k(0)|, \quad \theta_k^2 = |g'_k(x_k)|. \quad (1.10)$$

When x is rational, of the form $x = u/v$ with a pair (u, v) of coprime integers, the sequence $\theta_k(x)$ is closely related to the sequence v_k of *remainders* that occurs in the execution of the Euclid algorithm on the pair (u, v) , and the equality $\theta_k(u/v) = (v_k/v)$ holds. This is why the remainder v_k is also called the *ending continuant*. The decomposition

$$h = g_k \circ \ell_k \quad \text{with} \quad g_k = h_{[m_1]} \circ h_{[m_2]} \circ \dots \circ h_{[m_k]}, \quad \ell_k = h_{[m_{k+1}]} \circ \dots \circ h_{[m_p]}$$

entails the following expressions for the beginning and ending continuants, as derivatives of LFT's

$$\frac{1}{v^2} = |h'(0)| = |g'_k(\ell_k(0))| \cdot |\ell'_k(0)|, \quad \frac{1}{q_k^2} = |g'_k(0)|, \quad \frac{1}{v_k^2} = |\ell'_k(0)|. \quad (1.11)$$

If we let $q_{-1} = 0$, $q_0 = 1$, $\theta_0 = 1$, $\theta_1 = x$, the two sequences (q_k) , (θ_k) satisfy the recursion formulae, for $k \in \llbracket 0, P(x) - 1 \rrbracket$,

$$q_{k+1} = m_{k+1}q_k + q_{k-1}, \quad \theta_{k+2} = \theta_k - m_{k+1}\theta_{k+1}. \quad (1.12)$$

1.5.2 The dynamical system and the plain transfer operator.

As the pair $(\mathcal{I}, S)$ defines a dynamical system, we strongly use the main tools of dynamical system theory. The transfer operator together with some of its extensions play a central role in our study. In this section we give its definition and introduce some of its variants. Section 1.8.2 describes the spectral properties of these operators that are useful in this paper.

The main study in dynamical systems concerns itself with the interplay between properties of the transformation S and properties of trajectories under iteration of the transformation. The behaviour of typical trajectories of dynamical systems is more easily explained by examining the flow of densities. We consider that the interval $\mathcal{I}$ is endowed with some density $g = g_0$.

The time evolution governed by the map S modifies the density, and the successive densities $g_0, g_1, g_2, \dots, g_n, \dots$ describe the global evolution of the system at time $t = 0, 1, 2, \dots$. For each inverse branch $h \in \mathcal{H}$,

$$\mathcal{H} = \left\{ h_{[m]} : x \mapsto \frac{1}{m+x}; \quad m \geq 1 \right\}.$$

the component operator $\mathbf{H}_{[h]}$ defined as

$$\mathbf{H}_{[h]}[g](x) = |h'(x)| \cdot g \circ h(x) .$$

expresses the part of the new density which is brought when one uses the branch h . Then, the operator

$$\mathbf{H} := \sum_{h \in \mathcal{H}} \mathbf{H}_{[h]} \quad (1.13)$$

is the density transformer (or the Perron-Frobenius operator) which expresses the new density g_1 as a function of the old density g_0 via the relation $g_1 = \mathbf{H}[g_0]$.

It proves convenient to add a (complex) parameter s , as Ruelle proposed it (Ruelle, 1978), and introduced the *transfer operator* $\mathbf{H}_s$ defined as

$$\mathbf{H}_s := \sum_{h \in \mathcal{H}} \mathbf{H}_{s,[h]}, \quad \text{with} \quad \mathbf{H}_{s,[h]}[g](x) = |h'(x)|^s \cdot g \circ h(x) .$$

As the equality $\mathbf{H}_1 = \mathbf{H}$ hold, the operator $\mathbf{H}_s$ provides an extension of the density transformer and it admits the general form

$$\mathbf{H}_s[g](x) := \sum_{h \in \mathcal{H}} |h'(x)|^s \cdot g \circ h(x) = \sum_{m \geq 1} \frac{1}{(m+x)^{2s}} \cdot g\left(\frac{1}{m+x}\right) .$$

In the constrained case, the Cantor set $\mathcal{J}^{[M]}$ is invariant by all the LFT's $h_{[m]}$ of the set $\mathcal{H}_M := \{h_{[m]}, m \leq M\}$, and the *constrained transfer operator* is defined as

$$\mathbf{H}_{M,s}[g](x) := \sum_{h \in \mathcal{H}_M} |h'(x)|^s \cdot g \circ h(x) = \sum_{m \leq M} \frac{1}{(m+x)^{2s}} \cdot g\left(\frac{1}{m+x}\right) . \quad (1.14)$$

For $M = \infty$, this is the “unconstrained” transfer operator $\mathbf{H}_s$, and the index M is omitted.

1.5.3 The transfer operator viewed as a generating operator.

The set $\mathcal{H}_M^k$ is defined as the k -th power of the initial set $\mathcal{H}_M$,

$$\mathcal{H}_M^k = \{h = h_1 \circ \dots \circ h_k; \ h_i \in \mathcal{H}_M, i \in \llbracket 1, k \rrbracket\} .$$

When $M = \infty$, this is the set of the inverse branches of S^k . For any $M \leq \infty$, and due to the multiplicative properties of derivatives, the k -th iterate of the operator $\mathbf{H}_{M,s}$ has exactly the same expression as the operator itself in (1.14), except that the sum is now taken over $\mathcal{H}_M^k$,

$$\mathbf{H}_{M,s}^k[g](x) := \sum_{h \in \mathcal{H}_M^k} |h'(x)|^s \cdot g \circ h(x) .$$

In a very general sense, the k -th iterate of the transfer operator describes the data after k iterations. The evolution of data during all the possible finite executions of the process involves the semi-group $\mathcal{H}_M^*$. This is why, for rational studies, we are led to work with the quasi-inverse $(I - \mathbf{H}_{M,s})^{-1}$ of the transfer operator, that will play a central rôle, as we see in Section 1.5.7.

1.5.4 Principles for our analysis.

Any parameter of interest is written as a sum of costs of the form

$$R_k := \tau(m_{k+1}) q_{k-1}^a q_k^b \theta_k^c \theta_{k+1}^d ,$$

The continuants q_k , the distances θ_k or the digits m_{k+1} are all defined as denominators of LFT's, as it appears in (1.10). And, for LFT's, there is a close connection between denominators and derivatives. This explains why the (usual) transfer operator is able to generate them in a separate way. However, the cost R_k , interpreted as a random variable, involves products of the variables $m_{k+1}, q_j, \theta_\ell$, and these variables are not independent. It is then necessary to (slightly) extend the (usual) transfer operator in order to generate such products. Our study is then based on two main facts.

Fact 1. There exist (extended) transfer operators for generating such products.

Fact 2. There exists a close relation between the transfer operators which are used for the study of such a cost in the real model and those used for the study of the same cost in the rational model.

1.5.5 Extended transfer operators.

We first make precise Fact 1. For unconstrained models, we use the various transfer operators whose component operators are described in Figure 1.7. The extended operator $\mathbf{H}_{(s,t)}$ was already introduced and used in many works of dynamical analysis [see for instance (Vallée, 1998)], and it is well adapted for generating together q_k and θ_k . Here, we also deal with the digit-cost $\tau(m_{k+1})$, and we introduce the *weighted operator* $\mathbf{H}_s^{(\tau)}$, already used in (Lhote and Vallée, 2008).

For constrained models, the generating operators have the same components operators as in the unconstrained case, but the sum runs over the set $\mathcal{H}_M := \{h_{[m]}; m \leq M\}$. To stress the dependence on M , we write

$$\mathbf{H}_{M,s}, \quad \mathbf{H}_{M,(s,t)}, \quad \mathbf{H}_{M,s}^{(\tau)}$$

for the various transfer operators which are written as the sum over $\mathcal{H}_M$ of component operators that are defined in Figure 1.7.

The main analytical properties of these operators (when they act on functions of class $\mathcal{C}^1$) are described, for instance, in (Vallée, 2006). The properties which are useful here are summarized in Section 1.8.

We now describe, in Sections 1.5.6 and 1.5.7 the main steps of dynamical analysis in each model (real or rational).

1.5.6 Dynamical analysis in the real model.

The mean value of a cost R_k on the set $\mathcal{J}^{[M]}$ is equal to

$$\mathbb{E}^{[M]}[R_k] = \int_{\mathcal{J}^{[M]}} R_k(x) \, d\nu_M(x) ,$$

Name	First definition of the component operator	Second definition of the component operator
$\mathbf{H}_s$	$ h'(x) ^s \cdot g \circ h(x)$	$\frac{1}{(m+x)^{2s}} g\left(\frac{1}{m+x}\right)$
$\mathbf{H}_{(s,t)}$	$ h'(x) ^s h'(y) ^t \cdot G(h(x), h(y))$	$\frac{1}{(m+x)^{2s} (m+y)^{2t}} G\left(\frac{1}{m+x}, \frac{1}{m+y}\right)$
$\mathbf{H}_s^{(\tau)}$	$ h'(x) ^s \tau\left(\frac{1}{h(0)}\right) \cdot g \circ h(x)$	$\frac{1}{(m+x)^{2s}} \tau(m) g\left(\frac{1}{m+x}\right)$

Figure 1.7 Definition of operators via their component operators. Remark that g denotes a function of one variable, and G a function of two variables. The second column is relative to the LFT h , while the third one is relative to $h = h_{[m]}$.

where ν_M is the Hausdorff measure of $\mathcal{J}^{[M]}$ defined in Section 1.3.1. To estimate the asymptotic behaviour of this mean value, we proceed with three main steps.

Step 1. We look for alternative forms of the cost R_k which involve the transfer operators of the underlying dynamical system.

Step 2. Using dominant spectral properties of transfer operators [see Section 1.8.2] near the value σ_M – equal to the Hausdorff dimension of $\mathcal{J}^{[M]}$ – leads to asymptotic estimates for the mean value $\mathbb{E}^{[M]}[R_k]$ in terms of dominant spectral objects.

Step 3. The speed of convergence is related to subdominant spectral properties [see Section 1.8.2] and we obtain the remainder estimates (exponential with respect to k) of our four theorems in the real models.

1.5.7 Dynamical analysis in the rational case.

We wish to evaluate the mean value of a cost $R_k(u/v)$ which depends on parameters of the continued expansion of the rational real u/v at depth k . When the index k depends on the depth $P(u/v)$ via an admissible function K , this random variable only depends on u/v , and we denote it by $R_{<K>}$ or simply by R . We shall consider the following two Dirichlet series for the rational model, respectively related to this cost R or the unitary cost $R = 1$,

$$S_R^{[M]}(s) := \sum_{(u/v) \in \Omega^{[M]}} \frac{R(u/v)}{v^{2s}} = \sum_{n \geq 1} \frac{a_n}{n^{2s}}, \quad S_1^{[M]}(s) := \sum_{(u/v) \in \Omega^{[M]}} \frac{1}{v^{2s}} = \sum_{n \geq 1} \frac{b_n}{n^{2s}}. \quad (1.15)$$

As the coefficients a_n and b_n are respectively equal to

$$a_n := \sum_{(u/n) \in \Omega^{[M]}} R(u, n), \quad b_n := \sum_{(u/n) \in \Omega^{[M]}} 1,$$

the expectation $\mathbb{E}_N^{[M]}[R]$ involves partial sums of a_n, b_n under the form

$$\mathbb{E}_N^{[M]}[R] = \frac{\Phi_R^{[M]}(N)}{\Phi_1^{[M]}(N)}, \quad \text{with } \Phi_R^{[M]}(N) := \sum_{n \leq N} a_n, \quad \Phi_1^{[M]}(N) := \sum_{n \leq N} b_n. \quad (1.16)$$

We then proceed with three main steps, which define the general method of the dynamical analysis method described for instance in (Vallée, 2006):

Step 1. We first describe an alternative form for the generating function $S_1^{[M]}(s)$ of the set $\Omega^{[M]}$: with the first relation given in (1.11), the relation holds,

$$S_1^{[M]}(s) = (I - \mathbf{H}_{M,s})^{-1}[1](0). \quad (1.17)$$

We also look for an alternative form of the Dirichlet series $S_R^{[M]}(s)$, which involves the (various) transfer operators of the underlying dynamical system.

Step 2. Using dominant spectral properties of transfer operators [see Section 1.8.2], we isolate a “dominant” part in the series $S_R^{[M]}(s)$, and thus a dominant pole, located at $s = \pi_M$. For the cost $R = 1$, with Eq. (1.17), π_M equals the Hausdorff dimension σ_M of $\mathcal{S}^{[M]}$, and this is also the case for any balanced cost, whereas for unbalanced costs, with imbalance v , π_M equals $\sigma_M(v, \delta)$ [See Theorem 1.4.4]. Now, there is a close relation between the residue of $S_R^{[M]}(s)$ at $s = \pi_M$ and the asymptotic behaviour of its coefficients, given by the estimate

$$\Phi_R^{[M]}(N) \sim \text{Res} \left[\frac{S_R^{[M]}(s)}{2s} N^{2s}; s = \pi_M \right].$$

which involves the sum $\Phi_R^{[M]}(N)$ defined in (1.16). Then, the denominator in Eq. (1.16) is of order $\Theta(N^{2\sigma_M})$, whereas the estimate for the numerator differs for the balanced and unbalanced costs.

(a) For balanced costs, according to the order of the pole $\pi_M = \sigma_M$ (simple or double), the numerator is of order $\Theta(N^{2\sigma_M})$ or $\Theta(N^{2\sigma_M} \log N)$ and the expectation is of constant order or of logarithmic order.

(b) For unbalanced costs, the pole π_M is always simple, and the numerator is of order $\Theta(N^{2\sigma_M(v, \delta)})$ and the expectation is of order $\Theta(N^{2(\sigma_M(v, \delta) - \sigma_M)})$.

Step 3. The previous holds as soon as we have some extra knowledge about the Dirichlet series $S_R^{[M]}(s)$ when s is close to the vertical line $\Re s = \pi_M$, with large $|\Im s|$. There are now two cases for the remainder terms [see Section 1.8.4].

(a) For balanced costs, when M becomes large, the dominant singularity $\pi_M = \sigma_M$ is close to 1, and, with results à la Dolgopyat (Dolgopyat, 1998) which are only proven to hold when $\Re s$ is close to 1, we show that the Dirichlet series $S_R^{[M]}(s)$ is of polynomial growth for $|\Im s| \rightarrow \infty$ on a convenient vertical strip near $\Re s = \pi_M$. We then use a version of the Landau theorem, and this provides precise remainder terms of order $N^{-\gamma}$ where γ is related to the width of this vertical strip.

(b) For unbalanced costs, the dominant singularity π_M is no longer close to 1.

However, the Dirichlet series $S_R^{[M]}(s)$ is analytic on the vertical line $\Re s = \pi_M$ (except at $s = \pi_M$). We then use a Tauberian theorem which does not provide any explicit remainder term.

1.5.8 Generating operators for a cost.

We expect similar results in the two probabilistic models (real model and rational model). Such a similarity is due to the Fact 2, stated in Section 1.5.4, that we make now more precise:

Definition 1.5.1 Consider a cost R that is studied in the real model via a sequence of costs (R_k) or in the rational model via an admissible function of the depth $p \mapsto F(p)$ and the Dirichlet generating function $S_R^{[M]}(s)$. The sequence of functions $\mathbf{R}_{M,s}^{[k]}$ forms a *basic sequence* for the cost R if one has:

$$\begin{aligned} \text{in the real case} \quad \mathbb{E}^{[M]}[R_k] &= \int_{\mathcal{S}^{[M]}} \mathbf{R}_{M,\sigma_M}^{[k]}(u) \, d\nu_M(u) \quad \text{for any } k \in \mathbb{N} \\ \text{in the rational case} \quad S_R^{[M]}(s) &= \sum_{p \geq 1} \mathbf{H}_{M,s}^{p-F(p)-1} \left[\mathbf{R}_{M,s}^{[F(p)]} \right] (0) . \end{aligned}$$

The existence of a basic sequence is the beginning part of our analysis. We exhibit such a basic sequence, in Lemma 1.6.1 for balanced costs, and in Lemma 1.7.1 in the unbalanced framework.

1.5.9 Plan of the next two sections.

The next two sections are devoted to outline the proofs for our four theorems. Section 1.6 deals with the three balanced parameters—covered space, discrepancy and Arnold measure—. The balance f equals 1 for discrepancy and covered space and equals 2 for Arnold measure. Section 1.7 is devoted to the study of parameter $q_k^b \theta_k^c$. This is an unbalanced cost except in the particular case when $b = c$. The distances, whose expression is recalled in Figure 1.5, are closely related to the sequence θ_k and define unbalanced costs, of imbalance equal to 1.

In each section, we first describe the basic sequence, then, we follow the general scheme that is described in Sections 1.5.6 (for the real models) and 1.5.7 (for the rational models). In this short version, we mainly focus on Steps 1 and 2 of the general scheme. The details for Step 3 are provided in the long version.

1.6 Balanced costs.

First, we provide in Section 1.6.1 the expression of the basic sequence which generates an elementary cost. It involves “beginning” operators (only in the rational model), “ending” operators and a “middle” operator (in both models). Then, as the

beginning and ending operators intervene with large powers, Section 1.6.2 first uses their dominant spectral properties. Then, Section 1.6.3 focusses on the middle operator, and leads to a further classification of costs into two subclasses –extremal costs, or ordinary costs– which will be of crucial importance in the sequel, and explains the difference between the first two theorems and the third one.

1.6.1 Expression of the basic sequence.

We first show the existence of a basic sequence for each balanced cost,

$$R_k = \tau(m_{k+1}) (q_k \theta_k)^f \left(\frac{q_{k-1}}{q_k} \right)^a \left(\frac{\theta_{k+1}}{\theta_k} \right)^d. \quad (1.18)$$

First, we observe that the four variables $q_{k-1}, q_k, \theta_k, \theta_{k+1}$ are related via the Bezout identity which expresses q_{k-1} as a function of $q_k, \theta_k, \theta_{k+1}$, namely $q_{k-1} = (1 - \theta_k q_k) / \theta_{k+1}$. We then obtained an alternative expression for the balanced cost described in (1.18) as

$$R_k = \left[(q_k \theta_k)^f \left(\frac{1}{q_k \theta_k} - 1 \right)^a \right] \cdot \left[\tau(m_{k+1}) \left(\frac{\theta_{k+1}}{\theta_k} \right)^{d-a} \right], \quad (1.19)$$

which decomposes into two factors, the first one which is a function of the balanced product $q_k \theta_k$, and the second one which highly depends on the digit m_{k+1} or via the ratio θ_{k+1} / θ_k which is of order $\Theta(1/m_{k+1})$ (see Section 1.5.1).

The following result is mainly based on a refinement of the expressions provided in (1.10) and (1.11).

Lemma 1.6.1 *Consider a balanced cost defined with a balance f and a triple (τ, a, d) , and written as in Eq.(1.18). The following holds:*

- (i) *This cost admits a basic sequence which involves the weighted operator $\mathbf{H}_{M,(s,t)}^{(\tau)}$ defined in Figure 1.7 under the form*

$$\mathbf{R}_{M,s}^{[k]} := \mathbf{H}_{M,(s+(d-a)/2)}^{(\tau)} \left[\mathbf{L}_{M,(s,a,f)}^{[k]} \right],$$

$$\text{with } \mathbf{L}_{M,(s,a,f)}^{[k]}(x) := \sum_{g \in \mathcal{H}_M^k} |g'(x)|^s \left| \frac{g'(x)}{g'(0)} \right|^{f/2} \left(\left| \frac{g'(x)}{g'(0)} \right|^{-1/2} - 1 \right)^a.$$

- (ii) *The function $\mathbf{L}_{M,(s,a,f)}^{[k]}$ is $O(x^a)$ for $x \rightarrow 0$, and the basic sequence admits another useful expression*

$$\mathbf{R}_{M,s}^{[k]} := \mathbf{H}_{M,(s+d/2)}^{(\tau)} \left[\widehat{\mathbf{L}}_{M,(s,a,f)}^{[k]} \right], \quad \text{with } \widehat{\mathbf{L}}_{M,(s,a,f)}^{[k]}(x) := \frac{1}{x^a} \mathbf{L}_{M,(s,a,f)}^{[k]}(x).$$

- (iii) There is also another expression for the function $\mathbf{L}^{[k]}$ which involves k -th iterates of transfer operators, together with the “section” $\Pi : \mathcal{C}^1(\mathcal{J}^2) \rightarrow \mathcal{C}^1(\mathcal{J})$ for which $\Pi[f](x) = f(x, 0)$,

$$\mathbf{L}_{M,(s,a,f)}^{[k]} := \Pi \left[\sum_{j=0}^a \binom{a}{j} (-1)^{a-j} \mathbf{H}_{M,(s+(f-j)/2, -(f-j)/2)}^k [1] \right]. \quad (1.20)$$

Remark. The similarity of the patterns between Eq. (1.19) and the expression of Assertion (i) is striking. The weighted operator $\mathbf{H}_{M,(s+(d-a)/2)}^{(\tau)}$ generates the second factor while the function $\mathbf{L}_{M,(s,a,f)}^{[k]}$ generates the first factor. Assertion (ii) is interesting because it splits the roles of the triple (a, d, τ) into two groups: the pair (τ, d) for the middle operator, and the integer a for the ending part. Assertion (iii) describes the behaviour of the ending part in terms of k -th iterates of operators.

Proof

Assertion (i). We begin by the real case, then we study the rational case.

Real case. We use a refined decomposition. On each fundamental interval of depth $k+1$, of the form $g \circ h(\mathcal{J})$, with $g \in \mathcal{H}_M^k, h \in \mathcal{H}_M$, the real x is written $x = g \circ h(u)$, and the measure ν_M is defined via the characteristic equality

$$d\nu_M(g \circ h(u)) = |(g \circ h)'(u)|^{\sigma_M} d\nu_M(u) = |g'(h(u))|^{\sigma_M} \cdot |h'(u)|^{\sigma_M} d\nu_M(u)$$

when $g, h \in \mathcal{H}_M^*$. Furthermore, the relations (1.10) entail expressions of $q_k, \theta_k, \theta_{k+1}, m_{k+1}$ which involve the derivatives of g and h , namely

$$\begin{aligned} \frac{1}{q_k^2} &= |g'(0)|, & \theta_k^2 &= |g'(h(u))|, \\ \theta_{k+1}^2 &= |(g \circ h)'(u)| = |g'(h(u))| \cdot |h'(u)|, & \frac{1}{m_{k+1}^2} &= |h'(0)|. \end{aligned}$$

Taking the sum over $h \in \mathcal{H}_M$ gives rise to the “middle” operator $\mathbf{H}_{M,(\sigma_M+(d-a)/2)}^{(\tau)}$. Taking the sum over LFT’s $g \in \mathcal{H}_M^k$ gives rise to the function $\mathbf{L}_{M,(\sigma_M,a,f)}^{[k]}$. Finally, we obtain the expression given in Assertion (i).

Rational case. The similar refined decomposition of the LFT $h_1 \circ \dots \circ h_p$ with

$$g = h_1 \circ h_2 \circ \dots \circ h_k, \quad h := h_{k+1}, \quad \ell = h_{k+2} \circ h_{k+3} \circ \dots \circ h_p$$

now entails the following relations

$$\begin{aligned} \frac{1}{v^2} &= \frac{1}{v_0^2} = |g'(h \circ \ell(0))| \cdot |h'(\ell(0))| \cdot |\ell'(0)|, & \frac{1}{q_k^2} &= |g'(0)|, \\ \frac{1}{v_k^2} &= |(h \circ \ell)'(0)| = |h'(\ell(0))| \cdot |\ell'(0)|, & \frac{1}{v_{k+1}^2} &= |\ell'(0)|, & \frac{1}{m_{k+1}^2} &= |h'(0)|. \end{aligned}$$

Taking the sum over LFT’s f gives rise to the “middle” operator $\mathbf{H}_{M,(s+(d-a)/2)}^{(\tau)}$.

Taking the sum over LFT's $g \in \mathcal{H}_M^k$ gives rise to the function $\mathbf{L}_{M,(s,a,f)}^{[k]}$. Finally, the sum over LFT's $\ell \in \mathcal{H}_M^{p-k-1}$ gives the “beginning” operator of the basic sequence.

Assertion (ii). As the function g' is non zero on $\mathcal{J}$, the function $x \mapsto |g'(x)|^{-1/2}$ belongs to $\mathcal{C}^1(\mathcal{J})$ and

$$|g'(x)|^{-1/2} = |g'(0)|^{-1/2} + O(x), \quad \left| \frac{g'(x)}{g'(0)} \right|^{-1/2} - 1 = O(x) .$$

Then the function $\widehat{\mathbf{L}}_{M,(s,a,f)}^{[k]}$ belongs to the space $\mathcal{C}^1(\mathcal{J})$, and Eq. (1.6.1) is a consequence of the equality $|h(x)|^{1/2} = |h'(x)|$ that holds for all the LFT's $h \in \mathcal{H}$.

Assertion (iii). Eq. (1.20) follows from the expression $\mathbf{L}_{M,(s,a,f)}^{[k]}$ given in Assertion (i), with the binomial formula, the definitions of $\mathbf{H}_{M,(s+t,-t)}$ and section Π . $\square$

We have performed Step 1 of the dynamical analysis, as it is described in Sections 1.5.6 and 1.5.7. We have now⁷ to perform Step 2. The expressions provided by Lemma 1.6.1 involve ending and beginning operators which appear with large powers and we first use in Section 1.6.2 their dominant spectral properties near the real axis, described in Section 1.8.2. In Section 1.6.3, we deal with the middle operator.

1.6.2 Using dominant spectral properties.

With Assertion (iii) of Lemma 1.6.1, and for s near the real axis, the dominant part [see Section 1.8.2] of “ending” operators involved in the basic sequence provides the estimate

$$\mathbf{L}_{M,(s,a,f)}^{[k]} \sim \lambda_M^k(s) \Phi_{M,(s,a,f)} ,$$

where $\lambda_M(s)$ is the dominant eigenvalue, and the function $\Phi_{M,(s,a,f)}$ is expressed with the section Π of eigenfunctions $\varphi_{M,(s+t,-t)}$. Moreover, with Assertion (ii) of Lemma 1.6.1, the function $\Phi_{M,(s,a,f)}$ is $O(x^a)$ for $x \rightarrow 0$ and

$$\widehat{\Phi}_{M,(s,a,f)}(x) := (1/x^a) \Phi_{M,(s,a,f)}(x) \quad (1.21)$$

$$= (1/x^a) \sum_{j=0}^a \binom{a}{j} (-1)^{a-j} \Pi[\varphi_{M,(s+(f-j)/2, -(f-j)/2}](x) . \quad (1.22)$$

This entails, with Assertion (ii) of Lemma 1.6.1, asymptotic estimates for the function $\mathbf{R}_{M,s}^{[k]}$ which now involve the function $\widehat{\Phi}_{M,(s,a,f)}$,

$$\mathbf{R}_{M,s}^{[k]} \sim \lambda_M^k(s) \mathbf{J}_{M,s}, \quad \text{with} \quad \mathbf{J}_{M,s} := \mathbf{H}_{M,(s+d/2)}^{(\tau)} [\widehat{\Phi}_{M,(s,a,f)}] . \quad (1.23)$$

Real model. In the real case, using the equality $\lambda_M(\sigma_M) = 1$, the previous estimate (1.23) at $s = \sigma_M$ gives rise to the asymptotic estimate

$$\mathbb{E}^{[M]}[R_k] \sim \int_{\mathcal{J}^{[M]}} \mathbf{J}_{M,\sigma_M}(u) \, d\nu_M(u) . \quad (1.24)$$

⁷ We recall that Step 3 is not performed in this short version

Rational model. In the rational case, we also use dominant spectral properties for the “beginning” operator, for s near the real axis, [see Section 1.8.2],

$$\mathbf{H}_{M,s}^k[g](0) \sim \lambda_M^k(s) \varphi_{M,s}(0) \mathbf{Q}_{M,s}[g], \quad (1.25)$$

Collecting all the leading estimates obtained in (1.23, 1.25) gives rise to a “leading” term for the Dirichlet series $S_R^{[M]}(s)$, of the form

$$\varphi_{M,s}(0) \cdot \Sigma_M(s) \cdot \Psi_M(s)$$

$$\text{with} \quad \Sigma_M(s) := \sum_{p \geq 1} \lambda_M^p(s) = \frac{\lambda_M(s)}{1 - \lambda_M(s)} \quad \text{and} \quad \Psi_M(s) := \mathbf{Q}_{M,s}[\mathbf{J}_{M,s}]. \quad (1.26)$$

The other part $H_M(s)$ of the Dirichlet series $S_R^{[M]}(s)$, which gathers all the other terms, will be actually a remainder term, due to the fact that K is an admissible function of the depth and it remains analytic when the series $\Sigma_M(s)$ becomes singular, namely at $s = \sigma_M$ for which the denominator $1 - \lambda_M(s)$ is zero. Then, the value

$$A_M := \Psi_M(\sigma_M) \quad (1.27)$$

which involves the function Ψ_M defined in (1.26) intervenes in the residue of $S_R^{[M]}(s)$ at $s = \sigma_M$ and plays a central role in the analysis in the rational model.

Comparing the leading terms in the two models. We remark that the term $\Psi_M(\sigma_M)$ which intervenes in the rational case, also intervenes in the real case. Indeed, with Sections 1.8.2 and 1.8.3, the projector $\mathbf{Q}_{M,\sigma_M}$ satisfies

$$\mathbf{Q}_{M,\sigma_M}[g] = \int_{\mathcal{J}^{[M]}} g(t) d\nu_M(t) \quad \text{so that} \quad \Psi_M(\sigma_M) = \int_{\mathcal{J}^{[M]}} \mathbf{J}_{M,\sigma_M}(u) d\nu_M(u),$$

coincides with the constant defined in (1.24). This explains the strong similarity between the analyses in the two models and the central role played by the constant A_M defined in (1.27).

1.6.3 Role of the middle operator. Classification of pairs (τ, d) .

It remains to deal with the middle operator $\mathbf{H}_{M,(s+d/2)}^{(\tau)}$. We first focus on the case when the digit-cost τ satisfies $\tau(m) \sim \lambda m^e$ when $m \rightarrow \infty$. Then, the middle operator is closely related to the (possibly) truncated hybrid ζ function $\zeta_M(2s+d, -e; x)$ where $\zeta_M(s, t; x)$ is defined as

$$\zeta_M(s, t; x) := \sum_{m=1}^M \frac{1}{(m+x)^s} \frac{1}{m^t}.$$

More precisely, since the inequality $1 + d - e \geq 0$ holds, the difference

$$\mathbf{H}_{M,(s+d/2)}^{(\tau)}[g](x) - \lambda \zeta_M(2s+d, -e; x)g(0)$$

defines an operator which is bounded (uniformly with respect to M) when $\Re s$ is close to 1 (even though each term is itself not uniformly bounded). Then, the middle

operator is “closely related” to the truncated hybrid function $\zeta_M(2s+d, -e; x)$ itself “closely related” to the truncated zeta function $\zeta_M(2s+d-e)$. When $\Re s$ is close to 1, there are two cases for the pair (e, d) and this leads to the following definition:

Definition 1.6.2 Consider an elementary cost R_k defined as in (1.18).

- (i) When the digit-cost $\tau(m)$ is $O(m^e)$ with $e \leq d$, the cost R_k is said to be *ordinary*.
- (ii) When the digit-cost $\tau(m)$ is $\Theta(m^e)$ with $e = 1 + d$, the cost R_k is said to be *extremal*. This is only possible when $a = 0$.
- (iii) A digit-cost of the form $\tau(m) = m^e$ is said to be *standard*.

We now explain the specificities of the two cases:

Ordinary cost. For any constraint M (possibly infinite), the truncated zeta functions $\zeta_M(2\sigma + d - e)$ are uniformly bounded with respect to M when σ is close to 1. The middle operator is bounded (uniformly with respect to M). The sequence A_M tends to $A_\infty := \Psi(1)$ when $M \rightarrow \infty$ with a speed of order $O(1/M)$. The Dirichlet series $S^{[M]}(s)$ has a simple pole at $s = \sigma_M$ for any M (finite or infinite).

Extremal cost. For $M = \infty$, the function $\zeta(2\sigma + d - e)$ is singular at $\sigma = 1$, and, for any $M \leq \infty$, the middle operator has the same behaviour as the truncated zeta function $\zeta_M(2s + d - e)$ [see Section 1.8.5]. It then creates a pole at $s = 1$ for $M = \infty$ and is no longer uniformly bounded with respect to M when $M \rightarrow \infty$. Furthermore, $A_\infty := \Psi(1) = \infty$, and $A_M := \Psi_M(\sigma_M)$ is of order $\log M$. The Dirichlet series $S_R^{[M]}(s)$ has a simple pole at $s = \sigma_M$ for finite M (with a residue of order $\log M$) and a double pole at $s = 1$ for $M = \infty$ which gives rise to a logarithmic term for the mean value.

1.6.4 The basic result for a balanced cost.

These statements will be made precise in the following theorem, which is a central step of the proof of the first three theorems. This result is completely proven in (Cesaratto and Vallée, 2014). Here, we have only explained the dominant terms.

Theorem 1.6.3 *There are three positive real numbers $\gamma, \theta, \rho < 1$, such that, for any balanced cost R_k defined in Eq. (1.18), the following holds and involves the constant $A_M = A_M[R]$ defined in Eq. (1.27):*

- (a) *If the cost R_k is ordinary, then, for any integer $M \leq \infty$, the mean values satisfy*

$$\mathbb{E}^{[M]}[R_k] = A_M + O(\rho^k), \quad \mathbb{E}_N^{[M]}[R] = A_M + O(N^{-\gamma}).$$

Furthermore, when $M \rightarrow \infty$, the sequence A_M tends to A_∞ and satisfies

$$A_M = A_\infty + O\left(\frac{1}{M}\right).$$

- (b) *If the cost R_k is extremal, then, for any finite integer M , the mean values satisfy*

$$\mathbb{E}^{[M]}[R_k] = A_M + O(\rho^k), \quad \mathbb{E}_N^{[M]}[R] = A_M + O(N^{-\theta/\log M}).$$

Furthermore, when $M \rightarrow \infty$, the sequence A_M is of logarithmic order, and, if the cost R_k is standard, there exists a constant $\hat{r}$ for which

$$A_M = \log_2 M + \hat{r} + O\left(\frac{1}{M}\right).$$

(c) If the cost (R_k) is extremal and $M = \infty$, the mean value $\mathbb{E}[R_k]$ is infinite, the mean value $\mathbb{E}_N[R]$ is of logarithmic order and, if the cost R_k is standard, there exists a constant r for which the mean value $\mathbb{E}_N[R]$ satisfies

$$\mathbb{E}_N[R] = \log_2 N + r + O(N^{-\gamma}).$$

1.6.5 End of the proof of theorems for balanced parameters.

We explain how Theorem 1.6.3 is applied to the proof of our first three theorems.

For Theorem 1.4.1, when the truncation T is at a boundary position, the digit m_{k+1} does not intervene, the integer e equals 0, and parameters $X_k^\pm$ only involve ordinary (and standard) costs R_k [see Figure 1.5]. We are in the case (a) of Theorem 1.6.3, and it remains to compute the exact values of the limit $x_\infty^\pm$, which will be done in the next section.

For Theorem 1.4.2, we consider a truncation at position $\mu \in]0, 1[$ and the two types of costs may a priori appear. However, for the covered space, all the costs are ordinary; we directly deal with the costs $\tau(m) := \lfloor 1 + \mu(m-1) \rfloor^e$ and we use case (a) of Theorem 1.6.3. We now obtain constants A_M which depend on μ , and the sum of all these constants give rise to the constant $s_M(\mu)$. This leads to the proof.

For Theorem 1.4.3, we deal with a truncation at position $\mu \in]0, 1[$, and the other two parameters [discrepancy and Arnold measure]. There now exist both ordinary and extremal costs R_k . For ordinary costs, we directly deal with the costs $\tau(m) := \lfloor 1 + \mu(m-1) \rfloor^e$, as previously, and we apply case (a) of Theorem 1.6.3. For extremal costs, we decompose the digit-cost

$$\lfloor 1 + \mu(m-1) \rfloor^e = \mu^e m^e + \rho(m) \quad \text{with} \quad \rho(m) = \Theta(m^{e-1}). \quad (1.28)$$

The first part gives rise to the leading extremal part of the cost R_k , (which is colinear to a standard extremal cost) and the second part gives rise to a remainder cost which is now ordinary. We apply case (a) of Theorem 1.6.3 to the remainder cost, and cases (b) and (c) of the same theorem to the leading extremal cost. The polynomial $\underline{X}_k^{(\mu)}$ which collects all the leading extremal parts of cost $R_k^{(\mu)}$ equals in each case

$$\begin{aligned} \underline{\Delta}_k^{(\mu)} &= \mu m_{k+1} q_k \theta_k - \mu^2 m_{k+1}^2 q_k \theta_{k+1} \\ \underline{A}_k^{(\mu)} &= \mu m_{k+1} q_k^2 \theta_k^2 - 2\mu^2 m_{k+1}^2 q_k^2 \theta_k \theta_{k+1} + \mu^3 m_{k+1}^3 q_k^2 \theta_{k+1}^2. \end{aligned}$$

These expressions, together with Parts (b) and (c) of Theorem 1.6.3, give rise to the leading polynomials (with respect to μ) which occur in Theorem 1.4.3.

1.6.6 Computation of constants $x_\infty^\pm$ in Theorem 1.4.1.

The constants which occur in the statement of Theorem 1.4.1 for $M = \infty$ are obtained by summing each constant $\Psi(1)$ brought by each elementary cost R_k which appears in the decomposition of each parameter (discrepancy, Arnold constant and covered space). Such a constant $\Psi(1)$ is the integral of a function $\widehat{\Phi}_{(1,a,f)}$ for some pairs (a, f) with $0 \leq a \leq f$ and $f = 1, 2$. Any such function $\widehat{\Phi}_{(1,a,f)}$ is expressed in Eq. (1.21) with the dominant eigenfunction φ_1 and the sections Π of the dominant eigenfunctions $\varphi_{(3/2,-1/2)}$, $\varphi_{(2,-2)}$. Section 1.8.3 provides an explicit expression for the functions φ_1 , $\Pi[\varphi_{(3/2,-1/2)}]$ and $\Pi[\varphi_{(2,-2)}]$. This yields an explicit expression for the function $u \mapsto \widehat{\Phi}_{(1,a,f)}(u)$ as a linear combination of the functions $(1+u)^{-b}$ with $b := 1, 2, 3$. The integral on such functions leads to the constants of Figure 1.6.

1.7 Unbalanced costs.

We here outline the proof of Theorem 1.4.4 which deals with costs $q_k^b \theta_k^c$. Remark that the case $b = c$ gives rise to balanced costs which have been already studied in the previous section. Section 1.2.8 explains why we may expect an asymptotic exponential behaviour for the expectation $\mathbb{E}[q_k^b \theta_k^c]$ in the case when c and b are not equal. We show here that the imbalance $v := c - b$ plays a central rôle.

1.7.1 Expression of the basic sequence.

The following lemma describes the basic sequence of functions relative to the general cost $q_k^b \theta_k^c$. As the equalities $a = d = e = 0$ hold, this expression is simpler than the previous one of Lemma 1.6.1.

Lemma 1.7.1 *Each cost $R_k = q_k^b \theta_k^c$ admits a basic sequence*

$$\mathbf{R}_{M,s}^{[k]}(x) = \mathbf{H}_{M,(s+c/2,-b/2)}^k[1](x, 0)$$

and

$$\text{in the real model} \quad \mathbb{E}^{[M]}[R_k] = \int_{\mathcal{J}^{[M]}} \mathbf{H}_{M,(\sigma_M+c/2,-b/2)}^k[1](u, 0) d\nu_M(u)$$

$$\text{in the rational model} \quad S_R^{[M]}(s) = \sum_{p \geq 1} \mathbf{H}_{M,s}^{p-F(p)} \circ \mathbf{H}_{M,(s+c/2,-b/2)}^{F(p)}[1](0, 0) .$$

The proof follows the same lines as in Lemma 1.6.1. For the real case, one deals with $g \in \mathcal{H}_M^k$, and, in the rational case, with the decomposition $h = g \circ \ell$.

1.7.2 Step 2 in the real model.

We denote the imbalance $c - b$ by v . When the inequality $v > t_M$ holds, the operator $\mathbf{H}_{M,(s+c/2,-b/2)}$ is well-defined at $s = \sigma_M$, and the spectral decomposition at $s = \sigma_M$

leads to the estimate

$$\mathbf{H}_{M,(\sigma_M+c/2,-b/2)}^k[1](x) \sim \lambda_M^k(\sigma_M + v/2) \varphi_{M,(\sigma_M+c/2,-b/2)}(x),$$

which gives rise to the estimate

$$\mathbb{E}^{[M]}[R_k] \sim \lambda_M^k(\sigma_M + v/2) \int_0^1 \varphi_{M,(\sigma_M+c/2,-b/2)}(u) dv_M(u).$$

1.7.3 Step 2 in the rational model.

The Dirichlet series $S_R^{[M]}(s)$ is convergent in the halfplane $\Re s > s_M$. For any complex number s with $\Re s > s_M$, and close enough to the real axis, we use, as in (1.25), the dominant spectral behaviour of the operator $\mathbf{H}_{M,s}^k$. This gives rise to a dominant part for the Dirichlet series $S_R^{[M]}(s)$, in the same vein as in (1.26). We obtain

$$S_R^{[M]}(s) = \varphi_{M,s}(0) \cdot \Sigma_M(s) \cdot \Psi_M(s) + H_M(s)$$

where the first term collects all the dominant terms

$$\Sigma_M(s) := \sum_p \lambda_M^{p-F(p)}(s) \lambda_M^{F(p)}(s + v/2), \quad \Psi_M(s) := \mathbf{Q}_{M,s}[x \mapsto \varphi_{M,(s+c/2,-b/2)}(x)],$$

and the “remainder” term $H_M(s)$ collects the three other terms, each of them containing at least one subdominant term.

In the present case, the admissible function intervenes in the leading term, and this is why we consider, as in (Daireaux and Vallée, 2004; Lhote and Vallée, 2008), particular admissible functions $F(p)$ of the form $F(p) = \lfloor \delta p \rfloor$, with $\delta \in \mathbb{Q} \cap [0, 1]$, already defined in Section 1.3.3. Consider indeed a rational $\delta := B/C$ defined with two coprime integers B, C which satisfy $B \leq C$. Then, the series Σ_M is written as

$$\Sigma_M(s) = \ell_M(s, \delta) \sum_{p \geq 1} L_M^{Cp}(s, \delta) = \frac{\ell_M(s, \delta, v)}{1 - L_M^C(s, \delta, v)}, \quad \text{with}$$

$$\ell_M(s, \delta, v) = \sum_{j=0}^{C-1} \lambda_M^{j-\lfloor \delta j \rfloor}(s) \lambda_M^{\lfloor \delta j \rfloor}(s + v/2), \quad L_M(s, \delta, v) = \lambda_M^{1-\delta}(s) \lambda_M^\delta(s + v/2).$$

The poles of the Dirichlet series $\Sigma_M(s)$ are brought by the zeroes of the map $s \mapsto L_M^C(s, \delta, v) - 1$ where C is a positive integer. The map $s \mapsto L_M(s, \delta, v)$ is an extension of a map which has been already studied in (Lhote and Vallée, 2008). And, the long version of this chapter (Cesaratto and Vallée, 2014) provides a proof of all the properties which are stated in Theorem 1.4.4 for the unique real zero $\sigma_M(v, \delta)$ of the equation $L_M(s, \delta, v) - 1$.

1.8 Summary of functional analysis.

We consider a (possibly infinite) integer M , and we denote by $\mathbb{G}_{M,s}$ a generic element of the set $\mathcal{G}_{M,s}$, defined as

$$\mathcal{G}_{M,s} := \{\mathbf{H}_{M,s}\} \cup \{\mathbf{H}_{M,(s+t,-t)}; t \in \mathbb{R}\}.$$

1.8.1 Generalities.

Operators in $\mathcal{G}_{M,s}$ may act on functions of one or two variables. The integer q denotes the number of variables. It always equals 2 for the operator $\mathbb{G}_{M,s}$ except for the (plain) operator $\mathbf{H}_{M,s}$, where it equals 1. More precisely, we consider the Banach spaces $\mathcal{C}^1(\mathcal{I}^q)$, endowed with the following norms defined from the sup-norm on $\mathcal{I}^q$ denoted by $\|\cdot\|_0$,

$$\|f\|_1 = \|f\|_0 + \|f'\|_0 \quad \text{for } q = 1, \quad \|F\|_1 = \|F\|_0 + \|DF\|_0 \quad \text{for } q = 2.$$

We will also use another norm, the $(1, \tau)$ norm, defined later in (1.31).

There are three cases for the constraint M : the two possible cases $M < \infty$, $M = \infty$, but we are also interested in the “transition” when $M \rightarrow \infty$.

There are two important real numbers which depend on the integer M .

(i) The convergence abscissa s_M defines the (half)-plane $\Re s > s_M$ where the operator $\mathbb{G}_{M,s}$ is well-defined: This abscissa s_M equals $-\infty$ for $M < \infty$ and equals $1/2$ for $M = \infty$.

(ii) The real σ_M is the real s for which the dominant eigenvalue $\lambda_M(s)$ of the operator $\mathbb{G}_{M,s}$ equals 1.

The operators $\mathbb{G}_{M,s} \in \mathcal{G}_{M,s}$ share many properties with the plain operator $\mathbf{H}_{M,s}$. We describe their main analytical properties, in particular their spectral properties, and focus on the behaviour of these operators when the parameter s equals σ_M .

1.8.2 Dominant spectral properties and spectral decomposition.

Each operator $\mathbf{H}_{M,s}$ [when s is close to the real axis] or $\mathbf{H}_{M,(s,t)}$ [when $s+t$ is close to the real axis] is quasi-compact and has spectral dominant properties.

The operator $\mathbf{H}_{M,s}$ has a unique dominant eigenvalue $\lambda_M(s)$, which is simple. This is also the dominant eigenvalue of the adjoint operator $\mathbf{H}_{M,s}^*$. The dominant eigenvector of $\mathbf{H}_{M,s}$ is denoted by $\varphi_{M,s}$ and the dominant eigenmeasure of the adjoint $\mathbf{H}_{M,s}^*$ is denoted by $\mathbf{Q}_{M,s}$, and there is a normalization condition $\mathbf{Q}_{M,s}[\varphi_{M,s}] = 1$. For s close to the real axis, one has

$$\mathbf{H}_{M,s}^k[g](x) \sim \lambda_M^k(s) \varphi_{M,s}(x) \mathbf{Q}_{M,s}[g]. \quad (1.29)$$

The operator $\mathbf{H}_{M,(s+t,-t)}$ has a unique dominant eigenvalue $\lambda_M(s+t, -t)$, which is simple and equal to $\lambda_M(s)$. This is also the dominant eigenvalue of the adjoint operator $\mathbf{H}_{M,(s+t,-t)}^*$. The dominant eigenvector of $\mathbf{H}_{M,(s+t,-t)}$ is denoted by $\varphi_{M,(s+t,-t)}$ and the dominant eigenmeasure of the adjoint $\mathbf{H}_{M,(s+t,-t)}^*$ is denoted by $\mathbf{Q}_{M,(s+t,-t)}$, and there is a normalization condition $\mathbf{Q}_{M,(s+t,-t)}[\varphi_{M,(s+t,-t)}] = 1$. For s close to the real axis, one has

$$\mathbf{H}_{M,(s+t,-t)}^k[G](x, y) \sim \lambda_M^k(s) \varphi_{M,(s+t,-t)}(x, y) \mathbf{Q}_{M,(s+t,-t)}[G]. \quad (1.30)$$

In Eq. (1.29) or (1.30), the quasi-compactness of operators entails that the remainder term is of order $|\lambda_M(s)|^k O(\rho^k)$, with $\rho < 1$ and a hidden constant that is uniform with respect to s , for s close enough to the real axis, and uniform with respect to M , for M large enough.

1.8.3 Special values at $s = \sigma_M$.

For any M possibly infinite, the equation $\lambda_M(s) - 1$ has a unique solution on the real axis, located at $s = \sigma_M$. The value σ_M is the Hausdorff dimension of the constrained set $\mathcal{J}^{[M]}$. The dominant eigenmeasure of the adjoint operator $\mathbf{H}_{M,s}^*$ at $s = \sigma_M$ coincides with the Hausdorff measure ν_M of the set $\mathcal{J}^{[M]}$.

For $M = \infty$, the eigenfunction φ_1 is the Gauss density and $-\lambda'(1)$ is the entropy

$$\varphi_1(x) = \frac{1}{\log 2} \left(\frac{1}{1+x} \right), \quad \mathbf{Q}_1[f] = \int_{\mathcal{J}} f(w) dw, \quad \text{and} \quad \lambda'(1) = -\frac{\pi^2}{6 \log 2}.$$

The sections of the eigenfunctions $\varphi_{(2,-1)}$ and $\varphi_{(3/2,-1/2)}$ satisfy

$$\begin{aligned} 3 \log 2 \cdot \Pi[\varphi_{(2,-1)}](x) &= (1+x)^{-1} + (1+x)^{-2} + (1+x)^{-3}, \\ 2 \log 2 \cdot \Pi[\varphi_{(3/2,-1/2)}](x) &= (1+x)^{-1} + (1+x)^{-2}. \end{aligned}$$

For $M \rightarrow \infty$, the dominant spectral objects of the constrained operators $\mathbf{H}_{M,s}^*$ tend to the spectral objects of the unconstrained one $\mathbf{H}_s^*$ with a speed of order $O(1/M)$.

1.8.4 For s far from the real axis.

They are two cases, according as $\sigma := \Re s$ be close to 1 or any $\sigma > s_M$.

For any $M \geq 2$, for any $\sigma > s_M$, on the vertical line $\Re s = \sigma$, with $s \neq \sigma$, one has,

$$\|\mathbb{G}_{M,s}[F]\|_0 < \lambda_M(\sigma) \|F\|_0, \quad \text{for any } F \in \mathcal{C}^1(\mathcal{J}^q).$$

[Dolgopyat bounds when σ is close to 1]. They use the $(1, \tau)$ norm, defined as

$$\|F\|_{(1,\tau)} := \|F\|_0 + \frac{1}{|\tau|} \|F\|_1. \quad (1.31)$$

For any $\tau_0 > 0$, there exist $\alpha, \beta > 0$, $M_0 > 0$, $K > 0$, for which, when s belongs to the part of the vertical strip $|\Re s - 1| \leq \beta$, with $\tau := \Im s$ satisfying $|\tau| > \tau_0 > 0$, the norm $(1, \tau)$ of the n -th iterate of any operator $\mathbb{G}_{M,s} \in \mathcal{G}_{M,s}$ satisfies, for $M \geq M_0$

$$\|\mathbb{G}_{M,s}^n\|_{(1,\tau)} \leq K \cdot \gamma^n \cdot |\tau|^\alpha, \quad \text{for } n \geq 1.$$

1.8.5 Basic properties of the ζ function.

The weighted operator $\mathbf{H}_{M,s}^{(\tau)}$, used to study the “middle” digit, is quite close (when $\tau(m) = m^e$) to the (possibly truncated) Riemann zeta function $\zeta_M(s - e)$. The plain Riemann ζ function is analytic on $\Re s > 1$, has a pole at $s = 1$, with a residue equal

to 1. It is of polynomial growth (with respect to $\Im s$) when $\Re s$ is close to 1. For $M > \infty$ and for real $s < 1$ with $|s - 1| \log M \leq 1$, the truncated ζ function satisfies

$$\zeta_M(s) = \log M + \gamma + O(|1 - s| \log^2 M) .$$

1.9 Conclusion and open problems.

Our results precisely describe the pseudo-randomness of a random Kronecker sequence $\mathcal{K}_{\langle T \rangle}(\alpha)$ with five parameters, for two-distance truncations and in various probabilistic models (real versus rational, constrained or not constrained).

Extension to three-distance truncations. We only deal here with two-distance truncations. For any three-distance truncation, and for all the four parameters, *except the discrepancy*, there exist general formulae (of the same type as the present ones) which express the parameter $X_{\langle T \rangle}$ as a function of the truncation T and the four continuants $q_k, q_{k-1}, \theta_k, \theta_{k+1}$ (for a truncation of index k). Any three-distance truncation, may be described with two positions: the principal position μ , already used here, and another (auxiliary) position. This is why a similar study can be conducted in this case, with similar expected results, (and heavier computations).

The special case of the discrepancy. The situation is completely different for the discrepancy. For three-distance truncations, there does not exist a formula which expresses $\Delta_{\langle T \rangle}$ for a truncation of index k solely with $T, q_k, q_{k-1}, \theta_k, \theta_{k+1}$. However, the paper (Baxa and Schoissengeier, 1994) provides estimates that relate the discrepancy $\Delta_{\langle T \rangle}(\alpha)$ for an index k to the average $A_k := (1/k) \sum_{i=1}^k m_i$ of the first k digits in the continued fraction expansion of α . We are thus led to consider constrained models of another type, which deal with the sets $\mathcal{J}^{[M]}$ of real numbers for which each average A_k is bounded by some constant M . We may also consider their rational counterparts $\tilde{\Omega}_N^{[M]}$. In these models, we may expect a logarithmic behaviour for the mean discrepancy. In the paper (Cesaratto and Vallée, 2006), we already studied the set $\mathcal{J}^{[M]}$ and provided estimates for its Hausdorff dimension, with tools of dynamical analysis.

Quadratic numbers. Other particular Kronecker sequences $\mathcal{K}(\alpha)$, associated with quadratic irrational numbers α , are also interesting to study from our probabilistic point of view. Such sequences are usually classified as the most random, because of their “low” worst-case discrepancy. For dynamical analysis, there is a strong parallelism between rational and irrational quadratic numbers, as it is shown for instance in (Vallée:1998): we just replace the quasi-inverse $(I - \mathbf{H}_s)^{-1}$ by the zeta function of the dynamical system. It is then surely possible to conduct similar analyses among quadratic irrational numbers, with similar expected results.

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Notation index

$A_{\langle T \rangle}$ (Arnold measure), 10
 $V_{\langle T \rangle}$ (covered space), 8
 $D_{\langle T \rangle}, \Delta_{\langle T \rangle}$ (discrepancy), 9
 $\Gamma_{\langle T \rangle}$ (distances in a Kronecker sequence), 7
 σ_M (Hausdorff dimension), 15
 ν_M (Hausdorff measure), 15
 $\mathbf{H}_s$ (transfer operator), 3
 $\mathcal{K}(\alpha)$ (Kronecker sequence), 1, 4
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