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A quasi-linear time algorithm for computing modular polynomials in dimension 2

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Abstract

We propose to generalize the work of Régis Dupont for computing modular polynomials in dimension 2 to new invariants. We describe an algorithm to compute modular polynomials for invariants derived from theta constants and prove under some heuristics that this algorithm is quasi-linear in its output size. Some properties of the modular polynomials defined from quotients of theta constants are analyzed. We report on experiments with our implementation.

An isogeny is a morphism between two abelian varieties that is surjective and has a finite kernel. It is an important notion for the theorical study of abelian varieties, but also for cryptographic applications because it allows one to transfer the discrete logarithm problem from a variety, where the problem is difficult, to an isogenous variety, where it may be easier.

The computation of an isogeny could mean several things: given a maximal isotropic subgroup of the ℓ-torsion, to be able to compute the isogenous variety; to calculate the image of a point by an isogeny; to check if two abelian varieties are isogenous and if so compute an isogeny. But what interests us here is the computation of all the isogenous varieties (for a fixed degree) of a given variety and this can be done with modular polynomials.

These polynomials also have other applications. In dimension 1, they are the key (SEA) for speeding up the algorithm of Schoof for counting the number of points on an elliptic curve (see [12, 42]), for constructing elliptic curves with a known number of points by complex multiplication (see [2, 17, 45]) and for the computation of the endomorphism ring of elliptic curves (see [4]). They can be computed in quasi-linear time (see [6, 13]).

In dimension 2, these polynomials play the same role, but are harder to compute. They could also speed up the CRT-algorithm (see [11]) to compute class fields of degree 4 CM-fields, which would lead to faster algorithms to construct cryptographically secure Jacobians of hyperelliptic curves.

An algorithm to compute the modular polynomials in dimension 2 has been introduced by Régis Dupont (see [10]) in 2006. Using it, he managed to compute the polynomials parameterizing $(2,2)$-isogenies, but these polynomials are so big that he could not compute them for $(3,3)$-isogenies. This is due to the fact that he used Igusa invariants (see Definition 13).

In this article, we will present a generalization of his algorithm allowing one to compute modular polynomials with invariants f_1, f_2, f_3 derived from theta constants, for a congruence subgroup Γ of the symplectic group Γ_2. We present results with Streng invariants (see Definition 28) and quotients of theta constants. The algorithm proceeds by evaluation/interpolation: without knowing the polynomials we are still able to evaluate them on any values and if we do so on sufficiently many values, we can interpolate to recover the polynomials. The most important difference with the dimension 1 case is that we have to interpolate trivariate rational

functions rather than univariate polynomials. This adds difficulties in the evaluation step; we need to choose the arguments in which to evaluate in a specific way. For this, we will see that we need to be able to find Ω modulo Γ from $f_1(\Omega)$, $f_2(\Omega)$ and $f_3(\Omega)$.

To do that, we deduce from $f_1(\Omega)$, $f_2(\Omega)$ and $f_3(\Omega)$ the Igusa invariants $j_1(\Omega)$, $j_2(\Omega)$, $j_3(\Omega)$ and then we apply Mestre's algorithm to obtain a hyperelliptic curve with these Igusa invariants. Using Thomae's formula, numerical integration and the Borchardt mean, it is possible to obtain Ω modulo Γ_2, under Conjecture 23. Once we have Ω modulo Γ_2, we have to find Ω modulo Γ. This can be done thanks to the functional equation of the theta constants of Proposition 8. It remains to compute the products of Definition 27 which ends the evaluation step.

All the computation are done with multiprecision floating point numbers. Explicit bounds on the size of the coefficients of the modular polynomials are unknown in dimension 2 (this is already a hard problem in dimension 1). Thus our algorithm is heuristic. Under the heuristics and Conjecture 23, we have shown that our algorithm is quasi-linear (Theorem 34). In practice, we may double the precision until we manage to find a sufficient precision to compute the polynomials. We underline the fact that the computations have been done at high precision so that we required a fast algorithm to compute the theta constants. This algorithm uses the Borchardt mean.

We first applied the algorithm of Dupont with Streng invariants instead of Igusa invariants to compute modular polynomials. The invariants of Streng are equivalent to the Igusa ones in the sense that they describe the same moduli space up to birational equivalence (and indeed, there are formula to switch from ones to the others: see (5)). We managed to compute the modular polynomials parameterizing $(2,2)$- and $(3,3)$-isogenies. The reason why this was possible is that the use of Streng invariants produce much smaller polynomials in terms of degrees and sizes of the coefficients, and thus the precision of the computation is smaller (as similarly noticed by Streng in [44] for the computation of class polynomials). For example, the modular polynomials for $p = 2$ with Streng invariants fill 2.1 MB compared to 57 MB with Igusa invariants.

We have then applied our generalized algorithm to $b'_i(\Omega) = \frac{\theta_i(\Omega/2)}{\theta_0(\Omega/2)}$ for $i = 1, 2, 3$, which are modular functions for the group $\Gamma(2,4)$, and computed the modular polynomials with these invariants for $p = 3$, $p = 5$ and $p = 7$. As these polynomials fill respectively 175 KB, 200 MB and 29 GB, we have no hope of computing the modular polynomials for larger p. The polynomials found are much smaller than those with Igusa or Streng invariants. For comparison, for $p = 3$ they fill 175 KB while those with Streng invariants fill 890 MB. We explain this by the fact that these polynomials have symmetries (Theorem 44) and are sparse (Theorem 45).

The remainder of this article is organised as follows. In the first section, we recall the theory of abelian varieties, of theta constants and of modular polynomials. In the second section, we explain how to interpolate multivariate rational functions and we do a complexity analysis. This section is independent of the others. The algorithm to compute the modular polynomials is given in the third section and some applications of it are described in the fourth section. The fifth section is dedicated to a deeper examination of these polynomials (in particular the symmetries appearing and an interpretation of the denominators of the coefficients of these polynomials). In Section 6 we discuss our implementation of the algorithm and finally in Section 7 we give some examples of hyperelliptic curves over finite field with isogenous Jacobians that we managed to compute using our modular polynomials.

1 Theory of modular polynomials

The Siegel upper half-space $\mathcal{H}_g$ for dimension g is the set of $g \times g$ symmetric matrices over the complex numbers with positive definite imaginary part. It is a moduli space for principally polarized abelian varieties (see [3, Proposition 8.1.2]). Indeed, a principally polarized abelian variety is a torus $\mathbb{C}^g/(\Omega\mathbb{Z}^{2g} + \mathbb{Z}^{2g})$ for $\Omega \in \mathcal{H}_g$ (which is called a *period matrix*).

Let I_g denote the identity matrix of size g and $J = \left(\begin{smallmatrix} 0 & I_g \\ -I_g & 0 \end{smallmatrix}\right)$. We define the symplectic group of dimension $2g$ as $\mathrm{Sp}_{2g}(\mathbb{Z}) = \{\gamma \in \mathrm{Gl}_{2g}(\mathbb{Z}) : {}^t\gamma J \gamma = J\}$. It can easily be shown that if $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, then $\gamma \in \mathrm{Sp}_{2g}(\mathbb{Z})$ if and only if the following three equalities are verified

$$\begin{array}{rcl} {}^tAC = {}^tCA & & A\,{}^tB = B\,{}^tA \\ {}^tBD = {}^tDB & \text{which are equivalent to} & D\,{}^tC = C\,{}^tD \\ {}^tDA - {}^tBC = I_g & & A\,{}^tD - B\,{}^tC = I_g \end{array} \tag{1}$$

because $\mathrm{Sp}_{2g}(\mathbb{Z})$ is closed under transposition of matrices. Moreover, $\mathrm{Sp}_{2g}(\mathbb{Z})$ acts (from the left) on $\mathcal{H}_g$ by $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)\Omega = (A\Omega + B)(C\Omega + D)^{-1}$, the matrix $C\Omega + D$ being invertible for $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_{2g}(\mathbb{Z})$ and $\Omega \in \mathcal{H}_g$.

The quotient space $\mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathcal{H}_g$ is a moduli space for isomorphism classes of principally polarized abelian varieties of dimension g (see [3, Theorem 8.2.6]).

We define $\Gamma_g = \mathrm{Sp}_{2g}(\mathbb{Z})$. Note that $-I_{2g}$ acts trivially on $\mathcal{H}_g$, so that some authors prefer to consider the projective symplectic group.

Proposition 1. *The group Γ_g is generated by J and the $\frac{g(g+1)}{2}$ matrices*

$$M_{i,j} = \begin{pmatrix} I_g & m_{i,j} \\ 0 & I_g, \end{pmatrix}$$

where $m_{i,j}$ is the matrix of size g all entries of which are 0 except for those at (i,j) and (j,i), which are equal to 1.

Proof. This is a direct consequence of [31, Proposition 6, pages 41–42]. □

Let $\mathcal{F}_g \subseteq \mathcal{H}_g$ be such that $\Omega = (\Omega_{u,v})_{u,v\in[1,g]}$ is in $\mathcal{F}_g$ if and only if Ω verifies

1. $|\Re(\Omega_{u,v})| \leq \frac{1}{2}$ for each $u, v \in [1, g]$;
2. the matrix $\Im(\Omega)$ is reduced in the sense of Minkowski (see [31, Chapter I.2] for the definition);
3. for each $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_g$, $|\det(C\Omega + D)| \geq 1$.

Remark 2. The third point has in principle to be verified for each matrix of Γ_g. However for the dimensions 1 and 2 this condition has to be verified only for a well-known finite set (of cardinality 1 in dimension 1 and 19 in dimension 2, see [31, Proposition 3, p.33] and [22]).

The set $\mathcal{F}_g$ is a *fundamental domain* in the sense that for all $\Omega \in \mathcal{H}_g$, there exists $\gamma \in \Gamma_g$ such that $\gamma\Omega \in \mathcal{F}_g$ and γ is unique if $\gamma\Omega$ is an inner point of $\mathcal{F}_g$.

Definition 3. *Let Γ be a subgroup of finite index of Γ_g and $k \in \mathbb{Z}$. A* Siegel modular form *of weight k for Γ is a function $f : \mathcal{H}_g \to \mathbb{C}$ such that*

1. *f is holomorphic on $\mathcal{H}_g$;*
2. *$\forall \gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$ and $\Omega \in \mathcal{H}_g$, $f(\gamma\Omega) = \det(C\Omega + D)^k f(\Omega)$;*
3. *in the case $g = 1$, f has to be holomorphic at the cusps (see for example [40, Definition 2.3.1] for the definition).*

Definition 4. *Let Γ a subgroup of finite index of Γ_g. A function $f : \mathcal{H}_g \to \mathbb{C}$ is a* Siegel modular function *for Γ if and only if there are two Siegel modular forms f_1 and f_2 for Γ of the same weight and such that $f = \frac{f_1}{f_2}$.*

From another point of view, a complex torus is an abelian variety if and only if it can be embedded into a projective space. This embedding can be done using theta functions. We will only focus on the classical theta functions because they provide a projective coordinate system for the principally polarized abelian varieties and because these functions can easily be handled computationally.

Definition 5. *Let $\Omega \in \mathcal{H}_g$ and let z be a vector of $\mathbb{C}^g$. The Riemann theta function is the function*

$$\theta : \mathbb{C}^g \times \mathcal{H}_g \to \mathbb{C}, \quad (z, \Omega) \mapsto \sum_{n \in \mathbb{Z}^g} \exp(i\pi\, {}^t n \Omega n + 2i\pi\, {}^t n z)$$

and for $a, b \in \mathbb{Q}^g$, the (classical) theta function with characteristic (a, b) is

$$\begin{aligned} \theta \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](z, \Omega) &= \textstyle\sum_{n \in \mathbb{Z}^g} \exp(i\pi\, {}^t(n+a)\Omega(n+a) + 2i\pi\, {}^t(n+a)(z+b)) \\ &= \exp(i\pi\, {}^t a \Omega a + 2i\pi\, {}^t a(z+b))\theta(z + \Omega a + b, \Omega). \end{aligned}$$

These functions converge absolutely and uniformly on every compact subset of $\mathbb{C}^g \times \mathcal{H}_g$ due to the fact that the imaginary part of Ω is positive definite.

Proposition 6. *Let $\Omega \in \mathcal{H}_g$ and $n \in \mathbb{N}$. The functions $f : \mathbb{C}^g \to \mathbb{C}$ satisfying for each $z \in \mathbb{C}^g$ and each $m', m'' \in \mathbb{Z}^g$,*

$$f(z + \Omega m' + m'') = f(z) \exp(-i\pi n\, {}^t m' \Omega m' - 2i\pi n\, {}^t z m')$$

are said to be functions of level n. They form a vector space of dimension n^g denoted by R_n^Ω.

Proof. See [37, Section II.1]. □

Theorem 7 (Lefschetz)**.** *For $n \geq 3$, any linearly independent set of $k \geq n^g$ functions of level n provides an embedding of $\mathbb{C}^g/(\Omega\mathbb{Z}^{2g} + \mathbb{Z}^{2g})$ into $\mathbb{P}^{k-1}(\mathbb{C})$. For $n = 2$, the functions of level 2 map only to $(\mathbb{C}^g/(\Omega\mathbb{Z}^{2g} + \mathbb{Z}^{2g}))/\sim$, where $\sim$ is the equivalence relation such that $z \sim -z$.*

Proof. See [36, Page 29]. □

Several bases for R_n^Ω are well-known. We use $\mathscr{F}_n = \left\{\theta \left[\begin{smallmatrix} 0 \\ b \end{smallmatrix}\right](z, \frac{\Omega}{n}), b \in \frac{1}{n}\mathbb{Z}^g/\mathbb{Z}^g\right\}$, and $\mathscr{F}_{(n,1)^n} = \left\{\theta \left[\begin{smallmatrix} 0 \\ b \end{smallmatrix}\right](z, \Omega)^n, b \in \frac{1}{n}\mathbb{Z}^g/\mathbb{Z}^g\right\}$. Theta functions have a lot of properties. The following two are useful for an implementation of these functions. For a matrix X, denote by X_0 the vector composed of the diagonal entries of X.

Proposition 8. *Let* $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_g$, $e' = \frac{1}{2}({}^tAC)_0$ *and* $e'' = \frac{1}{2}({}^tDB)_0$. *Then for all vectors* a, b *in* $\mathbb{Q}^g$, z *in* $\mathbb{C}^g$ *and* Ω *in* $\mathcal{H}_g$ *we have*

$$\begin{aligned}\theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](\gamma z, \gamma\Omega) = \quad & \zeta_\gamma \sqrt{\det(C\Omega + D)} \exp\left(i\pi\, {}^tz(C\Omega + D)^{-1}Cz\right) \theta\left[{}^t\gamma \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) + \left(\begin{smallmatrix} e' \\ e'' \end{smallmatrix}\right)\right](z, \Omega) \\ & \cdot \exp(-i\pi\, {}^taA\, {}^tBa) \exp(-i\pi\, {}^tbC\, {}^tDb) \exp(-2i\pi\, {}^taB\, {}^tCb) \\ & \cdot \exp(-2i\pi\, {}^t({}^tAa + {}^tCb + e')e'')\end{aligned}$$

where ζ_γ *is an eighth root of unity depending only on* γ.

Proof. See [30, Chapter 5, Theorem 2] or [8, Proposition 3.1.24]. □

Remark 9. The eighth root of unity and the square root do not depend on the characteristic. As we will always consider quotients of theta functions, we do not need to know the root and the determination of the square root.

Note that a matrix acts on the characteristic. The next proposition ([37, Page 123]) allows us to speak about permutations.

Proposition 10. *For each* $\alpha, \beta \in \mathbb{Z}^g$ *and* $a, b \in \mathbb{Q}^g$, $\theta\left[\begin{smallmatrix} a+\alpha \\ b+\beta \end{smallmatrix}\right](z, \Omega) = \exp(2i\pi\, {}^ta\beta)\theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](z, \Omega)$.

The *theta constants* of level n are the theta functions of level n evaluated at $z = 0$. In the following, we will focus on the theta constants of genus 2 with characteristic in $\{0, \frac{1}{2}\}^2$.

To simplify the notation we define for all $a = \left(\begin{smallmatrix} a_0 \\ a_1 \end{smallmatrix}\right)$ and $b = \left(\begin{smallmatrix} b_0 \\ b_1 \end{smallmatrix}\right)$ in $\{0, 1\}^2$

$$\theta_{b_0+2b_1+4a_0+8a_1}(\Omega) := \theta_{a,b}(\Omega) := \theta\left[\begin{smallmatrix} a/2 \\ b/2 \end{smallmatrix}\right](0, \Omega).$$

We have the property that $\theta_{a,b}(\Omega) = (-1)^{{}^tab}\theta_{a,b}(\Omega)$ so that of the 16 theta constants, 6 are identically zero (we say that they are odd) and we denote $\mathcal{P} = \{0, 1, 2, 3, 4, 6, 8, 9, 12, 15\}$ the subscripts of the even theta constants.

The next proposition ([30, Chapter IV, Theorem 1]) establishes a relation between the $\theta_i^2(\Omega)$ for $i = 0, \ldots, 15$ and the $\theta_i(\Omega/2)$ for $i = 0, \ldots, 3$.

Proposition 11 (Duplication formula)**.** *For all* $a, b \in \{0, 1\}^2$ *and* $\Omega \in \mathcal{H}_2$, *we have*

$$\theta_{a,b}^2(\Omega) = \frac{1}{4} \sum_{b_1+b_2 \equiv b \bmod 2} (-1)^{{}^tab_1} \theta_{0,b_1}(\Omega/2)\theta_{0,b_2}(\Omega/2).$$

Proposition 12. *Let* $\Omega \in \mathcal{H}_2$ *and* $\Omega' \in \mathcal{F}_2$ *be in the same class for the action of* Γ_2. *Then either the matrix* Ω' *is diagonal and then exactly one of the even theta constants evaluated in* Ω *vanishes and at the same time* $\theta_{15}(\Omega') = 0$, *or* Ω' *is not diagonal and no even theta constant vanishes in* Ω *(nor in* Ω'*).*

Proof. See [10, Proposition 6.5 and Corollary 6.1]. □

Let

$$h_4 = \sum_{i \in \mathcal{P}} \theta_i^8, \quad h_6 = \sum_{60 \text{ triples } (i,j,k) \in \mathcal{P}^3} \pm(\theta_i\theta_j\theta_k)^4,$$

$$h_{10} = \prod_{i \in \mathcal{P}} \theta_i^2, \quad h_{12} = \sum_{15 \text{ tuples } (i,j,k,l,m,n) \in \mathcal{P}^6} (\theta_i\theta_j\theta_k\theta_l\theta_m\theta_n)^4,$$

$$\text{and} \quad h_{16} = \frac{1}{3}(h_{12}h_4 - 2h_6h_{10}).$$

(see for example [44, 10, 50] for the exact definition). The functional equation of Proposition 8 shows that h_i is a Siegel modular form of weight i for the group Γ_2.

Definition 13. *We call Igusa invariants or j-invariants the functions j_1, j_2, j_3 defined by*

$$j_1 := \frac{h_{12}^5}{h_{10}^6}, \quad j_2 := \frac{h_4 h_{12}^3}{h_{10}^4}, \quad j_3 := \frac{h_{16} h_{12}^2}{h_{10}^4}.$$

Theorem 14. *The field K of Siegel modular functions in dimension 2 is $\mathbb{C}(j_1, j_2, j_3)$.*

Proof. See [29]. □

Generically, by [28], two principally polarized abelian surfaces are isomorphic if and only if they have the same j-invariants.

Let Γ be a subgroup of Γ_2 of index k. Denote by $\mathbb{C}_\Gamma$ the field of meromorphic functions of $\mathcal{H}_2$ invariant under the action of Γ (it is the function field of $\Gamma\backslash\mathcal{H}_2$). In particular, $\mathbb{C}_{\Gamma_2} = K$. By [20], $\mathbb{C}_\Gamma$ is a finite algebraic extension of degree k of $\mathbb{C}_{\Gamma_2}$.

Let f be a modular function, $\gamma \in \Gamma_2$ and p a prime number. We define the matrix $\gamma_p := \left(\begin{smallmatrix} A & pB \\ C/p & D \end{smallmatrix}\right)$ and the functions f^γ, f_p and f_p^γ from $\mathcal{H}_2 \to \mathbb{C}$ by $f^\gamma(\Omega) = f(\gamma\Omega)$, $f_p(\Omega) = f(p\Omega)$ and $f_p^\gamma(\Omega) = f(p\gamma\Omega)$ respectively. Let $\Gamma_0(p) := \left\{ \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_2 : C \equiv 0 \bmod p \right\}$.

The three functions $j_{\ell,p} := (j_\ell)_p$ are invariant under the group $\Gamma_0(p)$. Indeed, if $\gamma \in \Gamma_0(p)$, then $p\gamma\Omega = \gamma_p(p\Omega)$ so that $j_{\ell,p}^\gamma(\Omega) = j_\ell(p\gamma\Omega) = j_\ell(\gamma_p(p\Omega)) = j_\ell(p\Omega) = j_{\ell,p}(\Omega)$. In other words, Ω is equivalent to $\gamma\Omega$ for $\gamma \in \Gamma_2$, but that does not mean that $p\Omega$ is equivalent to $p\gamma\Omega$: it is the case only if γ is in $\Gamma_0(p)$.

Let C_p be a set of representatives of the quotient $\Gamma_2/\Gamma_0(p)$. The period matrices of the (p,p)-isogenous varieties of a variety Ω are the $p\gamma\Omega$ for $\gamma \in C_p$ (by Theorem 3.2 of [5]).

Proposition 10.1 of [10] gives C_p for each p and it tells us that $[\Gamma_2 : \Gamma_0(p)] = p^3 + p^2 + p + 1$.

Lemma 15. *For a prime p, $\mathbb{C}_{\Gamma_0(p)}$ equals $K(j_{\ell,p})$ for $\ell = 1, 2, 3$.*

Proof. See [5, Lemma 4.2]. □

Note that the functions j_ℓ have poles at $\Omega \in \mathcal{H}_2$ such that $h_{10}(\Omega) = 0$. This happens when $\theta_i(\Omega) = 0$ for some i. By Proposition 12, if $\Omega' \in \mathcal{F}_2$ is equivalent to Ω, then Ω' is diagonal. We deduce that Ω corresponds to a product of elliptic curves. So the functions $j_{\ell,p}$ have poles at $\Omega \in \mathcal{H}_2$ corresponding to varieties that are (p,p)-isogenous to a product of elliptic curves.

Let $\Phi_{1,p}(X) = \prod_{\gamma \in C_p}(X - j_{1,p}^\gamma)$. It is the minimal polynomial of $j_{1,p}$ over K. As the functions $j_{2,p}$ and $j_{3,p}$ are contained in $K(j_{1,p}) = K[j_{1,p}]$ by Lemma 15, we define $\Phi_{2,p}(X)$, and $\Phi_{3,p}(X)$ to be the monic polynomials in $K[X]$ of degree less than $\deg(\Phi_{1,p}(X))$ satisfying $j_{2,p} = \Phi_{2,p}(j_{1,p})$ and $j_{3,p} = \Phi_{3,p}(j_{1,p})$.

Furthermore we have for $\ell = 2, 3$ that $\Phi_{\ell,p}(j_{1,p}) = \Psi_{\ell,p}(j_{1,p})/\Phi'_{1,p}(j_{1,p})$ where

$$\Psi_{\ell,p}(X) = \sum_{\gamma \in C_p} j_{\ell,p}^\gamma \prod_{\gamma' \in C_p \backslash \{\gamma\}} (X - j_{1,p}^{\gamma'}).$$

Definition 16. *Let p be a prime number. We call $\Phi_{1,p}(X)$, $\Psi_{2,p}(X)$ and $\Psi_{3,p}(X)$ the modular polynomials for j_1, j_2 and j_3.*

For any prime p, the modular polynomials $\Phi_{1,p}(X)$, $\Phi_{2,p}(X)$, $\Phi_{3,p}(X)$ lie in the ring $\mathbb{Q}(j_1, j_2, j_3)[X]$ (see [5, Theorem 5.2]). This is also the case for $\Psi_{\ell,p}(X)$ for $\ell = 2, 3$ so that we will focus on $\Phi_{1,p}(X)$, $\Psi_{2,p}(X)$ and $\Psi_{3,p}(X)$. The evaluation map $\mathbb{C}(j_1, j_2, j_3) \to \mathbb{C}$ sending j_i to $j_i(\Omega)$ maps these polynomials to polynomials in $\mathbb{C}[X]$. The meaning of $\Phi_{1,p}(X)$ is that its roots evaluated at $\Omega \in \mathcal{H}_2$ are the j_1-invariants of the principally polarized abelian surfaces that are (p,p)-isogenous to the variety Ω. Moreover, if x is such a root, then $(x, \Phi_{2,p}(x), \Phi_{3,p}(x))$ are the j-invariants of a principally polarized abelian surface (p,p)-isogenous to a variety with invariants $(j_1(\Omega), j_2(\Omega), j_3(\Omega))$.

Denote by $\mathcal{L}_p$ the locus of all the principally polarized abelian surfaces which are (p,p)-isogenous to a product of elliptic curves. This locus $\mathcal{L}_p$ is a 2-dimensional algebraic subvariety of the 3-dimensional moduli space $\Gamma_2 \backslash \mathcal{H}_2$ and can be parameterized by an equation $L_p = 0$ for a polynomial L_p in $\mathbb{Q}[j_1, j_2, j_3]$.

Lemma 17. *The denominators of the coefficients of $\Phi_{1,p}(X)$, $\Psi_{2,p}(X)$ and $\Psi_{3,p}(X)$ are all divisible by the polynomial L_p.*

Proof. See [5, Lemma 6.2]. □

We are particularly interested in the denominators of the modular polynomials because they are at the cause of many difficulties when compute these poynomials.

2 Interpolation

We explain in this section how to interpolate multivariate polynomials and rational fractions, which will be needed to compute modular polynomials by evaluation and interpolation. The problem is the following: we assume that we have an algorithm f such that for any $x_1, \ldots, x_n \in \mathbb{C}$ it returns the value $P(x_1, \ldots, x_n)$ (or a floating point approximation thereof), where P is an unknown multivariate polynomial or rational fraction $P(X_1, \ldots, X_n)$ with complex coefficients. We want to find P.

We denote $\mathcal{M}(d)$ the time to multiply polynomials of degree less than or equal to d with coefficients having N bits and $\mathcal{M}'(N)$ the time complexity to multiply two integers of N bits. By [48, Corollary 8.19], we have that $\mathcal{M}(d) \in O(d \log d \, \mathcal{M}'(N))$ if we use the FFT and if we assume that $N \in \Omega(\log d)$, which is necessary to distinguish between the different roots of unity used in the FFT. Moreover, $\mathcal{M}'(N) \in O(N \log N \log\log N)$ (see [43]).

Following the basic idea of [10], we work out all the details and give a complexity analysis.

2.1 Interpolation of a multivariate polynomial

The problem of interpolating a univariate polynomial P is well-known and can be solved by Lagrange's or Newton's method, which need $\deg(P) + 1$ evaluations. The complexity of fast interpolation is $O(\mathcal{M}(\deg(P)) \log(\deg(P)))$ (see [48, Section 10, Corollary 10.12]).

In the case of a bivariate polynomial $P(X, Y)$, we notice that it can be written in the following way

$$P(X,Y) = \sum_{i=0}^{d_X} \left(\sum_{j=0}^{d_Y} c_{i,j} Y^j \right) X^i = \sum_{i=0}^{d_X} c_i(Y) X^i.$$

We can compute $P(X, y)$ for a fixed y by evaluating $P(x_i, y)$ for $i = 1, \ldots, d_X + 1$ and interpolating. The ℓ-th coefficient of this polynomial is $c_\ell(y)$, which is a univariate polynomial. It can be obtained if one has computed $c_\ell(y_j)$ for $d_Y + 1$ values y_j.

Thus to obtain $P(X, Y)$ we proceed as follows. For j from 1 to $d_Y + 1$, fix y_j and choose $d_X + 1$ values x_i (one may choose the same values for different j), then evaluate $P(x_i, y_j)$ and interpolate a univariate polynomial to find $P(X, y_j)$. Finally, for each $\ell = 0, \ldots, d_X$, interpolate $c_\ell(Y)$. Hence interpolating a bivariate polynomial needs $(d_Y + 1)(d_X + 1)$ evaluations and the complexity for the interpolation is

$$(d_Y + 1)O(\mathcal{M}(d_X)\log(d_X)) + (d_X + 1)O(\mathcal{M}(d_Y)\log(d_Y)) \subseteq \tilde{O}(d_X d_Y N).$$

The interpolation of a trivariate polynomial can be done in a similar way. We write it as

$$P(X, Y, Z) = \sum_{i=0}^{d_X} \left(\sum_{j=0}^{d_Y} \left(\sum_{k=0}^{d_Z} c_{i,j,k} Z^k \right) Y^j \right) X^i = \sum_{i=0}^{d_X} \left(\sum_{j=0}^{d_Y} c_{i,j}(Z) Y^j \right) X^i = \sum_{i=0}^{d_X} c_i(Y, Z) X^i.$$

If, for fixed y and z, we evaluate $P(x_i, y, z)$, $i = 1, \ldots, d_X + 1$, and then interpolate, we obtain $P(X, y, z)$ and the ℓ-th coefficient is $c_\ell(y, z)$. This gives us a method to evaluate $c_\ell(Y, Z)$ for different values and we can use what we said about bivariate polynomials to find $c_\ell(Y, Z)$.

Thus we proceed as follows. For j from 1 to $d_Y + 1$ and for k from 1 to $d_Z + 1$ we evaluate $P(x_i, y_j, z_k)$ for $d_X + 1$ values x_i and then we do $(d_Y + 1)(d_Z + 1)$ interpolations to obtain all the $P(X, y_j, z_k)$. Each of the $d_X + 1$ coefficients $c_\ell(Y, Z)$ is a bivariate polynomial and we have given above the complexity to obtain it. All in all, we will do $(d_X + 1)(d_Y + 1)(d_Z + 1)$ evaluations and the complexity for the interpolation is

$$(d_Y + 1)(d_Z + 1)O(\mathcal{M}(d_X)\log(d_X)) + (d_X + 1)(d_Z + 1)O(\mathcal{M}(d_Y)\log(d_Y)) +$$

$$(d_X + 1)(d_Y + 1)O(\mathcal{M}(d_Z)\log(d_Z)) \subseteq \tilde{O}(d_X d_Y d_Z N).$$

We can generalise this improved algorithm recursively to the case of a polynomial in n variables $X_1, \ldots, X_n$. It takes $\prod_{i=1}^{n}(d_{X_i} + 1)$ evaluations. The complexity for the interpolation of a polynomial in n variables is

$$\sum_{i=1}^{n} \prod_{\substack{j=1 \\ j \neq i}}^{n} (d_{X_j} + 1) O(\mathcal{M}(d_{X_i}) \log d_{X_i}) \subseteq \tilde{O}\left(\prod_{i=1}^{n} d_{X_i} N \right)$$

Note the symmetry which means that the ordering of the variables does not matter.

2.2 Interpolation of a multivariate rational fraction

We begin with the univariate case: $F(X) = \frac{A(X)}{B(X)}$, with $A(X) = \sum_{i=0}^{d_X^A} A_i X^i \in \mathbb{C}[X]$ and $B(X) = \sum_{i=0}^{d_X^B} B_i X^i \in \mathbb{C}[X]$. We look for the solution with minimal degrees. Each pair (A, B) is then defined only up to a multiplicative constant.

Let $n = d_X^A + d_X^B + 1$. Writing $A(X) - F(X)B(X) = 0$ induces us to proceed with linear algebra: it suffices to evaluate F in $n + 1$ values x_i and to find the coefficients A_i and B_i by

solving the following linear system

$$\begin{pmatrix} 1 & x_1 & \dots & x_1^{d_X^A} & -F(x_1) & -F(x_1)\cdot x_1 & \dots & -F(x_1)\cdot x_1^{d_X^B} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \dots & x_n^{d_X^A} & -F(x_n) & -F(x_n)\cdot x_n & \dots & -F(x_n)\cdot x_n^{d_X^B} \end{pmatrix} \cdot \begin{pmatrix} A_0 \\ \vdots \\ A_{d_X^A} \\ B_0 \\ \vdots \\ B_{d_X^B} \end{pmatrix} = \begin{pmatrix} 0 \\ \\ \vdots \\ \\ 0 \end{pmatrix}$$

This method is easy to implement, but its complexity is bad. Another solution consists in using Cauchy interpolation (see [48, Section 5.8]) with the fast Euclidean algorithm ([48, Section 11]), which produces an algorithm of complexity $O(\mathcal{M}(n)\log(n))$. The number of evaluations is n.

We explain it briefly. Let k and m be such that $\deg A < k$ and $\deg B \leq m-k$. Take $x_1, \dots, x_m \in \mathbb{C}$ and $y_i = F(x_i)$ for $1 \leq i \leq m$. Let f be an interpolating polynomial. We look at polynomials $r(X)$ and $t(X)$ such that for any i, $r(x_i) = t(x_i)F(x_i)$; this implies ($r(x_i) = t(x_i)y_i = t(x_i)f(x_i)$ if and only if $r \equiv tf \bmod (X - x_i)$ for all i) and by the Chinese remainder theorem, it is equivalent to ask that $r \equiv tf \bmod g$, where $g = \prod_{i=1}^{m}(X - x_i)$.

We use then the extended euclidean algorithm on g and f. Let r_j, s_j, t_j be the j-th row of the algorithm, where j is minimal such that $\deg r_j < k$ (namely $r_1 = g, r_2 = f, s_1 = 1, s_2 = 0, t_1 = 0, t_2 = 1$ and $r_\ell = gs_\ell + ft_\ell$ for each row ℓ). By Corollary 5.18 of [48, Section 5.8], r_j and t_j verify $r_j(x_i) = t_j(x_i)y_i$ and $t_j(x_i) \neq 0$ for all i.

Thus, it suffices to compute this row to interpolate the fraction F. It is possible to compute a single row with the fast Euclidean algorithm.

We study now the bivariate case $F(X,Y) = \frac{A(X,Y)}{B(X,Y)}$, with

$$A(X,Y) = \sum_{i=0}^{d_X^A}\sum_{j=0}^{d_Y^A} c_{i,j}^A X^i Y^j = \sum_{i=0}^{d_X^A} c_i^A(Y) X^i \in \mathbb{C}[X,Y]$$

and similarly for $B(X,Y)$. One could use linear algebra, but the complexity would be very bad. Thus we would like to proceed as in the bivariate case for polynomials, namely by fixing values y_j and computing the fractions $F(X, y_j)$ and then by interpolating the coefficients as polynomials in Y.

If $F(X,Y) \in \mathbb{Q}[X,Y]$, then for each rational fraction found, one can force the numerator and the denominator to have content 1, but because of the multiplicative constant, this will not work, as shown by the next example.

Example 18. Assume that we are searching $F(X,Y) = \frac{3X^2Y^2+Y+2}{3XY+3}$ and that we find $F(X,1) = \frac{X^2+1}{X+1}$, $F(X,2) = \frac{12X^2+4}{6X+3}$, $F(X,3) = \frac{27X^2+5}{9X+3}$ and also $F(X,5) = \frac{75X^2+7}{15X+3}$. For the greater coefficient of the numerator $c_2^A(Y)$: if we interpolate with $y_i = 2,3,5$, which gives us $c_2^A(y_i) = 12, 27, 75$, we find $3Y^2$ which is correct; on the other side, with $y_i = 1,2,3$ giving $c_2(y_i) = 1, 12, 27$, we obtain the polynomial $2Y^2 + 5Y - 6$, which is completely wrong, because of the simplification between the numerator and the denominator in the case $Y = 1$.

Remark 19. For some values of Y, there is a simplification by a polynomial; for example $F(X,-5) = -5X - 1$. This is something that is immediately noticed because the degrees in X are smaller. We assume that every time it happens, we drop the instance.

In the example, if we had fixed the coefficient of degree 0 of the denominator of the univariate rational fractions computed at some value (3 for instance), the simplification would not have been a problem. This is not true in general.

Example 20. This time write $F(X,1) = \frac{X^2+1}{X+1}$, $F(X,2) = \frac{3X^2+1}{\frac{3}{2}X+\frac{3}{4}}$, $F(X,3) = \frac{\frac{27}{5}X^2+1}{\frac{9}{5}X+\frac{3}{5}}$, ..., where we always fix the coefficient of degree 0 of the numerator at 1. Then we deduce by interpolation that $c_0^A(Y) = 1$, which is wrong. Indeed, when we divide by a constant to obtain the coefficient 1, it is as if we had divided by $Y+2$.

Thus the difficulty is that we have to normalize while being sure that the i-th coefficient of the numerator and the denominator of each fraction in X comes from the evaluation of the same polynomial in Y.

This normalization is easy to obtain in the very particuliar case where we already know one of the $c_i^A(Y)$ or $c_i^B(Y) \in \mathbb{C}[Y]$ (different from 0): we only have to multiply the solution found by the constant which gives us the good evaluation for the known c_i. We can then obtain (using Cauchy interpolation) the fraction with $n(d_Y+1)$ evaluations and the complexity of the interpolation is

$$(d_Y+1)O(\mathcal{M}(n)\log(n)) + (n+1)O(\mathcal{M}(d_Y)\log(d_Y)) \subseteq \tilde{O}(d_X d_Y N)$$

where $d_X = \max(d_X^A, d_X^B)$, $d_Y = \max(d_Y^A, d_Y^B)$ and $n = d_X^A + d_X^B + 1 \leq 2d_X + 1$.

Example 21. We continue the preceding example. Assume we know $c_0^A(Y) = Y+2$. We have $c_0^A(1) = 3$ and instead of the fraction $\frac{X^2+1}{X+1}$ we take $\frac{3X^2+3}{3X+3}$. We also have $c_0^A(2) = 4$ and we write $F(X,2) = \frac{12X^2+4}{6X+3}$ and so on.

In general, an idea to avoid this difficulty is to consider the fraction $F'(X,Y) = F(X,YX) = \frac{A'(X,Y)}{B'(X,Y)}$ because in this case, $c_0^{B'}(Y)$ is a constant. If it is not 0, we can choose to fix it to be 1 and then the previous argument (one $c_i^{B'}(Y)$ known) holds. Thus we have $F(X,YX)$ and we substitute Y by Y/X to obtain $F(X,Y)$. Since $d_X^{A'} = d_T^A$ and $d_X^{B'} = d_T^B$, where the subscript T stands for the total degree, the complexity is $\tilde{O}(d_T d_Y N)$ (and $d_T = \max(d_T^A, d_T^B)$).

Note also that in the particular case where the coefficient of degree zero of $c_0^B(Y)$ is 0, this method does not work. To overcome this difficulty we can consider $F(X+r, Y+s)$ instead of $F(X,Y)$ for some values r and s such that this coefficient will not be zero.

We study now the trivariate case. We want to interpolate $F(X,Y,Z) = \frac{A(X,Y,Z)}{B(X,Y,Z)}$ with $A(X,Y,Z)$, $B(X,Y,Z)$ in $\mathbb{C}[X,Y,Z]$. Denote $d_T = \max(d_T^A, d_T^B)$ (and similarly for d_X, d_Y and d_Z) and $n = d_T^A + d_T^B + 1$. As in the bivariate case, we compute $F(X,XY,XZ)$ and then substitute Y by Y/X and Z by Z/X to obtain $F(X,Y,Z)$. We explain how to compute $F(X,XY,XZ)$ recursively:

1. Suppose we are able to compute $F(X,XY,zX)$ for a fixed $z \in \mathbb{C}$. Then we only need d_Z+1 evaluations in z_i to interpolate (as polynomials) each coefficient in Z and find $F(X,XY,XZ)$. The number of coefficients is bounded above by $(n+1)(d_Y+1)$ so that the interpolation complexity for this step is $(n+1)(d_Y+1)O(\mathcal{M}(d_Z)\log(d_Z))$.

2. To obtain $F(X,XY,zX)$ for a fixed z, it suffices to apply the interpolation algorithm in the bivariate case. We will do this step d_Z+1 times so that the complexity is $(d_Z+1)((d_Y+1)O(\mathcal{M}(n)\log(n)) + (n+1)O(\mathcal{M}(d_Y)\log(d_Y)))$.

In doing this, the number of evaluations will be $n(d_Y+1)(d_Z+1)$ and the final interpolation complexity is $\tilde{O}(d_T d_Y d_Z N)$. (In the special case where we already know one of the $c_i^A(Y,Z)$ or $c_i^B(Y,Z)$, the complexity will be $\tilde{O}(d_X d_Y d_Z N)$).

An improvement of this algorithm is obtained in noting that there is the possibility to substitute Y by Y/X in the second step to find $F(X,Y,zX)$ and to compute $F(X,Y,XZ)$ in the first one. Thus, the number of coefficients in the first step will be bounded above by $(n'+1)(d_Y+1)$ where n' is $\deg_X^A(F(X,Y,zX))+\deg_X^B(F(X,Y,zX))+1$, which is $\leq n$, which allows one to reduce the number of interpolations. The complexity is then

$$(d_Y+1)(d_Z+1)O(\mathcal{M}(n)\log(n))+(n+1)(d_Z+1)O(\mathcal{M}(d_Y)\log(d_Y))+$$

$$(n'+1)(d_Y+1)O(\mathcal{M}(d_Z)\log(d_Z))\subseteq\tilde{O}(d_T d_Y d_Z N).$$

We can generalize this recursively to the case of a rational fraction F with m variables $X_1,\ldots,X_m$. We find

$$\prod_{i=2}^{m}(d_{X_i}+1)O(\mathcal{M}(n)\log(n))+\sum_{j=2}^{m}\prod_{\substack{i=2\\i\neq j}}^{m}(d_{X_i}+1)n(j)O(\mathcal{M}(d_{X_j})\log(d_{X_j}))\subseteq\tilde{O}(d_T\prod_{i=2}^{m}d_{X_i}N)$$

where $n=d_T^A+d_T^B+1$ and $n(j)$ is one plus the degree in X_1 of the numerator plus the degree in X_1 of the denominator of $F(X_1,X_2,\ldots,X_{j-1},X_jX_1,\ldots,X_mX_1)$.

Note that all these formulae for the complexity in the case of rational fractions are asymmetric so that the choice of the order of the variables is important. The formulae suggest that it is preferable to take X_1 as the variable with the largest degree. In that case, $n\leq 6d_{X_1}+1$ and the complexity of the interpolation is then $\tilde{O}(\prod_{i=1}^{m}d_{X_i}N)$.

3 Evaluation

We have seen that the modular polynomials lie in the ring $\mathbb{Q}(j_1,j_2,j_3)[X]$ so that we have to interpolate trivariate rational fractions to compute them. Using the method of interpolation of a rational fraction F exposed in the preceding section requires one to evaluate it at the points $F(x_i,x_iy_j,x_iz_k)$, where there exists $\Omega\in\mathcal{H}_2$ such that $(j_1(\Omega),j_2(\Omega),j_3(\Omega))=(x_i,x_iy_j,x_iz_k)$. We present here the method presented in [10] to deduce a matrix $\Omega\in\mathcal{H}_2$ from its j-invariants and then we present a way to extend this algorithm for other invariants.

3.1 Computing modular polynomials with the j-invariants

In practice, the modular polynomials have large coefficients and degrees. We use floating point multiprecision to compute them and the letter N will designate this precision in bits. We have an input $(x,y,z)\in\mathbb{C}^3$ and we are looking for $\Omega\in\mathcal{H}_2$ such that $(j_1(\Omega),j_2(\Omega),j_3(\Omega))=(x,y,z)$.

The key to do this is to look at the Borchardt mean. Let $(z_k)_{k\in\{1,2,3\}}\in\mathbb{C}^3$. We define the Borchardt sequence for $k\in\{1,2,3\}$ by

$$u_0^{(0)}=1\qquad\text{and}\qquad u_k^{(0)}=z_k$$

and recursively for all $n\geq 0$

$$u_0^{(n+1)}=\frac{1}{4}\sum_{k=0}^{3}u_k^{(n)}\qquad\text{and}\qquad u_k^{(n+1)}=\frac{1}{4}\sum_{k_1+k_2\equiv k \bmod 4}v_{k_1}^{(n)}v_{k_2}^{(n)}$$

where $v_0^{(n)}$ is any square root of $u_0^{(n)}$ and $v_k^{(n)} = 0$ if $v_0^{(n)} = 0$ or $u_k^{(n)} = 0$, otherwise $v_k^{(n)}$ is the square root of $u_k^{(n)}$ such that $\left|v_0^{(n)} - v_k^{(n)}\right| \leq \left|v_0^{(n)} + v_k^{(n)}\right|$ and with $\Im\left(v_k^{(n)}/v_0^{(n)}\right) > 0$ if there is equality.

This sequence converges to a unique complex number called the Borchardt mean and denoted by $B_2((z_k)_{k\in\{1,2,3\}})$. Let $b_i(\Omega) := \theta_i^2(\Omega)/\theta_0^2(\Omega)$ for $1 \leq i \leq 15$. We have

Proposition 22 ([10], Proposition 9.1)**.** *For all $\tau \in \mathcal{F}_2$, $B_2(b_1(\tau), b_2(\tau), b_3(\tau)) = \frac{1}{\theta_0^2(\tau)}$.*

Note that from this proposition and the ten even $b_i(\Omega)$, we can deduce all the $\theta_i^2(\Omega) = b_i(\Omega)\theta_0^2(\Omega)$ at the working precision (with some loss of precision).

Conjecture 23 ([10], Conjecture 9.1)**.** *With the notation of Proposition 1 we have, for all $\tau \in \mathcal{F}_2$ and for all $\gamma \in \{(JM_{1,1})^2, (JM_{1,2})^2, (JM_{2,2})^2\}$: $B_2(b_1(\gamma\tau), b_2(\gamma\tau), b_3(\gamma\tau)) = \frac{1}{\theta_0^2(\gamma\tau)}$.*

If this conjecture is true, it can easily be shown that for $\tau = \left(\begin{smallmatrix} \tau_1 & \tau_3 \\ \tau_3 & \tau_2 \end{smallmatrix}\right)$

$$\tau_1 = \frac{\imath}{\theta_4^2(\tau) B_2\left(\frac{\theta_0^2(\tau)}{\theta_4^2(\tau)}, \frac{\theta_6^2(\tau)}{\theta_4^2(\tau)}, \frac{\theta_2^2(\tau)}{\theta_4^2(\tau)}\right)}, \tag{2}$$

$$\tau_2 = \frac{\imath}{\theta_8^2(\tau) B_2\left(\frac{\theta_9^2(\tau)}{\theta_8^2(\tau)}, \frac{\theta_0^2(\tau)}{\theta_8^2(\tau)}, \frac{\theta_1^2(\tau)}{\theta_8^2(\tau)}\right)}, \tag{3}$$

and

$$\tau_3^2 - \tau_1\tau_2 = \frac{1}{\theta_0^2(\tau) B_2\left(\frac{\theta_8^2(\tau)}{\theta_0^2(\tau)}, \frac{\theta_4^2(\tau)}{\theta_0^2(\tau)}, \frac{\theta_{12}^2(\tau)}{\theta_0^2(\tau)}\right)}. \tag{4}$$

If τ is in the fundamental domain, the Minkowski reduction implies that $\Im(\tau_3) \geq 0$ which allows us to extract the good square root and obtain τ. Thus, it remains to show how to deduce from $(x, y, z) \in \mathbb{C}^3$, the ten $b_i(\Omega)$ where $\Omega \in \mathcal{F}_2$ is such that $(j_1(\Omega), j_2(\Omega), j_3(\Omega)) = (x, y, z)$. This can be done in four steps.

1. The first one is to use Mestre's algorithm (see for example [34]) at precision N to find a genus 2 curve $Y^2 = f(X)$ over $\mathbb{C}$ with f having degree 6 whose Igusa invariants are (x, y, z).

2. Once we have f, it is easy to deduce the set E of roots of f at precision N and from this set we use Thomae's formula (see [46]). Recall that this formula allows one to obtain the fourth power of the theta constants from an ordering of the roots of f (which corresponds to a choice of the basis of the homology group of the Riemann surface of the hyperelliptic curve).

   The problem here is that the functions b_i are not invariant under the symplectic group Γ_2 (but for a subgroup as we will see later). This means that for two matrices equivalent under the action of Γ_2 (namely they have the same j-invariants), the evaluation of the b_i in these matrices produces differents results. Hence the theta constants found with Thomae's formula gives us $b_i^2(\gamma\Omega)$ for some unknown $\gamma \in \Gamma_2$.

3. Now use a numerical integration technique (see for example [9, 47, 35]) at low precision N' with the same choice of the basis of the homology group to find the period matrix $\gamma\Omega$ that we reduce into the fundamental domain to obtain Ω at precision N' and γ.

Compute $b_i(\Omega)$ at precision N' (with some algorithm to compute theta constants). We do not use a numerical integration technique at precision N because it is too slow and it would increase the complexity of the algorithm.

4. Using the functional equation of Proposition 8 on $b_i^2(\gamma\Omega)$ with the matrix γ^{-1} allows one to obtain the $b_i^2(\Omega)$ at precision N and knowing $b_i(\Omega)$ at precision N' is enough to deduce the good square root and obtain $b_i(\Omega)$ at precision N.

Assumption 24. Note that we make the assumption that the numerical integration technique provides some $\gamma\Omega$ with γ small enough such that it can be correctly reduced in the fundamental domain at the precision N'.

We thus obtain the following algorithm.

Algorithm 25: Ω from $(j_1(\Omega), j_2(\Omega), j_3(\Omega))$

Data: $(x, y, z) = (j_1(\Omega), j_2(\Omega), j_3(\Omega))$ for some unknown $\Omega \in \mathcal{F}_2$, the working precision N and a smaller precision N'

Result: Ω

1 Use Mestre's algorithm to obtain a hyperelliptic curve $Y^2 = f(X)$ at precision N;
2 Deduce the ten $b_i(\Omega)$ at precision N using some numerical integration technique at precision N';
3 Use Proposition 22 to obtain the square of the theta constants at the working precision;
4 Use (2), (3), (4) to compute Ω at precision N (with some loss).

The second step is Algorithm 12 of [10] and the third and fourth are Algorithm 13. They have complexity $O(\mathcal{M}'(N))$ and $O(\mathcal{M}'(N)\log(N))$ (where $\mathcal{M}'(N)$ is the time complexity to multiply two integers of N bits) so that the algorithm is in $\tilde{O}(N)$.

Remark 26. Starting from $(1, b_1(\gamma\tau), b_2(\gamma\tau), b_3(\gamma\tau))$ and an approximation or τ, it is possible to compute a Borchardt sequence where the squares roots are choosen to be the $\frac{\theta_i(2^n\gamma\tau)}{\theta_0(\gamma\tau)}$ at each step. Using the duplication formula (Proposition 11), it can easily be proved that this will converge to $\frac{1}{\theta_0^2(\gamma\tau)}$. Thus it should be possible to not rely on the conjecture while maintenaing the same complexity, as already stated in the first variant page 200 of [10] and in the remark after Theorem 12 of [18].

The conjecture has been tested and verified numerically by Dupont for many millions of random matrices. We underline that it is easy to test if the matrix Ω found at the end has the good j-invariants or not.

3.2 New invariants for the modular polynomials

We begin in giving a generalization of the modular polynomials in order to have the possibility to use other invariants. In genus 1, this goes back to the works of Schläfli and Weber (see [41] and [13, Section 4.2 and 4.3]).

We consider only the congruence subgroups $\Gamma \subseteq \Gamma_2$, namely the groups with $\Gamma(n) = \{M \in \Gamma_2 : M \equiv \pm Id_4 \bmod n\} \subseteq \Gamma$ for some n. If n is minimal with this property, we say that n is the level of Γ. Let Γ be a congruence subgroup and f_1, f_2, f_3 be three modular functions

which are generators for the function field of $\Gamma\backslash\mathcal{H}_2$. Let p be a prime number such that the level of Γ is prime to p. Let C_p be a set or representatives of $\Gamma/(\Gamma\cap\Gamma_0(p))$.

Definition 27. *The modular polynomials for these data are, for $\ell = 2, 3$,*

$$\Phi_{1,p}(X) = \prod_{\gamma\in C_p}(X - f_{1,p}^{\gamma}) \qquad \textit{and} \qquad \Psi_{\ell,p}(X) = \sum_{\gamma\in C_p} f_{\ell,p}^{\gamma} \prod_{\gamma'\in C_p\backslash\{\gamma\}}(X - f_{1,p}^{\gamma'}).$$

We will sometimes write $\Phi_{1,p}(X, f_1, f_2, f_3)$ instead of $\Phi_{1,p}(X)$ and similarly for $\Psi_{\ell,p}(X)$. While the interpolation phase is still the same, the evaluation is slightly different: this time we have to find $\Omega\in\mathcal{H}_2$ from a triple $(x_1, x_2, x_3)\in\mathbb{C}$ such that $f_i(\Omega) = x_i$. Of course, this step depends on the three functions, but we will still give a general algorithm. On the other side, the computation of $\Phi_{1,p}(X, f_1(\Omega), f_2(\Omega), f_3(\Omega))$ and of $\Psi_{\ell,p}(X, f_1(\Omega), f_2(\Omega), f_3(\Omega))$ for some Ω does not change. We can apply the same algorithm and they have the same complexity (except of course for the evaluation of the $f_i(p\gamma\Omega)$).

As in the dimension 1 case, we have tried to look at modular functions that would produce smaller modular polynomials than those with the j-invariants.

The first we tried are the invariants used by Streng in his thesis [44] to obtain smaller class polynomials. These invariants for Γ_2 are defined to have the minimal power of h_{10} in the denominators.

Definition 28. *We call Streng invariants the functions i_1, i_2, i_3 defined by*

$$i_1 := \frac{h_4h_6}{h_{10}}, \quad i_2 := \frac{h_4^2h_{12}}{h_{10}^2}, \quad i_3 := \frac{h_4^5}{h_{10}^2}.$$

We will also say that these are j-invariants. The context will make it clear if we are speaking of Streng invariants or of Igusa invariants. This is justified by the next theorem. Note that it is easy to deduce from the Igusa invariants the Streng ones and vice versa. Indeed, we have

$$i_1 = \frac{j_2(j_2 - 3j_3)}{2j_1}, \quad i_2 = \frac{j_2^2}{j_1}, \quad i_3 = \frac{j_2^5}{j_1^3} \quad \text{and} \quad j_1 = \frac{i_2^5}{i_3^2}, \quad j_2 = \frac{i_2^3}{i_3}, \quad j_3 = \frac{i_2^2(i_2 - 2i_1)}{3i_3}. \tag{5}$$

Theorem 29. *The field K of Siegel modular functions in dimension 2 is $K = \mathbb{C}(j_1, j_2, j_3) = \mathbb{C}(i_1, i_2, i_3)$.*

Moreover, we also have the properties for $\ell = 1, 2, 3$ and a prime p that the three $i_{\ell,p}$ are invariants under the group $\Gamma_0(p)$ and that $\mathbb{C}_{\Gamma_0(p)} = K(i_{\ell,p})$ (the proof is similar to the one for the invariants of Igusa).

Thus to compute the modular polynomials, the difference with the invariants of Igusa is small: from $(i_1(\Omega), i_2(\Omega), i_3(\Omega))$ to obtain Ω, it is sufficient to use (5) to deduce the triple $(j_1(\Omega), j_2(\Omega), j_3(\Omega))$ and then to use Algorithm 25. The computation of the $i_\ell(p\gamma\Omega)$ (for a prime p, $\ell = 1, 2, 3$ and $\gamma\in C_p$) is equivalent by (5) to the computation of the $j_\ell(p\gamma\Omega)$.

The modular polynomials with Streng invariants are much smaller in terms of degrees and precision of the coefficients than those with Igusa invariants so that the interpolation step can be done more rapidly and the number of times we use Algorithm 25 is also much smaller (see the next section).

Other invariants can be obtained by using the theta constants. This is motivated by the fact that the j-invariants are defined in terms of the theta constants.

Let $\Gamma(2,4) = \left\{ \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_2 : \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \equiv I_4 \bmod 2 \text{ and } B_0 \equiv C_0 \equiv 0 \bmod 4 \right\}$, which is a normal subgroup of Γ_2. It is well-known that the $b_i(\Omega) := \theta_i^2(\Omega)/\theta_0^2(\Omega)$ are modular functions for the group $\Gamma(2,4)$. Actually, Theorem 1 of [33] states that the field $\mathbb{C}_{\Gamma(2,4)}$ of modular functions belonging to $\Gamma(2,4)$ is $\mathbb{C}(b_1, ..., b_{15})$.

Define for $i = 1, 2, 3$ the functions $b'_i(\Omega) := \frac{\theta_i(\Omega/2)}{\theta_0(\Omega/2)}$. From these three functions, it is easy to deduce the ten b_i using the duplication formula (Proposition 11). The converse is also true because we have

$$\begin{aligned} b'_1 &= (b_1 + b_9)(1 + b_4 + b_8 + b_{12})^{-1}, \\ b'_2 &= (b_2 + b_6)(1 + b_4 + b_8 + b_{12})^{-1}, \\ b'_3 &= (b_3 + b_{15})(1 + b_4 + b_8 + b_{12})^{-1}. \end{aligned} \tag{6}$$

Thus we consider the b'_i which allows us to handle three generators instead of ten.

Proposition 30. *Let $p > 2$ be a prime number. The classes of $\Gamma(2,4)/(\Gamma_0(p) \cap \Gamma(2,4))$ are in bijection with the classes of $\Gamma_2/\Gamma_0(p)$.*

Proof. Consider the map $\phi : \Gamma(2,4) \to \Gamma_2/\Gamma_0(p)$ with kernel $\Gamma_0(p) \cap \Gamma(2,4)$. The surjectivity comes from the Chinese remainder theorem and the fact that $\mathrm{Sp}(4,\mathbb{Z}) \to \mathrm{Sp}(4,\mathbb{Z}/4p\mathbb{Z})$ is surjective (the proof of which is analogous to [32, Section 6.1]). □

Proposition 31. *For a prime $p > 2$, $\mathbb{C}_{\Gamma(2,4)\cap\Gamma_0(p)}$ equals $\mathbb{C}_{\Gamma(2,4)}(b'_{i,p})$ for every $i = 1, 2, 3$.*

Proof. The proof is similar to that of Theorem 4.2 of [5]. One has to use the isomorphism between $\Gamma(2,4)/(\Gamma(2,4)\cap\Gamma(p))$ and $\Gamma_2/\Gamma(p)$ which comes from the Chinese remainder theorem and the surjectivity of $\mathrm{Sp}(4,\mathbb{Z}) \to \mathrm{Sp}(4,\mathbb{Z}/4p\mathbb{Z})$. □

Proposition 32. *The modular polynomials for b'_1, b'_2 and b'_3 lie in the ring $\mathbb{Q}(b'_1, b'_2, b'_3)[X]$. More generally, it is also the case for any functions derived from the theta constants.*

Proof. This comes from the fact that, for any $\Omega = \left(\begin{smallmatrix} \Omega_1 & \Omega_2 \\ \Omega_2 & \Omega_3 \end{smallmatrix}\right)$, the functions b'_1, b'_2 and b'_3 have a Laurent series expansion in $q_i = \exp(2\imath\pi\Omega_i)$ with rational coefficients. We conclude with a similar proof of Theorem 5.2 of [5]. □

We want to deduce Ω from $(b'_1(\Omega), b'_2(\Omega), b'_3(\Omega)) = (x_1, x_2, x_3)$. The first thing to do is to deduce from (x_1, x_2, x_3) the Igusa invariants of Ω. This can be done easily by calculating the ten $b_i(\Omega)$ from the three x_i with the duplication formula and then by using the definition of the j-invariants (see Definition 13). We then execute Algorithm 25 to deduce a period matrix $\Omega' \in \mathcal{H}_2$ from the j-invariants. Unfortunately, this Ω' is equivalent to Ω in the sense that they have the same j-invariants, but this does not imply that $b'_i(\Omega') = x_i$ because the functions b_i are invariants for the group $\Gamma(2,4)$ (and not Γ_2).

To overcome this difficulty, we have to consider the cosets of $\Gamma_2/\Gamma(2,4)$. To find the good Ω modulo $\Gamma(2,4)$, we can take all the representatives γ of this quotient and evaluate the three $b'_i(\gamma\Omega')$ at low precision. The triple nearest to (x_1, x_2, x_3) gives a matrix γ' and then we use the functional equation of Proposition 8 to obtain $b'_i(\gamma'\Omega') = b'_i(\Omega)$ at the working precision N. But this index is large so that this method is slow. Indeed, $[\Gamma_2 : \Gamma(2,4)] = 11520$. The next method we propose is faster.

Another solution consists of precomputing the action (permutations and constants) of the representatives of $\Gamma_2/\Gamma(2,4)$ using the functional equation and then comparing the three $b'_i(\Omega)$

with the three $b_i'(\Omega')$ to deduce the action and finally to find a representative γ in the precalculation which gives us $\gamma\Omega' = \Omega$. The time spent for each γ is thereby negligible.

More precisely, we know the three $b_i'(\Omega)$ and thus the ten even $b_i(\Omega)$ (recall that it is equivalent by (6) to have the three b_i') and Ω' (at the working precision). We compute the ten $b_i(\Omega')$ and we are looking for γ such that $\gamma\Omega' = \Omega$. Use the functional equation to obtain $\theta_k^2(\gamma\Omega') = \zeta_\gamma^2 \det(...)\imath^{\epsilon(\gamma,k)}\theta_\ell^2(\Omega')$ with $\epsilon(\gamma,k) \in \{0,1,2,3\}$. As the b_i are quotients of theta constants, we can already forget about ζ_γ^2 and $\det(...)$. We will say in this case that k is sent to ℓ by the action of γ. If 0 is sent to 0, then the sets A of the ten $b_i(\Omega) = b_i(\gamma\Omega')$ and B of the ten $b_i(\Omega')$ are equal up to permutation and fourth roots of unity. It is easy to compare these two sets to deduce the action of the matrix γ. But the difficulty is that 0 is not always sent to 0 and thus $b_i(\gamma\Omega')$ can not be written as a root of unity times $b_j(\Omega')$ (if i is sent to j), but as a root times a quotient of squares of theta constants evaluated at Ω'. However note that there is a c such that c is sent to 0 and a d such that 0 is sent to d. Thus we have $b_c(\gamma\Omega') = \imath^{\epsilon(\gamma,c)-\epsilon(\gamma,0)}\, b_d(\Omega')^{-1}$ and comparing A with B up to a fourth root of unity it is possible to find $b_d(\Omega')$. Then we have $b_k(\gamma\Omega') = \imath^{\epsilon(\gamma,k)-\epsilon(\gamma,0)}\frac{\theta_\ell^2(\Omega')}{\theta_d^2(\Omega')} = \imath^{\epsilon(\gamma,k)-\epsilon(\gamma,0)} b_\ell(\Omega') b_d(\Omega')^{-1}$ and it is enough to multiply the set A by $b_d(\Omega')^{-1}$ and compare it to B to deduce the action of γ.

This method can also be used to modify Step 3 of Algorithm 25. Indeed, in the case that we cannot choose the basis of the homology group for the numerical integration, we obtain the period matrix Ω at low precision, but we do not know the matrix γ such that $\gamma\Omega$ is the period matrix coming from Thomae's formula. As explained above by comparing $b_i(\Omega)$ at low precision and $b_i(\gamma\Omega)$ at the working precision, we can still deduce $b_i(\Omega)$ at the working precision.

This is what we did in practice. We used the code of Pascal Molin (see [35]) for the numerical integration technique and we noticed that, given the roots of a polynomial of degree 6, it returns a period matrix of the form $\gamma''\Omega''$, where $\Omega'' \in \mathcal{F}_2$ and γ'' always seems to have $-1, 0, 1$ as coefficients. Thus we never had any problem with the reduction in the fundamental domain of $\gamma''\Omega''$ (recall Assumption 24).

3.3 Complexity analysis

Let f_1, f_2, f_3 be three modular functions for a congruence subgroup Γ of Γ_2 generating the function field of Γ. Let p be a prime number which is prime to the level of Γ and C_p be a set of representatives of $\Gamma/(\Gamma \cap \Gamma_0(p))$. We have explained in the preceding section how to find Ω from $(f_1(\Omega), f_2(\Omega), f_3(\Omega))$ and we have then to evaluate the modular polynomials (see Definition 27) for a prime p at Ω, which means we have to compute for $\ell = 2, 3$:

$$\Phi_{1,p}(X, f_1(\Omega), f_2(\Omega), f_3(\Omega)) \qquad \text{and} \qquad \Psi_{\ell,p}(X, f_1(\Omega), f_2(\Omega), f_3(\Omega))$$

(and each coefficient of these polynomials is the evaluation at Ω of a trivariate rational fraction in f_1, f_2, f_3 that we have to interpolate). To do that, we compute first $f_{\ell,p}^{\gamma}(\Omega)$ for all $\gamma \in C_p$ and $\ell = 1, 2, 3$. Let $q = p^3 + p^2 + p + 1$ the degree of $\Phi_{1,p}(X)$. The evaluation of $\Phi_{1,p}(X)$ at Ω can be obtained in $O(\mathcal{M}(q)\log q)$ using a subproduct tree (see [48, Section 10.1]). The two other polynomials can be obtained with the same complexity using fast interpolation (see [48, Section 10.2]).

We summarize what we have explained through the following algorithm.

Algorithm 33: Evaluation of the modular polynomials

Data: $f_1(\Omega), f_2(\Omega), f_3(\Omega)$, a subgroup Γ of Γ_2 such that $\mathbb{C}_\Gamma = \mathbb{C}(f_1, f_2, f_3)$, a prime p prime to the level of Γ, a set C_p of representatives of $\Gamma/(\Gamma \cap \Gamma_0(p))$ and the precalculation of the action of Γ_2/Γ and precisions N and N'.

Result: $\Phi_{1,p}(X, f_1(\Omega), f_2(\Omega), f_3(\Omega))$ and $\Psi_{\ell,p}(X, f_1(\Omega), f_2(\Omega), f_3(\Omega))$ at precision N (with some loss) for $\ell = 2, 3$.

1 Deduce the j-invariants $j_i(\Omega)$ from $f_i(\Omega)$;
2 Use Mestre's algorithm to obtain a hyperelliptic curve $Y^2 = f(X)$ at precision N;
3 Deduce the ten $b_i(\Omega)$ at precision N using numerical integration at precision N';
4 Invert the functions to find Ω' with the good j-invariants at precicision N;
5 Compare (permutations and signs) the three $f_i(\Omega)$ with the three $f_i(\Omega')$;
6 Deduce a representative γ of this action using the precalculation;
7 Compute $\Omega = \gamma\Omega'$;
8 Compute the $f_{i,p}^{\gamma}(\Omega)$ at precision N for all $\gamma \in C_p$;
9 Compute $\Phi_{1,p}(X, f_1(\Omega), f_2(\Omega), f_3(\Omega))$ at precision N using a subproduct tree;
10 Using fast interpolation, compute $\Psi_{\ell,p}(X, f_1(\Omega), f_2(\Omega), f_3(\Omega))$;

The complexity of the algorithm depends on the complexity of the evaluation of the f_i at some Ω. Let $q = p^3 + p^2 + p + 1$. In the case of the theta constants and functions derived from them (as the j-invariants), Steps 1 to 7 are of complexity $O(\mathcal{M}'(N)\log(N))$ (by [10, Theorem 9.3]), Step 8 is of complexity $O(q\mathcal{M}'(N)\log(N))$ (by [18, Theorem 12] under Conjecture 23), Step 9 $O(\mathcal{M}(q)\log(q))$ and Step 10 $O(\mathcal{M}(q)\log(q))$ so that the complexity of this algorithm with functions derived from the theta constants is $O(q\mathcal{M}'(N)\log(N) + \mathcal{M}(q)\log(q)) \subseteq \tilde{O}(p^3 N)$. In practice, the limiting step is Step 8 (see Section 6).

Suppose f_1 is the variable which has the largest degree among all the numerators and denominators of the coefficients of the modular polynomials. Denote by d_T^A (resp. d_T^B) the maximum of the total degrees of the numerators (resp. denominators) of the coefficients of the three modular polynomials and by d_{f_1} (resp. d_{f_2}, d_{f_3}) the maximum exponent of the variable f_1 (resp. f_2, f_3) appearing in one of the coefficients of these three polynomials. Let $n = d_T^A + d_T^B \leq 6d_{f_1}$. To obtain the modular polynomials, Algorithm 33 will be executed $(n+1)(d_{f_2}+1)(d_{f_3}+1)$ times and we will interpolate $3q$ rational fractions. The complexity to compute the modular polynomials is then

$$(n+1)(d_{f_2}+1)(d_{f_3}+1)\tilde{O}(p^3 N) + \tilde{O}(np^3 d_{f_2} d_{f_3} N) \subseteq \tilde{O}(d_{f_1} d_{f_2} d_{f_3} p^3 N).$$

Note that we suppose we know the degrees of all the trivariate rational fractions to use the Cauchy interpolation with the extended Euclidean algorithm. We discuss in Section 6 how to find these degrees.

As we do not have any explicit bounds on the size of the coefficients of the modular polynomials, we assume that it is sufficient to use a floating point precision of $O(N)$, where N is the size of the largest coefficient, to do all the computations so that after the interpolation step it is possible to correctly round the coefficients. We also assume Assumption 24.

Theorem 34 (under Conjecture 23 and the heuristics of the preceding paragraph)**.** *Let f_1, f_2, f_3 be three modular functions derived from the theta constants for a congruence subgroup Γ of Γ_2 generating the function field $\mathbb{C}_\Gamma$. Let p be a prime number prime to the level*

of Γ. *Then, under the previous assumptions, the modular polynomials for these data can be computed in* $\tilde{O}(d_{f_1}d_{f_2}d_{f_3}p^3N)$ *time, if the degrees in the* f_i *of all the coefficients (numerators and denominators) of the three modular polynomials are known and if* $d_{f_1} = \max(d_{f_1}, d_{f_2}, d_{f_3})$ *and where* N *is the size of the largest coefficient of these modular polynomials.*

Remark 35.
- The conjecture is used for the complexity, but it does not affect the correctness of the algorithm because it is easy to check whether the period matrix found is the good one or not at each evaluation step.
- For the complexity $\tilde{O}(d_{f_1}d_{f_2}d_{f_3}p^3N)$ to be quasi-linear in the output size, we must assume that the average size of the coefficients of the modular polynomials is in $\Omega(N)$.

4 Computational results

We present in this section the modular polynomials we have computed with Streng invariants and with the b'_i. The experimental findings given in this section and proved in the next one are used to optimize the implementation of the computation of the modular polynomials (see Section 6).

4.1 Modular polynomials with the invariants of Streng

With the algorithm we have presented in Section 3.1, Régis Dupont [10] has calculated the modular polynomials with Igusa invariants for $p = 2$ but because of the large size of the coefficients and the large degrees of the rational fractions in j_1, j_2 and j_3, he has calculated only the denominators for $p = 3$ and the degrees of the rational fractions.

We begin with some notations to compare the results found between the Igusa and the Streng invariants (see Definitions 13 and 28). For $p = 2$, the number of isogenies is $p^3 + p^2 + p + 1 = 15$. Denote for $\ell = 2, 3$

$$\Phi_{1,2}(X) = X^{15} + \sum_{i=0}^{14} \frac{A_{1,i}(i_1, i_2, i_3)}{B_{1,i}(i_1, i_2, i_3)} X^i \qquad \text{and} \qquad \Psi_{\ell,2}(X) = \sum_{i=0}^{14} \frac{A_{\ell,i}(i_1, i_2, i_3)}{B_{\ell,i}(i_1, i_2, i_3)} X^i.$$

We consider the quotient $A_{j,i}/B_{j,i}$ as the i-th coefficient of the j-th modular polynomial. The numerator and the denominator of each coefficient are polynomials in $\mathbb{Z}[i_1, i_2, i_3]$.

We recall that Dupont found that the denominators of the three polynomials were of the form $1428 j_1^{\alpha} D_2(j_1, j_2, j_3)^6$ for some integer α ranging between 5 and 21 and D_2 of degrees 5, 7 and 5 in respectively j_1, j_2, and j_3 (see [10, Pages 225–226] for these results and for the definition of D_2). With Streng invariants, we have found that the denominators are of the form $c i_3^{\alpha} D'_2(i_1, i_2, i_3)$ for $\Phi_{1,2}$ and of the form $c i_3^{\alpha} (D'_2(i_1, i_2, i_3))^2$ for the others, where c is a constant in $\mathbb{Z}$, α varies from 0 to 3, and

$$\begin{aligned} D'_2 = (&24576 i_3 i_1^5 + (96 i_2^3 - 4608 i_3 i_2) i_1^4 + (-6220800 i_3 i_2 - 12288 i_3^2) i_1^3 + (-23328 i_2^4 - 48 i_3 i_2^3 + \\ &1088640 i_3 i_2^2 + 2304 i_3^2 i_2 + 24883200 i_3^2) i_1^2 + (93312 i_3 i_2^3 + 419904000 i_3 i_2^2 - 5909760 i_3^2 i_2 + (1536 i_3^3 - \\ &8398080000 i_3^2)) i_1 + (1417176 i_2^5 - 5832 i_3 i_2^4 + (6 i_3^2 - 94478400 i_3) i_2^3 + 287712 i_3^2 i_2^2 + (-288 i_3^3 + \\ &1154736000 i_3^2) i_2 + (-248832 i_3^3 + 755827200000 i_3^2))) \end{aligned}$$

is irreducible. It is clearl that the exponents of D_2 and D'_2 are related to the exponent of h_{10} in

the definition of the different j-invariants. Denote by $d_{i,j,\ell}$ the degree of the numerator of the ℓ-th coefficient of the i-th modular polynomial in i_j (see Definition 27) and $\alpha_{i,\ell}$ the exponent of j_3 appearing in the denominator of the ℓ-th coefficient of the i-th polynomial. The degrees found are written in Table 1.

ℓ	$d_{1,1,\ell}$	$d_{1,2,\ell}$	$d_{1,3,\ell}$	$\alpha_{1,\ell}$	$d_{2,1,\ell}$	$d_{2,2,\ell}$	$d_{2,3,\ell}$	$\alpha_{2,\ell}$	$d_{3,1,\ell}$	$d_{3,2,\ell}$	$d_{3,3,\ell}$	$\alpha_{3,\ell}$
0	25	11	11	3	30	17	15	3	33	17	16	3
1	23	11	11	3	28	17	15	3	31	17	16	3
2	23	11	11	3	28	17	15	3	31	17	16	3
3	21	11	11	3	26	17	15	3	29	17	16	3
4	21	11	11	3	26	17	15	3	29	17	16	3
5	20	11	10	3	25	17	14	3	28	17	15	3
6	20	11	10	3	25	17	14	3	28	17	15	3
7	18	10	9	2	23	17	14	3	26	17	15	3
8	18	10	9	2	23	16	13	2	26	16	14	2
9	16	10	8	2	21	15	12	2	24	15	13	2
10	16	8	7	1	21	15	12	2	24	15	13	2
11	15	8	7	1	20	13	11	1	23	13	12	1
12	15	7	7	1	20	13	11	1	23	13	12	1
13	11	6	5	0	16	12	10	1	20	12	11	1
14	8	5	4	0	13	11	8	0	16	11	9	0

Table 1: Degrees of the numerators of the modular polynomials with Streng invariants for $p = 2$

The degrees of the numerators of the coefficients of the modular polynomials found by Dupont for the Igusa invariants vary from 37 to 60 in j_1, from 50 to 75 in j_2 and from 33 to 50 in j_3 for $\Phi_{1,2}(X)$ while they do not exceed 25 with Streng invariants. The size of the integers in the former case is bounded by 210 decimal digits and by 105 in the latter case. Moreover, the three polynomials computed by Dupont (and accessible at his website) fill 57 MB and the others 2.1 MB. Thus the Streng invariants provide smaller modular polynomials in terms of degree, precision and total space.

We also managed to compute the three modular polynomials with Streng invariants for $p = 3$. The number of isogenies is 40. The denominators have the same properties as described before: they are of the form $ci_3^{\alpha}(D_3'(i_1, i_2, i_3))^2$ for $\Phi_{1,3}$ and of the form $ci_3^{\alpha}(D_3'(i_1, i_2, i_3))^4$ for the others. The common part D_3' is an irreducible polynomial which occurs with degrees 13, 10 and 8 in respectively i_1, i_2 and i_3. Dupont has found that the denominators with the Igusa invariants are of the form $cj_1^{\alpha}D_3(j_1, j_2, j_3)^{18}$, where D_3 has degrees 14, 20 and 13 in respectively j_1, j_2 and j_3. We present some degrees of the numerators in Table 2.

The degrees are far smaller than those with the Igusa invariants which range from 243 to 420. We do not know the size of the integers of the polynomial with Igusa invariants, but in the case of Streng invariants we have found that they can reach 550 decimal digits. The three polynomials fill 890 MB.

4.2 Modular polynomials with the b_i'

We have computed the modular polynomials for b_i' for $p = 3, 5$ and 7 (see (6) for their definition). Note that for $p = 2$, these polynomials do not exist because $\Gamma(2, 4) \cap \Gamma_0(2) = \Gamma(2, 4)$.

ℓ	$d_{1,1,\ell}$	$d_{1,2,\ell}$	$d_{1,3,\ell}$	$\alpha_{1,\ell}$	$d_{2,1,\ell}$	$d_{2,2,\ell}$	$d_{2,3,\ell}$	$\alpha_{2,\ell}$	$d_{3,1,\ell}$	$d_{3,2,\ell}$	$d_{3,3,\ell}$	$\alpha_{3,\ell}$
0	61	32	32	4	87	52	48	4	92	52	49	4
1	61	32	31	4	87	52	47	4	92	52	48	4
2	61	32	31	4	87	52	47	4	92	52	48	4
$\vdots$		$\vdots$	$\vdots$			$\vdots$	$\vdots$			$\vdots$	$\vdots$	
37	41	22	21	1	67	43	37	1	72	43	39	1
38	36	21	19	0	62	42	36	1	67	42	37	1
39	31	20	17	0	57	41	33	0	62	41	35	0

Table 2: Degrees of the numerators of the modular polynomials with Streng invariants for $p = 3$

This time there is only one common denominator D_p for all the coefficients of the three polynomials (there are no constants and no powers of one of the b'_i). For example, we have

$D_3 = 1024b_3'^6b_2'^6b_1'^{10} - ((768b_3'^8 + 1536b_3'^4 - 256)b_2'^8 + 1536b_3'^8b_2'^4 - 256b_3'^8)b_1'^8 + (1024b_3'^6b_2'^{10} + (1024b_3'^{10} + 2560b_3'^6 - 512b_3'^2)b_2'^6 - (512b_3'^6 - 64b_3'^2)b_2'^2)b_1'^6 - (1536b_3'^8b_2'^8 + (-416b_3'^4 + 32)b_2'^4 + 32b_3'^4)b_1'^4 - ((512b_3'^6 - 64b_3'^2)b_2'^6 - 64b_3'^6b_2'^2)b_1'^2 + 256b_3'^8b_2'^8 - 32b_3'^4b_2'^4 + 1.$

For $p = 5$ (resp. $p = 7$), the denominator occurs with exponents 70 (resp. 226) in the three b'_i. These three denominators have interesting properties. They are symmetric, the exponents of the b'_i are always even and there are relations modulo 2 and 4 between the exponents of each monomial. We have also noted similar properties for the numerators. In particular, we have noted that for $p = 3$ and 5, $\Psi_{2,p}(X, b'_1, b'_2, b'_3) = \Psi_{3,p}(X, b'_1, b'_3, b'_2)$ and $\Phi_{1,p}(X, b'_1, b'_2, b'_3) = \Phi_{1,p}(X, b'_1, b'_3, b'_2)$. Moreover, the total degrees for the denominators are 24, 120 and 226, which always seems to be $p^3 - p$. We will prove all this in the next section.

Table 3 shows a few of the degrees for $p = 3$. This table can be compared with the results found with the j-invariants (see Table 2). The notation is similar as before.

ℓ	$d_{1,1,\ell}$	$d_{1,2,\ell}$	$d_{1,3,\ell}$	$d_{2,1,\ell}$	$d_{2,2,\ell}$	$d_{2,3,\ell}$
0	40	10	10	37	13	12
1	37	12	12	36	15	14
2	38	14	14	37	17	16
3	39	16	16	36	19	18
4	36	16	16	35	19	18
$\vdots$		$\vdots$			$\vdots$	
35	21	16	16	22	19	18
36	20	16	16	19	19	18
37	17	16	16	16	17	16
38	14	14	14	15	15	14
39	13	12	12	12	13	12

Table 3: Degrees of the numerators of the modular polynomials with the b'_i for $p = 3$

Table 4 indicates the minimal and maximal degrees each of the b'_i do take for the differents modular polynomials for $p = 5$ and 7.

min-max of	b'_1	b'_2	b'_3
$\Phi_{1,5}$	75-156	70-92	70-92
$\Psi_{2,5}$	72-155	75-97	72-94
$\Phi_{1,7}$	233-400	226-272	226-272
$\Psi_{2,7}$	230-397	233-279	230-276

Table 4: Degrees of the numerators of the modular polynomials with the b'_i for $p = 5, 7$

The integers have about 10, 60 and 190 decimal digits for respectively $p = 3$, 5 and 7. The three polynomials fill 270 KB for $p = 3$ (which is 3000 times smaller than the total space of the modular polynomials with Streng invariants for $p = 3$), and 305 MB for $p = 5$ while only the two first fill 29 GB for $p = 7$ (we do not have computed the third because it would have taken too much time and we have assumed that there is the same symmetry as in the cases $p = 3$ and $p = 5$, so that the third polynomial can be deduced from the second one). Compared to the polynomials found with the invariants of Streng for $p = 3$, these invariants produce smaller polynomials in terms of degree, precision and total space.

5 Analysis of the results

5.1 Humbert surfaces

In this section we will examine the meaning of the denominators appearing in the different modular polynomials. The principal tool we use is the notion of Humbert surface, which have been studied in [24].

Let $\Delta \equiv 0, 1 \bmod 4$ and $\Delta > 0$. We call the Humbert surface H_Δ of discriminant Δ the irreducible surface of matrices which are equivalent to some $\Omega = \left(\begin{smallmatrix} \Omega_1 & \Omega_2 \\ \Omega_2 & \Omega_3 \end{smallmatrix}\right)$ in $\Gamma_2 \backslash \mathcal{H}_2$ satisfying $k\Omega_1 + \ell\Omega_2 - \Omega_3 = 0$ where k and ℓ are determined uniquely by $\Delta = 4k + \ell$ and $\ell \in \{0, 1\}$.

These surfaces are of particular interest for us because of the next proposition which states that $\mathcal{L}_p = H_{p^2}$.

Proposition 36. *Let m be a positive integer. Then the Humbert surface H_{m^2} is the moduli space for isomorphism classes of principally polarized abelian surfaces which split as a product of two elliptic curves via an isogeny of degree m^2.*

Proof. See [24, Proposition 2.14]. □

For each discriminant Δ there is an irreducible polynomial $L_\Delta(j_1, j_2, j_3)$ whose zero set is the Humbert surface of discriminant Δ. Thus, by Lemma 17, $L_{p^2}(j_1, j_2, j_3)$ divides the denominators of the modular polynomials with the Igusa invariants. The exponent to which $L_{p^2}(j_1, j_2, j_3)$ appears in the denominator seems to depend on the exponent of the h_{10} in the definition of the j-invariants. A heuristic reason for the factor j_1^α in the denominator of a coefficient of a modular polynomial is to compensate for the case where $h_{12}(\Omega) = 0$ (recall Definition 13). With Streng invariants, there is a factor i_3^α to compensate for the case where $h_4(\Omega) = 0$ (recall Definition 28). Note that j_1 (resp. i_3) has the greatest exponent of h_{12} (resp h_4) in its definition among j_1, j_2, j_3 (resp i_1, i_2, i_3).

Moreover, a formula for the degree of these surfaces exists. Let

$$a_{p^2} := 24 \sum_{\substack{x\in\mathbb{Z},\\ 4|(p^2-x^2)}} \sigma_1\left(\frac{p^2-x^2}{4}\right) + 12p^2 - 2$$

with $\sigma_1(n) = \sum_{d|n} d$ the sum of positive divisors function. Then

Theorem 37. *The degree of any Humbert surface of discriminant p^2 can be obtained by the formula*

$$v(p^2)\deg(H_{p^2}) + 5 = \frac{a_{p^2}}{2} \qquad \textit{where} \qquad v(p^2) = \begin{cases} 1/2 & \textit{if } p = 2 \\ 1 & \textit{otherwise.} \end{cases}$$

Proof. See [24, Theorem 3.8]. □

Applying this formula gives $\deg(H_4) = 60$ and $\deg(H_9) = 120$. Here, the degree of the surfaces is the degree of the homogenous form of L_Δ with weight $(4, 6, 10, 12)$ for the functions $(h_4, h_6, h_{10}, h_{12})$ (see [27, Pages 170–172]). We have then substituted the j-invariants of the common denominator for $p = 2$ and $p = 3$ by their definition in terms of the h_i and multiplied by a power of h_{10} to homogenize. The degree we have found for $p = 2$ is 100 (resp. 300) with the invariants of Streng (resp. of Igusa), but there is a factor h_4^{10} (resp. h_{12}^{20}) and we have $100 - 40 = 60$ (resp. $300 - 240 = 60$). This factor can be explained by the fact that the j-invariants are zero when $h_4 = 0$ (resp. $h_{12} = 0$). For $p = 3$, we have found (for Streng invariants) that the degree is 200 and there is a factor h_4^{20}. We have then $200 - 80 = 120$.

We study now what happens for our modular polynomials with the theta constants. We also have a formula for the degree due to the work of Runge ([39], see also [24]) who considered finite covers of $\Gamma_2\backslash\mathcal{H}_2$ for the study of Humbert surfaces because of the large degrees and coefficients of the polynomial with the j-invariants. Define $\Gamma^*(2, 4)$ to be the largest normal subgroup of $\Gamma(2, 4)$ which does not contain the matrix $\mathrm{diag}(-1, 1, -1, 1)$. The natural projection $\pi : \Gamma^*(2, 4)\backslash\mathcal{H}_2 \to \Gamma_2\backslash\mathcal{H}_2$ is a finite map. We say that each component of $\pi^{-1}(H_{p^2})$ in $\Gamma^*(2, 4)\backslash\mathcal{H}_2$ is a Humbert component and it is possible to define an order $v_i'(p^2)$ for each irreducible Humbert component $F_{p^2,i}$. Since $\Gamma^*(2, 4)$ is normal, these components have the same degree. Moreover by [39], any irreducible component of the covering of H_{p^2} is given by the zero set of a single irreducible polynomial.

Proposition 38. *The degree of any Humbert component $F_{p^2,i}$ in $\Gamma^*_{2,4}\backslash\mathcal{H}_2$ is given by the formula*

$$a_{p^2} = 10(1 + \deg(F_{p^2,i})).$$

Proof. See [24, Proposition 3.9]. □

Proposition 39. *Let $p > 2$ be a prime number. The degree of $F_{p^2,i}$ is $p^3 - p$.*

Proof. From the degree formula above, we have that $a_{p^2} = 10(1 + \deg(F_{p^2,i}))$ and from the definition of a_{p^2} it suffices to prove that $\sum_{x>0} \sigma_1(\frac{p^2-x^2}{4}) = (5p^3 - 6p^2 - 5p + 6)/24$. The left-hand side can be rewritten as $\frac{1}{2}\sum_{k=1}^{p} \sigma_1(k)\sigma_1(p-k)$ and the result comes from the equality $\frac{1}{2i\pi}G_2'(\Omega) = \frac{5}{6}G_4(\Omega) - 2G_2(\Omega)^2$ of [51] where G_i is the *ith* Eisenstein series (in genus 1). (Moreover, using the fact that σ_1 is multiplicative, it can be shown that for all $p > 2$, the degree of $F_{4p^2,i}$ is also $p^3 - p$, but we do not need this result). □

In our case, we use $\Gamma(2,4)$ and not $\Gamma^*(2,4)$, but we noted that the total degrees found for $p = 3, 5$ and 7 is always $p^3 - p$. The reason for this is that the degree formula depends on the number of Humbert components and the order of some isotropy subgroup and these numbers are equal for the groups $\Gamma^*(2,4)$ and $\Gamma(2)$ (see [24]) so that it is the case for $\Gamma(2,4)$ because $\Gamma^*(2,4) < \Gamma(2,4) < \Gamma(2)$. Thus the degree formula of a component of discriminant p^2 for the group $\Gamma(2,4)$ is the same as those for the group $\Gamma^*(2,4)$, namely $p^3 - p$. Note that the definition of degree here is the total degree of the polynomial because the $\theta_i(\tau/2)$ are Siegel modular forms of weight 1 for $\Gamma(2,4)$.

Consider this time the locus $\mathcal{L}'_p$ of all the principally polarized abelian surfaces modulo $\Gamma(2,4)$ that are (p,p)-isogenous to a principally polarized abelian surface Ω which is isogenous to a product of two elliptic curves by the $(2,2)$-isogeny $\Omega \to \Omega/2$ and such that $\theta_0(\Omega/2) = 0$ (recall that $b'_i(\Omega) := \theta_i(\Omega/2)/\theta_0(\Omega/2)$ and Proposition 12).

Proposition 40. *The denominators of the modular polynomials for the functions b'_1, b'_2, b'_3 are divisible by a polynomial L'_p in $\mathbb{Q}[b'_1, b'_2, b'_3]$ describing the preceding locus.*

Proof. We adapt the proof of lemma 6.2 of [5]. Let $\Omega \in \Gamma(2,4)\backslash\mathcal{H}_2$ which is (p,p)-isogenous to Ω' such that $\theta_0(\Omega'/2) = 0$. Let c be a coefficient of the polynomial $\Phi_{1,p}$. For some $\gamma \in \Gamma(2,4)/(\Gamma(2,4) \cap \Gamma_0(p))$, $b'_{1,p}(\gamma\Omega)$ is infinite. The evaluation of c at Ω is a symmetric expression in the $b'^{\gamma}_{1,p}(\Omega)$'s. Generically, there is no algebraic relation between these values and the evaluation of c at Ω is therefore infinite. Since the $b'_i(\Omega)$ are finite, the numerator of c is finite. We conclude that the denominator of c must vanish at Ω, which means that c is divisible by a polynomial describing the locus. The proof for $\Phi_{\ell,p}$, $\ell = 2, 3$ proceeds similarly. □

We have noticed that for $p = 3, 5$ and 7, the coefficients of the three modular polynomials with the b'_i always have L'_p has denominator (unlike the case with the j-invariants where there also is a factor j_1 or i_3, as explained in Section 4). This justified the following conjecture, which will be used in the next sections.

Conjecture 41. *The polynomial L'_p is the denominator of all the coefficient of the three modular polynomials.*

5.2 Symmetries

As mentioned above (Section 4.2), we have noticed for $p = 3, 5$ that $\Psi_{2,p}(X, b'_1, b'_2, b'_3) = \Psi_{3,p}(X, b'_1, b'_3, b'_2)$ and for $p = 3, 5, 7$ that $\Phi_{1,p}(X, b'_1, b'_2, b'_3) = \Phi_{1,p}(X, b'_1, b'_3, b'_2)$. These symmetries have the following meaning. For each variety $\Omega \in \Gamma(2,4)$ and $p\Omega$ having invariants $(b'_1(\Omega),\ b'_2(\Omega),\ b'_3(\Omega))$ and $(b'_{1,p}(\Omega),\ b'_{2,p}(\Omega),\ b'_{3,p}(\Omega))$, there exists a variety with invariants $(b'_1(\Omega), b'_3(\Omega), b'_2(\Omega))$ and such that one of its (p,p)-isogenous varieties has invariants $(b'_1(p\Omega), b'_3(p\Omega), b'_2(p\Omega))$.

A proof of this can be obtained by looking at the action of some matrices. Indeed, we have that $\Phi_{1,p}$ is the minimal polynomial of $b_{1,p}$, which means it is the unique polynomial such that for all $\Omega \in \mathcal{H}_2$, $\Phi_{1,p}(x, b'_1(\Omega), b'_2(\Omega), b'_3(\Omega)) = 0$ if and only if $x = b'_{1,p}(\Omega)$; hence $\Phi_{1,p}(b'_{1,p}(\gamma\Omega),\ b'_1(\gamma\Omega),\ b'_2(\gamma\Omega),\ b'_3(\gamma\Omega)) = 0$ for all $\gamma \in \Gamma_2$. What we are looking for is a matrix that fixes b'_1 and $b'_{1,p}$ and interchanges b'_2 with b'_3 and $b'_{2,p}$ with $b'_{3,p}$. This action on $\Phi_{1,p}(X)$ would provide a unitary polynomial with the same roots and degree as $\Phi_{1,p}(X)$ and since $\Phi_{1,p}(X)$ is a minimal polynomial, they both have to be equal.

Assume that we have the symmetry for $\Phi_{1,p}$. Then by Definition 27 we have $b'_{\ell,p} = \Psi_{\ell,p}(b'_{1,p})/\Phi'_{1,p}(b'_{1,p})$ for $\ell = 2, 3$. We use this action on $\Psi_{2,p}(X)$ which gives us

$$b'_{3,p} = \Psi_{2,p}(b'_{1,p}, b'_1, b'_3, b'_2)/\Phi'_{1,p}(b'_{1,p}, b'_1, b'_2, b'_3),$$

so that

$$\Psi_{2,p}(b'_{1,p}, b'_1, b'_3, b'_2) = b'_{3,p}\Phi'_{1,p}(b'_{1,p}, b'_1, b'_2, b'_3) = \Psi_{3,p}(b'_{1,p}, b'_1, b'_2, b'_3).$$

Firstly, the search is done among the representatives of $\Gamma_2/\Gamma(2,4)$ because $\Gamma(2,4)$ fixes the b'_i. A representative of the unique class such that $(b'^{\gamma}_1, b'^{\gamma}_2, b'^{\gamma}_3) = (b'_1, b'_3, b'_2)$ is

$$\gamma = \begin{pmatrix} 1 & -3 & -2 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -4 & -5 & 1 \end{pmatrix}.$$

Secondly, we look for a matrix γ' in $\Gamma(2,4)$ such that $\gamma\gamma' \in \Gamma_0(p)$. For $p = 3, 5$ and 7 we can take for γ' respectively

$$\begin{pmatrix} -5 & 24 & -12 & 12 \\ -2 & 19 & -12 & 8 \\ 0 & 6 & -5 & 2 \\ -2 & 4 & 0 & 3 \end{pmatrix}, \begin{pmatrix} -7 & 6 & 4 & 2 \\ 0 & -7 & 2 & 0 \\ 0 & 10 & -3 & 0 \\ 10 & -8 & -6 & -3 \end{pmatrix} \text{ and } \begin{pmatrix} 13 & 12 & -16 & -6 \\ -10 & -3 & 10 & 4 \\ 56 & 14 & -55 & -22 \\ 30 & -40 & -12 & -7 \end{pmatrix}.$$

Recall that for a matrix X, we denote by X_0 the vector composed of the diagonal entries of X.

Lemma 42. *Let $M = \left(\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right) \in \Gamma_2/\Gamma(2,4)$ and $M' \in \Gamma(2,4)$ such that $MM' \in \Gamma_0(p)$, for some prime $p > 2$. Then $(MM')_p$ is in the same equivalence class as M for all $p \equiv 1 \bmod 4$. For $p \equiv 3 \bmod 4$, this is the case if we have the additional properties $(A'\,{}^tB')_0 \equiv 0 \bmod 2$ and $(C'\,{}^tD')_0 \equiv 0 \bmod 2$.*

Proof. Let $MM' = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. We study under which conditions $(MM')_p M^{-1} \in \Gamma(2,4)$ or, equivalently, when $(MM')_p(MM')^{-1}$ is in $\Gamma(2,4)$. We have

$$\begin{pmatrix} A & pB \\ C/p & D \end{pmatrix} \begin{pmatrix} {}^tD & -{}^tB \\ -{}^tC & {}^tA \end{pmatrix} = \begin{pmatrix} A\,{}^tD - pB\,{}^tC & -A\,{}^tB + pB\,{}^tA \\ C/p\,{}^tD - D\,{}^tC & -C/p\,{}^tB + D\,{}^tA \end{pmatrix},$$

where $\left(\begin{smallmatrix} {}^tD & -{}^tB \\ -{}^tC & {}^tA \end{smallmatrix}\right)$ is the inverse of M by Equation (1). As $p \equiv 1 \bmod 2$, this product is the identity modulo 2. Now for $p \equiv 1 \bmod 4$, we have $-A\,{}^tB + pB\,{}^tA \equiv C/p\,{}^tD - D\,{}^tC \equiv 0 \bmod 4$ (recall that this product is in Γ_2) so that $(MM')_p(MM')^{-1} \in \Gamma(2,4)$. For $p \equiv 3 \bmod 4$, we have $-A\,{}^tB + pB\,{}^tA \equiv 2A\,{}^tB \bmod 4$ and $C/p\,{}^tD - D\,{}^tC \equiv 2C\,{}^tD \bmod 4$. Thus to be in $\Gamma(2,4)$, we want $(A\,{}^tB)_0 \equiv (C\,{}^tD)_0 \equiv 0 \bmod 2$. Finally note that $M' \equiv I_4 \bmod 2$ and we deduce the lemma. □

By this lemma, we have that $(\gamma\gamma')_p$ is in the same equivalence class as γ for any prime $p > 2$, hence the permutation $(b'^{\gamma\gamma'}_{1,p}, b'^{\gamma\gamma'}_{2,p}, b'^{\gamma\gamma'}_{3,p}) = (b'_{1,p}, b'_{3,p}, b'_{2,p})$. Moreover the surjectivity of $\mathrm{Sp}_4(\mathbb{Z}) \to \mathrm{Sp}_4(\mathbb{Z}/4p\mathbb{Z})$ and the Chinese remainder theorem prove that the matrix γ' always exists. Thus there are these symmetries for all prime $p > 2$ (see Theorem 44).

By the above we have also proved that the denominator is always symmetric in b'_2 and b'_3. To prove that L'_p is also symmetric in b'_1 and b'_2 (resp. b'_1 and b'_3), we use the matrices

$$\gamma_{410} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \qquad \text{and} \qquad \gamma_{8316} = \begin{pmatrix} 1 & 0 & 0 & 2 \\ -3 & 1 & 2 & -2 \\ -4 & 0 & 1 & -5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which fixes b'_3 (resp. b'_2) and interchanges b'_1 with b'_2 (resp. with b'_3).

The action of γ_{410} (resp. γ_{8316}) on L'_p provides an irreducible polynomial and with the same roots as L'_p, which are still in $\mathcal{L}'_p$ by the following lemma. Hence this polynomial is L'_p and thus it is symmetric.

Lemma 43. *Let $\Omega \in \mathcal{L}'_p$, $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_2/\Gamma(2,4)$ and γ' such that γ_p is in the same equivalence class of γ'. Suppose that the action of γ' on the theta constants sends $\{0,4,8,12\}$ to itself. Then $\gamma\Omega$ is in $\mathcal{L}'_p$.*

Proof. For $\Omega \in \Gamma(2,4)\backslash\mathcal{H}_2$ to be in $\mathcal{L}'_p$ means that there exists $M \in \Gamma(2,4)/(\Gamma(2,4) \cap \Gamma_0(p))$ satisfying $\theta_0(pM\Omega/2) = 0$. Let $M' \in \Gamma(2,4)/(\Gamma(2,4) \cap \Gamma_0(p))$ be such that $(M'\gamma)M^{-1} \in \Gamma_0(p)$. There exists then $\gamma'' \in \Gamma_0(p)$ with $M'\gamma = \gamma''M$. We have, using the duplication formula (Proposition 11)

$$\theta_0(pM'\gamma\Omega/2) = \theta_0(p\gamma''M\Omega/2) = \theta_0(\gamma''_p(pM\Omega)/2) = \sum_{i\in\{0,4,8,12\}} \theta_i^2(\gamma''_p(pM\Omega)).$$

Moreover, $\gamma''_p = (M'\gamma M^{-1})_p$ is in the same equivalence class as γ_p, namely γ' by hypothesis (recall that $\Gamma(2,4)$ is a normal subgroup). The action of γ' sends $\{0,4,8,12\}$ to itself, so that

$$\sum_{i\in\{0,4,8,12\}} \theta_i^2(\gamma''_p(pM\Omega)) = \zeta^2_{\gamma''_p} \det(...) \sum_{i\in\{0,4,8,12\}} \theta_i^2(pM\Omega) = \zeta^2_{\gamma''_p} \det(...)\theta_0(pM\Omega/2) = 0.$$

□

We have proved

Theorem 44. *Let $p > 2$ be a prime number. The modular polynomials for b'_1, b'_2 and b'_3 satisfy*

$$\Phi_{1,p}(X, b'_1, b'_2, b'_3) = \Phi_{1,p}(X, b'_1, b'_3, b'_2) \qquad \text{and} \qquad \Psi_{2,p}(X, b'_1, b'_2, b'_3) = \Psi_{3,p}(X, b'_1, b'_3, b'_2).$$

Morover, the polynomial L'_p is symmetric.

5.3 Relations modulo 2 and 4

We study now the different relations modulo 2 and 4 between the exponents of the b'_i in each coefficient. Consider the numerator of the ℓ-th coefficient of the m-th modular polynomial for $m = 1$ or 2 (we have seen that the third polynomial can be deduced from the second one), whose monomials are of the form $c_{ijk} b'^i_1 b'^j_2 b'^k_3$. We have found that for $p = 3, 5$ and 7, if $c_{ijk} \neq 0$, then

$$\begin{aligned} i &\equiv \ell + m + 1 \bmod 2 \\ i + j &\equiv -p\ell \bmod 4 \\ j + k &\equiv p(m-1) \bmod 4 \end{aligned} \tag{7}$$

and with similar notation, we always have

$$i \equiv j \equiv k \equiv 0 \bmod 2 \qquad \text{and} \qquad i + j \equiv j + k \equiv 0 \bmod 4 \tag{8}$$

for the denominators. These equalities are determined by the existence of some matrices γ with the property that $b'_i(\gamma\Omega) = i^{\alpha_i} b'_i(\Omega)$ and $b'_{i,p}(\gamma\Omega) = i^{\beta_i} b'_{i,p}(\Omega)$ with α_i and β_i in $\{0,1,2,3\}$. We will denote the action of such matrices by the vector $(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$.

With the same arguments as before, we deduce that such an action produces a polynomial with the same roots and degrees as $\Phi_{1,p}(X)$ (resp. $\Psi_{2,p}(X)$) which is then $\Phi_{1,p}(X)$ (resp. $\Psi_{2,p}(X)$) up to a constant. As $p^3+p^2+p+1 \equiv 0 \bmod 4$ for any prime $p>2$ and as the leading coefficient of $\Phi_{1,p}(X)$ is $X^{p^3+p^2+p+1}$, we conclude that such an action does not change $\Phi_{1,p}(X)$. This is not the case of $\Psi_{2,p}(X)$ which is of degree p^3+p^2+p in X.

The matrix

$$\gamma_{134} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 2 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

acts by $(-1,1,1,-1,1,1)$ for all p by the functional equation of Proposition 8 and Lemma 42.

Using lemma 43 shows that this matrix preserves the Humbert component (and $(\gamma_{134})_p$ is in the same equivalence class of γ_{134} by Lemma 42). Thus we obtain a polynomial with the same roots and degrees as L'_p: it is a multiple of L'_p. As the latter is irreducible, it contains at least one monomial where there is not b'_1 so that the matrix does not change this monomial and the constant is thus 1. As L'_p is symmetric, we deduce that it has even exponents in b'_1, b'_2 and b'_3.

If we assume Conjecture 41, then we just proved that the action of γ_{134} on the numerators do not depend on its action on the denominator. Then, on the numerators of $\Phi_{1,p}$, the action of γ_{134} shows that $i+\ell$ is always even. For $\Psi_{2,p}(X)$, we have to determine the constant which appears. The leading coefficient of this polynomial is $\sum_{\gamma\in C_p} b_{2,p}'^{\gamma} X^{p^3+p^2+p}$. Consider now the minimal polynomial $\prod_{\gamma\in C_p}(X-b_{2,p}'^{\gamma})$ of b'_2 and note that it is invariant by the preceding action, which is thus also the case of $\sum_{\gamma\in C_p} b_{2,p}'^{\gamma}$. We deduce that the constant is -1 (because of the $X^{p^3+p^2+p}$), namely

$$\sum_{\gamma\in C_p} b_{2,p}'^{\gamma\gamma_{134}} (b_{1,p}'^{\gamma_{134}})^{p^3+p^2+p} = -\sum_{\gamma\in C_p} b_{2,p}'^{\gamma} (b'_{1,p})^{p^3+p^2+p}.$$

We have thus shown the first of the three equalities of (7).

For the other two, we have to consider the matrices

$$\gamma_{141} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & 1 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \gamma_{21} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}.$$

Their action for $p \equiv 1 \bmod 4$ are respectively $(\imath,\imath,1,\imath,\imath,1)$ and $(1,\imath,\imath,1,\imath,\imath)$ and for $p \equiv 3 \bmod 4$ it is $(\imath,\imath,1,-\imath,-\imath,1)$ and $(1,\imath,\imath,1,-\imath,-\imath)$ because in this case $(\gamma_{141})_p$ and $(\gamma_{21})_p$ are equivalent to

$$\gamma_{1886} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 1 & -1 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \gamma_{155} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 3 & 0 & -1 \end{pmatrix}.$$

On L'_p, the action of γ_{141} does not change the Humbert component by Lemma 43, so that $L'_p(\imath b'_1, \imath b'_2, b'_3)$ is a multiple of L'_p.

As L'_p is irreducible, there is a monomial without b'_1, which is then of the form $c b_2'^i b_3'^j$ for some constant c. We have already shown that $i \equiv j \equiv 0 \bmod 2$ so that $c(\imath b'_2)^i b_3'^j = \pm c b_2'^i b_3'^j$. If it is equal, then the action of γ_{141} fixes L'_p. Otherwise $i \equiv 2 \bmod 4$ and as L'_p is symmetric, we also have the monomials $c b_1'^i b_2'^j$ and $c b_3'^i b_2'^j$. Now look at the latter: $c b_3'^i (\imath b'_2)^j = \pm c b_3'^i b_2'^j$. If it is not equal, then $j \equiv 2 \bmod 4$ and then $c(\imath b'_1)^i (\imath b'_2)^j = c b_1'^i b_2'^j$. In all cases, the action of γ_{141} fixes L'_p. We can adapt this proof on γ_{121} and deduce (8).

We use similar arguments on $\Phi_{1,p}(X)$ and $\Psi_{\ell,p}(X)$ to prove (7). Thus we obtain

Theorem 45. *Let $p > 2$ be a prime number. Then the polynomial L'_p satisfy (8). Moreover, if we assume the Conjecture 41, the numerators of the two first modular polynomials verifies (7).*

6 Implementation

6.1 External packages

Dupont presented two algorithms to compute theta functions. The first one uses the definition as sums of exponentials and it computes $\theta_i(\Omega)$ for $i = 0, 1, 2, 3$, $\Omega \in \mathcal{F}_2$ at precision N with a complexity of $O(\mathcal{M}'(N)N)$. The second one uses Newton lifts and the Borchardt mean and is in $O(\mathcal{M}'(N)\log(N))$ under Conjecture 23. It computes $\theta_i^2(\Omega)/\theta_0^2(\Omega)$, $i = 1, 2, 3$. These algorithms have been studied and implemented by Enge and Thomé in [18, 19]. Using finite differences, they proved that the complexity to compute the squares of the theta constants is in $O(\mathcal{M}'(N)\log(N))$ under conjecture 23.

We used the cmh library written in C for the evaluation of the square of the theta functions (we also recovered from it the implementation of Mestre's algorithm and some other functions that were already written in GP) and we used the pari-gnump software [14] for switching between number types from the GNU multiprecision ecosystem (GMP, MPFR and MPC [23, 25, 16]) and corresponding types in Pari/GP to be able to use the algorithm of cmh with GP.

There are two reasons for which the algorithms to compute the theta constants are defined for Ω only in the fundamental domain. The first one is for the convergence and the second is because we can use the functional equation of Proposition 8 to obtain the theta constants at $\Omega \in \mathcal{H}_2$ from the theta constants at $\Omega' \in \mathcal{F}_2$. We have implemented an algorithm to compute the squares of the theta constants for any matrix in $\mathcal{H}_2$ with GP [1].

For Algorithm 33, we need a method to reduce some $\Omega \in \mathcal{H}_2$ into the fundamental domain. We implemented the standard method (see [22, 10]). We also used the code of Pascal Molin [35] to compute $\Omega \in \mathcal{H}_2$ corresponding to a given hyperelliptic curve equation.

Moreover, we have to know the cosets of $\Gamma(2,4)/(\Gamma_0(p) \cap \Gamma(2,4))$ for some primes p. They are naturally calculated beforehand. A generalization of Algorithm 2 of [10] to dimension 2 allows one to compute, for subgroups $\Gamma' \subset \Gamma$ of Γ_2, the representatives of the classes of Γ/Γ' and a set of generators of Γ' from a set of generators of Γ and from a function which decides if a matrix lies in Γ' or not. We apply it twice: first on $\Gamma = \Gamma_2$ and $\Gamma' = \Gamma(2,4)$, then on $\Gamma = \Gamma(2,4)$ and $\Gamma' = \Gamma_0(p) \cap \Gamma(2,4)$. Another solution consists in using Proposition 10.1 of [10] which provides a set of representatives of $\Gamma_2/\Gamma_0(p)$ for all $p \geq 2$. We have to multiply each representative by a matrix in $\Gamma_0(p)$ such that the resulting matrix is in $\Gamma(2,4)$, which is possible by the Chinese remainder theorem.

6.2 Evaluation and interpolation

Until now we have presented the algorithm from a theoretical point of view. In practice, we proceed as follows. Since we want to use fast interpolation, it is necessary to know the degrees of the coefficients in the three invariants f_1, f_2 and f_3. For example, let $F(f_1, f_2, f_3)$ be one of the coefficients we want to compute. To obtain the total degree of the numerator and of the denominator of F, it is enough to compute the matrices Ω in the Siegel space with Algorithm 33 such that $(f_1(\Omega),\ f_2(\Omega),\ f_3(\Omega)) = (x_i, x_i y, x_i z)$ for some x_i and fixed y and z, to evaluate $F(x_i, x_i y, x_i z)$ and then to do the interpolation of a univariate rational fraction.

This also gives upper bounds for the degrees in f_1, f_2 and f_3. To obtain the degrees in f_1 (and similarly in the others), we can compute $F(x_i, y, z)$ and interpolate, but this will not give the a correct answer every time (even if we assume that the precision is correct and that we have enough x_i). Indeed, some simplifications may occur. Thus, to be sure of the result, it is preferable to evaluate and interpolate for many values of y and z and also for $F(X+r, y+s, z+t)$ for some values of r, s and t.

Once we have this information, we have two choices for how to proceed. The first consists in doing sufficiently many evaluations to compute all the coefficients (in X) of the three modular polynomials with interpolation of rational fractions. An evaluation means the computation of the modular polynomials at Ω such that $(f_1(\Omega), f_2(\Omega), f_3(\Omega))$ is of the form $(x_i, x_i y_j, x_i z_k)$. Otherwise we focus first on only one coefficient (the one with the lowest total degree) to compute the common denominator and then we do sufficiently many evaluations (here of the form (x_i, y_j, z_k)) to compute the other coefficients using interpolations of multivariate polynomials. We can speak about polynomials because we can multiply each evaluation by the evaluation of the denominator (and in the case of the Streng invariants, also by an exponent of i_3).

In the first case, the number of evaluations will depend on the maximal total degree of the three polynomials, while in the second case, the total degree will intervene only for the coefficient with lowest degrees. Moreover, the precision needed to interpolate rational fractions is greater than those to interpolate polynomials (and the complexity of an evaluation of the modular polynomials at some matrices of $\mathcal{H}_2$ depends on the precision) and it is easier to interpolate polynomials than rational fractions. For the second choice, the degree tables suggest focusing on the coefficient of highest degree (in X) of $\Phi_{1,p}(X)$.

One can choose to take integer values for the invariants. The matrix Ω with these invariants and also the invariants of the isogenous varieties will not take integer values, but each coefficient of the evaluated modular polynomials will be a rational number. Thus it could be possible at each evaluation to find these rational numbers using continued fractions (if the working precision is good enough). The interpolation phase could then be done using exact values. However when doing this, the precision needed in practice will increase and the time of evaluation too. It is preferable to take floating point values for the invariants and reconstruct the rational numbers once the polynomials have been interpolated at the working precision to find the exact coefficients.

6.3 Timings

Note that in the evaluation there are two steps: given $(f_1(\Omega), f_2(\Omega), f_3(\Omega))$ find Ω and then evaluate the modular polynomials at Ω. The last one takes most of the time (at large enough precision). For example for $p = 5$ and 7 at precision 1000 decimal digits it takes 0.5 seconds to compute Ω from the $b'_i(\Omega)$ and the computation of the two polynomials $\Phi_{1,p}(X, b'_1(\Omega), b'_2(\Omega), b'_3(\Omega))$ and $\Psi_{2,p}(X, b'_1(\Omega), b'_2(\Omega), b'_3(\Omega))$ take 12 and 30 seconds for respectively $p = 5$ and $p = 7$ (this difference is due to the number of isogenies: 156 for one and 400 for the other).

We focus now on the computation of the modular polynomials with Streng invariants (recall the results of Section 4.1). We proceed with the second method which is not always faster (because it requires two evaluation steps), but it has the advantage of providing the denominator which is the origin of most of the difficulties when computing modular polynomials. Moreover, we do the interpolation of univariate rational fraction with linear algebra because it is fast enough.

In level 2, the largest total degree, of the numerator of the coefficient of degree 14 of $\Phi_{1,2}(X)$ is 9 and that of the denominator D_2' is 7. To compute the denominator it is enough to do $(9+7+2)(5+1)(4+1) = 540$ evaluations. Once we have computed them, we do $(33+1)(17+1)(16+1) = 10404$ evaluations to compute the numerators (see Table 1). All of this can be done at a precision of 100 decimal digits. An evaluation takes around 1.33 second so that the denominator can be computed in around 12 minutes and all the polynomials in 4 hours (on one processor).

In level 3, the total degrees are 35 for both the numerator (of the coefficient of degree 39 of $\Phi_{1,3}$) and the denominator. The denominator can be computed with $(35+35+2)(20+1)(17+1) = 27216$ evaluations in 17 hours at precision 300 and then all the numerators with $(92+1)(52+1)(49+1) = 246450$ evaluations (see Table 2) in around 30 days at precision 1000. (The difference in precision here comes from the fact that the integers of the denominator are much smaller than the integers of the numerators). The interpolation phase takes around 1 hour.

To compute the modular polynomials with the b_i', we can use the results found in Sections 4.2 and 5. In particular, we only have to compute the first two modular polynomials.

For $p = 3$, the total degrees are 25 and 24 for the numerator and the denominator of the 39-th coefficient. It takes around $(25+24+2)(12+1)(12+1)/32 \approx 270$ evaluations to obtain the denominator and around $(40+1)(19+1)(18+1)/32 \approx 487$ for the numerators (see Table 3). We used 100 decimal digits for the precision and then an evaluation takes approximately 0.6 seconds so that the (two and thus the three) modular polynomials can be obtained in less than 10 minutes (the interpolation phase is negligible).

For $p = 5$, the total degrees are 121 and 120 for the numerator and the denominator of the 155-th coefficient. The theoretical numbers of evaluations for the denominator and the numerators are $(121+120+2)(72+1)(72+1)/32 < 40500$ and $(156+1)(97+1)(94+1)/32 < 46000$ (see Table 4). They can be done at precision 1000 decimal digits where each evaluation takes roughly 12 seconds. The polynomials can be calculated in less than 12 days (on one processor). The interpolation can be done in less than 2 hours.

For $p = 7$, we have computed at first the common denominator because of memory space (the two first polynomials fill 29 GB). Moreover we found that the leading coefficient of the denominators in b_1' is respectively $2^{10}b_2'^6b_3'^6b_1'^{10}$ and $2^{70}b_2'^{10}b_3'^{10}b_1'^{70}$, so that we conjectured it would be of the same kind for $p = 7$. Through some experimentations, we found that it was $2^{226}b_2'^{38}b_3'^{38}b_1'^{226}$. Knowing this monomial allows one to interpolate as explained in the second paragraph after Remark 19, which reduces the number of evaluations because this number depends on the degree in b_1' instead of the total degree.

The degrees of the 399-th coefficient are 233 and 226 (and the total degrees are 337 and 336 so that the gain is significant). The number of evaluations for the denominator was around $(233+226+2)(226+1)(226+1)/32 < 727000$ and for the numerators of the two modular polynomials around $(400+1)(279+1)(276+1)/32 < 972000$ (see Table 4). For the denominator, we managed to compute it in less than 700 days at precision 2000 and for the numerators in around 2000 days at precision 3000. The interpolation time was around a week. It is negligible compared to the evaluation time.

Finally note that each evaluation is independent of the others so that the computation of modular polynomials is highly parallelizable. The interpolation of a coefficient is independent of the interpolation of the others so that the interpolation step is also parallelizable. Moreover, it is possible to parallelize the interpolation of a single coefficient.

7 Examples of isogenous curves

The main purpose of the modular polynomials is to find hyperelliptic curves with isogenous Jacobians, in particular over a finite field. We give some examples with the different polynomials we have computed. Note that the algorithm we have presented is heuristic because we have no bounds on the precision loss and we have no proof that the polynomials we found are correct. We could do interval arithmetic; what we do instead is to heuristically check for correctness on additional random values not yet used during the evaluation/interpolation algorithm: for some $\Omega \in \mathcal{H}_2$, we have to verify that

$$\Phi_{1,p}(f_{1,p}(\Omega), f_1(\Omega), f_2(\Omega), f_3(\Omega)) = 0$$

and that for $\ell = 2, 3$

$$f_{\ell,p}(\Omega) = \Psi_{\ell,p}(f_{1,p}(\Omega), f_1(\Omega), f_2(\Omega), f_3(\Omega))/\Phi'_{1,p}(f_{1,p}(\Omega), f_1(\Omega), f_2(\Omega), f_3(\Omega)).$$

With one high precision computation, one can be virtually certain that the result is correct.

The Jacobians of the following curves are $(3, 3)$-isogenous varieties. We computed the curves using the modular polynomials with Streng invariants. The first ones over $\mathbb{F}_{5261}$:

$$\begin{aligned} Y^2 &= 272X^5 + 4278X^4 + 4297X^3 + 4063X^2 + 1069X + 2998, \\ Y^2 &= 695X^5 + 2322X^4 + 3115X^3 + 4588X^2 + 1453X + 655 \end{aligned}$$

and the following ones over $\mathbb{F}_{2534267893}$:

$$\begin{aligned} Y^2 &= 1774507961X^6 + 48872812X^5 + 2028583210X^4 + 1092030439X^3 + \\ &\quad 671225738X^2 + 2233670825X + 608155867, \\ Y^2 &= 1927466494X^6 + 2286039407X^5 + 1720123333X^4 + 87910848X^3 + \\ &\quad 2422852850X^2 + 183139891X + 825611194. \end{aligned}$$

We also give two examples of curves with $(5, 5)$-isogenous Jacobians computed using the modular polynomials with the b'_i. Over $\mathbb{F}_{101}$:

$$\begin{aligned} Y^2 &= 27X^5 + 71X^4 + 91X^3 + 59X^2 + 5X + 14, \\ Y^2 &= 29X^5 + 26X^4 + 38X^3 + 20X^2 + 7X + 51 \end{aligned}$$

and over $\mathbb{F}_{4294967311}$:

$$\begin{aligned} Y^2 &= 2420332800X^5 + 3653091983X^4 + 2536585478X^3 + 2805510580X^2 + \\ &\quad 159741347X + 2690010753, \\ Y^2 &= 4076826784X^5 + 2616936853X^4 + 3748957676X^3 + 1209100179X^2 + \\ &\quad 3172892980X + 1266950302. \end{aligned}$$

Finally, we give two pairs of curves with $(7, 7)$-isogenous Jacobians, computed using the modular polynomials with the b'_i. Over $\mathbb{F}_{10009}$:

$$\begin{aligned} Y^2 &= 4826X^5 + 471X^4 + 2876X^3 + 5411X^2 + 7948X + 1308, \\ Y^2 &= 7218X^5 + 7699X^4 + 7011X^3 + 7103X^2 + 1845X + 4087 \end{aligned}$$

and over $\mathbb{F}_{3452678353}$:

$$\begin{aligned} Y^2 &= 393356368X^5 + 1698662093X^4 + 471351782X^3 + 448279016X^2 + \\ &\quad 1342046779X + 3241061457, \\ Y^2 &= 2171506943X^5 + 2231412358X^4 + 2005208933X^3 + 580698082X^2 + \\ &\quad 306153493X + 474327543. \end{aligned}$$

The motivated reader can check the curves we have constructed with the modular polynomials: it is enough to verify that the curves have the same zeta functions.

The polynomials are accessible at the adress: `http://www.math.u-bordeaux1.fr/~emilio/`.

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