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On the proper orientation number of bipartite graphs

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Abstract

An orientation of a graph $G$ is a digraph $D$ obtained from $G$ by replacing each edge by exactly one of the two possible arcs with the same endvertices. For each $v \in V(G)$, the indegree of $v$ in $D$, denoted by $d_D^-(v)$, is the number of arcs with head $v$ in $D$. An orientation $D$ of $G$ is proper if $d_D^+(u) \neq d_D^-(v)$, for all $uv \in E(G)$. The proper orientation number of a graph $G$, denoted by $\overline{\chi}(G)$, is the minimum of the maximum indegree over all its proper orientations. It is well-known that $\overline{\chi}(G) \leq \Delta(G)$, for every graph $G$. In this paper, we first prove that $\overline{\chi}(G) \leq \left\lfloor \frac{\Delta(G) + \sqrt{\Delta(G)}}{2} \right\rfloor + 1$ if $G$ is a bipartite graph, and $\overline{\chi}(G) \leq 4$ if $G$ is a tree. We then prove that deciding whether $\overline{\chi}(G) \leq \Delta(G) - 1$ is an NP-complete problem. We also show that it is NP-complete to decide whether $\overline{\chi}(G) \leq 2$, for planar subcubic graphs $G$. Moreover, we prove that it is NP-complete to decide whether $\overline{\chi}(G) \leq 3$, for planar bipartite graphs $G$ with maximum degree 5.

Keywords: proper orientation, graph colouring, bipartite graph, hardness.

1. Introduction

In this paper, all graphs are simple, that is without loops and multiple edges. We follow standard terminology as used in [1].

An orientation $D$ of a graph $G$ is a digraph obtained from $G$ by replacing each edge by just one of the two possible arcs with the same endvertices. For each $v \in V(G)$, the indegree of $v$ in $D$, denoted by $d_D^-(v)$, is the number of arcs with head $v$ in $D$. We use the notation $d^-(v)$ when the orientation $D$ is clear from the context. The orientation $D$ of $G$ is proper if $d^-(u) \neq d^-(v)$, for all $uv \in E(G)$. An orientation with maximum indegree at most $k$ is called a $k$-orientation. The proper orientation number of a graph $G$, denoted by $\overline{\chi}(G)$, is the minimum integer $k$ such that $G$ admits a proper $k$-orientation. This graph parameter was introduced by Ahadi and Dehghan [2]. It is well-defined for any graph $G$ since one can always obtain a proper $\Delta(G)$-orientation (see [2]). In other words, $\overline{\chi}(G) \leq \Delta(G)$. Note that every proper orientation of a graph $G$ induces a proper vertex colouring of $G$. Thus, $\overline{\chi}(G) \geq \chi(G) - 1$. Hence, we have the following sequence of inequalities: $\omega(G) - 1 \leq \chi(G) - 1 \leq \overline{\chi}(G) \leq \Delta(G)$.

These inequalities are best possible in the sense that, for a complete graph $K$, $\omega(K) - 1 = \chi(K) - 1 = \overline{\chi}(K) = \Delta(K)$. However, one might expect better upper bounds on some parameters by taking a convex

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combination of two others. Reed [3] showed that there exists \( \epsilon_0 > 0 \) such that \( \chi(G) \leq \epsilon_0 \cdot \omega(G) + (1 - \epsilon_0) \Delta(G) \) for every graph \( G \) and conjectured the following.

**Conjecture 1** (Reed [3]). For every graph \( G \), \( \chi(G) \leq \left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil \).

If true, this conjecture would be tight. Johansson [4] settled Conjecture 1 for \( \omega(G) = 2 \) and \( \Delta(G) \) sufficiently large.

Likewise, one may wonder if similar upper bounds might be derived for the proper orientation number.

**Problem 1.**

(a) Does there exist a positive \( \epsilon_1 \) such that \( \chi(G) \leq \epsilon_1 \cdot \omega(G) + (1 - \epsilon_1) \Delta(G) \)?

(b) Does there exist a positive \( \epsilon_2 \) such that \( \chi(G) \leq \epsilon_2 \cdot \chi(G) + (1 - \epsilon_2) \Delta(G) \)?

Observe that both questions are intimately related. Indeed if the answer to (a) is positive for \( \epsilon_1 \), then the answer to (b) is also positive for \( \epsilon_1 \). On the other hand, if the answer to (b) is positive for \( \epsilon_2 \), then the answer to (a) is also positive for \( \epsilon_1 = \epsilon_0 \cdot \epsilon_2 \) by the above-mentioned result of Reed.

In Section 2, we answer Problem 1 positively in the case of bipartite graphs by showing that: if \( G \) is bipartite, then \( \chi(G) \leq \left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil + 1 \). We also argue that this bound is tight for \( \Delta(G) \in \{2, 3\} \).

In Section 3, we prove that \( \chi(T) \leq 4 \), for every tree \( T \). Moreover, we show that \( \chi(T) \leq 3 \) if \( \Delta(T) \leq 6 \), and \( \chi(T) \leq 2 \) if \( \Delta(T) \leq 3 \). We also argue that all these bounds are tight.

In Section 4, we study the computational complexity of computing the proper orientation number of a bipartite graph. In their seminal paper, Ahadi and Dehghan proved that it is \( \text{NP}\)-complete to decide whether \( \chi(G) = 2 \) for planar graphs \( G \). We first improve their reduction and show that it is \( \text{NP}\)-complete to decide whether \( \chi(G) \leq 2 \), for planar subcubic graphs \( G \). Moreover, we prove that deciding whether \( \chi(G) \leq \Delta(G) - 1 \) is an \( \text{NP}\)-complete problem for general graphs \( G \). Finally, we show that it is also \( \text{NP}\)-complete to decide whether \( \chi(G) \leq 3 \) for planar bipartite graphs \( G \) with maximum degree 5.

Due to space limitation, we omit the proofs of these results.

**2. General upper bound**

**Theorem 1.** Let \( G \) be a bipartite graph and let \( k \) be a positive integer. If \( \Delta(G) > 2k + \frac{\sqrt{\Delta(G)eM}+1}{2} \), then \( \chi(G) \leq \Delta(G) - k \).

**Sketch of proof.** In order to prove this theorem, we describe an algorithm (see Algorithm 1) that produces a proper \((\Delta(G) - k)\)-orientation. Let \( G = (X \cup Y, E) \) be a bipartite graph as in the statement of Theorem 1. The algorithm consists of two phases.

The first phase (lines 1 to 8 in Algorithm 1) produces an orientation, not necessarily proper, of the edges of \( G \) in such a way that the indegree of each vertex in \( X \) is at most \( k \) and the indegree of each vertex in \( Y \) is at most \( \Delta(G) - k \). It proceeds as follows. We first orient all edges \( xy \in E(G) \) from \( x \) to \( y \), where \( x \in X \) and \( y \in Y \). Then we define \( k \) matchings as described subsequently.

Let \( G_1 = G \), and let \( M_1 \) be a matching in \( G_1 \) that covers all vertices of maximum degree. For each \( i \in \{2, \ldots, k\} \), let \( G_i \) be the graph obtained from \( G_{i-1} \) by removing the edges in \( M_{i-1} \), that is \( G_i = G_{i-1} \setminus M_{i-1} \), and let \( M_i \) be a matching in \( G_i \) that covers all vertices of degree \( \Delta(G_i) \). Such a \( M_i \) exists since it is well known that every bipartite graph \( H \) has a proper \( \Delta(H) \)-edge-colouring. Clearly, we have \( \Delta(G_i) = \Delta(G_{i-1}) - 1 \), for each \( i \in \{2, 3, \ldots, k\} \). Let \( M := \bigcup_{i=1}^{k} M_i \). Observe that if a vertex has degree \( \Delta(G) - k + j \) in \( G \), where \( j \in \{1, 2, \ldots, k\} \), then it is incident to at least \( j \) edges in \( M \). Hence, for all \( j \in \{1, 2, \ldots, k\} \) and for each vertex \( y \) in \( Y \) of degree \( \Delta(G) - k + j \) in \( G \), we reverse the orientation of exactly \( j \) edges in \( M \) incident to \( y \). This ends the first phase.

The second phase reverses the orientation of some edges in \( E(G) \setminus M \), step by step, in order to obtain a \((\Delta(G) - k)\)-orientation. This orientation is proper under the assumption of Theorem 1.
Algorithm 1: Proper Orientation of Bipartite Graphs

\begin{algorithm}
\begin{algorithmic}
\Input{Bipartite graph $G = (X \cup Y, E)$ and $k \in \mathbb{N}$ s.t. $\Delta(G) > 2k + \frac{\sqrt{1+8k}+1}{2}$.}
\Output{Proper $(\Delta(G) - k)$-orientation for $G$.}
\State $G_1 \leftarrow G$
\State Orient all edges in $G$ from $X$ to $Y$
\For{$i = 1, \ldots, k$}
\State $M_i \leftarrow$ matching of $G_i$ saturating all vertices of degree $\Delta(G_i)$
\State $G_{i+1} \leftarrow G_i - M_i$
\EndFor
\State $M \leftarrow \bigcup_{i=1}^{k} M_i$
\ForAll{$y \in Y$}
\State $\bar{X} \leftarrow X$
\While{$\exists x \in X$ s.t. $|N_{\leq \ell}(x)| \geq \ell - d^-(x)$ and $|N_{\geq \ell}(x)| \leq \ell - d^-(x)$}
\State $\bar{Y} \leftarrow$ set of $\ell - d^-(x)$ vertices of highest indegree in $N_{\leq \ell}(x)$
\State $\bar{X} \leftarrow \bar{X} \setminus \{x\}$
\EndWhile
\EndFor
\EndFor
\end{algorithmic}
\end{algorithm}

Theorem 2. If $G$ is a bipartite graph, then $\chi^\Delta(G) \leq \left\lceil \frac{\Delta(G)+\sqrt{\Delta(G)}}{2} \right\rceil + 1$.

\textit{Sketch of proof.} By Theorem 1, for every $k \in \mathbb{N}$, if $\Delta(G) > 2k + \frac{\sqrt{1+8k}+1}{2}$, then $\chi^\Delta(G) \leq \Delta(G) - k$. In order to obtain a good upper bound for $\chi^\Delta(G)$, we must find the largest positive integer $k$ such that the condition of Theorem 1 holds for a given graph $G$.

Solving the inequality for $k$, we obtain that $k < \frac{\Delta(G)-\sqrt{\Delta(G)}}{2}$. Since $k$ is integer, we conclude that $\chi^\Delta(G) \leq \Delta(G) - \left\lceil \frac{\Delta(G)-\sqrt{\Delta(G)}}{2} \right\rceil + 1$, and the result follows.

Note that if $G$ is bipartite and $\Delta(G) \in \{2, 3, 4\}$, then the bound of Theorem 2 is equal to the trivial upper bound $\chi^\Delta(G) \leq \Delta(G)$. For $\Delta(G) = 1$ and $\Delta(G) = 2$, this bound is tight due to the paths with 2 and 4 vertices, respectively. In addition, there exists a bipartite graph $G$ with $\Delta(G) = 3$ and $\chi^\Delta(G) = 3$.

3. Trees

Theorem 3. If $T$ is a tree, then the following statements hold:

(1) if $\Delta(T) \leq 3$, then $\chi^\Delta(T) \leq 2$;
(2) if $\Delta(T) \leq 6$, then $\chi^\Delta(T) \leq 3$;
(3) $\chi^\Delta(T) \leq 4$.

\textit{Sketch of proof.} We prove the three statements by using similar arguments. For $i \in \{1, 2, 3\}$, we consider a minimal counter-example $M_i$ to statement $(i)$ with respect to the number of vertices, and derive a contradiction that implies that no counter-example exists. Since $M_i$ is a minimal counter-example, we have $\chi^\Delta(M_i) > i + 1$, but $\chi^\Delta(T) \leq i + 1$, for any proper subtree $T$ of $M_i$. We use the latter fact to derive a proper $(i + 1)$-orientation of $M_i$, which contradicts $\chi^\Delta(M_i) > i + 1$. }
The three statements of the theorem are tight in the following sense: there is a tree with maximum
dergree 4 and proper orientation number 3, and a tree with maximum degree 7 and proper orientation
number 4.

4. \textit{NP}-completeness

Ahadi and Dehgan [2] showed that it is \textit{NP}-complete to decide whether \(\chi^+(G) \leq 2\) for planar graphs \(G\) by using a reduction from the \textsc{Planar 3-SAT} problem. We first improve this result by showing that it is \textit{NP}-complete to decide whether the proper orientation number of planar subcubic graphs is at most 2.

\textbf{Theorem 4.} The following problem is \textit{NP}-complete:
\begin{itemize}
\item \textbf{Input:} A planar graph \(G\) with \(\Delta(G) = 3\) and \(\delta(G) = 2\).
\item \textbf{Question:} \(\chi^+(G) \leq 2\)?
\end{itemize}

\textit{Sketch of proof.} We show a reduction from the problem of deciding whether a planar 3-SAT formula is satisfiable. It is known that the \textsc{Planar 3-SAT} problem is \textit{NP}-complete [5]. Let \(\phi = (X, C)\) be an instance of this problem, where \(X = \{x_1, \ldots, x_n\}\) is the set of variables and \(C = \{C_1, \ldots, C_m\}\) is the set of clauses. Using the variable and clause gadgets depicted in Figures 1(a) and 1(b), respectively, we construct a planar graph \(G'(\phi)\) such that \(\chi^+(G'(\phi)) \leq 2\) if, and only if, \(\phi\) is satisfiable. \qed

Figure 1: The variable and clause gadgets.

Recall that \(\chi^+(G) \leq \Delta(G)\), for any graph \(G\). On the other hand, the following theorem shows that, for any integer \(k \geq 3\), it is already \textit{NP}-complete to determine whether \(\chi^+(G) < k\), for graphs \(G\) with \(\Delta(G) = k\).

\textbf{Theorem 5.} Let \(k\) be an integer such that \(k \geq 3\). The following problem is \textit{NP}-complete:
\begin{itemize}
\item \textbf{Input:} A graph \(G\) with \(\Delta(G) = k\) and \(\delta(G) = k - 1\).
\item \textbf{Question:} \(\chi^+(G) \leq k - 1\)?
\end{itemize}

Finally, we show that it is \textit{NP}-complete to decide whether \(\chi^+(G) \leq 3\), for planar bipartite graphs \(G\).

\textbf{Theorem 6.} The following problem is \textit{NP}-complete:
\begin{itemize}
\item \textbf{Input:} A planar bipartite graph \(G\) with \(\Delta(G) = 5\).
\item \textbf{Question:} \(\chi^+(G) \leq 3\)?
\end{itemize}

\textit{Sketch of proof.} We show a reduction from the problem of deciding whether a planar monotone 3-SAT formula is satisfiable. This problem was recently shown to be \textit{NP}-complete [6]. The idea of our reduction is roughly the same as in Theorems 4 and 5. \qed
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