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On the proper orientation number of bipartite graphs

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Abstract

An orientation of a graph \( G \) is a digraph \( D \) obtained from \( G \) by replacing each edge by exactly one of the two possible arcs with the same endvertices. For each \( v \in V(G) \), the indegree of \( v \) in \( D \), denoted by \( d_D^-(v) \), is the number of arcs with head \( v \) in \( D \). An orientation \( D \) of \( G \) is proper if \( d_D^-(u) \neq d_D^-(v) \), for all \( uv \in E(G) \). The proper orientation number of a graph \( G \), denoted by \( \chi(G) \), is the minimum of the maximum indegree over all its proper orientations. It is well-known that \( \chi(G) \leq \Delta(G) \), for every graph \( G \). In this paper, we first prove that \( \chi(G) \leq \left( \Delta(G) + \sqrt{\Delta(G)} \right) / 2 \) if \( G \) is a bipartite graph, and \( \chi(G) \leq 4 \) if \( G \) is a tree.

We then prove that deciding whether \( \chi(G) \leq \Delta(G) - 1 \) is an \( \mathsf{NP} \)-complete problem. We also show that it is \( \mathsf{NP} \)-complete to decide whether \( \chi(G) \leq 2 \), for planar subcubic graphs \( G \). Moreover, we prove that it is \( \mathsf{NP} \)-complete to decide whether \( \chi(G) \leq 3 \), for planar bipartite graphs \( G \) with maximum degree 5.

Keywords: proper orientation, graph colouring, bipartite graph, hardness.

1. Introduction

In this paper, all graphs are simple, that is without loops and multiple edges. We follow standard terminology as used in [1].

An orientation \( D \) of a graph \( G \) is a digraph obtained from \( G \) by replacing each edge by just one of the two possible arcs with the same endvertices. For each \( v \in V(G) \), the indegree of \( v \) in \( D \), denoted by \( d_D^-(v) \), is the number of arcs with head \( v \) in \( D \). We use the notation \( d^-(v) \) when the orientation \( D \) is clear from the context. The orientation \( D \) of \( G \) is proper if \( d^-(u) \neq d^-(v) \), for all \( uv \in E(G) \). An orientation with maximum indegree at most \( k \) is called a \( k \)-orientation. The proper orientation number of a graph \( G \), denoted by \( \chi(G) \), is the minimum integer \( k \) such that \( G \) admits a proper \( k \)-orientation. This graph parameter was introduced by Ahadi and Dehghan [2]. It is well-defined for any graph \( G \) since one can always obtain a proper \( \Delta(G) \)-orientation (see [2]). In other words, \( \chi(G) \leq \Delta(G) \). Note that every proper orientation of a graph \( G \) induces a proper vertex colouring of \( G \). Thus, \( \chi(G) \geq \chi(G) - 1 \). Hence, we have the following sequence of inequalities: \( \omega(G) - 1 \leq \chi(G) - 1 \leq \chi(G) \leq \Delta(G) \).

These inequalities are best possible in the sense that, for a complete graph \( K \), \( \omega(K) - 1 = \chi(K) - 1 = \chi(K) = \Delta(K) \). However, one might expect better upper bounds on some parameters by taking a convex...
combination of two others. Reed [3] showed that there exists $\epsilon_0 > 0$ such that $\chi(G) \leq \epsilon_0 \cdot \omega(G) + (1 - \epsilon_0) \Delta(G)$ for every graph $G$ and conjectured the following.

**Conjecture 1** (Reed [3]). For every graph $G$, $\chi(G) \leq \left\lceil \frac{\Delta(G) + 1 + \omega(G)}{2} \right\rceil$.

If true, this conjecture would be tight. Johansson [4] settled Conjecture 1 for $\omega(G) = 2$ and $\Delta(G)$ sufficiently large.

Likewise, one may wonder if similar upper bounds might be derived for the proper orientation number.

**Problem 1.**

(a) Does there exist a positive $\epsilon_1$ such that $\chi(G) \leq \epsilon_1 \cdot \omega(G) + (1 - \epsilon_1) \Delta(G)$?

(b) Does there exist a positive $\epsilon_2$ such that $\chi(G) \leq \epsilon_2 \cdot \chi(G) + (1 - \epsilon_2) \Delta(G)$?

Observe that both questions are intimately related. Indeed if the answer to (a) is positive for $\epsilon_1$, then the answer to (b) is also positive for $\epsilon_1$. On the other hand, if the answer to (b) is positive for $\epsilon_2$, then the answer to (a) is also positive for $\epsilon_1 = \epsilon_0 \cdot \epsilon_2$ by the above-mentioned result of Reed.

In Section 2, we answer Problem 1 positively in the case of bipartite graphs by showing that: if $G$ is bipartite, then $\chi(G) \leq \left\lceil \frac{\Delta(G) + \sqrt{\Delta(G)}}{2} \right\rceil + 1$. We also argue that this bound is tight for $\Delta(G) \in \{2, 3\}$.

In Section 3, we prove that $\chi(G) \leq 4$, for every tree $T$. Moreover, we show that $\chi(G) \leq 3$ if $\Delta(T) \leq 6$, and $\chi(G) \leq 2$ if $\Delta(T) \leq 3$. We also argue that all these bounds are tight.

In Section 4, we study the computational complexity of computing the proper orientation number of a bipartite graph. In their seminal paper, Ahadi and Dehghan proved that it is $\mathcal{NP}$-complete to decide whether $\chi(G) = 2$ for planar graphs $G$. We first improve their reduction and show that it is also $\mathcal{NP}$-complete to decide whether $\chi(G) \leq 2$, for planar subcubic graphs $G$. Moreover, we prove that deciding whether $\chi(G) \leq \Delta(G) - 1$ is an $\mathcal{NP}$-complete problem for general graphs $G$. Finally, we show that it is also $\mathcal{NP}$-complete to decide whether $\chi(G) \leq 3$ for planar bipartite graphs $G$ with maximum degree 5.

Due to space limitation, we omit the proofs of these results.

**2. General upper bound**

**Theorem 1.** Let $G$ be a bipartite graph and let $k$ be a positive integer. If $\Delta(G) > 2k + \sqrt{2k^2 - 1}$, then $\chi(G) \leq \Delta(G) - k$.

**Sketch of proof.** In order to prove this theorem, we describe an algorithm (see Algorithm 1) that produces a proper $(\Delta(G) - k)$-orientation. Let $G = (X \cup Y, E)$ be a bipartite graph as in the statement of Theorem 1. The algorithm consists of two phases.

The first phase (lines 1 to 8 in Algorithm 1) produces an orientation, not necessarily proper, of the edges of $G$ in such a way that the indegree of each vertex in $X$ is at most $k$ and the indegree of each vertex in $Y$ is at most $\Delta(G) - k$. It proceeds as follows. We first orient all edges $xy \in E(G)$ from $x$ to $y$, where $x \in X$ and $y \in Y$. Then we define $k$ matchings as described subsequently.

Let $G_1 = G$, and let $M_1$ be a matching in $G_1$ that covers all vertices of maximum degree. For each $i \in \{2, \ldots, k\}$, let $G_i$ be the graph obtained from $G_{i-1}$ by removing the edges in $M_{i-1}$, that is $G_i = G_{i-1} \setminus M_{i-1}$, and let $M_i$ be a matching in $G_i$ that covers all vertices of degree $\Delta(G_i)$. Such a $M_i$ exists since it is well known that every bipartite graph $H$ has a proper $\Delta(H)$-edge-colouring. Clearly, we have $\Delta(G_i) = \Delta(G_{i-1}) - 1$, for each $i \in \{2, 3, \ldots, k\}$. Let $M := \bigcup_{i=1}^k M_i$. Observe that if a vertex has degree $\Delta(G) - k + j$ in $G$, where $j \in \{1, 2, \ldots, k\}$, then it is incident to at least $j$ edges in $M$. Hence, for all $j \in \{1, 2, \ldots, k\}$ and for each vertex $y$ in $Y$ of degree $\Delta(G) - k + j$ in $G$, we reverse the orientation of exactly $j$ edges in $M$ incident to $y$. This ends the first phase.

The second phase reverses the orientation of some edges in $E(G) \setminus M$, step by step, in order to obtain a $(\Delta(G) - k)$-orientation. This orientation is proper under the assumption of Theorem 1. □
Algorithm 1: Proper Orientation of Bipartite Graphs

\[
\begin{align*}
\textbf{Input:} & \quad \text{Bipartite graph } G = (X \cup Y, E) \text{ and } k \in \mathbb{N} \text{ s.t. } \Delta(G) > 2k + \frac{\sqrt{1+8k+1}}{2}.
\textbf{Output:} & \quad \text{Proper } (\Delta(G) - k)\text{-orientation for } G.
G_1 & \leftarrow G \\
& \text{Orient all edges in } G \text{ from } X \text{ to } Y \\
& \text{for } i = 1, \ldots, k \text{ do} \\
& \quad M_i \leftarrow \text{matching of } G_i \text{ saturating all vertices of degree } \Delta(G_i) \\
& \quad G_{i+1} \leftarrow G_i - M_i \\
& \quad M \leftarrow \bigcup_{i=1}^{k} M_i \\
& \text{foreach } y \in Y \text{ do} \\
& \quad \text{reverse the orientation of max}\{0; d_G(y) - \Delta(G) + k\} \text{ edges of } M \text{ incident to } y \\
& \quad \tilde{X} \leftarrow X \\
& \text{for } \ell = \Delta(G) - k - 1, \ldots, 2 \text{ do} \\
& \quad \text{while } \exists x \in X \text{ s.t. } |N_{\leq \ell}(x)| \geq \ell - d^{-}(x) \text{ and } |N_{= \ell}(x)| \leq \ell - d^{-}(x) \text{ do} \\
& \quad \quad \tilde{Y} \leftarrow \text{set of } \ell - d^{-}(x) \text{ vertices of highest indegree in } N_{\leq \ell}(x) \\
& \quad \quad \text{foreach } y \in \tilde{Y} \text{ do} \\
& \quad \quad \quad \text{Reverse the orientation of } xy \text{ (i.e. re-orient } xy \text{ towards } x) \\
& \quad \quad \tilde{X} \leftarrow \tilde{X} \setminus \{x\}
\end{align*}
\]

Theorem 2. If \( G \) is a bipartite graph, then \( \bar{\chi}(G) \leq \left\lfloor \frac{\Delta(G)+\sqrt{\Delta(G)}}{2} \right\rfloor + 1. \)

\textit{Sketch of proof.} By Theorem 1, for every \( k \in \mathbb{N} \), if \( \Delta(G) > 2k + \frac{\sqrt{1+8k+1}}{2} \), then \( \bar{\chi}(G) \leq \Delta(G) - k \). In order to obtain a good upper bound for \( \bar{\chi}(G) \), we must find the largest positive integer \( k \) such that the condition of Theorem 1 holds for a given graph \( G \).

Solving the inequality for \( k \), we obtain that \( k < \frac{\Delta(G)-\sqrt{\Delta(G)}}{2} \). Since \( k \) is integer, we conclude that \( \bar{\chi}(G) \leq \Delta(G) - \left\lfloor \frac{\Delta(G)-\sqrt{\Delta(G)}}{2} \right\rfloor + 1 \), and the result follows.

Note that if \( G \) is bipartite and \( \Delta(G) \in \{2, 3, 4\} \), then the bound of Theorem 2 is equal to the trivial upper bound \( \bar{\chi}(G) \leq \Delta(G) \). For \( \Delta(G) = 1 \) and \( \Delta(G) = 2 \), this bound is tight due to the paths with 2 and 4 vertices, respectively. In addition, there exists a bipartite graph \( G \) with \( \Delta(G) = 3 \) and \( \bar{\chi}(G) = 3 \).

3. Trees

Theorem 3. If \( T \) is a tree, then the following statements hold:

(1) if \( \Delta(T) \leq 3 \), then \( \bar{\chi}(T) \leq 2 \);
(2) if \( \Delta(T) \leq 6 \), then \( \bar{\chi}(T) \leq 3 \);
(3) \( \bar{\chi}(T) \leq 4 \).

\textit{Sketch of proof.} We prove the three statements by using similar arguments. For \( i \in \{1, 2, 3\} \), we consider a minimal counter-example \( M_i \) to statement (i) with respect to the number of vertices, and derive a contradiction that implies that no counter-example exists. Since \( M_i \) is a minimal counter-example, we have \( \bar{\chi}(M_i) > i + 1 \), but \( \bar{\chi}(T) \leq i + 1 \), for any proper subtree \( T \) of \( M_i \). We use the latter fact to derive a proper \( (i + 1) \)-orientation of \( M_i \), which contradicts \( \bar{\chi}(M_i) > i + 1 \). \( \square \)
The three statements of the theorem are tight in the following sense: there is a tree with maximum degree 4 and proper orientation number 3, and a tree with maximum degree 7 and proper orientation number 4.

4. NP-completeness

Ahadi and Dehgan [2] showed that it is NP-complete to decide whether \( \chi^r(G) \leq 2 \) for planar graphs \( G \) by using a reduction from the PLANAR 3-SAT problem. We first improve this result by showing that it is NP-complete to decide whether the proper orientation number of planar subcubic graphs is at most 2.

**Theorem 4.** The following problem is NP-complete:
**Input:** A planar graph \( G \) with \( \Delta(G) = 3 \) and \( \delta(G) = 2 \).
**Question:** \( \chi^r(G) \leq 2 ? \)

Sketch of proof. We show a reduction from the problem of deciding whether a planar 3-SAT formula is satisfiable. It is known that the PLANAR 3-SAT problem is NP-complete [5].

Let \( \phi = (X, \mathcal{C}) \) be an instance of this problem, where \( X = \{x_1, \ldots, x_n\} \) is the set of variables and \( \mathcal{C} = \{C_1, \ldots, C_m\} \) is the set of clauses. Using the variable and clause gadgets depicted in Figures 1(a) and 1(b), respectively, we construct a planar graph \( G'(\phi) \) such that \( \chi^r(G'(\phi)) \leq 2 \) if, and only if, \( \phi \) is satisfiable.

![Figure 1: The variable and clause gadgets.](image)

Recall that \( \chi^r(G) \leq \Delta(G) \), for any graph \( G \). On the other hand, the following theorem shows that, for any integer \( k \geq 3 \), it is already NP-complete to determine whether \( \chi^r(G) < k \), for graphs \( G \) with \( \Delta(G) = k \).

**Theorem 5.** Let \( k \) be an integer such that \( k \geq 3 \). The following problem is NP-complete:
**Input:** A graph \( G \) with \( \Delta(G) = k \) and \( \delta(G) = k - 1 \).
**Question:** \( \chi^r(G) \leq k - 1 ? \)

Finally, we show that it is NP-complete to decide whether \( \chi^r(G) \leq 3 \), for planar bipartite graphs \( G \).

**Theorem 6.** The following problem is NP-complete:
**Input:** A planar bipartite graph \( G \) with \( \Delta(G) = 5 \).
**Question:** \( \chi^r(G) \leq 3 ? \)

Sketch of proof. We show a reduction from the problem of deciding whether a planar monotone 3-SAT formula is satisfiable. This problem was recently shown to be NP-complete [6]. The idea of our reduction is roughly the same as in Theorems 4 and 5.
References