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On the proper orientation number of bipartite graphs

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\section*{Abstract}

An orientation of a graph $G$ is a digraph $D$ obtained from $G$ by replacing each edge by exactly one of the two possible arcs with the same endvertices. For each $v \in V(G)$, the indegree of $v$ in $D$, denoted by $d_D^-(v)$, is the number of arcs with head $v$ in $D$. An orientation $D$ of $G$ is proper if $d_D^-(v) \neq d_D^+(v)$, for all $uv \in E(G)$. The proper orientation number of a graph $G$, denoted by $\overline{\chi}(G)$, is the minimum of the maximum indegree over all its proper orientations. It is well-known that $\overline{\chi}(G) \leq \Delta(G)$, for every graph $G$. In this paper, we first prove that $\overline{\chi}(G) \leq \left\lfloor \left( \Delta(G) + \sqrt{\Delta(G)} \right) / 2 \right\rfloor + 1$ if $G$ is a bipartite graph, and $\overline{\chi}(G) \leq 4$ if $G$ is a tree. We then prove that deciding whether $\overline{\chi}(G) \leq \Delta(G) - 1$ is an $\mathcal{NP}$-complete problem. We also show that it is $\mathcal{NP}$-complete to decide whether $\overline{\chi}(G) \leq 2$, for planar subcubic graphs $G$. Moreover, we prove that it is $\mathcal{NP}$-complete to decide whether $\overline{\chi}(G) \leq 3$, for planar bipartite graphs $G$ with maximum degree 5.

\textbf{Keywords:} proper orientation, graph colouring, bipartite graph, hardness.

\section{Introduction}

In this paper, all graphs are simple, that is without loops and multiple edges. We follow standard terminology as used in [1].

An orientation $D$ of a graph $G$ is a digraph obtained from $G$ by replacing each edge by just one of the two possible arcs with the same endvertices. For each $v \in V(G)$, the indegree of $v$ in $D$, denoted by $d_D^-(v)$, is the number of arcs with head $v$ in $D$. An orientation $D$ of $G$ is proper if $d_D^-(v) \neq d_D^+(v)$, for all $uv \in E(G)$. An orientation with maximum indegree at most $k$ is called a $k$-orientation. The proper orientation number of a graph $G$, denoted by $\overline{\chi}(G)$, is the minimum integer $k$ such that $G$ admits a proper $k$-orientation. This graph parameter was introduced by Ahadi and Dehghan [2]. It is well-defined for any graph $G$ since one can always obtain a proper $\Delta(G)$-orientation (see [2]). In other words, $\overline{\chi}(G) \leq \Delta(G)$. Note that every proper orientation of a graph $G$ induces a proper vertex colouring of $G$. Thus, $\overline{\chi}(G) \geq \chi(G) - 1$. Hence, we have the following sequence of inequalities: $\omega(G) - 1 \leq \chi(G) - 1 \leq \overline{\chi}(G) \leq \Delta(G)$.

These inequalities are best possible in the sense that, for a complete graph $K$, $\omega(K) - 1 = \chi(K) - 1 = \overline{\chi}(K) = \Delta(K)$. However, one might expect better upper bounds on some parameters by taking a convex

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combination of two others. Reed [3] showed that there exists $\epsilon_0 > 0$ such that $\chi(G) \leq \epsilon_0 \cdot \omega(G) + (1 - \epsilon_0) \Delta(G)$ for every graph $G$ and conjectured the following.

**Conjecture 1** (Reed [3]). For every graph $G$, $\chi(G) \leq \left\lfloor \frac{\Delta(G)+1+\omega(G)}{2} \right\rfloor$.

If true, this conjecture would be tight. Johansson [4] settled Conjecture 1 for $\omega(G) = 2$ and $\Delta(G)$ sufficiently large.

Likewise, one may wonder if similar upper bounds might be derived for the proper orientation number.

**Problem 1.**

(a) Does there exist a positive $\epsilon_1$ such that $\chi(G) \leq \epsilon_1 \cdot \omega(G) + (1 - \epsilon_1) \Delta(G)$?

(b) Does there exist a positive $\epsilon_2$ such that $\chi(G) \leq \epsilon_2 \cdot \chi(G) + (1 - \epsilon_2) \Delta(G)$?

Observe that both questions are intimately related. Indeed if the answer to (a) is positive for $\epsilon_1$, then the answer to (b) is also positive for $\epsilon_1$. On the other hand, if the answer to (b) is positive for $\epsilon_2$, then the answer to (a) is also positive for $\epsilon_1 = \epsilon_0 \cdot \epsilon_2$ by the above-mentioned result of Reed.

In Section 2, we answer Problem 1 positively in the case of bipartite graphs by showing that: if $G$ is bipartite, then $\chi(G) \leq \left\lfloor \frac{\Delta(G)+\Delta(G)}{2} \right\rfloor + 1$. We also argue that this bound is tight for $\Delta(G) \in \{2, 3\}$.

In Section 3, we prove that $\chi(T) \leq 4$, for every tree $T$. Moreover, we show that $\chi(T) \leq 3$ if $\Delta(T) \leq 6$, and $\chi(T) \leq 2$ if $\Delta(T) \leq 3$. We also argue that all these bounds are tight.

In Section 4, we study the computational complexity of computing the proper orientation number of a bipartite graph. In their seminal paper, Ahadi and Dehghan proved that it is $\text{NP}$-complete to decide whether $\chi(G) = 2$ for planar graphs $G$. We first improve their reduction and show that it is $\text{NP}$-complete to decide whether $\chi(G) \leq 2$, for planar subcubic graphs $G$. Moreover, we prove that deciding whether $\chi(G) \leq \Delta(G) - 1$ is an $\text{NP}$-complete problem for general graphs $G$. Finally, we show that it is also $\text{NP}$-complete to decide whether $\chi(G) \leq 3$ for planar bipartite graphs $G$ with maximum degree 5.

Due to space limitation, we omit the proofs of these results.

## 2. General upper bound

**Theorem 1.** Let $G$ be a bipartite graph and let $k$ be a positive integer. If $\Delta(G) > 2k + \frac{\sqrt{8k^2 + 1}}{2}$, then $\chi(G) \leq \Delta(G) - k$.

**Sketch of proof.** In order to prove this theorem, we describe an algorithm (see Algorithm 1) that produces a proper $(\Delta(G) - k)$-orientation. Let $G = (X \cup Y, E)$ be a bipartite graph as in the statement of Theorem 1. The algorithm consists of two phases.

The first phase (lines 1 to 8 in Algorithm 1) produces an orientation, not necessarily proper, of the edges of $G$ in such a way that the indegree of each vertex in $X$ is at most $k$ and the indegree of each vertex in $Y$ is at most $\Delta(G) - k$. It proceeds as follows. We first orient all edges $xy \in E(G)$ from $x$ to $y$, where $x \in X$ and $y \in Y$. Then we define $k$ matchings as described subsequently.

Let $G_1 = G$, and let $M_i$ be a matching in $G_i$ that covers all vertices of maximum degree. For each $i \in \{2, \ldots, k\}$, let $G_i$ be the graph obtained from $G_{i-1}$ by removing the edges in $M_{i-1}$, that is $G_i = G_{i-1} \setminus M_{i-1}$, and let $M_i$ be a matching in $G_i$ that covers all vertices of degree $\Delta(G_i)$. Such a $M_i$ exists since it is well known that every bipartite graph $H$ has a proper $(\Delta(H))$-edge-colouring. Clearly, we have $\Delta(G_i) = \Delta(G_{i-1}) - 1$, for each $i \in \{2, 3, \ldots, k\}$. Let $M := \bigcup_{i=1}^k M_i$. Observe that if a vertex has degree $\Delta(G) - k + j$ in $G$, where $j \in \{1, 2, \ldots, k\}$, then it is incident to at least $j$ edges in $M$. Hence, for all $j \in \{1, 2, \ldots, k\}$ and for each vertex $y$ in $Y$ of degree $\Delta(G) - k + j$ in $G$, we reverse the orientation of exactly $j$ edges in $M$ incident to $y$. This ends the first phase.

The second phase reverses the orientation of some edges in $E(G) \setminus M$, step by step, in order to obtain a $(\Delta(G) - k)$-orientation. This orientation is proper under the assumption of Theorem 1. \qed
Algorithm 1: Proper Orientation of Bipartite Graphs

**Input:** Bipartite graph $G = (X \cup Y, E)$ and $k \in \mathbb{N}$ s.t. $\Delta(G) > 2k + \frac{\sqrt{1+8k}+1}{2}$.

**Output:** Proper $(\Delta(G) - k)$-orientation for $G$.

1. $G_1 \leftarrow G$
2. Orient all edges in $G$ from $X$ to $Y$
3. for $i = 1, \ldots, k$ do
   4. $M_i \leftarrow$ matching of $G_i$ saturating all vertices of degree $\Delta(G_i)$
   5. $G_{i+1} \leftarrow G_i - M_i$
   6. $M \leftarrow \bigcup_{j=1}^{k} M_j$
7. foreach $y \in Y$ do
   8. reverse the orientation of $\max\{0; d_G(y) - \Delta(G) + k\}$ edges of $M$ incident to $y$
9. $\bar{X} \leftarrow X$
10. for $\ell = \Delta(G) - k - 1, \ldots, 2$ do
11. while $\exists x \in X$ s.t. $|N_{\leq \ell}(x)| \geq \ell - d^-(x)$ and $|N_{= \ell}(x)| \leq \ell - d^-(x)$ do
12. $\bar{Y} \leftarrow$ set of $\ell - d^-(x)$ vertices of highest indegree in $N_{\leq \ell}(x)$
13. foreach $y \in \bar{Y}$ do
14. Reverse the orientation of $xy$ (i.e. re-orient $xy$ towards $x$)
15. $\bar{X} \leftarrow \bar{X} \setminus \{x\}$

**Theorem 2.** If $G$ is a bipartite graph, then $\overline{\chi}(G) \leq \left\lfloor \frac{\Delta(G)+\sqrt{\Delta(G)}}{2} \right\rfloor + 1$.

**Sketch of proof.** By Theorem 1, for every $k \in \mathbb{N}$, if $\Delta(G) > 2k + \frac{\sqrt{1+8k}+1}{2}$, then $\overline{\chi}(G) \leq \Delta(G) - k$. In order to obtain a good upper bound for $\overline{\chi}(G)$, we must find the largest positive integer $k$ such that the condition of Theorem 1 holds for a given graph $G$.

Solving the inequality for $k$, we obtain that $k < \frac{\Delta(G)-\sqrt{\Delta(G)}}{2}$. Since $k$ is integer, we conclude that $\overline{\chi}(G) \leq \Delta(G) - \left\lfloor \frac{\Delta(G)-\sqrt{\Delta(G)}}{2} \right\rfloor + 1$, and the result follows.

Note that if $G$ is bipartite and $\Delta(G) \in \{2, 3, 4\}$, then the bound of Theorem 2 is equal to the trivial upper bound $\overline{\chi}(G) \leq \Delta(G)$. For $\Delta(G) = 1$ and $\Delta(G) = 2$, this bound is tight due to the paths with 2 and 4 vertices, respectively. In addition, there exists a bipartite graph $G$ with $\Delta(G) = 3$ and $\overline{\chi}(G) = 3$.

3. Trees

**Theorem 3.** If $T$ is a tree, then the following statements hold:

1. if $\Delta(T) \leq 3$, then $\overline{\chi}(T) \leq 2$;
2. if $\Delta(T) \leq 6$, then $\overline{\chi}(T) \leq 3$;
3. $\overline{\chi}(T) \leq 4$.

**Sketch of proof.** We prove the three statements by using similar arguments. For $i \in \{1, 2, 3\}$, we consider a minimal counter-example $M_i$ to statement (i) with respect to the number of vertices, and derive a contradiction that implies that no counter-example exists. Since $M_i$ is a minimal counter-example, we have $\overline{\chi}(M_i) > i + 1$, but $\overline{\chi}(T) \leq i + 1$, for any proper subtree $T$ of $M_i$. We use the latter fact to derive a proper $(i + 1)$-orientation of $M_i$, which contradicts $\overline{\chi}(M_i) > i + 1$. 


The three statements of the theorem are tight in the following sense: there is a tree with maximum degree 4 and proper orientation number 3, and a tree with maximum degree 7 and proper orientation number 4.

4. \texttt{NP}-completeness

Ahadi and Dehgan [2] showed that it is \texttt{NP}-complete to decide whether $\overrightarrow{\chi}(G) \leq 2$ for planar graphs $G$ by using a reduction from the \textsc{Planar 3-SAT} problem. We first improve this result by showing that it is \texttt{NP}-complete to decide whether the proper orientation number of planar subcubic graphs is at most 2.

\textbf{Theorem 4.} The following problem is \texttt{NP}-complete:
\begin{itemize}
\item \textbf{Input}: A planar graph $G$ with $\Delta(G) = 3$ and $\delta(G) = 2$.
\item \textbf{Question}: $\overrightarrow{\chi}(G) \leq 2$?
\end{itemize}

\textit{Sketch of proof.} We show a reduction from the problem of deciding whether a planar 3-SAT formula is satisfiable. It is known that the \textsc{Planar 3-SAT} problem is \texttt{NP}-complete [5]. Let $\phi = (X, C)$ be an instance of this problem, where $X = \{x_1, \ldots, x_n\}$ is the set of variables and $C = \{C_1, \ldots, C_m\}$ is the set of clauses. Using the variable and clause gadgets depicted in Figures 1(a) and 1(b), respectively, we construct a planar graph $G'(\phi)$ such that $\overrightarrow{\chi}(G'(\phi)) \leq 2$ if, and only if, $\phi$ is satisfiable.

![Figure 1: The variable and clause gadgets.](image)

Recall that $\overrightarrow{\chi}(G) \leq \Delta(G)$, for any graph $G$. On the other hand, the following theorem shows that, for any integer $k \geq 3$, it is already \texttt{NP}-complete to determine whether $\overrightarrow{\chi}(G) \leq k$, for graphs $G$ with $\Delta(G) = k$.

\textbf{Theorem 5.} Let $k$ be an integer such that $k \geq 3$. The following problem is \texttt{NP}-complete:
\begin{itemize}
\item \textbf{Input}: A graph $G$ with $\Delta(G) = k$ and $\delta(G) = k - 1$.
\item \textbf{Question}: $\overrightarrow{\chi}(G) \leq k - 1$?
\end{itemize}

Finally, we show that it is \texttt{NP}-complete to decide whether $\overrightarrow{\chi}(G) \leq 3$, for planar bipartite graphs $G$.

\textbf{Theorem 6.} The following problem is \texttt{NP}-complete:
\begin{itemize}
\item \textbf{Input}: A planar bipartite graph $G$ with $\Delta(G) = 5$.
\item \textbf{Question}: $\overrightarrow{\chi}(G) \leq 3$?
\end{itemize}

\textit{Sketch of proof.} We show a reduction from the problem of deciding whether a planar monotone 3-SAT formula is satisfiable. This problem was recently shown to be \texttt{NP}-complete [6]. The idea of our reduction is roughly the same as in Theorems 4 and 5.
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