Load capacity of bodies

Reuven Segev*

Department of Mechanical Engineering, Ben-Gurion University, Beer-Sheva, Israel

Abstract

For the stress analysis in a plastic body Ω, we prove that there exists a maximal positive number C, the *load capacity ratio*, such that the body will not collapse under any external traction field t bounded by CY_0, where Y_0 is the yield stress. The load capacity ratio depends only on the geometry of the body and is given by

$$\frac{1}{C} = \sup_{w \in LD(\Omega)_D} \frac{\int_{\partial\Omega} |w|\, \mathrm{d}A}{\int_\Omega |\varepsilon(w)|\, \mathrm{d}V} = \|\gamma_D\|.$$

Here, $LD(\Omega)_D$ is the space of incompressible vector fields w for which the corresponding linear strains $\varepsilon(w)$ are assumed to be integrable and γ_D is the trace mapping assigning the boundary value $\gamma_D(w)$ to any $w \in LD(\Omega)_D$.

Keywords: Continuum mechanics; Stress analysis; Plasticity; Trace

1. Introduction

Consider a homogeneous isotropic elastic–perfectly plastic body Ω loaded by traction fields on its boundary. Let Y_0 be the yield stress, so, for example, the von-Mises yield condition for the stress $\sigma(x)$ at $x \in \Omega$ is

$$\sqrt{\tfrac{3}{2}}|\sigma(x) - \tfrac{1}{3}\operatorname{tr}\sigma(x) I|_{\mathrm{F}} = \sqrt{\tfrac{3}{2}}|\sigma_D(x)|_{\mathrm{F}} = Y_0,$$

where $|A|_{\mathrm{F}} = \sqrt{A_{ij}A_{ij}}$ is the Frobenius norm for matrices and $\sigma_D = \sigma - \frac{1}{3}\operatorname{tr}\sigma I$ is the stress deviator. We prove in this study that there exists a maximal positive number C, to which we refer as the *load capacity ratio,* such that the body will not collapse under any external traction field t bounded by CY_0. Thus, while the limit analysis factor of the theory of plasticity (e.g., [1,2]) pertains to a specific distribution of external loading, the load capacity ratio is independent of the distribution of external loading and implies that no collapse will occur for any field t on $\partial\Omega$ as long as

$$\operatorname*{ess\,sup}_{y \in \partial\Omega} |t(y)| < CY_0. \tag{1.1}$$

Collapse will occur for some t if the bound would be $C'Y_0$ with any $C' > C$.

The load capacity ratio depends only on the geometry of the body and we prove below that

$$\frac{1}{C} = \sup_{w \in LD(\Omega)_D} \frac{\int_{\partial\Omega} |w|\, \mathrm{d}A}{\int_\Omega |\varepsilon(w)|\, \mathrm{d}V} = \|\gamma_D\|. \tag{1.2}$$

Here, $LD(\Omega)_D$ is the space of incompressible (isochoric) integrable vector fields w that satisfy zero boundary conditions on an open subset Γ_0 of $\partial\Omega$ and for which the corresponding linear strains $\varepsilon(w)$ are assumed to be integrable. For $w \in LD(\Omega)_D$, we use $\|w\|_{LD} = \|\varepsilon(w)\|_1$ and indeed because of the boundary

* Tel.: +972 8 647 7043; fax: +972 8 647 2813.
E-mail address: rsegev@bgu.ac.il
URL: http://www.bgu.ac.il/~rsegev.

conditions it is a norm on $LD(\Omega)_D$. Finally, $\gamma_D\colon LD(\Omega)_D \longrightarrow L^1(\partial\Omega, \mathbb{R}^3)$ is the trace mapping assigning the boundary value $\gamma_D(w)$ to any $w \in LD(\Omega)_D$.

It should be mentioned that although we consider only loading by boundary traction for the sake of simplicity, body forces may be included in the analysis using the same methods as in [3]. (We indicate below some of the adaptations used when body forces are included.)

The notion of load capacity ratio is an application to plasticity of ideas presented in our previous work [3–5] where we consider stress fields on bodies whose maxima are the least. The general setting may be described as follows.

Let Ω represent the region occupied by the body in space so that the body is supported on a part Γ_0 of its boundary and let t be the external surface traction acting on the part Γ_t of its boundary. The mechanical properties of the body are not specified, and so, there is a class of stress fields that satisfy the equilibrium conditions with the external loading. (Clearly, distinct distributions of the mechanical properties within the body will result in general distinct equilibrating stress distributions.) Each equilibrating stress field in this class has its own maximal value, and we denote by σ_t^{opt} the least maximum.

Specifically, the magnitude of the stress field at a point is evaluated using a norm on the space of matrices. It is noted that yield conditions in plasticity usually use seminorms on the space of stress matrices rather than norms and the adaptation needed for plasticity will be described further below. By the maximum of a stress field we mean the essential supremum over the body of its magnitude, and later for plasticity, the essential supremum of the value of the yield function. Thus, we ignore excessive values on regions of zero volume. The traction fields that we admit are essentially bounded also. The set Ω is assumed to be open, bounded and its boundary is assumed to be smooth. Furthermore, it is assumed that Γ_t and Γ_0 are disjoint open subsets of the boundary whose closures cover the boundary, and that their closures intersect on a smooth curve.

Subject to these assumptions (see further details in Section 2) our first result is:

Theorem 1.1. (i) *The existence of stresses*: *Given an essentially bounded traction field t on Γ_t, there is a collection Σ_t of essentially bounded symmetric tensor fields*, *interpreted physically as stress fields*, *that represent t in the form*

$$\begin{aligned} &\int_{\Gamma_t} t \cdot w \,\mathrm{d}A = \int_{\Omega} \sigma_{ij}\varepsilon_{ij}(w)\,\mathrm{d}V, \\ &\sigma \in \Sigma_t, \quad w \in C^\infty(\overline{\Omega}, \mathbb{R}^3), \end{aligned} \tag{1.3}$$

where $\varepsilon(w) = \frac{1}{2}(\nabla w + \nabla w^{\mathrm{T}})$.

(ii) *The existence of optimal stress fields*: *Let*

$$\sigma_t^{\mathrm{opt}} = \inf_{\sigma\in\Sigma_t} \left\{ \operatorname{ess}\sup_{x\in\Omega} |\sigma(x)| \right\}. \tag{1.4}$$

Then, *there is a stress field $\widehat{\sigma} \in \Sigma_t$ such that*

$$\sigma_t^{\mathrm{opt}} = \operatorname{ess}\sup_{x\in\Omega} |\widehat{\sigma}(x)|. \tag{1.5}$$

(iii) *The expression for σ_t^{opt}*: *The optimum satisfies*

$$\sigma_t^{\mathrm{opt}} = \sup_{w\in C^\infty(\overline{\Omega},\mathbb{R}^3)} \frac{|\int_{\Gamma_t} t\cdot w\,\mathrm{d}A|}{\int_\Omega |\varepsilon(w)|\,\mathrm{d}V}, \tag{1.6}$$

where the magnitude of $\varepsilon(w)(x)$ is evaluated using the norm dual to the one used for the values of stresses.

Item (i) above is of theoretical interest. It is a representation theorem for the virtual work performed by the traction field using tensor fields that we naturally interpret as stresses. It should be noted that the existence of stress is not assumed here a priori. The expression for the representation by stresses turns out to be the principle of virtual work (1.3). Thus, the equilibrium conditions are derived mathematically on the basis of quite general assumptions. Item (i) also ensures us that the representing stress fields are also essentially bounded. Item (ii) states that the optimal value is actually attainable for some stress field and not just as a limit process.

Next, we consider *generalized stress concentration factors* for the given body. For a given external loading, traditional stress concentration factors are used by engineers to specify the ratio between the maximal stress in the body and the maximum nominal stress obtained using simplified formulas where various geometric irregularities are not taken into account. Regarding these nominal stresses as boundary traction fields, we formulate the notion of a stress concentration factor for a stress field σ in equilibrium with the traction t mathematically as the ratio between the maximal stress and the maximum traction. Specifically, we set

$$K_{t,\sigma} = \frac{\operatorname{ess}\sup_{x\in\Omega} |\sigma(x)|}{\operatorname{ess}\sup_{y\in\Gamma_t} |t(y)|}. \tag{1.7}$$

In particular, the optimal stress concentration factor for the given traction t is

$$K_t = \inf_{\sigma\in\Sigma_t}\{K_{t,\sigma}\} = \frac{\sigma_t^{\mathrm{opt}}}{\operatorname{ess}\sup_{y\in\Gamma_t} |t(y)|}. \tag{1.8}$$

Finally, realizing that engineers may be uncertain as to the nature of the external loading, we let the external loading vary and define the *generalized stress concentration factor*, a purely geometric property of the body Ω, as

$$K = \sup_t\{K_t\}, \tag{1.9}$$

where t varies over all essentially bounded traction fields. In other words, K is the worst possible optimal stress concentration factor. Using the result on optimal stresses, we prove straightforwardly the following:

Theorem 1.2. *The generalized stress concentration factor satisfies*

$$K = \sup_{w\in C^\infty(\overline{\Omega},\mathbb{R}^3)} \frac{\int_{\Gamma_t} |w|\,\mathrm{d}A}{\int_\Omega |\varepsilon(w)|\,\mathrm{d}V} = \|\gamma_0\|, \tag{1.10}$$

where γ_0 is the trace mapping for vector fields satisfying the boundary conditions on Γ_0.[1]

To prove the theorems we use standard tools of analysis and the theory of LD spaces given by [6–8]. Results analogous to Theorems 1.1 and 1.2 were presented in our earlier work cited above. In [4], a weaker form of equilibrium is assumed, and in all earlier work we did not consider boundary conditions for the displacements on Γ_0.

The following adaptations are made in order to consider body force fields in addition to the surface tractions. When, in addition, an essentially bounded body force field b is considered, we denote the external loading vector by $F = (t, b)$ and the stress concentration factor is redefined as

$$K_{F,\sigma} = \frac{\operatorname{ess\,sup}_{x\in\Omega} |\sigma(x)|}{\max\{\operatorname{ess\,sup}_{y\in\Gamma_t} |t(y)|, \operatorname{ess\,sup}_{x\in\Omega} |b(x)|\}}. \tag{1.11}$$

With Σ_F denoting the class of all stress fields in equilibrium with the pair $F = (t, b)$, one sets naturally

$$K_F = \sup_{\sigma\in\Sigma_F} K_{F,\sigma}. \tag{1.12}$$

The analog of Theorem 1.2, in particular Eq. (1.10), is now

$$K = \|\delta\| = \sup_{w\in LD(\Omega)_0} \frac{\int_\Omega |w|\,\mathrm{d}V + \int_{\Gamma_t} |\gamma_0(w)|\,\mathrm{d}A}{\int_\Omega |\varepsilon(w)|\,\mathrm{d}V}, \tag{1.13}$$

where $LD(\Omega)_0$ is the space of vector fields that vanish on Γ_0 and whose associated strains are integrable and δ is the mapping defined on $LD(\Omega)$ by $\delta(w) = (\gamma_0(w), w)$ (see [3] for additional details).

Next we turn to the adaptation needed for the application to plasticity. It is assumed that the yield function is a norm on the space of matrices applied to the deviatoric component of the stress matrix. Thus, it turns out that the same mathematical structure applies if we consider incompressible vector fields in the suprema of Eqs. (1.6) and (1.10). For example, the analog of Eq. (1.6) is

$$\sigma_t^{\mathrm{opt}} = \sup_{w\in LD(\Omega)_D} \frac{|\int_{\Gamma_t} t\cdot w\,\mathrm{d}A|}{\int_\Omega |\varepsilon(w)|\,\mathrm{d}V}, \tag{1.14}$$

where $LD(\Omega)_D$ is the collection of incompressible vector fields satisfying the boundary conditions and having integrable strains.

It turns out that optimal stresses are related to limit analysis of plasticity. In fact, the limit analysis factor λ^* (see Remark 5.2 and [1,2,9]) is simply given by

$$\lambda^* = \frac{Y_0}{\sigma_t^{\mathrm{opt}}}. \tag{1.15}$$

Furthermore, the expression for the optimal stress of Eq. (1.14) is implied mathematically by the results of Christiansen [1,2] and Temam and Strang [9]. This implies that the optimal stress fields do not require special designs of non-homogeneous material properties but occur for the frequently used models of elastic–plastic bodies. In particular, elastic–plastic material with $Y_0 = \sigma_t^{\mathrm{opt}}$ will attain the optimal stress field independently of the distribution of the external load.

We take advantage of these observations and introduce here the notion of load capacity ratio—a purely geometric property of the body. As described above, the load capacity ratio may be conceived as a universal limit design factor, which is independent of the distribution of the external loading. It immediately follows from its definition that

$$C = \frac{1}{K}. \tag{1.16}$$

Section 2 introduces the notation, assumptions and some background material. In particular, the space $LD(\Omega)$ of vector fields of integrable linear strains (see [6–8]) is described. Following some preliminary material concerning the boundary conditions in Section 3, the proof of the theorems is given in Section 4. The adaptation to plasticity theory, including the introduction of the load capacity ratio, is presented in Section 5.

It is noted that for structures, in particular, the finite dimensional models obtained by finite element approximation, the expression for C may be set as a convex optimization problem.

2. Notation and preliminaries

2.1. Basic variables

We consider a body under a given configuration in space. The space is modelled simply by $\mathbb{R}^3$ and the image of the body under the given configuration is the subset $\Omega \subset \mathbb{R}^3$. It is assumed that Ω is open and bounded and that it has a C^1 boundary $\partial\Omega$. Furthermore, there are two open subsets $\Gamma_0 \subset \partial\Omega$ and $\Gamma_t \subset \partial\Omega$ such that Γ_0 is the region where the body is supported and Γ_t is the region where the body is not supported so that a surface traction field t may be exerted on the body on Γ_t. Thus, it is natural to assume that Γ_0 and Γ_t are non-empty and disjoint, $\overline{\Gamma}_0 \cup \overline{\Gamma}_t = \partial\Omega$, and $\Lambda = \partial\Gamma_0 = \partial\Gamma_t$ is a differentiable one-dimensional submanifold of $\partial\Omega$. (The regularity assumptions may be generalized without affecting the validity of the constructions below.)

Basic objects in the construction are spaces of generalized velocity fields. A generic *generalized velocity field* (alternatively, *virtual velocity* or *virtual displacement*) will be denoted by w. In the sequel we consider a number of Banach spaces containing generalized velocities and a generic space of generalized velocities will be a denoted by $\mathbf{W}$. Generalized forces will be elements of the dual space $\mathbf{W}^*$. Thus, a *generalized force* F is a bounded linear functional $F\colon \mathbf{W} \longrightarrow \mathbb{R}$ such that $F(w)$ is interpreted as the virtual power (virtual work) performed by the force for the generalized velocity w. We recall that the dual norm of a linear functional F is defined as

$$\|F\| = \sup_{w\in\mathbf{W}} \frac{|F(w)|}{\|w\|}. \tag{2.1}$$

[1] Further details on γ_0 are described in Section 3.

2.2. *Virtual linear strains and stresses*

As an example for the preceding paragraph, consider the space $L^1(\Omega, \mathbb{R}^6)$ of L^1-symmetric tensor fields on Ω. A typical element $\varepsilon \in L^1(\Omega, \mathbb{R}^6)$ is interpreted as a *virtual linear strain field*. We will use $|\varepsilon(x)|$ to denote the norm of the matrix $\varepsilon(x)$. Various such norms are described in [5]. Thus,

$$\|\varepsilon\|_1 = \int_\Omega |\varepsilon(x)|\, \mathrm{d}V. \tag{2.2}$$

The dual space $L^1(\Omega, \mathbb{R}^6)^* = L^\infty(\Omega, \mathbb{R}^6)$ contains symmetric essentially bounded tensor fields σ that act on the linear strain fields by

$$\sigma(\varepsilon) = \int_\Omega \sigma(x)(\varepsilon(x))\, \mathrm{d}V. \tag{2.3}$$

Here, we use the same notation for the functional σ and the essentially bounded tensor field representing it and we regard the matrix $\sigma(x)$ as a linear form on the space of matrices so that $\sigma(x)(\varepsilon(x)) = \sigma(x)_{ij}\varepsilon(x)_{ji}$. Naturally, an element $\sigma \in L^\infty(\Omega, \mathbb{R}^6)$ is interpreted as a *stress field*. The dual norm of a stress field is given as

$$\|\sigma\| = \|\sigma\|_\infty = \operatorname{ess}\sup_{x\in\Omega} |\sigma(x)|. \tag{2.4}$$

Here, $|\sigma(x)|$ is calculated using the norm on the space of matrices which is dual to the one used for the evaluation of $|\varepsilon(x)|$ (see [5] for details). Thus, the choice of the space $L^1(\Omega, \mathbb{R}^6)$ for linear strains is natural when one is looking for the maximum of the stress tensor.

2.3. *The space of boundary velocity fields and boundary tractions*

As another example to be used later, consider the space $L^1(\Gamma_t, \mathbb{R}^3)$ of integrable vector fields on the "free" part of the boundary. Its dual space is

$$L^1(\Gamma_t, \mathbb{R}^3)^* = L^\infty(\Gamma_t, \mathbb{R}^3) \tag{2.5}$$

so that a generalized force in this case will be represented by an essentially bounded vector field t on Γ_t. Using the same notation for the functional and the vector field representing it, we have

$$t(u) = \int_{\Gamma_t} t(y)\cdot u(y)\, \mathrm{d}A \tag{2.6}$$

so that t may be interpreted as a traction field on Γ_t as expected. The dual norm of the traction field t is

$$\|t\| = \|t\|_\infty = \operatorname{ess}\sup_{y\in\Gamma_t} |t(y)| \tag{2.7}$$

again the relevant maximum.

2.4. *The space $LD(\Omega)$ and its elementary properties*

A central role in the subsequent analysis is played by the space $LD(\Omega)$ containing vector fields of integrable strains (see [6–8]). We summarize below its definition and basic relevant properties (see [7] for proofs and details).

2.4.1. *Definition*

For an integrable vector field $w \in L^1(\Omega, \mathbb{R}^3)$, let ∇w denote its distributional gradient and consider the corresponding linear strain (a tensor distribution)

$$\varepsilon(w) = \tfrac{1}{2}(\nabla w + \nabla w^{\mathrm{T}}). \tag{2.8}$$

The vector field w has an integrable strain if the distribution $\varepsilon(w)$ is an integrable symmetric tensor field, i.e., it belongs to $L^1(\Omega, \mathbb{R}^6)$. For the sake of simplifying the notation, we use ε for both the strain mapping here and its value in the example above. The space $LD(\Omega)$ is defined by

$$LD(\Omega) = \{w\colon \Omega \to \mathbb{R}^3;\ w \in L^1(\Omega, \mathbb{R}^3),\ \varepsilon(w) \in L^1(\Omega, \mathbb{R}^6)\}. \tag{2.9}$$

A natural norm is provided by

$$\|w\| = \|w\|_{LD} = \|w\|_1 + \|\varepsilon(w)\|_1 \tag{2.10}$$

and it induces on $LD(\Omega)$ a Banach space structure. Clearly, the linear strain mapping

$$\varepsilon\colon LD(\Omega) \longrightarrow L^1(\Omega, \mathbb{R}^6) \tag{2.11}$$

is linear and continuous.

2.4.2. *Approximations*

With the regularity assumption for $\partial\Omega$, the space of restrictions to Ω of smooth mappings in $C^\infty(\overline{\Omega}, \mathbb{R}^3)$ is dense in $LD(\Omega)$, so that any LD-vector field may be approximated by restrictions of smooth vector fields defined on the closure $\overline{\Omega}$.

2.4.3. *Trace mapping*

There is a unique continuous and linear trace mapping

$$\gamma\colon LD(\Omega) \longrightarrow L^1(\partial\Omega, \mathbb{R}^3) \tag{2.12}$$

satisfying the consistency condition

$$\gamma(u|_\Omega) = u|_{\partial\Omega} \tag{2.13}$$

for any continuous mapping $u \in C^0(\overline{\Omega}, \mathbb{R}^3)$. Furthermore, the trace mapping is surjective. Thus, although LD mappings are defined on the open set Ω, they have meaningful L^1 boundary values. (This property follows from the regularity of the boundary too.)

2.4.4. *Equivalent norm*

Let Γ be an open subset of $\partial\Omega$ and for $w \in LD(\Omega)$ let

$$\|w\|_\Gamma = \int_\Gamma |\gamma(w)|\, \mathrm{d}A + \|\varepsilon(w)\|_1, \tag{2.14}$$

then $\|w\|_\Gamma$ is a norm on $LD(\Omega)$ which is equivalent to the original norm defined in Eq. (2.10).

3. Constructions associated with the boundary conditions

3.1. The space $L^1(\partial\Omega, \mathbb{R}^3)_0$

Let $L^1(\partial\Omega, \mathbb{R}^3)_0 \subset L^1(\partial\Omega, \mathbb{R}^3)$ be the vector space of vector fields on $\partial\Omega$ such that for each $u \in L^1(\partial\Omega, \mathbb{R}^3)_0$, $u(y) = 0$ for almost all $y \in \Gamma_0$. It is noted that the restriction mapping

$$\rho_0 : L^1(\partial\Omega, \mathbb{R}^3) \longrightarrow L^1(\Gamma_0, \mathbb{R}^3), \quad \rho_0(u) = u|_{\Gamma_0}, \tag{3.1}$$

is linear and continuous. Thus, since

$$L^1(\partial\Omega, \mathbb{R}^3)_0 = \rho_0^{-1}\{0\}, \tag{3.2}$$

$L^1(\partial\Omega, \mathbb{R}^3)_0$ is a closed subspace of $L^1(\partial\Omega, \mathbb{R}^3)$.

The restriction mapping

$$\rho_t : L^1(\partial\Omega, \mathbb{R}^3)_0 \longrightarrow L^1(\Gamma_t, \mathbb{R}^3), \quad \rho_t(u) = u|_{\Gamma_t}, \tag{3.3}$$

is also linear and continuous. In addition, as $\partial\Gamma_0 = \partial\Gamma_t = \Lambda$ have zero area measure,

$$\int_{\partial\Omega} |u|\, \mathrm{d}A = \int_{\Gamma_t} |\rho_t(u)|\, \mathrm{d}A, \quad u \in L^1(\partial\Omega, \mathbb{R}^3)_0 \tag{3.4}$$

so that ρ_t is a norm-preserving injection.

Consider the zero extension mapping

$$e_0 : L^1(\Gamma_t, \mathbb{R}^3) \longrightarrow L^1(\partial\Omega, \mathbb{R}^3)_0 \tag{3.5}$$

defined by

$$e_0(u)(y) = \begin{cases} u(y) & \text{for } y \in \Gamma_t, \\ 0 & \text{for } y \notin \Gamma_t. \end{cases} \tag{3.6}$$

Clearly, $\rho_t \circ e_0$ is the identity on the space $L^1(\Gamma_t, \mathbb{R}^3)$. Moreover, for any $u \in L^1(\partial\Omega, \mathbb{R}^3)_0$, $e_0(\rho_t(u))(y) = u(y)$ almost everywhere (except for $y \in \Lambda$), so $e_0 \circ \rho_t$ is the identity on $L^1(\partial\Omega, \mathbb{R}^3)_0$. We conclude:

Lemma 3.1. *The mappings ρ_t and e_0 induce an isometric isomorphism of the spaces $L^1(\partial\Omega, \mathbb{R}^3)_0$ and $L^1(\Gamma_t, \mathbb{R}^3)$. The dual mappings e_0^* and ρ_t^* induce an isometric isomorphism of the spaces $L^1(\Gamma_t, \mathbb{R}^3)^*$ and $L^1(\partial\Omega, \mathbb{R}^3)_0^*$. Every element $t_0 \in L^1(\partial\Omega, \mathbb{R}^3)_0^*$ is represented uniquely by an essentially bounded $t \in L^\infty(\Gamma_t, \mathbb{R}^3)$ in the form*

$$t_0(u) = \int_{\Gamma_t} t \cdot u\, \mathrm{d}A. \tag{3.7}$$

3.2. The space $LD(\Omega)_0$ of velocity fields satisfying the boundary conditions

Recalling the definition of the equivalent norm on $LD(\Omega)$ in Eq. (2.14), we set $\Gamma = \Gamma_0$ in that equation. Henceforth, we will use on $LD(\Omega)$ the equivalent norm

$$\|w\| = \|w\|_{\Gamma_0} = \int_{\Gamma_0} |\gamma(w)|\, \mathrm{d}A + \|\varepsilon(w)\|_1. \tag{3.8}$$

Consider the vector subspace $LD(\Omega)_0$ defined by

$$LD(\Omega)_0 = \gamma^{-1}\{L^1(\partial\Omega, \mathbb{R}^3)_0\} \subset LD(\Omega). \tag{3.9}$$

Thus, $LD(\Omega)_0$ is the subspace containing vector fields on Ω whose boundary values vanish on Γ_0 almost everywhere. Since γ is continuous and $L^1(\partial\Omega, \mathbb{R}^3)_0$ is a closed subspace of $L^1(\partial\Omega, \mathbb{R}^3)$, $LD(\Omega)_0$ is a closed subspace of $LD(\Omega)$. Combining this with Lemma 3.1 we obtain immediately:

Lemma 3.2. *The mapping*

$$\gamma_0 = \rho_t \circ \gamma|_{LD(\Omega)_0} : LD(\Omega)_0 \longrightarrow L^1(\Gamma_t, \mathbb{R}^3) \tag{3.10}$$

is a linear and continuous surjection. Dually,

$$\gamma_0^* = (\gamma|_{LD(\Omega)_0})^* \circ \rho_t^* : L^\infty(\Gamma_t, \mathbb{R}^3) \longrightarrow LD(\Omega)_0^* \tag{3.11}$$

is a continuous injection.

Observing Eq. (3.8), for each $w \in LD(\Omega)_0$,

$$\|w\| = \|\varepsilon(w)\|_1. \tag{3.12}$$

Lemma 3.3. *The mapping*

$$\varepsilon_0 = \varepsilon|_{LD(\Omega)_0} : LD(\Omega)_0 \to L^1(\Omega, \mathbb{R}^6) \tag{3.13}$$

is an isometric injection.

Proof. Eq. (3.12) implies immediately that $\|w\| = \|\varepsilon(w)\|_1$ for all $w \in LD(\Omega)_0$. Being a linear isometry, the zero element is the only element that is mapped to zero, so ε_0 is injective. In addition to relying on the technical property (Section 2.4.4) of $LD(\Omega)$ to show that ε_0 is injective, it should be mentioned that this follows from the fact that for any vector field w on Ω, $\varepsilon(w) = 0$ only if w is a rigid vector field, i.e., if w is of the form $w(x) = a + b \times x$, $a, b \in \mathbb{R}^3$. Now, the only rigid vector field that vanishes on the open set Γ_0 is the zero vector field. □

4. The mathematical constructions

Let $t \in L^\infty(\Gamma_t, \mathbb{R}^3)$ be a traction field on the free part of the boundary. Then, $\gamma_0^*(t)$ is an element of $LD(\Omega)_0^*$ representing t. The basic properties of elements of $LD(\Omega)_0^*$ are as follows.

Lemma 4.1. *Each $S \in LD(\Omega)_0^*$ may be represented by some non-unique tensor field $\sigma \in L^\infty(\Omega, \mathbb{R}^6)$ in the form*

$$S = \varepsilon_0^*(\sigma) \quad \text{or} \quad S(w) = \int_\Omega \sigma(x)(\varepsilon_0(w)(x))\, \mathrm{d}V. \tag{4.1}$$

The dual norm of S satisfies

$$\|S\| = \inf_\sigma \|\sigma\|_\infty = \inf_\sigma \left\{ \operatorname*{ess\,sup}_{x \in \Omega} |\sigma(x)| \right\}, \tag{4.2}$$

where the infimum is taken over all tensor fields σ satisfying $S = \varepsilon_0^*(\sigma)$, i.e., tensor fields representing S. There is a $\widehat{\sigma} \in L^\infty(\Omega, \mathbb{R}^6)$ for which the infimum is attained.

Proof. By applying the Hahn–Banach theorem, the assertion follows from the fact that ε_0 is a linear and isometric injection as in Lemma 3.3 (see also [3]). □

Applying this lemma to $S=\gamma_0^*(t)$ one may draw the following conclusions.

Conclusion 4.2. Forces on the body given by essentially bounded surface tractions are represented by tensor fields on the body. These tensor fields are naturally interpreted as stress fields. The condition that a stress tensor field σ represents the surface traction t is

$$\gamma_0^*(t)=\varepsilon_0^*(\sigma), \tag{4.3}$$

and explicitly,

$$\int_{\Gamma_t} t\cdot\gamma_0(w)\,\mathrm{d}A=\int_\Omega \sigma(\varepsilon_0(w))\,\mathrm{d}V \tag{4.4}$$

for each vector field $w\in LD(\Omega)_0$, i.e., a vector field of integrable strain satisfying the boundary condition on Γ_0. This condition is just the principle of virtual work which is a weak form of the equation of equilibrium and the corresponding boundary conditions. Thus, we have derived both the existence of stresses and the equilibrium conditions analytically under mild assumptions.

It is noted that the subscript 0, only indicating the restriction of the various operations to fields satisfying the boundary conditions, may be omitted above. Also, as the restrictions of smooth vector fields on $\overline{\Omega}$ are dense in $LD(\Omega)$, it is sufficient to verify that the condition holds for smooth fields on $\overline{\Omega}$. For such fields, the integrand on the left may be replaced simply by $t\cdot w$.

Conclusion 4.3. There is an optimal stress field $\widehat{\sigma}$ representing t and

$$\|\gamma_0^*(t)\|=\|\widehat{\sigma}\|_\infty=\inf_\sigma\left\{\operatorname*{ess\,sup}_{x\in\Omega}|\sigma(x)|\right\}, \tag{4.5}$$

where the infimum is taken over all stress fields σ satisfying $\gamma_0^*(t)=\varepsilon_0^*(\sigma)$, i.e., all stress fields in equilibrium with t. Thus, the infimum on the right is the optimal maximal stress σ_t^{opt}. In addition, by the definition of the dual norm we have

$$\|\gamma_0^*(t)\|=\sup_{w\in LD(\Omega)_0}\frac{|\gamma_0^*(t)(w)|}{\|w\|} \tag{4.6}$$

$$=\sup_{w\in LD(\Omega)_0}\frac{|t(\gamma_0(w))|}{\|\varepsilon(w)\|_1}, \tag{4.7}$$

where in the last line we used Eq. (3.12). We conclude that

$$\sigma_t^{\mathrm{opt}}=\sup_{w\in LD(\Omega)_0}\frac{|\int_{\Gamma_t}t\cdot\gamma_0(w)\,\mathrm{d}A|}{\int_\Omega|\varepsilon(w)|\,\mathrm{d}V}. \tag{4.8}$$

Recalling that the restrictions of smooth mappings on $\overline{\Omega}$ are dense in $LD(\Omega)$ and that for such mappings the trace mapping is just the restriction, the optimal stress may be evaluated as

$$\sigma_t^{\mathrm{opt}}=\sup_w\frac{|\int_{\Gamma_t}t\cdot w\,\mathrm{d}A|}{\int_\Omega|\varepsilon(w)|\,\mathrm{d}V}, \tag{4.9}$$

where the supremum is taken over all smooth mappings in $C^\infty(\overline{\Omega},\mathbb{R}^3)$ that vanish on Γ_0.

It is noted that the value of σ_t^{opt} depends on the norm used for strain matrices.

We now turn to the simple proof of Theorem 1.2.

Proof. We had

$$\sigma_t^{\mathrm{opt}}=\sup_{w\in LD(\Omega)_0}\frac{|t(\gamma_0(w))|}{\|\varepsilon(w)\|_1}=\|\gamma_0^*(t)\|$$

so that

$$K=\sup_{t\in L^\infty(\Gamma_t,\mathbb{R}^3)}\frac{\sigma_t^{\mathrm{opt}}}{\|t\|}=\sup_{t\in L^\infty(\Gamma_t,\mathbb{R}^3)}\left\{\frac{\|\gamma_0^*(t)\|}{\|t\|}\right\}$$
$$=\|\gamma_0^*\|=\|\gamma_0\|, \tag{4.10}$$

where the last equality is the standard equality between the norm of a mapping and the norm of its dual (e.g., [10, pp. 191–192]). □

5. Load capacity for plastic bodies

The analysis we presented in the previous sections may be applied to the limit analysis of plastic bodies. While in the preceding analysis the magnitude of the stress at a point was represented by the norm of the stress matrix, for the analysis of plasticity, the relevant quantity is the value of the yield function. The yield function is usually taken as a seminorm on the space of matrices—a norm on the deviatoric component of the stress. The necessary adaptation is as follows.

5.1. Notation and preliminaries

We denote by π_P the usual projection of the space of matrices on the subspace $P=\{aI\mid a\in\mathbb{R}\}$, i.e.,

$$\pi_P(m)=\tfrac{1}{3}m_{ii}I, \tag{5.1}$$

and by π_D the projection on the subspace of deviatoric (traceless) matrices D so that

$$\pi_D(m)=m_D=m-\pi_P(m). \tag{5.2}$$

Thus, the pair (π_D,π_P) makes an isomorphism of the space of matrices with $D\oplus P$. We will therefore make the identification $\mathbb{R}^6=D\oplus P$ and $\mathbb{R}^{6*}=D^*\oplus P^*$. We will use the same notation $|\cdot|$ for both the norm on $\mathbb{R}^6$, whose elements are interpreted as strain values, and the dual norm on $\mathbb{R}^{6*}$, whose elements are interpreted as stress values (although the norms may be different in general). Thus, we assume that the yield function is of the form

$$Y(m)=|\pi_D(m)|. \tag{5.3}$$

For example, if we take $|\cdot|$ to be the Frobenius norm on $\mathbb{R}^{6*}$ we get the von-Mises yield criterion. In practical terms this means that the material will not yield at a point x if $Y(\sigma(x))<Y_0$ for some limiting yield stress value $Y_0\in\mathbb{R}^+$.

Thus, we will extend the foregoing discussion to the case where Y, evidently a seminorm, replaces the norm on the space of stress matrices. For the space of stress fields we will therefore have the seminorm $\|\cdot\|_Y$ defined by

$$\|\sigma\|_Y = \|\pi_D \circ \sigma\|_\infty = \operatorname{ess}\sup_{x\in\Omega} Y(\sigma(x)) = \operatorname{ess}\sup_{x\in\Omega} |\sigma_{D(x)}|. \tag{5.4}$$

The expression defining the optimal stress becomes

$$\sigma_t^{\text{opt}} = \inf_{\gamma^*(t)=\varepsilon^*(\sigma)} \|\sigma\|_Y = \inf_{\gamma^*(t)=\varepsilon^*(\sigma)} \{\|\pi_D \circ \sigma\|_\infty\}. \tag{5.5}$$

The condition for collapse is $\sigma_t^{\text{opt}} \geqslant Y_0$ and we use Ψ to denote the collapse manifold, i.e.,

$$\Psi = \{t \mid \sigma_t^{\text{opt}} = Y_0\}. \tag{5.6}$$

Remark 5.1. The expression

$$\sigma_t^{\text{opt}} = \inf_{\varepsilon^*(\sigma)=\gamma^*(t)} \|\sigma\|_Y$$

for the optimal stress may be reformulated as follows. Recalling that Y_0 denotes the yield stress, we write $\sigma=\sigma_1/\lambda$, $\|\sigma_1\|_Y=Y_0$ and noting that $\|\sigma/\|\sigma\|_Y\|_Y=1$, we are looking for

$$\sigma_t^{\text{opt}} = \inf_{\substack{\varepsilon^*(\sigma_1/\lambda)=\gamma^*(t),\\ \lambda\in\mathbb{R}^+,\sigma_1\in\partial B}} \|\sigma_1/\lambda\|_Y, \tag{5.7}$$

where B is the ball in $L^\infty(\Omega, D)$ of radius Y_0. Thus,

$$\sigma_t^{\text{opt}} = \inf_{\substack{\varepsilon^*(\sigma_1/\lambda)=\gamma^*(t),\\ \lambda\in\mathbb{R}^+,\sigma_1\in\partial B}} \frac{Y_0}{\lambda}, \tag{5.8}$$

$$\frac{Y_0}{\sigma_t^{\text{opt}}} = \sup\{\lambda \mid \exists\sigma_1 \in \partial B, \varepsilon^*(\sigma_1) = \gamma^*(\lambda t)\}. \tag{5.9}$$

Clearly, in the last equation ∂B may be replaced by $\overline{B}$ because if we consider σ with $\|\sigma\|_Y<1$, then $\sigma_1=\sigma/\|\sigma\|_Y$ is in ∂B and the corresponding λ will be multiplied by $\|\sigma\|_Y<1$. The unit ball B contains the stress fields essentially bounded by the yield stress and we are looking for the largest multiplication of the force for which there is an equilibrating stress field that is essentially bounded by Y_0. Thus, we are looking for

$$\frac{Y_0}{\sigma_t^{\text{opt}}} = \lambda^* = \sup\{\lambda \mid \exists\sigma \in B, \varepsilon^*(\sigma) = \gamma^*(\lambda t)\} \tag{5.10}$$

which is the limit analysis factor (e.g., [1,2,9]).

The expression defining the generalized stress concentration factor assumes the form

$$\begin{aligned} K &= \sup_t \frac{\sigma_t^{\text{opt}}}{\|t\|_\infty} = \sup_t \inf_{\gamma^*(t)=\varepsilon^*(\sigma)} \frac{\|\sigma\|_Y}{\|t\|_\infty} \\ &= \sup_t \inf_{\gamma^*(t)=\varepsilon^*(\sigma)} \frac{\|\pi_D\circ\sigma\|_\infty}{\|t\|_\infty}. \end{aligned} \tag{5.11}$$

For the application to plasticity, we use the term *load capacity* for $C=1/K$. Hence,

$$C = \frac{1}{\sup_t (\sigma_t^{\text{opt}}/\|t\|_\infty)} = \inf_t \frac{\|t\|_\infty}{\sigma_t^{\text{opt}}}. \tag{5.12}$$

For every loading t we set

$$t_\Psi = \frac{t}{\sigma_t^{\text{opt}}/Y_0} \tag{5.13}$$

so that using $\sigma_{\lambda t}^{\text{opt}} = \|\gamma^*(\lambda t)\| = \lambda\sigma_t^{\text{opt}}$ for any $\lambda>0$ one has

$$\sigma_{t_\Psi}^{\text{opt}} = Y_0, \quad \frac{\|t\|_\infty}{\sigma_t^{\text{opt}}} = \frac{\|t_\Psi\sigma_t^{\text{opt}}/Y_0\|_\infty}{\sigma_t^{\text{opt}}} = \|t_\Psi\|_\infty/Y_0. \tag{5.14}$$

It follows that for any t, t_Ψ belongs to the collapse manifold Ψ and the operation above is a projection onto the collapse manifold. Thus,

$$C = \inf_t \frac{\|t\|_\infty}{\sigma_t^{\text{opt}}} = \inf_{t_\Psi\in\Psi} \|t_\Psi\|_\infty/Y_0, \tag{5.15}$$

and indeed, $CY_0 = \inf_{t_\Psi\in\Psi}\|t_\Psi\|_\infty$ is the largest radius of a ball containing only surface forces for which collapse does not occur.

5.2. *Constructions associated with the extension to plasticity*

The top row of the following commutative diagram describes the various kinematic mappings we used in the case of a norm on the space of stresses as considered in the previous sections. The subspace $L^1(\Omega, D)$ of $L^1(\Omega, \mathbb{R}^6)$ contains incompressible strain fields and there is a natural projection $\pi_D^\circ\colon L^1(\Omega,\mathbb{R}^6)\to L^1(\Omega, D)$ given by $\pi_D^\circ(\chi)=\pi_D\circ\chi$. The inclusion of a subspace in a vector space will be generally denoted as ι. We will also use the notation

$$LD(\Omega)_D = \varepsilon_0^{-1}\{L^1(\Omega, D)\} \tag{5.16}$$

for the subspace of incompressible LD-vector fields. Thus, using ε_D and γ_D for the restrictions $\varepsilon_0|_{LD(\Omega)_D}$ and $\gamma_0|_{LD(\Omega)_D}$, respectively, we have the following commutative diagram:

$$\begin{array}{ccccc} L^1(\Gamma_t,\mathbb{R}^3) & \xleftarrow{\gamma_0} & LD(\Omega)_0 & \xrightarrow{\varepsilon_0} & L^1(\Omega,\mathbb{R}^6) \\ \| & & \uparrow\iota & & \iota\uparrow\downarrow\pi_D^\circ \\ L^1(\Gamma_t,\mathbb{R}^3) & \xleftarrow{\gamma_D} & LD(\Omega)_D & \xrightarrow{\varepsilon_D} & L^1(\Omega, D). \end{array} \tag{5.17}$$

The dual diagram is

$$\begin{array}{ccccc} L^\infty(\Gamma_t,\mathbb{R}^3) & \xrightarrow{\gamma_0^*} & LD(\Omega)_0^* & \xleftarrow{\varepsilon_0^*} & L^\infty(\Omega,\mathbb{R}^6) \\ \| & & \downarrow\iota^* & & \iota^*\downarrow\uparrow\pi_D^{\circ *} \\ L^\infty(\Gamma_t,\mathbb{R}^3) & \xrightarrow{\gamma_D^*} & LD(\Omega)_D^* & \xleftarrow{\varepsilon_D^*} & L^\infty(\Omega, D). \end{array} \tag{5.18}$$

Since ε_D is just a restriction of ε_0, it is still a linear, norm-preserving injection. Thus, the assertion of Lemma 4.1 and the subsequent conclusions hold where $LD(\Omega)_D$, ε_D and γ_D replace $LD(\Omega)_0$, ε_0 and γ_0, respectively. The expression for the optimal stress for the plasticity analysis is therefore

$$\sigma_t^{\mathrm{opt}} = \inf_{\gamma^*(t)=\varepsilon^*(\sigma)} \|\sigma\|_Y = \sup_{w\in LD(\Omega)_D} \frac{|\int_{\Gamma_t} t\cdot w\,\mathrm{d}A|}{\int_\Omega |\varepsilon(w)|\,\mathrm{d}V}. \tag{5.19}$$

Finally, the load capacity ratio is given by

$$\begin{aligned}\frac{1}{C} = K &= \sup_{t\in L^\infty(\Gamma_t,\mathbb{R}^3)} \frac{\sigma_t^{\mathrm{opt}}}{\|t\|_\infty} \\ &= \sup_{w\in LD(\Omega)_D} \frac{\int_{\Gamma_t} |w|\,\mathrm{d}A}{\int_\Omega |\varepsilon(w)|\,\mathrm{d}V} = \|\gamma_D\|. \end{aligned} \tag{5.20}$$

Remark 5.2. Our result (5.2) for the optimal stress associated with $S = \gamma_D^*(t)$ is

$$\sigma_S^{\mathrm{opt}} = \sup_{w\in LD(\Omega)_D} \frac{|S(w)|}{\|\varepsilon(w)\|_Y}. \tag{5.21}$$

For limit analysis in plasticity it is shown by Christiansen [1,2] and Temam and Strang [9] that the kinematic version of the limit load is equivalent to the statical version above, specifically,

$$\lambda^* = \inf_{S(w)=1} \left\{ \sup_{\sigma\in B} \left\{\varepsilon^*(\sigma)(w)\right\}\right\}. \tag{5.22}$$

We will show that the two expressions are equivalent for the setting of stress optimization.

Eq. (5.21) may be rewritten as

$$\sigma_S^{\mathrm{opt}} = \sup_{w\in LD(\Omega)_D, S(w)=1} \frac{1}{\|\varepsilon(w)\|_Y} \tag{5.23}$$

$$= \frac{1}{\inf_{S(w)=1}\|\varepsilon(w)\|_Y}, \tag{5.24}$$

where we used $\sup_x (1/x) = 1/\inf x$. We conclude that

$$\frac{1}{\sigma_S^{\mathrm{opt}}} = \inf_{S(w)=1} \|\varepsilon(w)\|_Y. \tag{5.25}$$

On the other hand, if we replace in the kinematic version of the limit load the requirement $\sigma\in B$ by the requirement $\|\sigma\|_Y \leqslant Y_0$ which is the analog in our setting, we obtain

$$\lambda^* = \inf_{S(w)=1} \left\{ \sup_{\|\sigma\|_Y\leqslant Y_0} \{\varepsilon^*(\sigma)(w)\}\right\} \tag{5.26}$$

$$= \inf_{S(w)=1} \left\{ \sup_{\|\sigma\|_Y\leqslant Y_0} \{\sigma(\varepsilon(w))\}\right\}. \tag{5.27}$$

Using $\|\varepsilon(w)\| = \sup_{\|\sigma\|\leqslant 1} |\sigma(\varepsilon(w))|$, we have

$$\lambda^* = Y_0 \inf_{S(w)=1} \|\varepsilon(w)\|_Y \tag{5.28}$$

so that indeed $\lambda^* = Y_0/\sigma_S^{\mathrm{opt}}$.

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