Mean curvature flow with obstacles: existence, uniqueness and regularity of solutions

Gwenael Mercier, Matteo Novaga

To cite this version:

Gwenael Mercier, Matteo Novaga. Mean curvature flow with obstacles: existence, uniqueness and regularity of solutions. 2014. <hal-01067821>

HAL Id: hal-01067821
https://hal.archives-ouvertes.fr/hal-01067821
Submitted on 24 Sep 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Mean curvature flow with obstacles:
existence, uniqueness and regularity of solutions

Gwenael Mercier∗, Matteo Novaga†

Abstract
We show short time existence and uniqueness of $C^{1,1}$ solutions to the mean curvature flow with obstacles, when the obstacles are of class $C^{1,1}$. If the initial interface is a periodic graph we show long time existence of the evolution and convergence to a minimal constrained hypersurface.

Contents

1 Introduction and main results 1

2 Mean curvature flow with a forcing term 3
  2.1 Evolution of geometric quantities 3
  2.2 The Monotonicity Formula 5

3 Proof of Theorem 1 7

4 Proof of Theorem 2 16

5 Proof of Theorem 3 20

A Viscosity solutions with obstacles 20
  A.1 Definition of viscosity solution 20
  A.2 Comparison principle 21
  A.3 Existence 22
  A.4 Regularity 24

1 Introduction and main results

Mean curvature flow is a prototypical geometric evolution, arising in many models from Physics, Biology and Material Science, as well as in a variety of mathematical problems. For such a reason, this flow has been widely studied in the past years, starting from the pioneristic work of K. Brakke [Bra78] (we refer to [GH86, Hui84, EH89, ES91, CGG91] for a far from complete list of references).

∗CMAP, École polytechnique, Palaiseau, France, email: gwenael.mercier@cmap.polytechnique.fr
†Dipartimento di Matematica, Università di Pisa, Largo Bruno Pontecorvo 5, 56127 Pisa, Italy; e-mail: novaga@dm.unipi.it
In some models, one needs to include the presence of hard obstacles, which the evolving surface cannot penetrate (see for instance \cite{ESV12} and references therein). This leads to a double obstacle problem for the mean curvature flow, which reads

\[ v = H \quad \text{on } M_t \cap U, \quad (1) \]

with constraint

\[ M_t \subset \bar{U} \quad \text{for all } t, \quad (2) \]

where \( v, H \) denote respectively the normal velocity and \( d \times \) times the mean curvature of the interface \( M_t \), and the open set \( U \subset \mathbb{R}^{d+1} \) represents the obstacle. Notice that, due to the presence of obstacles, the evolving interface is in general only of class \( C^{1,1} \) in the space variable, differently from the unconstrained case where it is analytic (see \cite{ISZ98}). While the regularity of parabolic obstacle problems is relatively well understood (see \cite{Sha08} and references therein), a satisfactory existence and uniqueness theory for solutions is still missing.

In \cite{ACN12} (see also \cite{Spa11}) the authors approximate such an obstacle problem with an implicit variational scheme introduced in \cite{ATW93, LS95}. As a byproduct, they prove global existence of weak (variational) solutions, and short time existence and uniqueness of regular solutions in the two-dimensional case. In \cite{Mer14} the first author adapts to this setting the theory of viscosity solutions introduced in \cite{CIL92, CGG91}, and constructs globally defined continuous (viscosity) solutions.

Let us now state the main results of this paper.

**Theorem 1.** Let \( M_0 \subset U \) be an initial hypersurface, and assume that both \( M_0 \) and \( \partial U \) are uniformly of class \( C^{1,1} \), with \( \text{dist}(M_0, \partial U) > 0 \). Then there exists \( T > 0 \) and a unique solution \( M_t \) to (1), (2) on \( [0, T) \), such that \( M_t \) is of class \( C^{1,1} \) for all \( t \in [0, T) \).

Notice that Theorem 1 extends a result in \cite{ACN12} to dimensions greater than two.

When the hypersurface \( M_t \) can be written as the graph of a function \( u(\cdot, t) : \mathbb{R}^d \rightarrow \mathbb{R} \), equation (1) reads

\[ u_t = \sqrt{1 + |\nabla u|^2} \text{ div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right). \quad (3) \]

If the obstacles are also graphs, the constraint (2) can be written as

\[ \psi^- \leq u \leq \psi^+, \quad (4) \]

where the functions \( \psi^\pm : \mathbb{R}^d \rightarrow \mathbb{R} \) denote the obstacles.

**Theorem 2.** Assume that \( \psi^\pm \in C^{1,1}(\mathbb{R}^d) \), and let \( u_0 \in C^{1,1}(\mathbb{R}^d) \) satisfy (4). Then there exists a unique (viscosity) solution \( u \) of (3), (4) on \( \mathbb{R}^d \times [0, +\infty) \), such that

\[ \|\nabla u(\cdot, t)\|_{L^\infty(\mathbb{R}^d)} \leq \max \left( \|u_0\|_{L^\infty(\mathbb{R}^d)}, \|\nabla \psi^\pm\|_{L^\infty(\mathbb{R}^d)} \right) \]

\[ \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^d)} \leq \left\| \sqrt{1 + |\nabla u_0|^2} \text{ div} \left( \frac{\nabla u_0}{\sqrt{1 + |\nabla u_0|^2}} \right) \right\|_{L^\infty(\mathbb{R}^d)} \]

for all \( t > 0 \). Moreover \( u \) is also of class \( C^{1,1} \) uniformly on \( [0, +\infty) \).

We observe that Theorem 2 extends previous results by Ecker and Huisken \cite{EH89} in the unconstrained case (see also \cite{CN13}).
Theorem 3. Assume that $u_0$ and $\psi^\pm$ are $Q$-periodic, with periodicity cell $Q = [0, L]^d$, for some $L > 0$. Then the solution $u(\cdot, t)$ of (3), (4) is also $Q$-periodic. Moreover there exists a sequence $t_n \to +\infty$ such that $u(\cdot, t_n)$ converges uniformly as $n \to +\infty$ to a stationary solution to (3), (4).

Our strategy will be to approximate the obstacles with “soft obstacles” modeled by a sequence of uniformly bounded forcing terms. Differently from [ACN12], where the existence of regular solution is derived from variational estimates on the approximating scheme, we obtain estimates on the evolving interface, in the spirit of [EH91a, EH91b, CNV11], which are uniform in the forcing terms.

Acknowledgements
We wish to thank to Antonin Chambolle for interesting discussions and useful comments on this work. The first author was partially supported by the ANR-12-BS01-0014-01 Project Geometra.

2 Mean curvature flow with a forcing term

2.1 Evolution of geometric quantities
Let $M$ be a complete orientable $d$-dimensional Riemannian manifold without boundary, let $F(\cdot, t) : M \to \mathbb{R}^{d+1}$ be a smooth family of immersions, and denote by $M_t$ the image $F(M, t)$. Since $M_t$ is orientable, we can write $M_t = \partial E(t)$ where $E(t)$ is a family of open subsets of $\mathbb{R}^{d+1}$ depending smoothly on $t$. We say that $M_t$ evolves by mean curvature with forcing term $k$ if

$$\frac{d}{dt}F(p, t) = -(H(p, t) + k(F(p, t)))\nu(p, t), \quad (5)$$

where $k : \mathbb{R}^{d+1} \to \mathbb{R}$ is a smooth forcing term, $\nu$ is the unit normal to $M_t$ pointing outside $E(t)$, and $H$ is (d times) the mean curvature of $M_t$, with the convention that $H$ is positive whenever $E(t)$ is convex.

We shall compute the evolution of some relevant geometric quantities under the law (5). We denote by $\nabla^S$, $\Delta^S$ respectively the covariant derivative and the Laplace-Beltrami operator on $M$. As in [Hui84], the metric on $M_t$ is denoted by $g_{ij}(t)$, it inverse is $g^{ij}(t)$, the scalar product (or any tensors contraction using the metric) on $M_t$ is denoted by $\langle \cdot, \cdot \rangle$ whereas the ambiant scalar product is $\langle \cdot, \cdot \rangle$, the volume element is $\mu_t$, and the second fundamental form is $A$. In particular we have $A(\partial_i, \partial_j) = h_{ij}$, where we set for simplicity $\partial_i = \partial \partial_i \mu_t$, and $H = h_{ii}$, using the Einstein notations (we implicitly sum every index which appears twice in an expression). We also denote by $\lambda_1, \ldots, \lambda_d$ the eigenvalues of $A$.

Notice that, in terms of the parametrization $F$, we have

$$g_{ij} = (\partial_i F, \partial_j F), \quad h_{ij} = - (\partial^2_{ij} F, \nu) \quad \text{for all } i, j \in \{1, \ldots, d\}. \quad (6)$$
Proposition 1. The following equalities hold:

\[
\frac{d}{dt} g_{ij} = -2(H + k) h_{ij} \tag{7}
\]
\[
\frac{d}{dt} \nu = \nabla^S (H + k) \tag{8}
\]
\[
\frac{d}{dt} \mu_t = -H(H + k) \mu_t \tag{9}
\]
\[
\frac{d}{dt} h_{ij} = \Delta^S h_{ij} + \nabla^S_i \nabla^S_j k - 2H h_{ij} g_{lm} h_{mj} - kg_{ml} h_{im} g_{jl} + |A|^2 h_{ij} \tag{10}
\]
\[
\frac{d}{dt} H = \Delta^S (H + k) + (H + k) |A|^2 \tag{11}
\]
\[
\frac{d}{dt} |A|^2 = \Delta^S |A|^2 + 2kg^{ij} g^{sl} g_{mn} h_{is} h_{lm} h_{nj} + 2|A|^4 - 2|\nabla^S A|^2 + 2 \langle A, (\nabla^S)^2 k \rangle. \tag{12}
\]

Proof. The proof follows by direct computations as in [Hui84, EH91b]. Recalling (6), we get

\[
\frac{d}{dt} g_{ij} = \frac{d}{dt} (\partial_i F, \partial_j F) = -(H + k) ((\partial_i \nu, \partial_j F) + (\partial_i F, \partial_j \nu)) = -2(H + k) h_{ij}
\]
\[
\frac{d}{dt} \nu = \left( \frac{d}{dt} \nu, \partial_i F \right) g^{ij} \partial_j F = -\left( \nu, \frac{d}{dt} \partial_i F \right) g^{ij} \partial_j F = \left( \nu, \partial_i ((H + k) \nu) \right) g^{ij} \partial_j F = \partial_i (H + k) g^{ij} \partial_j F = \nabla^S (H + k).
\]

The evolution of the measure on $M_t$

\[
\mu_t = \sqrt{\det[g]}
\]
is given by

\[
\frac{d}{dt} \sqrt{\det[g]} = \frac{d}{dt} \sqrt{\det[g]} \cdot \frac{\det[g] \cdot \text{Tr} \left( g^{ij} \frac{d}{dt} g_{ij} \right)}{2 \sqrt{\det[g]}}
\]
\[
= -\sqrt{\det[g]} \cdot (H + k) g^{ij} h_{ji} = -\mu_t (H + k).
\]

In order to prove (10) we compute (as usual, we denote the Christoffel symbols by $\Gamma^k_{ij}$)

\[
\frac{d}{dt} h_{ij} = -\frac{d}{dt} (\nu, \partial^2_{ij} F)
\]
\[
= -\left( \nabla^S (H + k), \partial^2_{ij} F \right) + \left( \partial^2_{ij} (H + k) \nu, \nu \right)
\]
\[
= \left( g^{kl} \partial_k (H + k) \partial_l F, \Gamma^k_{ij} \partial_k F - h_{ij} \nu \right)
\]
\[
+ \partial^2_{ij} (H + k) + (H + k) \left( \partial_j \left( h_{im} g^{ml} \partial_l F \right), \nu \right)
\]
\[
= \partial^2_{ij} (H + k) - \Gamma^k_{ij} \partial_k (H + k) + (H + k) h_{im} g^{ml} \left( \Gamma^k_{ij} \partial_k F - h_{ij} \nu, \nu \right)
\]
\[
= \nabla^S_i \nabla^S_j (H + k) - (H + k) h_{ij} g^{lm} h_{mj}. \tag{13}
\]

Using Codazzi’s equations, one can show that

\[
\Delta^S h_{ij} = \nabla^S_i \nabla^S_j H + H h_{id} g^{lm} h_{mj} - |A|^2 h_{ij}, \tag{14}
\]
so that (10) follows from (14) and (13). From (10) we deduce

\[
d\frac{d}{dt}H = \frac{d}{dt}g^{ij}h_{ij} = 2(H + k)g^{is}h_{sl}g^{lj}h_{ij} + g^{ij}(\nabla^i S \nabla^j S (H + k) - (H + k)h_{il}g^{lm}h_{mj}) = \Delta^S (H + k) + (H + k)|A|^2,
\]

which gives (11). In addition, we get

\[
\frac{d}{dt}|A|^2 = \frac{d}{dt}(g^{ik}g^{jl}h_{ij}h_{kl}) = 2\frac{d}{dt}g^{ik}h_{ij}h_{kl} + 2g^{ik}g^{jl} \frac{d}{dt}h_{ij}h_{kl} = 2(2(H + k)g^{is}h_{sl}g^{lj}h_{ij}h_{kl} + 2g^{ik}g^{jl}h_{ij}h_{kl}) = 2k(2(H + k)g_{st}h_{tl}g_{jm}h_{ij} - k|A|^2 h_{ij}) h_{kl}
\]

On the other hand, one has

\[
\Delta^S |A|^2 = 2\langle \Delta^S A, A \rangle + 2|\nabla^S A|^2 = 2g^{pq}g^{mn}h_{pm}\Delta^S h_{qn} + 2|\nabla^S A|^2.
\]

so that (12) follows from (16) and (15).

\[\Box\]

### 2.2 The Monotonicity Formula

We extend Huisken’s monotonicity formula [Hui90] to the forced mean curvature flow (5) (see also [CNV11, Section 2.2]).

Given a vector field \( \omega : M_t \to \mathbb{R}^{d+1} \), we let

\[\omega = (\omega, \nu) \nu, \quad \omega^T = \omega - \omega^T.\]

Letting \( X_0 \in \mathbb{R}^{d+1} \) and \( t_0 \in \mathbb{R} \), for \( (X, t) \in \mathbb{R}^{d+1} \times [t_0, +\infty) \) we define the kernel

\[
\rho(X, t) = \frac{1}{(4\pi(t - t_0))^d/2} \exp\left(-\frac{|X - X_0|^2}{4(t - t_0)}\right).
\]

A direct computation gives

\[
\frac{d\rho}{dt} = -\Delta^S \rho + \rho \left(\frac{(X_0 - X, (H + k)\nu)}{t_0 - t} - \frac{|X_0 - X|^2}{4(t_0 - t)^2}\right).
\]

**Proposition 2** (Monotonicity Formula).

\[
\frac{d}{dt} \int_{M_t} \rho = -\int_{M_t} \rho \left(\frac{H + k}{2} + \frac{(X - X_0, \nu)}{2(t_0 - t)}\right)^2 - k^2\frac{2}{4}.
\]

**Proof.** Recalling (9), we compute
\[
\frac{d}{dt} \int_{M_t} \rho = \int_{M_t} \frac{d}{dt} \rho - H(H + k)\rho \\
= \int_{M_t} \rho \left( -\frac{|X - X_0|^2}{4(t_0 - t)^2} + \frac{d}{dt} - \frac{(X - X_0, \nu)}{2(t_0 - t)}(H + k) - H(H + k) \right) \\
= -\int_{M_t} \rho \left( H\nu + \frac{X - X_0}{2(t_0 - t)} + \frac{k\nu}{2} \right)^2 - \frac{k^2}{4} + \int_{M_t} \frac{d}{dt} \rho + \int_{M_t} \rho \frac{(X - X_0, \nu)H}{2(t_0 - t)}
\]

We use the first variation formula: for all vector field \( Y \) on \( M_t \), we have
\[
\int_{M_t} \operatorname{div}_M Y = \int_{M_t} \langle H\nu, Y \rangle.
\]

As a result, with \( Y = \frac{\rho(X - X_0)}{2(t_0 - t)} \), we get
\[
\frac{d}{dt} \int_{M_t} \rho = -\int_{M_t} \rho \left( H\nu + \frac{X - X_0}{2(t_0 - t)} + \frac{k\nu}{2} \right)^2 - \frac{k^2}{4} + \int_{M_t} \rho \frac{(X - X_0, \nu)H}{2(t_0 - t)}
\]

In a similar way (see [EH89]) one can prove that for all functions \( f(X, t) \) defined on \( M_t \), one has
\[
\partial_t \int_{M_t} \rho f = \int_{M_t} \left( \frac{df}{dt} - \Delta^S f \right) \rho - \int_{M_t} f \rho \left( H + \frac{(X - X_0, \nu)}{2(t_0 - t)} + \frac{k^2}{2} - \frac{k^2}{4} \right).
\] (18)

Indeed, using (17)
\[
\frac{d}{dt} \int_{M_t} \rho f = \int_{M_t} f \frac{d\rho}{dt} + \frac{df}{dt} \rho - H(H + k)f\rho \\
= \int_{M_t} f \left( \frac{d\rho}{dt} - H(H + k)\rho \right) + \frac{df}{dt}\rho \\
= \int_{M_t} f \left( -\Delta^S \rho + \rho \frac{(X - X_0 - (H + k)\nu)}{t_0 - t} - \frac{1}{4} \|X - X_0\|^2 \right) - H(H + k)\rho \right) + \frac{df}{dt}\rho \\
= \int_{M_t} -\Delta^S \rho f + \frac{\rho \frac{(X - X_0 - (H + k)\nu)}{t_0 - t} - \frac{1}{4} \|X - X_0\|^2}{(t_0 - t)^2} - H(H + k)\rho \right) + \frac{df}{dt}\rho \\
= \int_{M_t} \rho \left( \frac{d}{dt} f - \Delta^S f \right) - \int_{M_t} f \rho \left( H + \frac{(X - X_0, \nu)}{2(t_0 - t)} + \frac{k^2}{2} - \frac{k^2}{4} \right).
\]

**Lemma 1.** Let \( f \) be defined on \( M_t \) and satisfy
\[
\frac{d}{dt} f - \Delta^S f \leq a \cdot \nabla^S f \quad \text{on } M_t
\] (19)

for some vector field \( a \) bounded on \([0, t_1]\). Then,
\[
\sup_{M_t, t \in [0, t_1]} f \leq \sup_{M_0} f.
\]
Proof. Denote by $a_0$ the bound on $a$, $k := \sup_{M_0} f$ and define $f_t = \max(f - t, 0)$. Assumption (19) implies
\[
\left(\frac{d}{dt} - \Delta^S\right) f_t^2 \leq 2 f_t a \cdot \nabla^S f_t - 2 |\nabla^S f_t|^2
\]
which, thanks to Young’s inequality, gives
\[
\left(\frac{d}{dt} - \Delta^S\right) f_t^2 \leq \frac{1}{2} a_0^2 f_t^2.
\]
Applying (18) to $f_t^2$, we get
\[
\frac{d}{dt} \int f_t^2 \rho \leq \frac{1}{2} (a_0^2 + \|k\|^2_{L^2}) \int f_t^2 \rho.
\]
Letting $l = \sup_{M_0} f$, so that $f_t \equiv 0$ on $M_0$, from (20) and the Gronwall’s Lemma we obtain that $f_t \equiv 0$ on $M_t$ for all $t \in (0, t_1]$, which gives thesis. \qed

3 Proof of Theorem 1

We now prove short time existence for the mean curvature flow with obstacles (1), (2). Let $M_0 = \partial E(0) \subset U$, where we assume that $U$, $E(0)$ are open sets with boundary uniformly of class $C^{1,1}$, with $\text{dist}(M_0, \partial U) > 0$. In particular, $M_0$ satisfies a uniform exterior and interior ball condition, that is, there is $R > 0$ such that, for every $x \in M_0$, one can find two open balls $B^+$ and $B^-$ of radius $R$ which are tangent to $M_0$ at $x$ and such that $B^+ \subset E(0)^c$ and $B^- \subset E(0)$. Let also $\Omega^- := E(0) \setminus \overline{U}$, and $\Omega^+ := E(0)^c \setminus (M_0 \cup \overline{U})$. Notice that $\Omega^\pm$ are open sets with $C^{1,1}$ boundaries, with $\text{dist}(\Omega^-, \Omega^+) > 0$. Let also
\[
k := 2N(\chi_{\Omega^-} - \chi_{\Omega^+})
\]
where $N$ is bigger than ($d$ times) the mean curvature of $\partial U$.

We want to show that equation (5), with $k$ as above, has a solution in an interval $[0, T)$. To this purpose, letting $\rho_\varepsilon$ be a standard mollifier supported in the ball of radius $\varepsilon$ centered at 0, we introduce a smooth regularization $k_\varepsilon = k * \rho_\varepsilon$ of $k$. Notice that $\|k_\varepsilon\|_{L^\infty} = 2N$, $k_\varepsilon(x) = 2N$ (resp. $k_\varepsilon(x) = -2N$) at every $x \in \Omega^-$ (resp. $x \in \Omega^-$) such that $\text{dist}(x, \partial U) \geq \varepsilon$, and $k_\varepsilon(x) = 0$ at every $x \in U$ such that $\text{dist}(x, \partial U) \geq \varepsilon$.

Using standard arguments (see for instance [EH91b, Theorem 4.1] and [EH91a, Prop. 4.1]) one can show existence of a smooth solution $M_t^\varepsilon$ of (5), with $k$ replaced by $k_\varepsilon$, on a maximal time interval $[0, T_\varepsilon)$.

Let now $\Omega_\varepsilon^\pm := \{x \in \Omega^\pm : \text{dist}(x, U) > \varepsilon\}$. The following result follows directly from the definition of $k_\varepsilon$.

Proposition 3. The hypersurfaces $\partial \Omega_\varepsilon^\pm$ are respectively a super and a subsolution of (5), with $k$ replaced with $k_\varepsilon$. In particular, by the parabolic comparison principle $M_t^\varepsilon$ cannot intersect $\partial \Omega_\varepsilon^\pm$.

We will show that we can find a time $T > 0$ such that for every $\varepsilon$, there exists a smooth solution of (5) (with $k$ replaced with $k_\varepsilon$) on $[0, T)$.

The following result will be useful in the sequel. We omit the proof which is a simple ODE argument.
Lemma 2. Let \( M_0 = \partial B_R(x_0) \) be a ball of radius \( R \leq 1 \) centered at \( x_0 \). Then, the evolution \( M_t \) by (5), with constant forcing term \( k = 2N \), is given by \( M_t = B_R(t)(x_0) \) with \( R(t) \geq \sqrt{R^2 - (4N + 2d)t} \). In particular, the solution exists at least on \([0, \frac{R^2}{4N + 2d}]\).

Proposition 4. There exists \( r > 0 \), a collection of balls \( B_i = B_r(x_i) \) of radius \( r \), and a positive time \( T_0 \) such that \( M_\epsilon \subset \bigcup_i B_i \) for every \( t \in [0, \min(T_0, T_\epsilon)) \). In addition, we can choose the balls \( B_i \) in such a way that, for every \( i \), there exists a unit vector \( \omega_i \in \mathbb{R}^{d+1} \) satisfying

\[
(\omega_i, \nu^+(x)) \geq \frac{1}{2} \quad \text{and} \quad (\omega_i, \nu^-(y)) \geq \frac{1}{2}
\]

for every \( x \in \partial \Omega^+ \cap B_{4r}(x_i) \) and \( y \in \partial \Omega^- \cap B_{4r}(x_i) \), where \( \nu^\pm \) is the outer normal to \( \Omega^\pm \).

As a result, \( \partial \Omega^\pm \cap B_{4r}(x_i) \) are graphs of some functions \( \psi^\pm_i : \mathbb{R}^d \to \mathbb{R} \cup \{\pm \infty\} \) over \( \omega_i^\perp \).
Notice also that $k$ is a $BV$ function and $Dk$ is a Radon measure concentrated on $\partial U$ such that
\[
(Dk, \omega_i) \geq \frac{|Dk|}{2} \text{ on } B_{4r}(x_i).
\]

Then, for every $x \in B_{2r}(x_i)$ and $\varepsilon$ sufficiently small (such that $\rho_\varepsilon(x) = 0$ as soon as $|x| \geq 2r$), we have
\[
(\nabla k_\varepsilon, \omega_i) = \left(\nabla \int_{\mathbb{R}^{d+1}} k(x-y)\rho_\varepsilon(y)dy, \omega_i \right)
= \int_{\mathbb{R}^{d+1}} (Dk(x-y), \omega_i) \rho_\varepsilon(y)dy
\geq \int_{\mathbb{R}^{d+1}} \frac{|Dk|(x-y)}{2} \rho_\varepsilon(y)dy
\geq \frac{|Dk| \ast \rho_\varepsilon}{2} \geq \frac{\nabla k_\varepsilon}{2}.
\]

\hfill \Box

In what follows, we will control the geometric quantities of $M_t^\varepsilon$ inside each ball $B_i$. As in [EH91a], we introduce a localization function $\phi_i$ as follows: let $\eta_i(x, t) = |x-x_i|^2 + (2d+\Lambda) t$ ($\Lambda$ be a positive constant that will be fixed later) and, for $R = 2r$, $\phi_i(x, t) = (R^2 - \eta_i(x, t))^+$.

We denote by $\phi_i$ the quantity $\phi_i(x, t)$, where $x = x(p, t)$ will be a generic point in $M_t$. Notice that there exists $T_1 = \frac{r^2}{2d+\Lambda}$ such that for all $t \in [0, \min(T, T_1)]$, $M_t^\varepsilon \subset \bigcup_i \{\phi_i > r^2\}$. (21)

As a result, we have the following

**Lemma 3.** Let $f$ be a smooth function defined on $M_t^\varepsilon$. Assume that there is a $C > 0$ such that $\phi_i f \leq C$ on $M_t^\varepsilon$ for all $t \leq \min(T, T_1)$ and $\forall i \in \mathbb{N}$.

Then,
\[
f \leq \alpha C \text{ on } M_t^\varepsilon \forall t \leq \min(T, T_1),
\]
where $\alpha$ depends only on the $C^{1,1}$ norm of $M_0$.

**Lemma 4.** Let $v := (\nu, \omega)^{-1}$. The quantity $v^2 \phi^2$ satisfies
\[
\left(\frac{d}{dt} - \Delta_S\right)\left(\frac{v^2 \phi^2}{2}\right) \leq \frac{1}{2} \left(\nabla^S(v^2 \phi^2), \frac{\nabla^S \phi^2}{\phi^2}\right) + \phi^2 v^3 \left(\nabla^S k_\varepsilon, \omega\right) + v^2 \phi(2k_\varepsilon(x, \nu) - \Lambda).
\]

**Proof.** In this proof and the proofs further, we use normal coordinates: we assume that $g_{ij}\delta_{ij}$ (Kronecker symbol) and that the Christoffel symbols $\Gamma_j^i_0$ vanish at the computation point.

We expand the derivatives
\[
\left(\frac{d}{dt} - \Delta_S\right)\left(\frac{v^2 \phi^2}{2}\right) = v^2 \left(\frac{d}{dt} - \Delta_S\right)\frac{\phi^2}{2} + \phi^2 \left(\frac{d}{dt} - \Delta_S\right)\frac{v^2}{2} - 2 \left(\nabla^S \frac{\phi^2}{2}, \nabla^S \frac{v^2}{2}\right).
\]

**First term.** We start computing
\[
\left(\frac{d}{dt} - \Delta_S\right)\left|\phi\right|^2 = -2k_\varepsilon(x, \nu) - 2d.
\]
Then,
\[
\left( \frac{d}{dt} - \Delta^S \right) \phi^2 = 2\phi(2k_\varepsilon (x - x_i, \nu) - \Lambda) - 2|\nabla^S |x|^2|^2. 
\]

**Second term.** We are interested in

\[
\frac{1}{2} \frac{d}{dt} (\omega, \nu)^2 = (\omega, \nu) \left( \frac{d}{dt} \nu, \omega \right) = (\omega, \nu) \left( \nabla^S (H + k_\varepsilon), \omega \right). 
\]

So,

\[
\frac{1}{2} \frac{d}{dt} (\omega, \nu)^{-2} = - (\omega, \nu)^{-3} \left( \nabla^S (H + k_\varepsilon), \omega \right). 
\]

On the other hand,

\[
\frac{1}{2} \Delta^S ((\omega, \nu)^{-2}) = (\omega, \nu)^{-1} \Delta^S (\omega, \nu)^{-1} - \left( \nabla^S (\omega, \nu)^{-1}, \nabla^S (\omega, \nu)^{-1} \right). 
\]

Let us note that

\[
\partial_{ij} \nu = \partial_i \left( h_{ij}g^{lm} \partial_m F \right) = \partial_i (h_{ij}) \delta_{tm} \partial_m F - h_{ij} \delta_{tm} (-h_{im} \nu) = \partial_i (h_{ij}) \partial_l F - \lambda_t^2 \delta_{ij} \nu. 
\]

We then get

\[
\Delta^S (\omega, \nu)^{-1} = \partial_{ii} (\omega, \nu)^{-1} = \partial_i \left( - (\omega, \partial_i \nu) (\omega, \nu)^{-2} \right) 
= - (\omega, \partial_i \nu) (\omega, \nu)^{-2} + 2 (\omega, \partial_i \nu)^2 (\omega, \nu)^{-3} 
= - (\omega, \nu)^{-2} (\partial_i h_{ij} \partial_l F - \lambda_t^2 \nu, \omega) + 2 (\omega, \nu)^{-3} (\omega, \lambda_t \partial_i F)^2. 
= - (\omega, \nu)^{-2} (\partial_l h_{ij} \partial_i F, \omega) + |A|^2 (v, \omega)^{-1} + 2 (\omega, \nu)^{-3} (\omega, \lambda_t \partial_i F)^2. 
\]

We also have

\[
\left( \nabla^S (\omega, \nu)^{-1}, \nabla^S (\omega, \nu)^{-1} \right) = (\omega, \nu)^{-4} (\omega, \partial_i \nu) (\omega, \partial_k \nu) 
= (\omega, \nu)^{-4} (\omega, h_{ka} g^{aw} \partial_l F)^2 = (\omega, \nu)^{-4} (\omega, \lambda_k \partial_l F)^2, 
\]

which leads to

\[
\left( \frac{d}{dt} - \Delta^S \right) \frac{v^2}{2} = -v^3 \left( \nabla^S (H + k_\varepsilon), \omega \right) + v^3 \partial_m (h_{ii}) (\omega, \partial_m F) 
- |A|^2 v^2 - 2v^2 \lambda_k^2 (\omega, \partial_k F)^2 - v^4 (\omega, \lambda_k \partial_l F)^2. 
\]

**Third term.** We notice, as in [EH91a] that $|\nabla^S \phi|^2 = 4\phi^2|\nabla^S |x|^2|^2$ and

\[
- (\nabla^S (v^2), \nabla^S \phi^2) = -3 (v \nabla^S (v), \nabla^S \phi^2) + \frac{1}{2} \left( \left( \nabla^S (v^2 \phi^2), \frac{\nabla^S \phi^2}{\phi^2} \right) - v^2 |\nabla^S \phi|^2 \right). 
\]

Then, Young’s inequality gives

\[
2 |v (\nabla^S v, \nabla^S \phi^2)| \leq 2 \phi^2 |\nabla^S v|^2 + \frac{1}{2 \phi^2} |\nabla^S \phi|^2 \leq 2 \phi^2 |\nabla^S v|^2 + 2v^2 |\nabla^S |x|^2|^2. 
\]
Hence,

\[-(\nabla^2_S(v^2), \nabla^2_S \phi^2) \leq -3\phi^2 |\nabla^2_S v^2|^2 - 3v^2 |\nabla^2_S |v|^2|^2 + \frac{1}{2} \left( \left( \nabla^2_S(v^2 \phi^2), \frac{\nabla^2_S \phi^2}{\phi^2} \right) - v^2 |\nabla^2_S \phi^2|^2 \right).\]

Summing the three terms, we get

\[\left( \frac{d}{dt} - \Delta^S \right) \left( \frac{v^2 \phi^2}{2} \right) \leq \frac{1}{2} \left( \nabla^2_S(v^2 \phi^2), \frac{\nabla^2_S \phi^2}{\phi^2} \right) - \phi^2 v^3 (\nabla^2_S k_\varepsilon, \omega) + v^2 \phi(2k_\varepsilon (v, \nu) - \Lambda).\]

For \( \gamma > 0 \), we let

\[\psi(v^2) := \frac{\gamma v^2}{1 - \gamma v^2}.\]

**Lemma 5.** For \( \varepsilon \leq r \), we have

\[\left( \frac{d}{dt} - \Delta^S \right) \frac{\phi^2 |A|^2 \psi(v^2)}{2} \leq \phi^2 \psi(v^2)(-\gamma |A|^4 - 2k_\varepsilon \sum_i \lambda_i^2 - 2 \langle A, (\nabla^2_S)k_\varepsilon \rangle)\]

\[-\phi^2 |A|^2 v^3 \psi'(v^2) (\nabla^2_S k_\varepsilon, \omega) - \phi^2 |A|^2 \sum_i (\lambda_i \omega^2)^2 \frac{2v^4 + \gamma v^6}{(1 - \gamma v^2)^2}.\]

**Proof.** We denote \( V = \frac{\phi^2 |A|^2 \psi(v^2)}{2} \) and compute

\[\left( \frac{d}{dt} - \Delta^S \right) \frac{\phi^2 |A|^2 \psi(v^2)}{2} = |A|^2 \psi(v^2) \left( \frac{d}{dt} - \Delta^S \right) \frac{1}{2} \phi^2 + \phi^2 \psi(v^2) \left( \frac{d}{dt} - \Delta^S \right) \frac{1}{2} |A|^2\]

\[+ \phi^2 |A|^2 \left( \frac{d}{dt} - \Delta^S \right) \frac{1}{2} \psi(v^2) - 2 \frac{1}{1/2 \nabla^2_S |A|^2, 1/2 \nabla^2_S \phi^2} \]

\[+ 2 \frac{1}{1/2 \nabla^2_S |A|^2, 1/2 \nabla^2_S \psi(v^2)} + 2 \frac{1}{1/2 \nabla^2_S \phi^2, 1/2 \nabla^2_S \psi(v^2)}\].

The two first terms have already been computed. Let us consider in the third one.

\[\frac{1}{2} \frac{d}{dt} \psi(v^2) = v \frac{du}{dt} \psi'(v^2) = -v^3 \psi'(v^2) \left( \nabla^2_S (H + k_\varepsilon), \omega \right),\]

\[\frac{1}{2} \Delta^S \psi(v^2) = \frac{1}{2} \partial_{ii} \psi(v^2) = \partial_{ii} \left( v \partial_{i} \psi' (v^2) \right) = v \Delta^S v \psi'(v^2) + 2v^2 |\nabla^2_S v|^2 \psi''(v^2) + |\nabla^2_S v|^2 \psi'(v^2)\]

\[= (3|\nabla^2_S v|^2 - v^3 (\partial_{ii}(h_{kk}))w^I + v^2 |A|^2) \psi'(v^2) + 2|\nabla^2_S v|^2 \psi''(v^2).\]

Hence

\[\left( \frac{d}{dt} - \Delta^S \right) \frac{1}{2} \psi(v^2) = -v^3 \psi'(v^2) \left( \nabla^2_S k_\varepsilon, \omega \right) - (3|\nabla^2_S v|^2 + v^2 |A|^2) \psi'(v^2) - 2v^2 |\nabla^2_S v|^2 \psi''(v^2).\]

As above, we want to conclude the proof using the weak maximum principle. So, we want to rewrite the last terms (which are gradient terms) using the gradient of \( V \). Let us expand \( \nabla^2_S V \).

\[\nabla^2_S \frac{\phi^2 |A|^2 \psi(v^2)}{2} = \phi^2 |A|^2 \frac{1}{2} \nabla^2_S \psi(v^2) + |A|^2 \frac{1}{2} \nabla^2_S \phi^2 + \phi^2 \psi(v^2) \frac{1}{2} \nabla^2_S |A|^2.\]
So,

\[
\left| \nabla^S \phi^2 |A|^2 \psi(v^2) \right|^2 = \phi^4 |A|^4 \left| \nabla^S \psi(v^2) \right|^2 / 4 + |A|^4 \psi^2(v^2) \left| \nabla^S \phi^2 \right|^2 / 4 + \phi^4 \psi^2(v^2) \left| \nabla^S |A|^2 \right|^2 / 4 \\
+ \phi^2 |A|^4 \psi(v^2) \left\langle \nabla^S \psi(v^2) \right\rangle, \left. \nabla^S \phi^2 \right\rangle + \phi^4 |A|^2 \psi(v^2) \left\langle \nabla^S \psi(v^2) \right\rangle, \left. \nabla^S |A|^2 \right\rangle \\
+ |A|^2 \psi^2(v^2) \phi^2 \left\langle \nabla^S \phi^2, \nabla^S |A|^2 \right\rangle.
\]

As a matter of fact,

\[
\frac{1}{\phi^2 |A|^2 \psi(v^2)} \left| \nabla^S \phi^2 |A|^2 \psi(v^2) \right|^2 = \phi^2 |A|^2 \left| \nabla^S \psi(v^2) \right|^2 / 4 \psi(v^2) + |A|^2 \psi(v^2) \left| \nabla^S \phi^2 \right|^2 / 4 \phi^2 \\
+ \phi^2 \psi(v^2) \left| \nabla^S |A|^2 \right|^2 / 4 |A|^2 + 2 |A|^2 \left\langle \nabla^S \psi(v^2) / 2, \nabla^S \phi^2 / 2 \right\rangle \\
+ 2 \phi^2 \left\langle \nabla^S \psi(v^2) / 2, \nabla^S |A|^2 / 2 \right\rangle + 2 \psi(v^2) \left\langle \nabla^S \phi^2 / 2, \nabla^S |A|^2 / 2 \right\rangle.
\]

We use the last equality to rewrite

\[
\left( \frac{d}{dt} - \Delta^S \right) \frac{\phi^2 |A|^2 \psi(v^2)}{2} = |A|^2 \psi(v^2) \left( \phi(2k_c (x, \nu) - \Lambda) - |\nabla^S x|^2 \right) \\
+ \phi^2 \psi(v^2) \left( - \left\langle \nabla^S A, \nabla^S A \right\rangle + |A|^4 - 2k_c g^a h_a h' + 2k_c \right) \\
+ \phi^2 |A|^2 \left( - \psi^2(v^2) \left\langle \nabla^S k_c \right\rangle, \omega \right) - 3 |\nabla^S v|^2 + 2 \psi |A|^2 \psi'(v^2) - 2v^2 |\nabla^S v|^2 \psi''(v^2) \\
- \frac{1}{\phi^2 |A|^2 \psi(v^2)} \left| \nabla^S \phi^2 |A|^2 \psi(v^2) \right|^2 \\
+ \phi^2 |A|^2 \left| \nabla^S \psi(v^2) \right|^2 / 4 \psi(v^2) + |A|^2 \psi(v^2) \left| \nabla^S \phi^2 \right|^2 / 4 \phi^2 + \phi^2 \psi(v^2) \left| \nabla^S |A|^2 \right|^2 / 4 |A|^2. \tag{33}
\]

Let us precise some terms:

\[
|\nabla^S \phi^2|^2 = 4 \phi^2 \cdot | - 2x^2|^2 = 4 \phi^2 (4|x|^2 - 4 (x, \nu)), \\
|\nabla^S \psi(v^2)|^2 = \psi'(v^2)^2 |\nabla^S v|^2 = 4 \psi'(v^2)^2 v^6 \sum_k (\lambda_k \omega^k)^2, \\
|\nabla^S |A|^2|^2 = 4 \sum_i (\partial_i (h_{ii}) \lambda_i)^2, \\
|\nabla^S A|^2 = \sum_{i,k,l} (\partial_i (h_{km}))^2.
\]

In addition, we have the obvious estimate

\[
|\nabla^S |A|^2|^2 \leq 4 |A|^2 \left| \nabla^S A \right|^2.
\]

So,

\[
\phi^2 |A|^2 \left| \nabla^S \psi(v^2) \right|^2 / 4 \psi(v^2) + |A|^2 \psi(v^2) \left| \nabla^S \phi^2 \right|^2 / 4 \phi^2 + \phi^2 \psi(v^2) \left| \nabla^S |A|^2 \right|^2 / 4 |A|^2 \\
\leq \phi^2 |A|^2 \psi(v^2)^2 v^6 \sum_k (\lambda_k \omega^k)^2 + 4 |A|^2 \psi(v^2) |x|^2 - (x, \nu)^2 + \phi^2 \psi(v^2) |\nabla^S |A|^2|^2.
\]
We plug this inequality into (33) and obtain

\[
\left( \frac{d}{dt} - \Delta_S \right) \frac{\phi^2 |A|^2 \psi(v^2)}{2} \leq |A|^2 \psi(v^2) \left( \phi(2k_\varepsilon (x, \nu) - \Lambda) - |\nabla^S x|^2 \right) \\
+ \phi^2 \psi(v^2) \left( - \nabla^S A, \nabla^S A \right) + |A|^4 - 2k_\varepsilon g^i s h_{st} g^t d h_{ij} h_{kl} - 2 \left( A, \nabla^2 k_\varepsilon \right) \\
+ \phi^2 |A|^2 \left( - v^3 \psi'(v^2) \left( \nabla^S k_\varepsilon , \omega \right) - (3|\nabla^S v|^2 + v^2 |A|^2) \psi'(v^2) - 2v^2 |\nabla^S v|^2 \psi''(v^2) \right) \\
- \frac{1}{\phi^2 |A|^2 \psi(v^2)} \left| \nabla^S \phi^2 |A|^2 \psi(v^2) \right|^2 \\
+ \phi^2 |A|^2 \frac{v^6 \psi'(v^2)^2}{\psi(v^2)} + 4 |A|^2 \psi(v^2) |\nabla^S A|^2.
\]

Let us regroup some terms (noting that $|\nabla^S v|^2 = v^4 \sum_i (\lambda_i \omega^i)^2$), we get

\[
\left( \frac{d}{dt} - \Delta_S \right) \frac{\phi^2 |A|^2 \psi(v^2)}{2} \leq |A|^2 \psi(v^2) \left( \phi(2k_\varepsilon (x, \nu) - \Lambda) \\
+ \phi^2 |A|^4 (\psi(v^2) - v^2 \psi'(v^2)) - 2\phi^2 \psi(v^2) k_\varepsilon g^i s h_{st} g^t d h_{ij} h_{kl} - 2\phi^2 \psi^2(v^2) \left( A, \nabla^2 k_\varepsilon \right) \\
- \phi^2 |A|^2 v^3 \psi'(v^2) \left( \nabla^S k_\varepsilon , \omega \right) - \frac{1}{\phi^2 |A|^2 \psi(v^2)} \left| \nabla^S \phi^2 |A|^2 \psi(v^2) \right|^2 \\
+ \phi^2 |A|^2 \sum_i (\lambda_i \omega^i)^2 \left( \frac{v^6 \psi'(v^2)^2}{\psi(v^2)} - 3v^4 \psi'(v^2) - 2v^6 \psi''(v^2) \right) .
\]

Then, we note that

\[
\frac{v^6 \psi'(v^2)^2}{\psi(v^2)} - 3v^4 \psi'(v^2) - 2v^6 \psi''(v^2) = -\frac{2v^4 + \gamma v^6}{(1 - \gamma v^2)^3} \leq 0
\]

and

\[
\psi(v^2) - v^2 \psi'(v^2) = -\gamma \psi^2(v^2) \leq 0.
\]

So,

\[
\left( \frac{d}{dt} - \Delta_S \right) \frac{\phi^2 |A|^2 \psi(v^2)}{2} \leq \phi^2 \psi(v^2) \left( -|A|^4 - 2k_\varepsilon \sum_i \lambda_i^2 \gamma - 2 \left( A, \nabla^2 k_\varepsilon \right) \\
- \phi^2 |A|^2 v^3 \psi'(v^2) \left( \nabla^S k_\varepsilon , \omega \right) - \phi^2 |A|^2 \sum_i (\lambda_i \omega^i)^2 \frac{2v^4 + \gamma v^6}{(1 - \gamma v^2)^3},
\]

what was expected. \( \square \)

We now show that $M_t$ can be locally written as a Lipschitz graph, with Lipschitz constant independent of $\varepsilon$.

**Proposition 5.** Let $\varepsilon \leq r$. Then, for every $t \in [0, \min(T_\varepsilon, T_1))$, $M_t \cap B_i$ can be written as a Lipschitz graph over $\omega^i$, with Lipschitz constant independent of $\varepsilon$.

**Proof.** We want to show that the quantity $(\nu, \omega_i)$ is bounded from below, or, equivalently, that $v := (\nu, \omega_i)^{-1}$ is bounded from above on every ball $B_i$. We want to estimate the quantity $v^2 \phi^2$ (we drop the explicit dependence on the index $i$) using Lemma 4.
We choose \( \Lambda \) such that the last term in (22) is nonpositive (take for instance \( \Lambda = 2NR \)). We also have to control
\[
v(\nabla^2 \kappa, \omega) = (\nu, \omega)^{-1}((\nabla \kappa, \omega) - (\nabla \kappa, \nu)(\nu, \omega)) = (\nu, \omega)^{-1}(\nabla \kappa, \omega) - (\nabla \kappa, \nu).
\]

Proposition 4 provides immediately
\[
(\nu, \omega)^{-1}(\nabla \kappa, \omega) - (\nabla \kappa, \nu) \geq (\nu, \omega)^{-1}\frac{|\nabla \kappa|}{2} - |\nabla \kappa|
\]
which is nonnegative as soon as \((\omega, \nu) \leq \frac{1}{\Phi} \frac{2}{3} \). From Lemma 4 and the weak maximum principle (see [PW84]), we obtain that \( \|v^2 \phi^2\|_\infty(t) \leq \max(\|v^2 \phi^2\|_\infty(0), 4R^2) \). Thanks to Lemma 3, this provides a uniform Lipschitz bound on the whole \( M_t \), for \( t \leq T_1 \). \( \square \)

Recalling Theorem 8.1 in [Hui84], from Proposition 5 it follows that, if \( T_\epsilon < T_1 \), the second fundamental form of \( M_t \) blows up as \( t \to T_2 \). Let us show that it does not happen.

**Proposition 6.** For every \( \epsilon \leq r \), there exists \( C_\epsilon > 0 \) such that
\[
\|A\|_{L^\infty(M_t)} \leq C_\epsilon \quad \text{for all} \ t \in [0, \min(T_\epsilon, T_1)).
\]

**Proof.** As in [EH91a], we are interested in the evolution of the quantity
\[
\frac{\phi^2|A|^2\psi(v^2)}{2}.
\]
Notice that
\[
|\lambda_i| = |\lambda_i| \leq 1 \frac{2\alpha}{2}\lambda_i + \frac{\alpha}{2} \lambda_i^2.
\]
Choosing \( \alpha \) such that \( \frac{2N}{\alpha} \leq \frac{\gamma}{2} \), one can write
\[
\left| -2k_v \phi^2 \psi(v^2) \sum_i \lambda_i^3 \right| \leq \phi^2 \psi(v^2) \left( \frac{\gamma}{2} |A|^4 + N\alpha |A|^2 \right).
\]
In addition, as soon as \( |A|^2 \geq 1 \), one has \( \langle A, \nabla^2 \kappa \rangle \leq |A|^2|\nabla^2 \kappa| \). One can also notice that as above, \( v(\nabla^2 \kappa, \omega) \geq 0 \) as soon as \( v \geq 2 \). On the other hand, if \( v \leq 2 \), one has
\[
v^3 |(v^2) = \frac{\psi(v) v}{1 - \gamma v} \leq 4\psi(v) \text{ for } \gamma \text{ sufficiently small}.
\]
So, anyway, if \( |A| \geq 1 \),
\[
\left( \frac{d}{dt} - \Delta S \right) \frac{\phi^2|A|^2\psi(v^2)}{2} \leq 2N\alpha \phi^2|A|^2\psi(v^2) + 4|\nabla^2 \kappa| \phi^2|A|^2\psi(v^2) + 8\phi^2|A|^2\psi(v^2) |\nabla^2 \kappa|.
\]
Finally, we apply the maximum principle to
\[
\tilde{A} := \exp \left[ -(2N\alpha + 4\|\nabla^2 \kappa\|_\infty + 8\|\nabla \kappa\|_\infty) t \right] \frac{\phi^2|A|^2\psi(v^2)}{2}
\]
which satisfies
\[
\left( \frac{d}{dt} - \Delta S \right) \tilde{A} \leq 0.
\]
It provides
\[
\forall t \leq \min(T_\epsilon, T_1), \quad \|\tilde{A}\|_\infty(t) \leq \|\tilde{A}\|_\infty(0)
\]
which shows that \( \phi^2|A|^2\psi(v^2) \) does not blow up.

Using Lemma 3 and choosing \( \gamma \) such that \( \psi(v^2) \) is bounded and remains far from zero, we know that \( |A| \) does not blow up for \( t \leq T_1 \). \( \square \)
Corollary 1. There exists $T_1$, depending only on the dimension, $\|k\|_\infty$ and the radius in the ball condition for $M_0$, such that there exists a solution $M_t^i$ of the mean curvature flow with forcing term $k_\varepsilon$ on $[0,T_1)$.

The surfaces $M_t^i$ are uniformly Lipschitz and every $M_t^i \cap B_i$ can be written as the graph of some function $u_t^i(x,t)$. All the $u_t^i$ are Lipchitz (in space) with a constant which depends neither on $i$ nor in $\varepsilon$. We want to show that they are also equicontinuous in time.

Proposition 7. The functions $u_t^i$ are Lipschitz continuous in $x$ and 1/2-Hölder continuous in $t$ on $B_i \times [0,T_1)$, uniformly with respect to $\varepsilon$ and $i$.

Proof. Let $\delta$ be fixed (we drop the index $\varepsilon$ in what follows), and let $t_0 \in [0,T_1)$. Let $x_0 \in M_i$ and $i$ such that $x_0 \in B_i$. Then, $(\nu(x_0),\omega_i) \leq C$ and $M_i$ is the graph of a function $u$ over $\omega_i^+$. Then, let $x_1 = x_0 + \delta \omega_i$. Thanks to the Lipschitz condition, there is a ball $B_{1/C\delta}(x_1)$ that does not touch $M_i$. Evolving by mean curvature with forcing term $k_\varepsilon$, this ball vanishes in a positive time $T_\delta > \omega(\delta) := \frac{\delta^2}{C^2(2d+1)}$ (note that $T_\delta$ does not depend on $\varepsilon$). By comparison principle, for $t \in [t_0,t_0 + \omega(\delta))$, $M_t$ does not go beyond $x_1$. That is equivalent to say that $u$ is 1/2-Hölder continuous in time, with a constant independent of $\varepsilon$.

We now pass to the limit as $\varepsilon$ goes to zero. By Proposition 7, the family $(u_t^i)$ is equi-Lipschitz in space and equi-continuous in time on $B_i \times [0,T_1)$. Therefore, by Arzelà–Ascoli’s Theorem one can find a sequence $\varepsilon_n \to 0$ and continuous functions $u_i$ such that, for every $i$, $u_t^i \xrightarrow{n \to \infty} u_i$ locally uniformly on $B_i \times [0,T_1)$.

Proposition 8. The functions $u_i$ are viscosity solutions of (3) on $B_i \times [0,T_1)$, with obstacles $U \cap B_i$ (see Appendix 5).

Proof. Thanks to Proposition 4, every $x \in B_i$ can be decomposed as $x = x' + z\omega_i$ with $z = (x',\omega_i)$. Then, there exists functions $\psi_i^\pm$ of class $C^{1,1}$ such that

$$U \cap B_i = \{(x',z) \in B_i : \psi_i^-(x') \leq z \leq \psi_i^+(x')\}.$$

For simplicity we shall drop the explicit dependence on the index $i$. Since $u^\varepsilon(x,0) = u_0(x)$ for all $\varepsilon$, and $u^{\varepsilon_n}$ converges uniformly to $u$ as $n \to +\infty$, it is clear that $u(x,0) = u_0(x)$.

Condition (45) immediately follows from Proposition 3. We now check that $u$ is a subsolution of (3). Let $(x_0,t_0) \in \mathbb{R}^d \times \mathbb{R}$ and $\varphi \in C^2$ such that $\psi^-(x_0,t_0) < u(x_0,t_0)$ and

$$(u - \varphi)(x_0,t_0) = \max_{(x,t) \in V} (u - \varphi)(x,t).$$

One can change $\varphi$ so that $(x_0,t_0)$ is a strict maximum point, and $u(x_0,t_0) = \varphi(x_0,t_0)$. Let $2\delta := u(x_0,t_0) - \psi^-(x_0,t_0)$. Thanks to the definition of $k_\varepsilon$, for all $\varepsilon \leq \delta$, we have $k_\varepsilon(x,\varphi(x,t)) \leq 0$ in a small neighborhood $V$ of $(x_0,t_0)$. Hence, for $\varepsilon$ sufficiently small $u^\varepsilon - \varphi$ attains its maximum in $V$ at $(x_\varepsilon,t_\varepsilon)$, with $(x_\varepsilon,t_\varepsilon) \to (x_0,t_0)$ as $\varepsilon \to 0$. Since $u^\varepsilon$ is a classical solution of (35), it is also a viscosity solution, therefore

$$\varphi_t - \sqrt{1 + |\nabla \varphi|^2} \text{div} \left( \frac{\nabla \varphi}{\sqrt{1 + |\nabla \varphi|^2}} \right) \leq \sqrt{1 + |\nabla \varphi|^2} k_\varepsilon(x,\varphi) \leq 0 \text{ at } (x_\varepsilon,t_\varepsilon).$$

Letting $\varepsilon \to 0$ we obtain that $u$ is a subsolution of (3). A similar argument shows that $u$ is also a supersolution of (3), and this concludes the proof. \qed
Conclusion of the proof of Theorem 1. The result in [PS07, Theorem 4.1] (see also Section A.4) applies, showing that the functions \( u_i \) are of class \( C^{1,1} \). As the uniform convergence \( u_i \to u \) implies the Hausdorff convergence of \( M_i \) to a limit \( M_t \) such that \( M_t \cap B_t = \text{graph}(u_i(t)) \), we built a \( C^{1,1} \) evolution to the mean curvature motion with obstacles on the time interval \([0, T_1]\). Thanks to [ACN12, Theorem 4.8 and Corollary 4.9] this evolution is also unique. This concludes the proof of Theorem 1.

4 Proof of Theorem 2

Let \( \psi^\pm \) be smooth functions such that \( \psi^\pm \to \psi^\pm \) as \( \varepsilon \to 0 \), uniformly in \( C^{1,1}(\mathbb{R}^d) \), and let \( N > 0 \) be such that
\[
N \geq \left\| \sqrt{1 + |\nabla u|^2} \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) \right\|_{L^\infty(\mathbb{R}^d)}
\]
for all \( \varepsilon > 0 \). (34)

We proceed as in Section 3 and we approximate (3), (4) with the forced mean curvature equation
\[
u_t = \sqrt{1 + |\nabla u|^2} \left[ \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) + k_\varepsilon(x, u) \right],
\]
where
\[
k_\varepsilon(x, u) = 2N \left( \chi \left( \frac{\psi^-_\varepsilon(x) - u}{\varepsilon} \right) - \chi \left( \frac{u - \psi^+_\varepsilon(x)}{\varepsilon} \right) \right),
\]
and \( \chi \) is a smooth increasing function such that \( \chi(s) \equiv 0 \) for all \( s \in (-\infty, 0] \), and \( \chi(s) \equiv 1 \) for all \( s \in [1, \infty) \). In particular \( \partial_u k_\varepsilon(x, u) \leq 0 \) for all \( (x, u) \).

Notice that \( k_\varepsilon \to g \) as \( \varepsilon \to 0 \), with
\[
k(x, u) = \begin{cases} 
2N & \text{if } u < \psi^-(x) \\
-2N & \text{if } u > \psi^+(x) \\
0 & \text{elsewhere}
\end{cases}
\]

Notice also that
\[
\frac{\partial k_\varepsilon}{\partial x_k}(x, u) + \frac{\partial k_\varepsilon}{\partial u}(x, u) \frac{\partial \psi^-_\varepsilon}{\partial x_k} = 0 \quad \text{if } u < \psi^+_\varepsilon \tag{36}
\]
\[
\frac{\partial k_\varepsilon}{\partial x_k}(x, u) + \frac{\partial k_\varepsilon}{\partial u}(x, u) \frac{\partial \psi^+_\varepsilon}{\partial x_k} = 0 \quad \text{if } u > \psi^-_\varepsilon.
\]

We denote by \( u_\varepsilon \) the solution of the approximate problem (35), which exists and is smooth for short times.

Proposition 9. The solution \( u_\varepsilon \) is defined for \( t \in [0, +\infty) \), and satisfies the estimates
\[
\|u_\varepsilon(\cdot, t)\|_{W^{1,\infty}(\mathbb{R}^d)} \leq C \quad \text{for all } t \in [0, +\infty) \tag{37}
\]
\[
\|u_\varepsilon(\cdot, t)\|_{W^{2,\infty}(\mathbb{R}^d)} \leq C(T) \quad \text{for all } t \in [0, T]. \tag{38}
\]

Proof. Estimate (37) follows from Proposition 5, choosing \( B_i = \mathbb{R}^{d+1}, \omega_i = \epsilon_{d+1} \) and \( \phi \equiv 1 \). Estimate (38) follows from (37) and Proposition 6. \( \square \)
In what follows, we use intrinsic derivatives on the graph $M_t := \{(x, u_\varepsilon(x,t))\}$, which will be denoted as above by an exponent $S$. The metric on $M_t$ is

$$g_{ij} = \delta_{ij} + \partial_i u_\varepsilon \partial_j u_\varepsilon$$

with inverse

$$g^{ij} = \delta_{ij} - \frac{\partial_i u_\varepsilon \partial_j u_\varepsilon}{1 + |\nabla u_\varepsilon|^2}.$$  

The tangential gradient of a function $f$ defined on $M_t$ is given by

$$(\nabla^S f)^i = g^{ij} \partial_j f - \frac{\partial_i u_\varepsilon \partial_j u_\varepsilon}{1 + |\nabla u_\varepsilon|^2} \partial_j f,$$

so that

$$(\nabla^S f, \nabla u_\varepsilon) = (\nabla f, \nabla u_\varepsilon) - \frac{\|\nabla u_\varepsilon\|^2}{1 + |\nabla u_\varepsilon|^2} (\nabla f, \nabla u_\varepsilon) = \frac{1}{1 + |\nabla u_\varepsilon|^2} (\nabla f, \nabla u_\varepsilon), \quad (39)$$

and

$$|\nabla^S f|^2 = \left( f_i - (u_\varepsilon)_i \sum_j \frac{(u_\varepsilon)_j f_j}{1 + |\nabla u_\varepsilon|^2} \right)^2$$

$$= |\nabla f|^2 + (u_\varepsilon)_i \left[ \frac{(\nabla u_\varepsilon, \nabla f)^2}{1 + |\nabla u_\varepsilon|^2} - 2 \frac{(u_\varepsilon)_i (u_\varepsilon)_j f_i f_j}{1 + |\nabla u_\varepsilon|^2} \right]$$

$$= |\nabla f|^2 + \frac{|\nabla u_\varepsilon|^2}{1 + |\nabla u_\varepsilon|^2} (\nabla u_\varepsilon, \nabla f)^2 - 2 \frac{(\nabla u_\varepsilon, \nabla f)^2}{1 + |\nabla u_\varepsilon|^2}$$

$$= |\nabla f|^2 - \frac{(\nabla u_\varepsilon, \nabla f)^2}{1 + |\nabla u_\varepsilon|^2} + \frac{(\nabla u_\varepsilon, \nabla f)^2}{(1 + |\nabla u_\varepsilon|^2)^2}. \quad (40)$$

In addition, the Laplace-Beltrami operator applied to $f$ is

$$\Delta^S f = g^{ij} f_{ij} = \Delta f - \frac{\partial_i u_\varepsilon \partial_j u_\varepsilon}{1 + |\nabla u_\varepsilon|^2} f_{ij} = \Delta f - \frac{(\nabla u_\varepsilon \nabla^2 f, \nabla u_\varepsilon)}{1 + |\nabla u_\varepsilon|^2}.$$  

**Proposition 10.** The quantity $\|\langle u_\varepsilon \rangle_t^2 \|_{\infty} (t)$ is nonincreasing in time. In particular,

$$\|\langle u_\varepsilon \rangle_t \|_{L^\infty(\mathbb{R}^d)} \leq \left\| \sqrt{1 + |\nabla u_\varepsilon|^2} \text{div} \left( \frac{\nabla u_\varepsilon}{\sqrt{1 + |\nabla u_\varepsilon|^2}} \right) \right\|_{L^\infty(\mathbb{R}^d)}.$$  

**Proof.** We compute

$$\frac{d}{dt} \frac{(u_\varepsilon)_t^2}{2} = (u_\varepsilon)_t \left[ \sqrt{1 + |\nabla u_\varepsilon|^2} \left( \text{div} \left( \frac{\nabla u_\varepsilon}{\sqrt{1 + |\nabla u_\varepsilon|^2}} \right) + k_\varepsilon(x, u_\varepsilon) \right) \right]_t.$$

Expanding this expression, we get

$$\frac{d}{dt} \frac{(u_\varepsilon)_t^2}{2} = (u_\varepsilon)_t \left[ \frac{\nabla (u_\varepsilon)_t \cdot \nabla u_\varepsilon}{\sqrt{1 + |\nabla u_\varepsilon|^2}} \left( \text{div} \left( \frac{\nabla u_\varepsilon}{\sqrt{1 + |\nabla u_\varepsilon|^2}} \right) + k_\varepsilon \right) \right.$$

$$+ \sqrt{1 + |\nabla u_\varepsilon|^2} \left( \text{div} \left( \frac{(u_\varepsilon)_t \cdot \nabla u_\varepsilon}{\sqrt{1 + |\nabla u_\varepsilon|^2}} - \frac{(\nabla u_\varepsilon)_t \cdot \nabla u_\varepsilon \nabla u_\varepsilon}{(1 + |\nabla u_\varepsilon|^2)^{3/2}} \right) + (u_\varepsilon)_t \partial u k_\varepsilon \right) \right].$$
Let us compute more explicitly the three terms of the expression above.

\[
(u_t)_t \frac{\nabla(u_t)_t \cdot \nabla u_e}{\sqrt{1 + |\nabla u_e|^2}} \left( \text{div} \left( \frac{-\nabla u_e}{\sqrt{1 + |\nabla u_e|^2}} \right) + k_e \right)
\]

\[
= \frac{\nabla (\frac{(u_t)^2}{2}) \cdot \nabla u_e}{\sqrt{1 + |\nabla u_e|^2}} \left( \frac{\Delta u}{\sqrt{1 + |\nabla u_e|^2}} - (u_e)_i (\nabla u_e)_i (1 + |\nabla u_e|^2)^{3/2} + k_e \right)
\]

\[
= \frac{\nabla (\frac{(u_t)^2}{2}) \cdot \nabla u_e}{\sqrt{1 + |\nabla u_e|^2}} \left( \frac{\Delta u_e}{1 + |\nabla u_e|^2} - \frac{\nabla u_e \cdot \nabla (\frac{(u_t)^2}{2})}{(1 + |\nabla u_e|^2)^2} + k_e \right),
\]

and

\[
(u_t)_t \text{div} \left( \frac{(\nabla u_e)_t \cdot \nabla u_e \nabla u_e}{(1 + |\nabla u_e|^2)^{3/2}} \right) = (u_t)_t \partial_i \left( \frac{(u_t)_t}{\sqrt{1 + |\nabla u_e|^2}} \right)
\]

\[
= \frac{(u_t)_t (u_t)_{tii}}{\sqrt{1 + |\nabla u_e|^2}} - \frac{1}{(1 + |\nabla u_e|^2)^{3/2}} (u_e)_i (u_t)_{tii} \nabla u_e \cdot (\nabla u_e)_i
\]

\[
= \frac{(u_t)_t \Delta (u_t)_t}{\sqrt{1 + |\nabla u_e|^2}} - \frac{1}{(1 + |\nabla u_e|^2)^{3/2}} (u_t)_t \partial_i \left( \frac{|\nabla u_e|^2}{2} \right)
\]

\[
= \frac{(u_t)_t \Delta (u_t)_t}{\sqrt{1 + |\nabla u_e|^2}} - \frac{1}{(1 + |\nabla u_e|^2)^{3/2}} \nabla \left( \frac{(u_t)^2}{2} \right) \cdot \nabla \left( \frac{|\nabla u_e|^2}{2} \right),
\]

Notice that

\[
\Delta^S \frac{(u_t)^2}{2} = \Delta \frac{(u_t)^2}{2} - \frac{\nabla u_e \cdot \nabla \left( \frac{(u_t)^2}{2} \right) \nabla u_e}{1 + |\nabla u_e|^2}
\]

\[
= (u_t)_t \Delta (u_t)_t + \frac{|(\nabla u_e)_t|^2}{2} - \frac{(u_e)_i (u_e)_j (u_e)_t (u_e)_t i j + (u_e)_i (u_e)_j (u_e)_t i j (u_e)_t j}{1 + |\nabla u_e|^2}.
\]

We then get

\[
\frac{d}{dt} \frac{(u_t)^2}{2} = \left( \frac{\nabla (\frac{(u_t)^2}{2}) \cdot \nabla u_e}{\sqrt{1 + |\nabla u_e|^2}} \right) k_e + \Delta^S \left( \frac{(u_t)^2}{2} \right) - 2 \left( \frac{\nabla \left( \frac{(u_t)^2}{2} \right) \cdot \nabla (\frac{|\nabla u_e|^2}{2})}{1 + |\nabla u_e|^2} \right)
\]

\[
+ 2 \left( \frac{\nabla (\frac{(u_t)^2}{2}) \cdot \nabla u_e \nabla (\frac{|\nabla u_e|^2}{2}) \cdot \nabla u_e}{(1 + |\nabla u_e|^2)^2} \right) + \left( \frac{(\nabla u_e \cdot (\nabla u_e))_t}{1 + |\nabla u_e|^2} \right) \partial_t k_e.
\]
Note that the last term is nonpositive by definition of $k_\varepsilon$.

In order to apply Lemma 1, we have to show the inequality
\[
- \frac{(\nabla u_\varepsilon, (\nabla u_\varepsilon)_t)^2}{1 + |\nabla u_\varepsilon|^2} + |(\nabla u_\varepsilon)_t|^2 \geq 0.
\]

It is enough to note that, since the solution exists for all times and it is smooth, the term $\nabla \left( \frac{|\nabla u_\varepsilon|^2}{2} \right)$ is bounded on each $[0, T]$ (the bound depends on $T$ and $\varepsilon$ but is enough to apply the lemma). In addition, every factor containing $\nabla (\nabla u_\varepsilon)_t$ also contains $\nabla u_\varepsilon$, hence the assumptions of Lemma 1 are satisfied for every $T > 0$, and this concludes the proof.

From Propositions 9 and 10, we deduce the following result.

**Proposition 11.** If $u_0$ is $C$-Lipschitz in space for some $C > 0$, and has bounded mean curvature, then the solution $u_\varepsilon$ of the approximate problem (35) is $C$-Lipschitz in space and Lipschitz in time with constant
\[
\| \sqrt{1 + |\nabla u_0|^2} \operatorname{div} \left( \frac{\nabla u_0}{\sqrt{1 + |\nabla u_0|^2}} \right) \|_{L^\infty(\mathbb{R}^d)}.
\]

Moreover, the following inequalities hold
\[
\psi^-_\varepsilon(x) - \varepsilon \leq u_\varepsilon(x, t) \leq \psi^+_\varepsilon(x) + \varepsilon.
\] (41)

**Proof.** The Lipschitz bounds of the solution are clear (it is Proposition 9 and 10).

In order to prove the second assertion, let us notice that by (34) and the definition of $k_\varepsilon$, we have
\[
k_\varepsilon(x, \psi^-_\varepsilon - \varepsilon) = 2N \geq \| \sqrt{1 + |\psi^-_\varepsilon|^2} \operatorname{div} \left( \frac{\psi^-_\varepsilon}{\sqrt{1 + |\psi^-_\varepsilon|^2}} \right) \|_{L^\infty(\mathbb{R}^d)},
\]
so that $\psi^-_\varepsilon - \varepsilon$ is a subsolution of (35). By the parabolic comparison principle (as in Proposition 3), we deduce that
\[
\psi^-_\varepsilon - \varepsilon \leq u_\varepsilon.
\]
The same arguments shows the other inequality in (41).

**Conclusion of the proof of Theorem 2.** Since the solutions $u_\varepsilon$ are equi-Lipschitz in space and time, they converge uniformly, as $\varepsilon \to 0$, to a limit function $u$ which is also Lipschitz continuous on $\mathbb{R}^d \times [0, +\infty)$.

Equation (41) yields
\[
\psi^- \leq u \leq \psi^+,
\]
and Proposition 8 gives that $u$ is a viscosity solution of (43).

Concerning the regularity of $u$, we proved that $(u_\varepsilon)_t$ and $\nabla u_\varepsilon$ are bounded on $[0, T]$, for any $T$ in the approximate problem. This gives a bound on the mean curvature of the approximate solution. This bound does not depend on $\varepsilon$ and remains true for the viscosity solution. As a result, the exact solution has bounded mean curvature and bounded gradient, which shows that $\Delta u$ is $L^\infty$ and, by elliptic regularity theory, $u$ is also in $W^{2,p}$ for any $p > 1$, and so $C^{1,\alpha}$ for every $\alpha < 1$ (see [Lun95] for details).

By Theorem 4 below, we can also directly apply to the solution $u$ a regularity result by Petrosyan and Shahgholian in [Sha08, PS07]. It follows that $u$ is in fact of class $C^{1,1}$, and this concludes the proof of Theorem 2.
5 Proof of Theorem 3

We compute the evolution of the area of the graph of $u$:

$$\frac{d}{dt} \int_Q \sqrt{1 + |\nabla u|^2} = \int_Q \frac{(\nabla u_t, \nabla u)}{\sqrt{1 + |\nabla u|^2}} = - \int_Q u_t \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right). \quad (42)$$

Notice that, for almost every $t > 0$, $u_t(t, x) = 0$ almost everywhere on the contact set. Indeed, for almost every $t$, $u_t$ exists for almost every $x \in Q$. If $u(x, t) = \psi^\pm(x)$, then $u - \psi^\pm$ reaches an extremum in $(x, t)$, which gives, $u_t(x, t) = 0$. In particular, from (42) we get

$$\frac{d}{dt} \int_Q \sqrt{1 + |\nabla u|^2} = - \int_Q u_t \left( \frac{u_t}{\sqrt{1 + |\nabla u|^2}} \right).$$

Integrating this equality in time, we obtain

$$\int_Q \sqrt{1 + |\nabla u|^2} \bigg|_0^T = \int_0^T \int_Q - \frac{u_t^2}{\sqrt{1 + |\nabla u|^2}},$$

which shows that

$$\int_0^T \int_Q u_t^2$$

is uniformly bounded in $T$. As a result $u_t \in L^2(\mathbb{R}^+, Q)$ so $u$ is in $H^1(Q, B_R)$.

Since $\|u_t\|_{L^2(Q)}$ is $L^2(\mathbb{R}^+)$, there exists a sequence $t_n \to \infty$ such that

$$\|u_t\|_{L^2(Q)(t_n)} \to 0.$$

In addition, $u(t_n)$ is equi Lipschitz and converges uniformly on compact sets to some $u_\infty$ which therefore satisfies in the viscosity sense

$$\sqrt{1 + |\nabla u|^2} \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0$$

with obstacles $\psi^\pm$ (see Appendix 5).

Remark. By [ISZ98], $u_{\text{min}}$ is analytic out of the (closed) contact set $\{u_{\text{min}} = \psi^\pm\}$.

A Viscosity solutions with obstacles

A.1 Definition of viscosity solution

Given an open subset $B$ of $\mathbb{R}^d$, let $u_0$, $\psi^+$ and $\psi^-$ be three Lipschitz functions $B \to \mathbb{R}$ such that

$$\psi^-(x, 0) \leq u_0(x) \leq \psi^+(x, 0).$$

We are interested in the viscosity solutions of the equation

$$u_t = \sqrt{1 + |\nabla u|^2} \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right), \quad u(x, 0) = u_0(x), \quad (43)$$

with the constraint

$$\psi^-(x) \leq u(x, t) \leq \psi^+(x). \quad (44)$$
**Definition 1** (see [CIL92, Mer14]). We say that a function \( u : B \times [0, T) \rightarrow \mathbb{R} \) is a viscosity subsolution of (43) if \( u \) satisfies the following conditions:

- \( u \) is upper semicontinuous;
- \( u(x,0) \leq u_0(x) \);
- \( \psi^- (x) \leq u(x,t) \leq \psi^+(x) \);
- for any \((x_0,t_0) \in \mathbb{R}^d \times \mathbb{R}^+ \) and \( \varphi \in C^2 \) such that \( u - \varphi \) has a maximum at \((x_0,t_0)\) and \( u(x_0,t_0) > \psi^-(x_0) \),
  \[ u_t \leq \sqrt{1 + |\nabla u|^2} \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right). \]  

(45)

Similarly, \( u \) is a viscosity supersolution of (43) if:

- \( u \) is lower semicontinuous;
- \( u(x,0) \geq u_0(x) \);
- (45) holds;
- for any \((x_0,t_0) \in \mathbb{R}^d \times \mathbb{R}^+ \) and \( \varphi \in C^2 \) such that \( u - \varphi \) has a minimum at \((x_0,t_0)\) and \( u(x_0,t_0) < \psi^+(x_0) \),
  \[ u_t \geq \sqrt{1 + |\nabla u|^2} \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right). \]  

(46)

We say that \( u \) is a viscosity solution of (43) if it is both a super and a subsolution.

### A.2 Comparison principle

In order to prove uniqueness of continuous viscosity solutions of (43), we shall prove a comparison principle between solutions following [GGIS91, Theorem 4] (see also [CGG91]).

**Proposition 12.** If \( u \) is a viscosity subsolution of (43) on \([0,T)\), \( v \) is a viscosity supersolution, if \( \psi^\pm \) are Lipschitz in space and if \( u(x,0) \leq v(x,0) \), then \( u(x,t) \leq v(x,t) \) for all \((x,t) \in \mathbb{R}^n \times [0,T)\).

**Proof.** We will check that the proof of [GGIS91, Theorem 2.1] can be extended to the obstacle case. Notice first that the assumptions (A.1) – (A.3) of [GGIS91, Theorem 2.1] are satisfied also in our case. Indeed, (A.1) comes directly from the Lipschitz bound on \( \psi^\pm \) and the constraint \( \psi^- \leq u, v \leq \psi^+ \) whereas (A.2) and (A.3) result from the assumed time zero comparison.

Let us show that [GGIS91, Proposition 2.3] also holds. Indeed, up to Equation (2.9) nothing changes. To continue the proof, using the same notation of [GGIS91, Proposition 2.3], we have to check that if

\[ \sup_{v} (w - \Psi) > 0, \]

then the supremum is reached in the complementary of the contact set \( \{ u = \psi^- \} \cup \{ v = \psi^+ \} \).
Indeed, notice that if \( u(x,t) = \psi^-(x) \), then, for all \( x, y, t, s \),
\[
u(x,t) - v(y,s) = \psi^-(x) - v(y,s) \leq \psi^-(y) + L(|x-y|) - v(y,s) \leq L(|x-y|)
\]
since \( v \geq \psi^- \). Hence, if \( u(x,t) = \psi^-(x) \), with \( K' > L \), we must have \( w - \Psi \leq 0 \), so the supremum of \( w - \Psi \) is attained in the complementary of \( \{ u = \psi^- \} \). One can show similarly that the supremum is reached in the complementary of \( \{ v = \psi^+ \} \). Hence Proposition 2.3 of [GGIS91] holds.

From Proposition 2.4 to Lemma 2.7 of [GGIS91], every result holds without changes. Concerning the proof of Theorem 2.1 of [GGIS91], the first assumption is
\[
\alpha = \limsup_{\theta \to 0} \{ w(t,x,y), |x-y| \leq \theta \} > 0.
\]
Then, Proposition 2.4 gives constants \( \delta_0 \) and \( \gamma_0 \) such that for all \( \delta \leq \delta_0, \gamma \leq \gamma_0 \) and \( \varepsilon > 0 \), there holds
\[
\Phi(\hat{x},\hat{y},\hat{t}) := \sup_{\mathbb{R}^d \times [0,T]} \Phi(x,y,t) > \frac{\alpha}{2}
\]
with
\[
\Phi(t,x,y) = u(x,t) - v(y,t) - \frac{|x-y|^4}{4\varepsilon} - \delta(|x|^2 + |y|^2) - \frac{\gamma}{T-t}.
\]

To conclude the proof, we only have to show that the maximum of \( \Phi \) is once again attained on the complementary of \( \{ u = \psi^- \} \cup \{ v = \psi^+ \} \). In the same way as for Proposition 2.3, if \( u(x,t) = \psi^-(x) \), we can write
\[
\Phi(t,x,y) = u(x,t) - v(y,t) - \frac{|x-y|^4}{4\varepsilon} - \delta(|x|^2 + |y|^2) - \frac{\gamma}{T-t} \leq \psi^-(y) + L|x-y| - v(y,t) \leq L|x-y|.
\]

Thanks to Proposition 2.5, \( |\hat{x} - \hat{y}| \to 0 \). So, with \( \varepsilon \) sufficiently small (one can reduce the quantity \( \varepsilon_0 \) given by Proposition 2.6), \( \Phi \) has its maximum out of \( \{ u = \psi^- \} \) (and similarly out of \( \{ v = \psi^+ \} \)), which enables the application of Lemma 2.7 and gives a contradiction as in [GGIS91].

\[\square\]

### A.3 Existence

In this subsection, we prove the following result:

**Proposition 13.** There exists a continuous viscosity solution to (43).

We follow [CIL92] to build a solution by means of the Perron’s method. Let us state an obvious but useful proposition and a key lemma for applying Perron’s method.

**Proposition 14.** Let \( u \) be a subsolution of the mean curvature motion for graphs (without obstacles) which satisfies \( u \leq u^+ \). Then, \( u_{ob} := u \vee u^- \) is a subsolution of (43) with obstacles (the same happens for \( v \) supersolution and \( v_{ob} = v \wedge v^+ \)).

In the sequel, we shall denote by \( u^*(\text{resp.} u_*) \) the upper (resp. lower) semicontinuous envelope of a function \( u \).

**Lemma 6.** Let \( \mathcal{F} \) be a family of subsolutions of (43). We define
\[
U(x,t) = \sup \{ u(x,t) \mid u \in \mathcal{F} \}.
\]

Then, \( U^* \) is a subsolution of (43).

The proof of the proposition and the lemma can be found in [CIL92], Lemma 4.2 (with obvious changes due to the parabolic situation and obstacles).
Construction of barriers  In the sequel, to claim that the initial condition is taken by the viscosity solution, we need to build barriers to sandwich the solution. More precisely, we want to build a subsolution \( w^- \) such that \((w^-)^*(x,0) = u_0(x)\) and a supersolution \( w^+ \) such that \((w^+)^*(x,0) = u_0(x)\). To show this claim, let us begin by a simple fact.

Let

\[
g_{a,b}^\alpha(x) = -\sum \alpha_i \frac{(x-a)_i^2}{\sqrt{1 + (x-a)_i^2}} + b \tag{47}
\]

for some \((a,b) \in \mathbb{R}^d \times \mathbb{R}\) and \(\alpha_i \geq 0\) such that \(g(x) \leq u_0(x)\). Note in particular that

\[
g_{a,b}^\alpha(x) \geq -\sum \alpha_i (x-a)_i^2 + b \text{ and } H(g_{a,b}^\alpha) \geq H(g_{a,b}^\alpha)|_{t=0} = -2\sum \alpha_i. \tag{48}
\]

Then, it is easy to show (using Proposition 14) that the function

\[
v(x,t) = \left( g_{a,b}^\alpha(x) + \left( 2 \sum_{i=1}^n \alpha_i + 3M \right) t \right) \vee \psi^-
\]

is a subsolution of (43). Indeed, the curvature of \(g_{a,b}^\alpha\) is smaller than \(2\sum \alpha_i\) and its gradient is bounded by 2 (so \(\sqrt{1 + |\nabla g|^2} \leq 3\)).

Thanks to Lemma 6, the function

\[
w^-(x,t) = \left( \sup_{\{\alpha_i,c\}} \left( g_{a,b}^\alpha(x) - 2\sum_{i=1}^n \alpha_i t - 3Mt \right) \vee \psi^- \right)^*
\]

is a subsolution of (43) (with obstacles).

It remains to show that \((w^-)^*(x,0) = u_0(x)\). To see this, notice that since \(u_0\) is Lipschitz and \(u_0 \geq \psi^-\), \(u_0(x) = w^-(x,0)\), yielding \(u_0(x) \leq (w^-)^*(x,0)\). But for all \(t \geq 0\), \(v(x,t) \leq u_0(x)\) so \(w^-(x,t) \leq u_0(x)\). By continuity of \(u_0\), \((w^-)^*(x,t) \leq u_0(x)\), which shows that \((w^-)^*(x,0) = u_0(x)\), and \(w^-\) is a low barrier for solutions of (43).

We build \(w^+\) in the same way.

Perron’s method  We use the classical Perron’s method to build a solution of (43) on \([0,T]\) for every \(t > 0\). Let us define

\[
W(x,t) = \sup \{u(x), \text{ | } u \text{ is a subsolution of (43) on } [0,T] \}.
\]

Since \(\psi^-\) is a subsolution, this set is non empty and \(W\) is well defined. Every subsolution is less that \(\psi^+\), so is \(W\).

Thanks to Lemma 6, \(W^*\) is a subsolution of (43) regardless the initial conditions. Applying the comparison principle (Proposition 12) to every subsolution \(u\) and \(w^+\) gives

\[
\forall x,t, \ W(x,t) \leq w^+(x,t).
\]

Considering the upper-semi-continuous envelopes, we get

\[
\forall x,t, \ W^*(x,t) \leq (w^+)^*(x,t)
\]

which immediately yields to

\[
W^*(x,0) = u_0(x).
\]

Then, \(W^*\) is a subsolution (with initial conditions), hence \(W^* = W\) which shows the upper semi-continuity of \(W\).

We want to prove that \(W\) is actually a solution of (43). In this order, let us prove the following
Lemma 7. Let $u$ be a subsolution of (43). If $u_\ast$ fails to be a supersolution (regardless initial conditions) at some point $(\hat{x}, \hat{t})$ then there exists a subsolution $u_\ast$ (regardless initial conditions) satisfying $u_\ast \geq u$ and sup $u_\ast - u > 0$ and such that $u(x, t) = u_\ast(x, t)$ for $|x - \hat{x}|, |t - \hat{t}| \leq \kappa$.

Proof. Let us assume that $u_\ast$ fails to be a supersolution at $(0, 1)$. Then there exists $(a, p, X) \in \mathcal{F}^{2,-} - u_\ast(0, 1)$ with

$$a + F(p, X) + k(0)\sqrt{1 + p^2} < 0.$$ 

Let us then define

$$u_{\delta, \gamma}(x, t) = u_\ast(0, 1) + \delta + (p, x) + a(t - 1) + \frac{1}{2} \left( Xx, x \right) - \gamma(|x|^2 + t - 1).$$

Thanks to the continuity of $F$ and $k$, $u_{\delta, \gamma}$ is a classical subsolution on $B_r(0, 1)$ of $u_t + F(Du, D^2u) + k(x)\sqrt{1 + |\nabla u|^2} = 0$ for $\delta, \gamma, r$ sufficiently small. By assumption,

$$u(x, t) \geq u_\ast(x, t) \geq u_\ast(0, 1) + a(t - 1) + (p, x) + \frac{1}{2} \left( Xx, x, + \right) a(|x|^2 + |t - 1|).$$

With $\delta = \gamma \frac{r^2 + x}{8},$ we get $u(x, t) > u_{\delta, \gamma}(x, t)$ for small $r$ and $|x|, |t - 1| \in [\frac{r}{2}, r]$. Reducing again $r$, we can assume that $u_{\delta, \gamma} < \psi^+$ on $B_r$. Thanks to Lemma 6,

$$\tilde{u}(x, t) = \begin{cases} \max(u(x, t), u_{\delta, \gamma}(x, t)) \text{ if } |x, t - 1| < r \\ u(x) \text{ otherwise} \end{cases}$$

is a subsolution of (43) (with no initial conditions).

Finally, this lemma combined with the definition of $W$ proves that $W$ is in fact a solution of (43) (the initial conditions were already checked).

A.4 Regularity

Proposition 15. The unique solution $u$ of (43) is Lipschitz in space, with the same constant as $u_0, \psi^\pm$.

Proof. We will prove that $u_z(x, t) = u(x + z, t) - L|z|$ is in fact a subsolution of (43). The Lipschitz bound is then straightforward (using the comparison principle).

To begin, we notice that $u(x + z, t) - L(|z|) \leq u^+(x, t)$ and $u(x + z, 0) - L|z| \leq u_0(x + z) - L|z| \leq u_0(x)$.

Assume now that $\varphi$ is any smooth function which is greater than $u_z$ with equality at $(\hat{x}, \hat{t})$. Then, either, $u_z(\hat{x}, \hat{t}) = \psi^-(\hat{x}, \hat{t})$ and nothing has to be done, or $u_z(\hat{x}, \hat{t}) > \psi^-(\hat{x}, \hat{t})$. In the second alternative, one can write

$$u(\hat{x} + t, \hat{t}) > \psi^-(\hat{x}) = \psi^-(\hat{x} + z) + (\psi^-(\hat{x}) - \psi^-(\hat{x} + z)),$$

so

$$u(\hat{x} + z, \hat{t}) > \psi^-(\hat{x} + z) + \psi^-(\hat{x}) - \psi^-(\hat{x} + z) + L|z| \geq u^-(\hat{x} + z, \hat{t}).$$

As $u$ is a subsolution at $(\hat{x} + z, \hat{t})$ and $u(x + z, t) \leq \varphi(x, t) + L|z|$ with equality at $(\hat{x} + z, \hat{t})$, one can write with $y = x + z, s = t$,

$$u(y, t) \leq \varphi(y - z, s) + L|z| := \phi(y, s),$$

24
with equality at \((\hat{y}, \hat{s})\) which gives
\[
\phi_t + F(D\phi(\hat{x}, \hat{t}), D^2\phi(\hat{x}, \hat{t})) \leq 0.
\]
Since the derivatives of \(\phi\) and \(\varphi\) are the same, we deduce
\[
\varphi_t + F(D\varphi, D^2\varphi) \leq 0,
\]
what was expected. \(\square\)

**Remark.** With the same arguments, one can prove that
\[
\forall \delta > 0, \quad \forall x, t, \quad |u(x, t + \delta) - u(x, t)| \leq \sup_x |u(x, \delta) - u(x, 0)|.
\]

We now present a general regularity result by Shahgholian [Sha08] which applies to viscosity solutions for parabolic equations with obstacles.

**Theorem 4** ([IPS07], Th. 4.1). Let \(Q^+ := \{ (x, t) \in \mathbb{R}^d \times \mathbb{R} : |x| < 1, t \in [0, 1) \}\) and \(H(u) = F(D^2 u, Du) - u_t\) where \(F\) is uniformly elliptic. Let \(u\) be a continuous viscosity solution of
\[
(u - \psi)H(u) = 0, \\
H(u) \leq 0, \\
u \geq \psi,
\]
in \(Q^+\), with boundary data
\[
u(x, t) = g(x, t) \geq \psi(x, t) \quad \text{on} \quad \{|x| = 1\} \cup \{ t = 0 \}.
\]
Assume that \(\psi \in C^{1,1}(Q^+)\) and \(g\) is continuous. Then, \(u \in C^{1,1}\) on every compact subset of \(Q^+\).

It has to be noticed \(H = F - \partial_t\) where \(F(D^2 u, Du) = -\sqrt{1 + |\nabla u|^2} \text{ div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right)\) satisfies all the assumptions of [Sha08], 1.3. Indeed, the uniform ellipticity is provided by the Lipschitz bound obtained in previous subsection.

Moreover, the viscosity solution \(u\) of (43) satisfies (49) and (50) on every cylinder \(Q_t^+(x_0) := \{|x - x_0| \leq r, \ t \in [t_0, t_0 + r]\}\) such that \(r\) is chosen sufficiently small in order to have either \(Q_t^+(x_0) \cap \{u = \psi^+\} = \emptyset\) or \(Q_t^+(x_0) \cap \{u = \psi^-\} = \emptyset\). In the second alternative, change every sign in the equations.

Applying Theorem 4 we get a \(C^{1,1}\) bound for \(u\) on every compact subset of \(Q_t^+(x_0)\). To show that \(u\) is \(C^{1,1}\) in the whole space, just cover \(\mathbb{R}^d \times \mathbb{R}^+\) with such \(Q_t^+(x_i)\).

**References**


26


