HAL open science

Resurgent methods and the first Painlevé equation

Eric Delabaere

▶ To cite this version:

Eric Delabaere. Resurgent methods and the first Painlevé equation. 2014. hal-01067086v1

HAL Id: hal-01067086
https://hal.science/hal-01067086v1

Preprint submitted on 22 Sep 2014 (v1), last revised 9 Nov 2015 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Eric DELABAERE

Complex differential and difference equations

Resurgent methods and the first Painlevé equation

September 22, 2014

Preface

These lecture notes are an extended form of a course given at a CIMPA master class held in LIMA, Perù, in the summer of 2008. The students that followed these lectures were already introduced to Gevrey and k-summability by Michèle Loday-Richaud, and to resurgence theory by David Sauzin, at an elementary level. My aim was merely to show the resurgent methods acting on an example and along that line, to extend the presentation of the resurgence theory of Jean Ecalle provided that the need.

The present lecture notes reflect this plan and this pedagogical point of view. The example that we follow along this course is the First Painlevé differential equation, or Painlevé I for short. Besides its simplicity, there are various reasons that justify this choice. One of them is the non-linearity, which is the field where the resurgence theory reveals its power. Another reason lies on the fact that resonances occur, a case which is scarcely found in the literature. Last but not least, the Painlevé equations and their transcendents appear today to be an inescapable knowledge in analysis for any young mathematician. It was thus certainly worthy to detail the complete resurgent structure for Painlevé I, a study that does not seem to have been performed before on any Painlevé equation.

I have tried to be as self-contained as possible, aiming at graduate students. Since this volume deals with ordinary non-linear differential equations, one begins with definitions and phenomena linked to the non-linearity. Special attention is then brought to Painlevé I and to its so-called tritruncated and truncated solutions that are constructed by proving the summability of the transseries solutions. One details the formal integral for Painlevé I and, equivalently, the formal transform that brings Painlevé I to its normal form. One analyzes the resurgent structure for Painlevé I through additional material in resurgence theory. As a rule, each chapter ends with some comments on possible extensions for which one provides references to the existing literature.

Acknowledgments: I warmly thank my student Julie Belpaume to whom I borrowed some materials used in this volume.

Angers, *Eric Delabaere*
September 2014

Contents

Chapter 1
Some elements about ordinary differential equations

Abstract This chapter is merely devoted to recalling usual notations and elementary results on ordinary differential equations (ODEs) in the complex domain. We give the fundamental existence theorem for Cauchy problems (Sect. 1.1). We detail the main differences between solutions of linear versus nonlinear ODEs, when the question of their analytic continuation is considered (Sect. 1.2). Finally we provide a short introduction to Painlevé equations (Sect. 1.3).

1.1 Ordinary differential equations in the complex domain

An ordinary differential equation (ODE) is a functional relation of the type

$$\mathcal{F}\big(x, \boldsymbol{u}(x), \boldsymbol{u}'(x), \cdots, \boldsymbol{u}^{(N)}(x)\big) = 0, \qquad \boldsymbol{u}^{(k)}(x) = \frac{\mathrm{d}^k \boldsymbol{u}}{\mathrm{d}x^k}(x) \in \mathbb{C}^m. \tag{1.1}$$

We refer to m as the **dimension** of the ODE. The **order** N of the ODE refers to the highest derivative considered in the equation.

This ODE of order N is said to be **solved in his highest derivative** if it is written as

$$\boldsymbol{u}^{(N)} = \boldsymbol{F}(x, \boldsymbol{u}, \cdots, \boldsymbol{u}^{(N-1)}). \tag{1.2}$$

1.1.1 The fundamental existence theorem

We recall the fundamental existence theorem for the Cauchy problem, for analytic ODEs (see, e.g. [20, 18, 24, 19]). We note $D(z, r) \subset \mathbb{C}$ the open disc centred on z and of radius r. For a given open domain $U \subset \mathbb{C}^m$ (i.e., a connected open set) we denote by $\mathcal{O}(U)$ the complex linear space of functions holomorphic in U.

A function f belongs to $\mathcal{O}(U)$ if f is continuous on $U \subset \mathbb{C}^m$ and holomorphic in each complex variable (Osgood theorem). As a matter of fact, it is enough to assume only the holomorphy in each complex variable without the continuity hypothesis (Hartogs theorem).

Theorem 1.1 (Cauchy problem). *Let $U \subset \mathbb{C}\times\mathbb{C}^m$ be an open domain and $\boldsymbol{F} : U \to \mathbb{C}^m$ a holomorphic vector function, $\boldsymbol{F} \in \mathcal{O}^m(U)$. Then, for every $(x_0, \boldsymbol{u}_0) \in U$ there exists a polydisc $D(x_0, \epsilon_0)\prod_{1\le i\le m} D(\boldsymbol{u}_{0i}, \epsilon_i) \subset U$ such that there exists a solution $\boldsymbol{u} : D(x_0, \epsilon_0) \to \prod_{1\le i\le m} \bar{D}(\boldsymbol{u}_{0i}, \epsilon_i)$ of the analytic ODE of order 1 and of dimension m*

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}x} = \boldsymbol{F}(x, \boldsymbol{u}) \tag{1.3}$$

which satisfies the initial value condition

$$\boldsymbol{u}(x_0) = \boldsymbol{u}_0 \tag{1.4}$$

and this solution is unique. Moreover $\boldsymbol{u}$ belongs to $\mathcal{O}(D(x_0, \epsilon_0))$ and also depends holomorphically on the initial value $\boldsymbol{u}_0$.

In what follows we shall consider essentially **scalar** ODEs, that it ODEs of dimension 1 and of order N. The theorem 1.1 translates to this case as well, since every ODE of order N and of dimension 1, once solved in his highest derivative, is equivalent to an ODE of order 1 and of dimension N : if $u = v_0$, $u' = v_1$, $\cdots$, $u^{(N-1)} = v_{N-1}$, the Cauchy problem

$$\begin{cases} u^{(N)} = F(x, u, \cdots, u^{(N-1)}) \\ \left(u(x_0), \cdots, u^{(N-1)}(x_0)\right) = \left(u_0, \cdots, u_0^{(N-1)}\right) \end{cases}$$

is equivalent to the Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} v_0 \\ \vdots \\ v_{N-2} \\ v_{N-1} \end{pmatrix} = \begin{pmatrix} v_1 \\ \vdots \\ v_{N-1} \\ F(x, v_0, \cdots, v_{N-1}) \end{pmatrix} \\ \begin{pmatrix} v_0 \\ \vdots \\ v_{N-1} \end{pmatrix}(x_0) = \begin{pmatrix} u_0 \\ \vdots \\ u_0^{(N-1)} \end{pmatrix}. \end{cases}$$

1.1.2 Some usual terminologies

The following terminologies are commonly used (see, e.g. [6]):

- The **general solution** of an ODE of order N and of dimension 1 is the set of all solutions determined in application of the Cauchy theorem 1.1. It depends on N arbitrary complex constants.
- A **particular** or **special** solution is a solution derived from the general solution when fixing a particular initial data.
- A **singular** solution is a solution which is not particular.

1.1.3 Algebraic differential equations

In a moment we shall concentrate on algebraic differential equations, these we define now.

For an open domain $D \subset \mathbb{C}$ we denote by $\mathcal{M}(D)$ the field of meromorphic functions in D.

The ODE (1.1) of order N and of dimension 1 is said to be **algebraic on a domain** D if $\mathcal{F} \in \mathcal{M}(D)[u, u', \cdots, u^{(N)}]$ that is, $\mathcal{F}$ is polynomial in $(u, u', \cdots, u^{(N)})$ with meromorphic coefficients in x.

An algebraic ODE is **rational** if it is of degree one in the highest derivative $u^{(N)}$, and **linear** (homogeneous) if $\mathcal{F}$ is a linear form in $(u, u', \cdots, u^{(N)})$.

1.2 On singularities of solutions of ordinary differential equations

We fix some notations that will be used in a moment.

Definition 1.1. Let $\lambda : [a,b] \subset \mathbb{R} \to \mathbb{C}$ be a path starting at $x_1 = \lambda(a)$ and ending at $x_2 = \lambda(b)$. If u is a (germ of) holomorphic function(s) at x_1 that can be analytically continued along λ, we note $\mathrm{cont}_\lambda u$ the resulting (germ of) holomorphic function(s) at x_2.

Denote by $\mathcal{O} = \bigsqcup_{x\in\mathbb{C}} \mathcal{O}_x$ the set of all germs of holomorphic functions. We equip $\mathcal{O}$ with its usual topology and with the projection $\mathfrak{q} : \begin{array}{c}\mathcal{O} \to \mathbb{C} \\ u \in \mathcal{O}_x \mapsto x \in \mathbb{C}\end{array}$ which associates to a germ its support [12, 9]. The space $\mathcal{O}$ becomes an étalé space, that is $\mathfrak{q}$ is a local homeomorphism. The analytic continuation of the germ $u \in \mathcal{O}_{x_1}$ along λ, if exists, is the image of the unique path $\Lambda : [a,b] \to \mathcal{O}$ such that $\Lambda(a) = u$ and whose projection by $\mathfrak{q}$ is λ : $\begin{tikzcd} & \mathcal{O} \arrow[dr, "\mathfrak{q}"] & \\ {[a,b]} \arrow[ur, "\Lambda"] \arrow[rr, "\lambda"'] & & \mathbb{C} \end{tikzcd}$. Wih this notation, $\mathrm{cont}_\lambda u = \Lambda(b)$.

We now consider an ODE of order N and of dimension 1,

$$\mathcal{F}\big(x, u(x), u'(x), \cdots, u^{(N)}(x)\big) = 0,$$

with $\mathcal{F} : U \to \mathbb{C}$ a holomorphic function on the open domain $U \subset \mathbb{C} \times \mathbb{C}^{N+1}$, $\mathcal{F} \in \mathcal{O}(U)$. Assume that $\Big(x_0, u_0, \cdots, u_0^{(N)}\Big) \in U$ and that

$$\begin{cases} \mathcal{F}\big(x_0, u_0, \cdots, u_0^{(N)}\big) = 0 \\ \partial_{N+2}\mathcal{F}\big(x_0, u_0, \cdots, u_0^{(N)}\big) \neq 0. \end{cases}$$

By the implicit function theorem, the Cauchy problem

$$\begin{cases} \mathcal{F}\big(x, u(x), u'(x), \cdots, u^{(N)}(x)\big) = 0 \\ \Big(u(x_0), \cdots, u^{(N)}(x_0)\Big) = \Big(u_0, \cdots, u_0^{(N)}\Big) \end{cases}$$

is locally equivalent to a Cauchy problem where the ODE is solved in its highest derivative. Theorem 1.1 thus provides a holomorphic solution u near $x = x_0$. We consider a path $\gamma : [a,b] \to \mathbb{C}$ from x_0 to x_1 in $\mathbb{C}$ and for $s \in [a,b]$

we denote $\gamma_s : [a, s] \to \mathbb{C}$ the restriction to $[a, s]$ of γ. Assume that u can be analytically continued along the path γ and that for every $s \in [a, b]$, the value at $\gamma(s)$ of the analytic continuation $\mathrm{cont}_{\gamma_s}\big(x, u, u', \cdots, u^{(N)}\big)$ along γ_s belongs to U. Then *the analytic continuation* $\mathrm{cont}_\gamma u$ *along* γ *of the solution* u *still satisfies the differential equation*, thanks to the uniqueness of the analytic continuation.

This property raises the question of describing the singularities of the analytic continuations of solutions of analytic ODEs, for instance for an algebraic differential equation defined on an open domain. As we shall see, appearance of singularities is quite different whether one considers linear or nonlinear ODEs.

1.2.1 Linear differential equations

For linear (homogeneous) ordinary differential equations, from the Cauchy existence theorem for linear differential equations (see, e.g. [35, 24, 19, 21]), the general solution has no other singularities than the so-called **fixed singularities** which arise from the coefficients of the ODE once solved for the highest derivative.

1.2.1.1 Example 1

We start with an equation where $x = 0$ is an irregular singular point of Poincaré rank 1,

$$x^2u' + u = 0, \qquad u(x) = C\mathrm{e}^{1/x}, \qquad C \in \mathbb{C}.$$

Here $x = 0$ is a fixed essential singularity for the general solution (but not for the particular solution $u(x) = 0$), which arises from the equation itself.

If $u \in \mathcal{O}(D(0, r)^\star)$ is a holomorphic function in the punctured disc, then u can be represented by its Laurent series expansion $\sum_{n\in\mathbb{Z}} a_n x^n$ which converges in $0 < |x| < r$. One says that 0 is an essential singularity if and only if the Laurent series expansion has an infinite number of $n < 0$ such that $a_n \neq 0$ or, equivalently, if u has no limit (finite or infinite) when $x \to 0$. A typical example is provided by the function $\mathrm{e}^{1/x}$.

1.2.1.2 Example 2

We consider the Airy equation,

$$u'' - xu = 0, \qquad u(x) = C_1 Ai(x) + C_2 Bi(x), \qquad C_1, C_2 \in \mathbb{C}.$$

Here Ai and Bi are the Airy's special functions of the first and second kind respectively. These are entire functions. When considered on the Riemann sphere $\overline{\mathbb{C}}$, $x = \infty$ appears as a fixed (essential) singularity for the general solution (except again for the particular solution $u(x) = 0$) which arises from the equation : $x = \infty$ is an irregular singular point of Poincaré rank 3/2.

More generally, for a linear ordinary differential equation

$$\sum_{k=0}^{N} a_k(x)u^{(k)} = 0, \qquad a_k(x) \in \mathcal{O}(D), \tag{1.5}$$

the general solution can be analytically continued as a multivalued function on $D \setminus S$, $S = \{\text{the zeros of } a_N\}$, or more precisely as a single valued holomorphic function once it is considered on a Riemann surface [12, 9] defined as a covering space, $\begin{array}{c}\mathcal{R}\\ \pi\downarrow\\ D\setminus S\end{array}$. In other words, the general solution is **uniformisable** (or also **stable**) [6] in the following sense : for any Cauchy data at $x_0 \in D \setminus S$ that determined a unique local solution u of (1.5) on a domain $U \subset D \setminus S$, one can find a domain $\mathscr{U} \subset \mathcal{R}$ such that $\pi|_{\mathscr{U}} : \mathscr{U} \to U$ is a homeomorphism, and a holomorphic function $\phi : \mathcal{R} \to \mathbb{C}$ so that $\phi|_{\mathscr{U}} = u \circ \pi|_{\mathscr{U}}$.

Then, for any domain $\mathscr{U}' \subset \mathcal{R}$ so that $\pi|_{\mathscr{U}'} : \mathscr{U}' \to U'$ is a homeomorphism, the function $\phi \circ (\pi|_{\mathscr{U}})^{-1}$ is still a holomorphic solution of (1.5) on U'.

1.2.2 Nonlinear differential equations

When nonlinear ODEs are concerned, beside the possibly fixed singularities arising from the equation, the general solution has as a rule other singularities which depend on the arbitrary coefficients : these are **movable singularities**.

1.2.2.1 Example 1

We start with the following nonlinear ODE,

$$xu' - u^2 = 0, \qquad \begin{array}{l} \text{general solution : } u(x) = \dfrac{1}{C - \log(x)}, \quad C \in \mathbb{C}. \\ \text{singular solution : } u(x) = 0 \end{array}$$

For the general solution, $x = 0$ is a fixed branch point singularity which comes from the equation. The general solution u is uniformisable : it should be considered as a function on the Riemann surface $\underset{\bullet}{\mathbb{C}}$ of the logarithm[1],

$$\underset{\bullet}{\mathbb{C}} = \{x = r\mathrm{e}^{\mathrm{i}\theta} \mid r > 0,\ \theta \in \mathbb{R}\}, \qquad \pi : x \in \underset{\bullet}{\mathbb{C}} \mapsto \overset{\bullet}{x} = r\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C}^\star.$$

One sees that the general solution u is meromorphic on $\underset{\bullet}{\mathbb{C}}$, with poles at $\pi^{-1}(\mathrm{e}^C)$: these are movable singularities, depending on the chosen coefficient C.

1.2.2.2 Example 2

The above example is just a special case of a more general rational ODE of order 1, the **Riccati equation**,

[1] We keep a notation of Ecalle, see definition 3.10. Of course $(\underset{\bullet}{\mathbb{C}}, \pi)$ can be thought of as the universal covering of the punctured space $\mathbb{C}^\star$.

$$u' = a_0(x) + a_1(x)u + a_2(x)u^2 \qquad a_i \in \mathcal{M}(D), \tag{1.6}$$

where $D \subset \mathbb{C}$ is a domain. By the change of unknown function

$$u = -\frac{1}{a_2(x)} \frac{\mathrm{d}}{\mathrm{d}x} \log v$$

the Riccati equation (1.6) is linearisable into the following linear ODE,

$$v'' + \left(\frac{a_2'(x)}{a_2(x)} - a_1(x) \right) v' + a_2(x)a_0(x)v = 0.$$

The general solution for this linear equation has (fixed) singularities located at the poles of $\dfrac{a_2'(x)}{a_2(x)} - a_1(x)$ and $a_2(x)a_0(x)$. We note $S \subset D$ this set of poles. Then the general solution of the Riccati equation (1.6) is uniformisable since it can be analytically continued as a meromorphic function on a Riemann surface defined as a covering over $D \setminus S$.
When the a_i belong to $\mathcal{O}(D)$, then the general solution of (1.6) is a meromorphic function on D [25].

1.2.2.3 Example 3

Another well known equation is the following algebraic nonlinear ODE of order 1, of degree 2 in its highest derivative, namely the **elliptic equation**:

$$u'^2 = 4u^3 - g_2 u - g_3, \qquad (g_2, g_3) \in \mathbb{C}. \tag{1.7}$$

A particular solution is provided by the Weierstrass p-function $\wp(x; g_2, g_3)$ which can be obtained as the inverse function of the elliptic integral of the first kind

$$x = \int_\infty^u \frac{dq}{\sqrt{4q^3 - g_2 q - g_3}}, \qquad \left(\frac{\mathrm{d}x}{\mathrm{d}u} \right)^2 = \frac{1}{4u^3 - g_2 u - g_3}.$$

(Just apply the inverse function theorem).
When the discriminant $\mathbb{D} = g_2^3 - 27g_3^2$ satisfies the condition $\mathbb{D} \neq 0$, the polynomial function $4u^3 - g_2 u - g_3 = 4(u - e_1)(u - e_2)(u - e_3)$ has 3 simple roots e_1, e_2, e_3. In that case the elliptic function $\wp(x; g_2, g_3)$ is a doubly periodic meromorphic function with double poles at the period lattice $m\omega_1 + n\omega_2$, $(n, m) \in \mathbb{Z}^2$, $\dfrac{\omega_1}{\omega_2} \notin \mathbb{R}$.
The period lattice can be described as follows : consider the elliptic curve $\mathcal{L} = \{(q, p) \in \mathbb{C}^2,\, p^2 = 4q^3 - g_2 q - g_3\}$ for $\mathbb{D} \neq 0$. The homology group $H_1(\mathcal{L}; \mathbb{Z})$ is a free $\mathbb{Z}$-module of rank 2 and we note γ_1 and γ_2 two cycles which generate $H_1(\mathcal{L}; \mathbb{Z})$. Then the period lattice is generated by the period integrals $\omega_1 = \displaystyle\int_{\gamma_1} \frac{dq}{p}$, $\omega_2 = \displaystyle\int_{\gamma_2} \frac{dq}{p}$ (equivalently $\omega_1 = 2\displaystyle\int_{e_1}^{e_2} \frac{dq}{\sqrt{4q^3 - g_2 q - g_3}}$, $\omega_2 = 2\displaystyle\int_{e_1}^{e_3} \frac{dq}{\sqrt{4q^3 - g_2 q - g_3}}$). The homology group $H_1(\mathcal{L}; \mathbb{Z})$ can be seen as a local system on $\mathbb{C}^2 \setminus \mathbb{D}$ (that is a locally constant sheaf of $\mathbb{Z}$-modules on $\mathbb{C}^2 \setminus \mathbb{D}$) from which one can deduce that $\omega_{1,2}$, viewed as functions of (g_2, g_3),

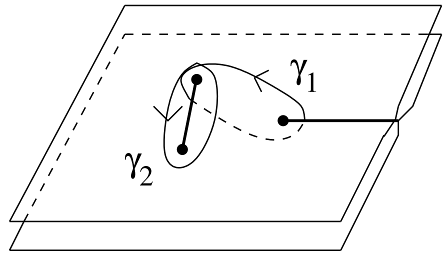


Fig. 1.1 The elliptic curve $\mathcal{L}$ viewed as the Riemann surface of $p = (4u^3 - g_2u - g_3)^{1/2}$. The homology classes of the cycles γ_1 and γ_2 drawn generate $H_1(\mathcal{L}; \mathbb{Z})$

can be analytically continued as "multivalued" analytic functions on $\mathbb{C}^2 \setminus \mathbb{D}$. When $\mathbb{D} = 0$ the solutions degenerate into simply periodic solutions, with a string of poles instead of a double array.

Conversely, starting from the period lattice with $\dfrac{\omega_1}{\omega_2} \notin \mathbb{R}$, the Weierstrass $\wp$-function can be obtained as

$$\wp(x; g_2, g_3) = x^{-2} + \sum_{\omega \neq 0} \{(x-\omega)^{-2} - \omega^{-2}\} = x^{-2} + g_2 \frac{x^2}{20} + g_3 \frac{x^4}{28} + \cdots$$

where the first summation extends over all $\omega = m\omega_1 + n\omega_2 \neq 0$, $(n, m) \in \mathbb{Z}^2$ while $g_2 = 60 \sum_{\omega \neq 0} \omega^{-4}$, $g_3 = 140 \sum_{\omega \neq 0} \omega^{-6}$.
The general solution of (1.7) is given by $\wp(x - x_0; g_2, g_3)$, since (1.7) is an autonomous ODE.
To go further on the nice properties of elliptic functions see, e.g. [33].

1.2.2.4 Example 4

Notice that singularities of differential equations may be isolated singularities such as poles, branch points of finite or infinite determinations, or essential singular points. They may be also essential singular lines, or even perfect sets of singular points. For instance, the general solution of the following Chazy equation of class III,

$$u^{(3)} - 2uu^{(2)} + 3u'^2 = 0, \tag{1.8}$$

is defined only inside or outside an open disc whose boundary is a natural movable boundary determined by the initial data [3, 4].

1.3 The Painlevé program, Painlevé property and Painlevé equations

At the end of the 19th century a list of special transcendental functions was known, most of them being obtained as solutions of linear algebraic differential equations.

> An algebraic function u in one complex variable x is a solution of a polynomial equation $P(x, u) = 0$, $P \in \mathbb{C}[x, u]$. A transcendental function u is a function which is not algebraic.

A challenging problem in analysis was thus to discover new transcendental functions defined by algebraic ODEs which cannot be expressed in term of solutions of linear algebraic ODEs : these new functions should thus be defined by non-linear algebraic differential equation [6, 8, 21].

For that purpose a systematic approach needs first to classify the ODEs under convenient criters. This is the goal of the so-called **Painlevé program** (see [6] and references therein) which consists in classifying all algebraic ODEs of first order, then second order, etc ..., whose general solution can be analytically continued as a *single valued* function[2]. In other words, no branch point is allowed. For instance the elliptic equation (1.7) or the Chazy equation (1.8) are such equations.

According to what we have seen, the Painlevé program splits into two problems:

- absence of fixed branch point for the general solution;
- absence of movable branch point for the general solution : this condition is the so-called **Painlevé property**.

In the literature, the term “Painlevé property” is sometimes used for the stronger property for the general solution of an ODE to be meromorphic, see [6]

Notice that the Painlevé property for an algebraic ODE $\mathcal{F}(x, u, u', \cdots, u^{(p)}) = 0$ defined on a domain $D \subset \mathbb{C}$ is preserved by:

- a holomorphic change of variable $x \in D \mapsto X = h(x)$, $h \in \mathcal{O}(D)$;
- a linear fractional change of the unknown with coefficient holomorphic in D (action of the homographic group),

$$u \mapsto U = \frac{a(x)u + b(x)}{c(x)u + d(x)},\ U \mapsto u = \frac{d(x)U - b(x)}{-c(x)U + a(x)},$$

$a, b, c, d \in \mathcal{O}(D)$, $ad - bc \neq 0$. Therefore, the classification in the Painlevé program is made up to these transformations.

Notice however that other actions preserving the Painlevé property can be considered, see [6, 7, 21].

1.3.1 ODEs of order one

We consider (nonlinear) ODEs of the form

$$\mathcal{F}\big(x, u, u'\big) = 0, \tag{1.9}$$

with $\mathcal{F} \in \mathcal{M}(D)[u, u']$. For that class of ODEs, the Painlevé program can be considered as being achieved and we mainly refer to [20, 18, 6, 21] for the classification.

In that case *no essential movable singular point can appear* ([20], Sect. 13.6). Therefore looking for ODEs of type (1.9) with the Painlevé property reduces in asking that the movable singular points are just poles.

When (1.9) is a rational ODE, then the class of ODE we are looking for is represented only by the Riccati equation (1.6). See [25], in particular the Malmquist-Yosida-Steinmetz type theorems.

[2] This condition can be weakened by asking the general solution to be only uniformisable.

The ODEs of type (1.9) of degree ≥ 2 in the highest derivative and satisfying the Painlevé property essentially reduce (up to the transformations mentioned above) to the elliptic equation (1.7). See [6, 20] for more precise statements.

1.3.2 ODEs of order two : the Painlevé equations

In contrast to what happens for algebraic ODEs of order one, essential movable singular points may exist when the order is ≥ 2, making the analysis more difficult. Nevertheless, the classification is known for at least algebraic equations of order two

$$\mathcal{F}\big(x, u, u', u''\big) = 0, \qquad \mathcal{F} \in \mathcal{M}(D)[u, u', u''] \tag{1.10}$$

which are rational, that is of degree one in u''. Such equations enjoying the Painlevé property reduce (up to transformation) to:

- equations which can be integrated by quadrature,
- or linear equations,
- or one of six ODEs known as the **Painlevé equations**, the first 3 being:

$$\begin{array}{ll} (P_I) & u'' = 6u^2 + x \\ (P_{II}) & u'' = 2u^3 + xu + \alpha \\ (P_{III}) & u'' = \dfrac{u'^2}{u} - \dfrac{u'}{x} + \dfrac{\alpha u^2 + \beta}{x} + \gamma u^3 + \dfrac{\delta}{u} \\ \cdots & \cdots \end{array} \tag{1.11}$$

For the complete list see, e.g. [20, 18, 6, 21]. In (1.11), $\alpha, \beta, \gamma, \delta$ are arbitrary complex constants. Each Painlevé equation can be derived from the "master equation" P_{VI} by some limit processes [21].

The Painlevé equations have beautiful properties, see e.g. [5, 21, 16]. One of them is the following one:

Theorem 1.2. *The general solution of the Painlevé equation P_J, $J = I, \cdots, VI$ admits no singular points except poles outside the set of fixed singularities.*

So the Painlevé equations have the Painlevé property, but moreover the general solution is free of movable essential singularities.
Notice that the Painlevé equation should be seen as defined on the Riemann sphere $\overline{\mathbb{C}}$. The set of fixed singular points S_J of P_J is a subset of $\{0, 1, \infty\}$. For instance S_I and S_{II} are just $\{\infty\}$, while $S_{III} = \{0, \infty\}$. Theorem 1.2 thus translates as follows : the general solution of P_J can be analytically continued as a meromorphic function on the universal covering of $\overline{\mathbb{C}} \setminus S_J$.

Theorem 1.2 can be proved in various ways. An efficient one uses the relationship between Painlevé equations and monodromy-preserving deformation of some Fuchsian differential equations [23, 22, 28, 21, 11].

The general (global) solutions of the Painlevé equations are called the **Painlevé transcendents**. This refers to the fact that, for generic values of the integration constants and of the parameters of the equations, these solutions cannot be written with elementary or classical transcendental functions, a question which has been completely solved only recently with the development of the modern nonlinear differential Galois theory (see [34] and references therein).

1.3.3 Painlevé equations and related topics

The renewed interest in Painlevé equations mainly came from theoretical physics in the seventies, with the study of PDEs of the soliton type (Boussinesq equation, Korteweg-de Vries KdV and modified Korteweg-de Vries equation mKdV, etc..): when linearized by inverse scattering transform [1], these PDEs give rise to ODEs with the Painlevé property. For instance, the Boussinesq equation $u_{tt} - u_{xx} - 6(u^2)_{xx} + u_{xxxx} = 0$ has a self-similar solution of the form $u(x,t) = w(x-t)$ where w is either an elliptic function or satisfies the first Painlevé equation. In the same lines, the (m)KdV hierarchy introduced by Lax in [27] (and already in substance in [26] after the work of Gardner *et al* [13] on the KdV equation), will later give rise to various **Painlevé hierarchies** which are thought of as higher-order Painlevé equations and much studied since. See for instance [29] and references therein, for an asymptotic study of the Jimbo-Miwa [22] and Flaschka-Newell [10] second Painlevé hierarchies [15].
Discrete (analogues of the) **Painlevé equations** are today the matter of an intensive research, after the pioneering work of Bessis *et al* [2] on the study of partition functions in quantum gravity, see for instance [14, 17] and references therein. Also non commutative extensions of integrable systems have recently attracted the attention of the specialists, with **non commutative** (analogues of the) **Painlevé equations** and their hierarchies as main examples, see e.g. [31]. Finally, we could hardly leave untold the important group-theoretic interpretation of Painlevé equations in the line of the work of Okamoto [30], see for instance [8] and references therein.

It is not our aim to say more about Painlevé equations in general except for the first Painlevé equation which is used in this course as field of experiments in asymptotic and resurgent analysis, and which is the matter for the next chapter.

References

1. M. Ablowitz, P. Clarkson, *Solitons, nonlinear evolution equations and inverse scattering.* London Mathematical Society Lecture Note Series, 149. Cambridge University Press, Cambridge, 1991.
2. D. Bessis, C. Itzykson, J. B. Zuber, *Quantum field theory techniques in graphical enumeration.* Adv. in Appl. Math. **1** (1980), no. 2, 109-157.
3. F. J. Bureau, *Sur des systèmes différentiels non linéaires du troisième ordre et les équations différentielles non linéaires associées.* Acad. Roy. Belg. Bull. Cl. Sci. (5) **73** (1987), no. 6-9, 335-353.
4. P. Clarkson, P. Olver, *Symmetry and the Chazy equation.* J. Differential Equations **124** (1996), no. 1, 225-246.
5. R. Conte (ed), *The Painlevé property. One century later.* CRM Series in Mathematical Physics. Springer-Verlag, New York, 1999.
6. R. Conte, *The Painlevé approach to nonlinear ordinary differential equations.* The Painlevé property, 77-180, CRM Ser. Math. Phys., Springer, New York, 1999.
7. R. Conte, M. Musette, *The Painlevé handbook.* Springer, Dordrecht, 2008.

8. E. Delabaere, M. Loday-Richaud (eds), *Théories asymptotiques et équations de Painlevé. [Asymptotic theories and Painlevé equations]* Séminaires et Congrés, **14**. Société Mathématique de France, Paris, 2006.
9. W. Ebeling, *Functions of several complex variables and their singularities.* Translated from the 2001 German original by Philip G. Spain. Graduate Studies in Mathematics, 83. American Mathematical Society, Providence, RI, 2007.
10. H. Flaschka, A. C. Newell, *Monodromy- and spectrum-preserving deformations. I.* Comm. Math. Phys. **76** (1980), no. 1, 65-116.
11. A.S. Fokas, A.R. Its, A.A. Kapaev, V.Y. Novokshenov, *Painlevé transcendents. The Riemann-Hilbert approach.* Mathematical Surveys and Monographs, 128. American Mathematical Society, Providence, RI, 2006.
12. O. Forster, *Lectures on Riemann Surfaces*, Graduate texts in mathematics; 81, Springer, New York (1981).
13. C. S. Gardner, J. M. Greene, M. D. Kruskal, R. M. Miura, *Methods for solving the Korteweg-de Vries equation.* Phys. Rev. Letters **19** (1967), 1095–1097.
14. B. Grammaticos, Y. Kosmann-Schwarzbach, T. Tamizhmani (eds), *Discrete integrable systems.* Lecture Notes in Physics, 644. Springer-Verlag, Berlin, 2004.
15. P. Gordoa, N. Joshi, A. Pickering, *Second and fourth Painlev hierarchies and Jimbo-Miwa linear problems.* J. Math. Phys. **47** (2006), no. 7, 073504, 16 pp.
16. V. Gromak, I. Laine, S. Shimomura, *Painlevé differential equations in the complex plane.* de Gruyter Studies in Mathematics, 28. Walter de Gruyter & Co., Berlin, 2002.
17. R.G. Halburd, R.J. Korhonen, *Meromorphic solutions of difference equations, integrability and the discrete Painlevé equations.* J. Phys. A **40** (2007), no. 6, R1-R38.
18. E. Hille, *Ordinary differential equations in the complex domain.* Pure and Applied Mathematics. Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1976.
19. Y. Ilyashenko, S. Yakovenko, *Lectures on analytic differential equations.* Graduate Studies in Mathematics, 86. American Mathematical Society, Providence, RI, 2008.
20. E.L. Ince, *Ordinary Differential Equations.* Dover Publications, New York, 1956.
21. K. Iwasaki, H. Kimura, S. Shimomura, M. Yoshida, *From Gauss to Painlevé. A modern theory of special functions.* Aspects of Mathematics, E16. Friedr. Vieweg & Sohn, Braunschweig, 1991.
22. M. Jimbo, T. Miwa, *Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. II.* Phys. D **2** (1981), no. 3, 407-448.
23. M. Jimbo, T. Miwa, K. Ueno, *Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. I. General theory and τ-function.* Phys. D **2** (1981), no. 2, 306-352.
24. M. Kohno, *Global analysis in linear differential equations.* Mathematics and its Applications, 471. Kluwer Academic Publishers, Dordrecht, 1999.
25. I. Laine, *Nevanlinna theory and complex differential equations.* de Gruyter Studies in Mathematics, 15. Walter de Gruyter & Co., Berlin, 1993.
26. P. Lax *Integrals of nonlinear equations of evolution and solitary waves.* Comm. Pure Appl. Math. **21** (1968), 467-490.
27. P. Lax *Almost periodic solutions of the KdV equation.* SIAM Rev. **18** (1976), no. 3, 351-375.
28. B. Malgrange, *Sur les déformations isomonodromiques.* Mathematics and physics (Paris, 1979/1982), 401-438, Progr. Math., **37**, Birkhäuser Boston, Boston, MA, 1983.
29. T. Morrison, *Asymptotics of higher-order Painlevé equations.* PhD thesis, The University of Sydney, School of Mathematics and Statistics, June 2009.
30. K. Okamoto, *Studies on the Painlevé equations. I. Sixth Painlevé equation $P_{\rm VI}$.* Ann. Mat. Pura Appl. (4) **146** (1987), 337-381.
31. V. Retakh, V. Rubtsov, *Noncommutative Toda chains, Hankel quasideterminants and the Painlev II equation.* J. Phys. A **43** (2010), no. 50, 505204, 13 pp.
32. D. Sauzin, *Introduction to 1-summability and the resurgence theory.* To appear.
33. C.L. Siegel, *Topics in complex function theory. Vol. I. Elliptic functions and uniformization theory.* Wiley Classics Library. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1988.
34. H. Umemura, *Galois theory and Painlevé equations.* In Théories asymptotiques et équations de Painlevé, 299-339, Sémin. Congr., **14**, Soc. Math. France, Paris, 2006.
35. W. Wasow, *Asymptotic expansions for ODE.* Reprint of the 1965 edition. Robert E. Krieger Publishing Co., Huntington, N.Y., 1976.

Chapter 2
The first Painlevé equation

Abstract This chapter aims at introducing the reader to properties of the first Painlevé equation and its general solution. The definition of the first Painlevé equation is recalled (Sect. 2.1). We precise how the Painlevé property translates for the first Painlevé equation (Sect. 2.2), a proof of which being postponed to an appendix. We explain how the first Painlevé equation arises also as a condition of isomonodromic deformations for a linear ODE (Sect. 2.3 and Sect. 2.4). Some symmetry properties are mentioned (Sect. 2.5). We spend some times in describing the asymptotic behaviour at infinity of the solutions of the first Painlevé equation and, in particular, we describe the truncated solutions (Sect. 2.6). We eventually briefly comment the importance of the first Painlevé transcendents for models in physics (Sect. 2.7).

2.1 The first Painlevé equation

We concentrate now on the first Painlevé equation,

$$(P_I)\ u'' = 6u^2 + x. \tag{2.1}$$

We notice that for every $x_0 \in \mathbb{C}$ and every $(u_0, u_0') \in \mathbb{C}^2$, theorem 1.1 ensures the existence of a unique solution of (2.1), holomorphic near x_0, satisfying the initial data $\big(u(x_0), u'(x_0)\big) = (u_0, u_0')$.

2.2 Painlevé property for the first Painlevé equation

As already mentioned, the first Painlevé equation satisfies the Painlevé property. We have the following more precise result.

Theorem 2.1. *Every solution of the Painlevé equation P_I can be analytically continued as a meromorphic function on $\mathbb{C}$ with only double poles.*

This theorem will be shown in appendix. We add the following result for completeness:

Theorem 2.2. *Every solution of (2.1) is a transcendental meromorphic function on $\mathbb{C}$ with infinitely many poles.*

Proof. We just give an idea of the proof. It is easy to see that every solution u of the first Painlevé equation (2.1) is a transcendental function. Otherwise, since u is meromorphic with double poles, u should be a rational function, $u(x) = \dfrac{P(x)}{Q(x)^2}$. Reasoning on the degrees of P and Q, one shows that this is impossible. So every solution u is a transcendental meromorphic function. It can be then derived from the Clunie lemma in Nevanlinna theory of meromorphic functions that necessarily u has an infinite set of poles [26, 13]. □

The above properties were well-known since Painlevé [37]. The following one was also known by Painlevé, however its complete proof has been given only recently [34], see also [5].

Theorem 2.3. *A solution of P_I cannot be described as any combination of solutions of first order algebraic differential equations and those of linear differential equations on $\mathbb{C}$.*

2.3 First Painlevé equation and isomonodromic deformations condition

Each Painlevé equation P_J is equivalent to a nonautonomous Hamiltonian system [35]. Concerning the first Painlevé equation this Hamiltonian system is given by the following **first Painlevé system**:

$$(\mathcal{H}_I) \begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}x} = \dfrac{\partial H_I}{\partial \mu} = \mu \\[2ex] \dfrac{\mathrm{d}\mu}{\mathrm{d}x} = -\dfrac{\partial H_I}{\partial u} = 6u^2 + x \end{cases}, \quad H_I(u,\mu,x) = \frac{1}{2}\mu^2 - 2u^3 - xu. \tag{2.2}$$

It is known [11, 36] that this Hamiltonian system arises as a **condition of isomonodromic deformations** of the following (Schlesinger type) second order linear ODE,

$$(\mathcal{SL}_I) \begin{cases} \dfrac{\partial^2 \Psi}{\partial z^2} = Q_I(z;u,\mu,x)\Psi \\[2ex] Q_I(z;u,\mu,x) = 4z^3 + 2xz + 2H_I(u,\mu,x) - \dfrac{\mu}{z-u} + \dfrac{3}{4(z-u)^2}, \end{cases} \tag{2.3}$$

In other words, u is solution of the first Painlevé equation (2.1) if and only if the monodromy data of (2.3) do not depend on x. We explain this point. Equation (2.3) has two fixed singularities $z = u, \infty$, so that any solution of (2.3) can be analytically continued on a Riemann surface over $\overline{\mathbb{C}} \setminus \{u, \infty\}$. The singular point $z = u$ is a regular singular point, and a local analysis easily shows that the monodromy at this point (see [33]) of any fundamental system of solutions of (2.3) does not depend on x. The other singular point $z = \infty$ is an irregular singular point. Thus the only nontrivial mondromy data of (2.3) are given by the Stokes multipliers at $z = \infty$.

The second order linear ODE (2.3) is equivalent to a first order linear ODE in dimension two. Each Stokes matrix is a two by two univalent matrix [28, 33], and thus depends on a sole complex coefficient called a Stokes multiplier.

In general these Stokes multipliers depend on x, except when Ψ satisfies the following isomonodromic deformation condition:

$$(\mathcal{D}_I)\ \frac{\partial \Psi}{\partial x} = A_I \frac{\partial \Psi}{\partial z} - \frac{1}{2}\frac{\partial A_I}{\partial z}\Psi, \qquad A_I = \frac{1}{2(z-u)} \tag{2.4}$$

The first Painlevé system (2.2) ensures the compatibility between equations (2.3) and (2.4) : solving a Painlevé equation is thus equivalent to solving an inverse monodromy problem (Riemann-Hilbert problem) [33, 17, 16, 23, 24, 38, 22, 20, 10].

We add another property : we mentioned that the asymptotics of (2.3) at $z = \infty$ are governed by some Stokes multipliers $s_i = s_i(u, \mu, x)$. It can be shown that the space of Stokes multipliers makes a complex manifold $\mathcal{M}_I$ of dimension 2. Also, for any point of $\mathcal{M}_I$ there exists a unique solution of the first Painlevé equation (2.1) for which the monodromy data of equation (2.3) are equal to the corresponding coordinates of this point [23].

2.4 Lax formalism

There is another fruitful alternative to get the Painlevé equations, however related to the previous one, based on the linear representations of integrable systems through the Lax formalism [27]. We exemplify this theory for Painlevé I, for which the so-called **Lax pair** A and B are the matrix operators given as follows [16]:

$$A = \begin{pmatrix} v(x) & 4\big(z-u(x)\big) \\ z^2 + u(x)z + u(x)^2 + x/2 & -v(x) \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 2 \\ z/2 + u(x) & 0 \end{pmatrix}.$$

To the matrix operator A one associates a first order ODE in the z variable, whose time evolution (the x variable) is governed by another first order ODE determined by the matrix operator B,

$$\begin{cases} \dfrac{\partial \Psi}{\partial z} = A\Psi \\ \dfrac{\partial \Psi}{\partial x} = B\Psi \end{cases} \tag{2.5}$$

The compatibility condition $\dfrac{\partial^2 \Psi}{\partial z \partial x} = \dfrac{\partial^2 \Psi}{\partial x \partial z}$ provides what is known as the **zero curvature condition** (or also Lax equation), namely $\frac{\partial A}{\partial x} - \frac{\partial B}{\partial z} = [B, A]$ where $[B, A] = BA - AB$ stands for the commutator. Expliciting this condition, one recovers the first Painlevé equation under the form $\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}x} = v \\ \dfrac{\mathrm{d}v}{\mathrm{d}x} = 6u^2 + x \end{cases}$.
From what we have previously seen, the zero curvature condition allows to think of (2.5) as an isomonodromic deformations condition for its first equation.

2.5 Symmetries

Here we would like to notice that the cyclic symmetry group of order five acts on the set of solutions (2.1). Indeed, introducing

$$\omega_k = \mathrm{e}^{\frac{2\mathrm{i}\pi}{5}k}, \qquad k = 0, \cdots, 4$$

then any solution u of (2.1) is mapped to another solution u_k through the transformation

$$u_k(x) = \omega_k^2 u\big(\omega_k x\big), \qquad k = 0, \cdots, 4.$$

In general u and u_k will be different solutions, an obvious exception being when u satisfies the initial data $u(0) = u'(0) = 0$.

2.6 Asymptotic at infinity

Our aim in this section is to describe all the possible behaviors at infinity of the solutions of the first Painlevé equation (2.1).
We first notice that $x = \infty$ is indeed a fixed singularity for P_I : making the change of variable $u(x) = \mathfrak{u}(t)$, $t = \dfrac{1}{x}$, equation (2.1) translates into $t^5\mathfrak{u}'' + 2t^4\mathfrak{u}' = 1 + 6t\mathfrak{u}^2$, where $t = 0$ appears as a (irregular) singular point.

We mention that, when analysing the asymptotics of solutions of differential equations at singular points, there are a great difference between linear and nonlinear ODEs. When a linear ODE is concerned, the asymptotics of every solution can be derived from the asymptotics of a fundamental system of solutions. For non linear ODEs some care has to be taken, since as a rule singular solutions may exist, which cannot be deduced from the general solution.

The study of all possible behaviors at infinity was first made by Boutroux [3, 4]. Various approaches can be used: a direct asymptotic approach in the line of Boutroux as in [14, 18, 21], or another one based on the relationship between the first Painlevé equation and a convenient Schlesinger type linear ODE as described in Sect. 2.3, see [23] (see also [24, 38, 25] for a semiclassical variant).

2.6.1 Dominant balance principle

Here we only want to give a rough idea of how to get the possible behaviors and, in the spirit of this course, we follow the viewpoint of asymptotic as in [14, 21, 18]. In this approach, for a given ODE, the first task is to determine the terms in the equation which are dominant and of comparable size when $x \to \infty$ along a path or a inside a sector. The reduced equation obtained by keeping only in the ODE these dominant terms gives the leading behavior.
One usual trick so as to guess the asymptotics of solutions of ODEs is thus the **dominant balance principle** [2]. A maximal dominant balance corresponds to the case where there is a maximal set of dominant terms of comparable size in the equation. As a rule, this gives rise to the general behavior. The remaining cases are called subdominant balances.

Here it is useful to introduce the following notations:

- $f \simeq g$ when $x \to \infty$ along a path if $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = Cte$, $Cte \in \mathbb{C}^\star$.
- $f \ll g$ when $x \to \infty$ along a path if $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.

The unique maximal balance that is possible for P_I consists in assuming that all the three terms in (2.1) are of comparable size when $x \to \infty$. In particular u^2 and x have comparable size, so that

$$u(x) = x^{\frac{1}{2}} O(1), \qquad x \to \infty.$$

We thus assume that

$$u(x) = x^{\frac{1}{2}} v\big(z(x)\big)$$

with $z(x) \to \infty$ and $v\big(z(x)\big) = O(1)$ when $x \to \infty$. If $z(x)$ behaves like a fractional power of x at infinity, then

$$\frac{z'(x)}{z(x)} \simeq \frac{z''(x)}{z'(x)} \simeq \frac{1}{x}$$

and this is what we will assume.
We also make the following remark : if $v\big(z\big)$ is an analytic function with an asymptotic expansion when $z \to \infty$, then one would have $\dfrac{v''(z)}{z^2} \ll \dfrac{v'(z)}{z} \ll v(z)$ for z near infinity, that is $\dfrac{v''\big(z(x)\big)}{z^2(x)} \ll \dfrac{v'\big(z(x)\big)}{z(x)} \ll v\big(z(x)\big)$ when $x \to \infty$. Here we will adjust the choice of $z(x)$ by adding the demand that

$$v\big(z(x)\big) \ll z(x) v'\big(z(x)\big) \ll z(x)^2 v''\big(z(x)\big) \quad \text{when} \quad x \to \infty.$$

These assumptions on v and $z(x)$ imply that

$$u'(x) = x^{-\frac{1}{2}} z(x) v'\big(z(x)\big) O(1) + o(1), \qquad u''(x) = x^{-\frac{3}{2}} z^2(x) v''\big(z(x)\big) O(1) + o(1).$$

Thus, if $v\big(z(x)\big) = v'\big(z(x)\big) = v''\big(z(x)\big) = O(1)$ and demanding that u'' and x have comparable size, one gets $z(x) = x^{\frac{5}{4}} O(1)$ as a necessary condition. This suggests with Boutroux [3, 4] to make the following transformation,

$$u(x) = \alpha x^{\frac{1}{2}} v(z), \qquad z = \beta x^{\frac{5}{4}}, \tag{2.6}$$

with $\alpha, \beta \neq 0$ some constants, under which equation (2.1) becomes:

$$v'' + \frac{v'}{z} - \frac{4}{25} \frac{v}{z^2} - \frac{96\alpha}{25\beta^2} v^2 - \frac{16}{25\alpha\beta^2} = 0.$$

With the following choice for α and β,

$$\alpha = \frac{\mathrm{e}^{\frac{\mathrm{i}\pi}{2}}}{\sqrt{6}}, \qquad \beta = \mathrm{e}^{\frac{5\mathrm{i}\pi}{4}} \frac{24^{\frac{5}{4}}}{30}, \tag{2.7}$$

one finally gets:

$$v'' = \frac{1}{2} v^2 - \frac{1}{2} - \frac{v'}{z} + \frac{4}{25} \frac{v}{z^2}. \tag{2.8}$$

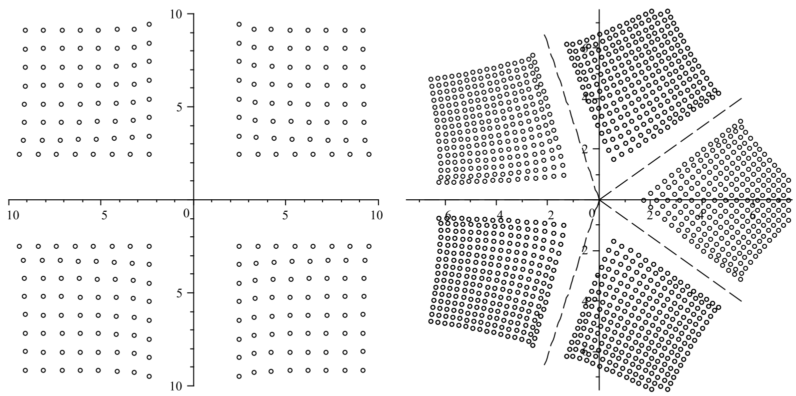


Fig. 2.1 Left hand side : approximate period lattices in each quadrants Q_i of z-plane. Right hand side, their images in the x-plane through the transformation $x \mapsto z$ defined by (2.6)-(2.7)

We now concentrate on this equation (2.8) and we examine the possible balances.

2.6.2 Maximal balance, elliptic function-type behavior

We consider the maximal balance case, that is we assume that v and its derivatives can be compared to unity. This means that equation (2.8) is asymptotic to the equation

$$v'' = \frac{1}{2}v^2 - \frac{1}{2}.$$

The solutions of this equation[1] are the functions $v(z) = 12\wp(z - z_0; \frac{1}{12}, g_3)$ where $\wp$ is the Weierstrass p-function (cf. Sect. 1.2.2) while z_0 and g_3 are two free complex parameters.
It can be shown [3, 4, 21] that this provides indeed the general behaviour of the Painlevé transcendents near infinity : for $|z|$ large enough in each open quadrants

$$Q_k = \{z \in \mathbb{C},\, k\frac{\pi}{2} < \arg z < (k+1)\frac{\pi}{2}\},\ k = 0, 1, 2, 3 \mod 4$$

the generic solution v of (2.8) has, for $|z|$ large enough, an approximate period lattice of poles, Fig. 2.1. In this domain, excluding small neighbourdhoods of poles, the asymptotics of such a generic solution v of (2.8) is governed by Weierstrassian elliptic functions. With Kruskal & Joshi [21] one can refer to this behavior as an *elliptic function-type* behavior.

This asymptotic behaviour translates for the Painlevé I transcendents through the transformation (2.6)-(2.7) into the asymptotics in the sectors

[1] Just multiply both sides of the equality by v', then integrate.

$$S_k = \{x \in \mathbb{C},\ -\pi + k\frac{2\pi}{5} < \arg x < -\pi + (k+1)\frac{2\pi}{5}\},\ k = 0,1,2,3,4 \mod 5. \tag{2.9}$$

We mention that when z approaches the real axis (*resp.* the imaginary axis) then, for $|z|$ large enough and in a small angular strip of width $O\Big((\log|z|)/|z|\Big)$, the solution v displays a *near oscillatory-type* behaviour with no poles, and one has $v(z) \to -1$ (*resp.* $v(z) \to +1$) when $|z| \to \infty$, see [21].

This means that the five special rays $\arg x = -\pi + k\frac{2\pi}{5}$, $k = 0, \cdots, 4$ play an important role in the asymptotics of the solutions of Painlevé I, the general solutions having lines of poles asymptotic to these rays.

2.6.3 Submaximal dominant balances, truncated solutions

We now consider submaximal dominant balances, that is when v or one of its derivatives differ from order unity. It can be shown [21] that the sole consistent case occurs when

$$v \simeq 1, \qquad v'' \ll 1.$$

This implies that equation (2.8) is now asymptotic to the equation

$$\frac{1}{2}v^2 - \frac{1}{2} = 0$$

that is $v(z) = \pm 1 + o(1)$. Examining this case leads to the following result:

Theorem 2.4. *The first Painlevé equation (2.1) has:*

- *five complex parameter families of solutions u, called* ***intégrales tronquées (truncated solutions)*** *after Boutroux, such that u is free of poles in two adjacent sectors S_k and S_{k+1} for $|x|$ large enough, with its asymptotics governed by*
$$u(x) = \Big(-\frac{x}{6}\Big)^{\frac{1}{2}}\Big(1 + O(x^{-\frac{5}{2}})\Big)$$
(for a convenient determination of the square root).
- *among the truncated solutions, five special solutions, the* ***intégrales tritronquées (tritruncated solutions)****, each of them being free of poles in four adjacent sectors $S_k, S_{k+1}, S_{k+2}, S_{k+3}$ for $|x|$ large enough.*

This theorem has various proofs (see for instance [19, 31, 32] for “nonconventional” approaches). We will see in this course how the resurgent analysis can be used to show theorem 2.4.

2.7 First Painlevé equation and physical models

As already said (Sect. 1.3.3), the Painlevé equations in general and the first Painlevé equation in particular, appear by similarity reductions of integrable PDEs. They play a significant role in others physical models, see e.g. [22]

and references therein for the first Painlevé equation. This includes the description of asymptotic regime in transition layers and caustic-type domain. We exemplify this fact with the focusing nonlinear Schrödinger equation $i\varepsilon\Psi_t + \frac{\varepsilon^2}{2}\Psi_{xx} + |\Psi|^2\Psi = 0$ (fNLS). It is shown in [9] that when considering the (so-called) dispersionless limit $\varepsilon \to 0$, the solutions (of convenient Cauchy problems) of (fNLS) are asymptotically governed by a tritruncated solution of the first Painlevé equation. In the same work, theoretical and numerical evidences led the authors to conjecture that tritruncated solutions of the first Painlevé equation have the following property, shown in [7] under the naming "the Dubrovin conjecture":

Proposition 2.1. *Each tritruncated solution of the first Painlevé equation is holomorphic on a full sector of the form* $\overset{\bullet}{\tilde{\boldsymbol{\mathfrak{s}}}}{}_0^{\infty}(I)$ *with* I *an arc of aperture* $|I| = 8\pi/5$.

See definition 3.7 for what means the sector $\overset{\bullet}{\tilde{\boldsymbol{\mathfrak{s}}}}{}_0^{\infty}(I)$.

Recently, resurgence theory spectacularly enters the realm of string theory and related models, as an efficient tool for making the connection between perturbative and non-perturbative effects. In particular, the first Painlevé equation was particularly adressed in [1] thanks of its physical interpretation in the context of 2D quantum gravity, when the so-called double-scaling limit is considered [8, 29, 30].

Appendix

The reader only interested in learning applications of resurgence theory may skip this appendix, where we show theorem 2.1 for completeness. We follow the proof given in [6]. See also [14, 15] and specially [13] with comments and references therein. We start with two lemmas.

Lemma 2.1. *If* u *is a solution of (2.1) which is holomorphic in a neighbourhood of* $x_0 \in \mathbb{C}$*, then the radius* R *of analyticity at* x_0 *satisfies* $R \geq 1/\rho$ *with*

$$\rho = \max\left(\Big|u(x_0)\Big|^{1/2}, \Big|\frac{u'(x_0)}{2}\Big|^{1/3}, \Big|u^2(x_0) + \frac{x_0}{6}\Big|^{1/4}, \Big|\frac{u(x_0)u'(x_0)}{2} + \frac{1}{24}\Big|^{1/5}\right).$$

Proof. If $u(x) = \displaystyle\sum_{k=0}^{\infty} c_k(x-x_0)^k \in \mathbb{C}\{x-x_0\}$ is solution (2.1) then

$$\begin{cases} c_0 = u(x_0), & c_1 = u'(x_0) \\ c_2 = 3c_0^2 + \dfrac{x_0}{2}, & c_3 = 2c_0c_1 + \dfrac{1}{6} \\ (k+1)(k+2)c_{k+2} = 6\sum_{m=0}^{k} c_m c_{k-m},\ k \geq 2 & \end{cases} \tag{2.10}$$

We note

$$\rho = \max\left(\Big|u(x_0)\Big|^{1/2}, \Big|\frac{u'(x_0)}{2}\Big|^{1/3}, \Big|u^2(x_0) + \frac{x_0}{6}\Big|^{1/4}, \Big|\frac{u(x_0)u'(x_0)}{2} + \frac{1}{24}\Big|^{1/5}\right),$$

so that for $0 \leq l \leq 3$

$$|c_l| \leq (l+1)\rho^{l+2}. \tag{2.11}$$

Assume that (2.11) is satisfied for every $0 \leq l \leq k+1$ for a given $k \geq 2$. Then by (2.10),

$$(k+1)(k+2)|c_{k+2}| \leq 6\sum_{m=0}^{k}(m+1)(k-m+1)\rho^{k+4} \leq (k+1)(k+2)(k+3)\rho^{k+4}.$$

The coefficients $\sum_{m=0}^{k}(m+1)(k-m+1)$ are those of the taylor expansions of $(1-x)^{-4}$ at the origin. Indeed, for $|x|<1$, $\frac{1}{1-x} = \sum_{k\geq 0} x^k$ so that $\frac{1}{(1-x)^2} = \sum_{k\geq 0}(k+1)x^k$. Therefore $\left(\frac{1}{(1-x)^2}\right)^2 = \sum_{k\geq 0}\left(\sum_{m=0}^{k}(m+1)(k-m+1)\right)x^k$.

We conclude that (2.11) is satisfied for every $l \geq 0$ and this implies that $R \geq \frac{1}{\rho}$, where R is the radius of convergence of the series expansion u. □

Lemma 2.2. *In a neighbourhood of any given point $\widetilde{x} \in \mathbb{C}$ there exists a one-parameter family of meromorphic solutions u of (2.1) having a pole at $\widetilde{x}$. Necessarily $\widetilde{x}$ is a double pole and u is given by the Laurent-series expansions*

$$u(x) = \frac{1}{(x-\widetilde{x})^2} - \frac{\widetilde{x}}{10}(x-\widetilde{x})^2 - \frac{1}{6}(x-\widetilde{x})^3 + c_4(x-\widetilde{x})^4 + \sum_{k\geq 6} c_k(x-\widetilde{x})^k$$

where $c_4 \in \mathbb{C}$ is a free parameter.

Proof. We are looking for a Laurent-series expansion $u(x) = \sum_{k=p}^{\infty} c_k(x-\widetilde{x})^k \in \mathbb{C}\{x-\widetilde{x}\}\left[\frac{1}{x-\widetilde{x}}\right]$ satisfying (2.1). Then necessarily $p \geq -2$, $c_{-2} = 1$ or 0, $c_{-1} = 0$. So either $\widetilde{x}$ is a regular point, otherwise

$$u(x) = \frac{1}{(x-\widetilde{x})^2} - \frac{\widetilde{x}}{10}(x-\widetilde{x})^2 - \frac{1}{6}(x-\widetilde{x})^3 + c_4(x-\widetilde{x})^4 + \sum_{k\geq 6} c_k(x-\widetilde{x})^k$$

where $c_4 \in \mathbb{C}$ is a free parameter, while for $k \geq 6$ the coefficients are polynomial functions of $(\widetilde{x}, \alpha)$. Indeed one has

$$(k-2)(k+5)c_{k+2} = 6\sum_{m=0}^{k} c_m c_{k-m}, \qquad k \geq 2.$$

We can define $\rho > 0$ (depending $(\widetilde{x}, \alpha)$) such that, for $0 \leq l \leq 5$,

$$|c_l| \leq \frac{1}{3}(l+1)\rho^{l+2}. \tag{2.12}$$

Assume that this property is satisfied for every c_l, $0 \leq l \leq k+1$ for a given $k \geq 4$. Then

$$(k-2)(k+5)|c_{k+2}| \leq \frac{2}{3}\sum_{m=0}^{k}(m+1)(k-m+1)\rho^{k+4} \leq \frac{1}{9}(k+1)(k+2)(k+3)\rho^{k+4}$$

and we conclude that $|c_{k+2}| \le \dfrac{1}{3}(k+3)\rho^{k+4}$. Therefore (2.12) is true for every $l \ge 0$ and the Laurent series expansion converges in the punctured dics $D(\widetilde{x}, 1/\rho)^{\star}$. $\square$

The following notations will now be used:

- $D_{x_0} \subset \mathbb{C}$ is an open disc, Ω is a discrete subset of D_{x_0} and $x_0 \in D_{x_0} \setminus \Omega$.
- u is a solution of (2.1) defines by some initial data at $x_0 \in D_{x_0} \setminus \Omega$ and u is meromorphic in $D_{x_0} \setminus \Omega$.
- $\lambda(a,b) : [0,1] \to D_{x_0} \setminus \Omega$ denotes a $\mathcal{C}^{\infty}$-smooth path in $D_{x_0} \setminus \Omega$ with endpoints $\lambda(a,b)(0) = a$ and $\lambda(a,b)(1) = b$. When $b \in \partial D_{x_0}$ it is assumed that $\lambda(a,b)$ is a path where b is removed (that is one considers the restriction to $[0,1[$ of $\lambda(a,b)$). Moreover we assume that the length of any subsegment $\lambda(c,d)$ of $\lambda(a,b)$ is less that $2|c-d|$.
  We mention that we use the same notation $\lambda(a,b)$ for the path and its image.
- $\widetilde{x} \in \partial D_{x_0}$ is a singular point for u.

Lemma 2.3. *Assume that* $u(x) = \displaystyle\sum_{k=-2}^{4} a_k (x-\widetilde{x})^k + O(|x-\widetilde{x}|^5)$ *when* $x \to \widetilde{x}$ *along* $\lambda(x_0, \widetilde{x})$*, with* $a_{-2} \neq 0$*. Then* u *is meromorphic at* $\widetilde{x}$ *and* u *is uniquely determined by* $(\widetilde{x}, a_4)$*.*

Proof. Since u is solution of (2.1) which is analytic at each point of the smooth path $\lambda(x_0, \widetilde{x})$ one has

$$u''(x) = 6u^2(x) + x = 6\left(\sum_{k=-2}^{4} a_k (x-\widetilde{x})^k + O(|x-\widetilde{x}|^5)\right)^2 + x$$

when $x \to \widetilde{x}$ along $\lambda(x_0, \widetilde{x})$. This implies that the asymptotic expansion is differentiable.

This is a consequence of the mean value theorem, $u(x) = u(x_0) + \displaystyle\int_{x_0}^{x} u'(s)\, ds$ along $\lambda(x_0, \widetilde{x})$ which is $\mathcal{C}^{\infty}$-smooth, and the uniqueness of the asymptotic expansion.

The same calculus as the one made in the proof of lemma 2.2 shows that

$$u(x) = \frac{1}{(x-\widetilde{x})^2} - \frac{\widetilde{x}}{10}(x-\widetilde{x})^2 - \frac{1}{6}(x-\widetilde{x})^3 + a_4(x-\widetilde{x})^4 + O(|x-\widetilde{x}|^5).$$

We note v the meromorphic solution of (2.1) obtained in lemma 2.2 with $c_4 = a_4$. We set

$$\begin{aligned} w(x) &= v(x) - (x-\widetilde{x})^{-2} = O(|x-\widetilde{x}|^2) \\ f(x) &= u(x) - v(x) = O(|x-\widetilde{x}|^5) \end{aligned}$$

and we want to show that $f = 0$. We have

$$f'' - \frac{12}{(x-\widetilde{x})^2} f = g, \qquad g = 12wf + 6f^2 = O(|x-\widetilde{x}|^7),$$

so that, integrating this linear equation,

$$f(x) = C_1(x-\widetilde{x})^{-3} + C_2(x-\widetilde{x})^4 \\ - 7(x-\widetilde{x})^{-3}\int_{\widetilde{x}}^{x}(s-\widetilde{x})^4 g(s)\,ds + \frac{(x-\widetilde{x})^4}{7}\int_{\widetilde{x}}^{x}(s-\widetilde{x})^{-3}g(s)\,ds.$$

Since $f(x) = O(|x-\widetilde{x}|^5)$ we get that f is solution of the fixed-point problem $f = \mathcal{N}(f)$ with

$$\mathcal{N}(f)(x) = -7(x-\widetilde{x})^{-3}\int_{\widetilde{x}}^{x}(s-\widetilde{x})^4 g(s)\,ds + \frac{(x-\widetilde{x})^4}{7}\int_{\widetilde{x}}^{x}(s-\widetilde{x})^{-3}g(s)\,ds.$$

For $x_1 \in \lambda(x_0,\widetilde{x})$ we consider the normed vector space $(\mathbb{B}, \|.\|)$,

$$\mathbb{B} = \{f \in \mathcal{C}^0(\lambda(x_1,\widetilde{x})),\ f = O(|x-\widetilde{x}|^5\}, \qquad \|f\| = \sup_{x\in\lambda(x_1,\widetilde{x})} |(x-\widetilde{x})^{-5}f(x)|.$$

We show later that $(\mathbb{B}, \|.\|)$ is a Banach space (lemma 2.4). Now for x_1 close enough from $\widetilde{x}$ (see lemma 2.4):

- the mapping $\mathcal{N}$ send the unit ball B of $\mathbb{B}$ into itself,
- the mapping $\mathcal{N} : B \to B$ is contractive.

Therefore the fixed-point problem $f = \mathcal{N}(f)$ has a unique solution in B by the contraction mapping theorem. Obviously this solution is $f = 0$ and therefore $u = v$. $\square$

Lemma 2.4. *With notations of the proof of lemma 2.3:* $(\mathbb{B}, \|.\|)$ *is a Banach space and the mapping* $\mathcal{N} : B \to B$ *is contractive.*

Proof.

1. $(\mathbb{B}, \|.\|)$ is a Banach space.
   Indeed, assume that (f_p) is a Cauchy sequence in $(\mathbb{B}, \|.\|)$,

   $$\forall\varepsilon,\ \exists p_0 : \forall p,q > p_0,\ \forall x \in \lambda(x_1,\widetilde{x}),\ |(x-\widetilde{x})^{-5}(f_p(x)-f_q(x))| < \varepsilon\ . \quad (2.13)$$

   Writing $g_p(x) = (x-\widetilde{x})^{-5}f_p(x)$, condition (2.13) implies that for every $x \in \lambda(x_1,\widetilde{x})$ the sequence $(g_p(x))$ is a Cauchy sequence, hence $g_p(x) \to g(x)$ in $\mathbb{C}$. Now making $q \to +\infty$ in (2.13) one sees that $g_p \to g$ uniformaly. Therefore $g \in \mathcal{C}^0(\lambda(x_1,\widetilde{x}))$ and is bounded on $\lambda(x_1,\widetilde{x})$. Thus $g = (x-\widetilde{x})^{-5}f$ with $f \in \mathbb{B}$.
2. The mapping $\mathcal{N}$ is contractive for x_1 close enough from $\widetilde{x}$.
   We introduce $\mathcal{N}_1(f)(x) = -7(x-\widetilde{x})^{-3}\int_{\widetilde{x}}^{x}(s-\widetilde{x})^4 g(s)\,ds$ and $\mathcal{N}_2(f)(x) = \dfrac{(x-\widetilde{x})^4}{7}\int_{\widetilde{x}}^{x}(s-\widetilde{x})^{-3}g(s)\,ds$ so that $\mathcal{N}(f) = \mathcal{N}_1(f) + \mathcal{N}_1(f)$.
   One can assume that for $s \in \lambda(x,\widetilde{x}))$, $|s-\widetilde{x}| \leq |x-\widetilde{x}|$. Also there exist $r > 0$ and $a > 0$ such that $|w(x)| \leq a|x-\widetilde{x}|^2$ when $|x-\widetilde{x}| \leq r$. We now assume that $|x_1-\widetilde{x}| \leq r$. For $f_1, f_2 \in B$ and $x \in \lambda(x_1,\widetilde{x})$ one has :

$$
\begin{aligned}
&\left|(x-\widetilde{x})^{-5}\Big(\mathcal{N}_1(f_1)-\mathcal{N}_2(f_2)\Big)\right| \\
&\le \left|-7(x-\widetilde{x})^{-8}\int_{\widetilde{x}}^{x}(s-\widetilde{x})^4\Big(12w(s)(f_1(s)-f_2(s))+6(f_1^2(s)-f_2^2(s))\Big)\,ds\right| \\
&\le 7|x-\widetilde{x}|^{-8}\Big(12a|x-\widetilde{x}|^{11}\|f_1-f_2\| \\
&\qquad\qquad +6|x-\widetilde{x}|^{14}\|f_1-f_2\|\|f_1+f_2\|\Big)\mathrm{Length}(\lambda(x,\widetilde{x})) \\
&\le 14|x-\widetilde{x}|^4\Big(12a+12|x-\widetilde{x}|^3\Big)\|f_1-f_2\|.
\end{aligned}
$$

The other term of $(x-\widetilde{x})^{-5}\Big(\mathcal{N}_2(f_1)-\mathcal{N}_2(f_2)\Big)$ is worked in a similar way. Choosing x_1 close enough from $\widetilde{x}$, one obtains the existence of a constant $0 < Cte < 1$ such that for $f_1, f_2 \in B$, $\|\mathcal{N}(f_1)-\mathcal{N}(f_2)\| \le Cte\|f_1-f_2\|$.
□

Lemma 2.5. *When $x \to \widetilde{x}$ along $\lambda(x_0,\widetilde{x})$ with $\widetilde{x} \in \partial D_{x_0}$ a singular point for u:*

1. $|u(x)|+|u'(x)| \to +\infty$,
2. u is unbounded.

Proof. • Point 1. The lemma 2.1 implies that $|u(x)|$ or $|u'(x)|$ has to be large for x near $\widetilde{x}$ which is a singular point.
• Point 2. If one multiplies (2.1) by u' and integrates, one gets

$$(u')^2 = 4u^3 + 2xu - 2\int_{x_0}^{x} u(s)\,ds + C \tag{2.14}$$

where $C \in \mathbb{C}$ is a constant. Therefore if u is bounded $x \to \widetilde{x}$ along $\lambda(x_0,\widetilde{x})$ then u' is bounded as well, which contradicts Point 1. □

Lemma 2.6. *When $x \to \widetilde{x}$ along $\lambda(x_0,\widetilde{x})$, with $\widetilde{x} \in \partial D_{x_0}$ a singular point for u, then:*

$$u^{-3}(x)\int_{x_0}^{x} u(s)\,ds \to 0, \qquad |u(x)| \to +\infty, \qquad |u'(x)| \to +\infty.$$

Proof. By lemma 2.5, we know that u is unbounded when $x \to \widetilde{x}$ along $\lambda(x_0,\widetilde{x})$, so that

$$\limsup_{x\to\widetilde{x}} |u(x)| = +\infty, \qquad \liminf_{x\to\widetilde{x}} |u^{-1}(x)| = 0.$$

We remind that $\limsup_{x\to\widetilde{x}} f(x) = \lim_{\varepsilon\to 0}\Big(\sup\big\{f(x),\, x \in \lambda(x_0,\widetilde{x}) \cap D(\widetilde{x},\varepsilon)\big\}\Big)$ while $\liminf_{x\to\widetilde{x}} f(x) = \lim_{\varepsilon\to 0}\Big(\inf\big\{f(x),\, x \in \lambda(x_0,\widetilde{x}) \cap D(\widetilde{x},\varepsilon)\big\}\Big)$.

Also, since for $x \in \lambda(x_0,\widetilde{x})$ one has

$$\left|u^{-3}(x)\int_{x_0}^{x} u(s)\,ds\right| \le |u^{-3}(x)|.\max_{\lambda(x_0,x)}|u|.\ \mathrm{Length}(\lambda(x_0,x)),$$

we get

$$\liminf_{x\to\widetilde{x}}\left\{\left|u^{-3}(x)\int_{x_0}^{x} u(s)\,ds\right|\right\} \le \liminf_{x\to\widetilde{x}}\left\{|u^{-3}(x)|.\max_{\lambda(x_0,x)}|u|.\ \mathrm{Length}(\lambda(x_0,x))\right\}.$$

The right hand side term vanishes because u is unbounded when $x \to \widetilde{x}$, thus

$$\liminf_{x\to\widetilde{x}} \left\{ \left| u^{-3}(x) \int_{x_0}^{x} u(s)\, ds \right| \right\} = 0. \tag{2.15}$$

In particular, for every $\gamma > 0$, for every $D(\widetilde{x}, \varepsilon)$, there exists $x \in \lambda(x_0, \widetilde{x}) \cap D(\widetilde{x}, \varepsilon)$ so that

$$\left| u^{-3}(x) \int_{x_0}^{x} u(s)\, ds \right| \leq \gamma.$$

Assumption : assume that $u^{-3}(x) \displaystyle\int_{x_0}^{x} u(s)\, ds \to 0$ is false, which translates into : there exists $\gamma > 0$ such that, for every $D(\widetilde{x}, \varepsilon)$, there exists $x \in \lambda(x_0, \widetilde{x}) \cap D(\widetilde{x}, \varepsilon)$ so that

$$\left| u^{-3}(x) \int_{x_0}^{x} u(s)\, ds \right| \geq \gamma.$$

By continuity, we see that for any $\gamma > 0$ small enough, there exists a sequence $x_n \to \widetilde{x}$, $x_n \in \lambda(x_0, \widetilde{x})$, such that

$$\left| \int_{x_0}^{x_n} u(s)\, ds \right| = \gamma \left| u^3(x_n) \right|. \tag{2.16}$$

The same arguments used in the proof of lemma 2.5 show that

$$\limsup_{n} |u(x_n)| = +\infty.$$

This means that there exists a subsequence (x_{n_k}) of (x_n) such that $|u(x_{n_k})| \to +\infty$. Therefore we can assume that

$$\lim_{n} |u(x_n)| = +\infty. \tag{2.17}$$

From (2.16) we see that

$$\lim_{n} \left| \int_{x_0}^{x_n} u(s)\, ds \right| = +\infty \tag{2.18}$$

while (2.17), (2.16) with $\gamma > 0$ small enough, and (2.14) imply that

$$\lim_{n} |u'(x_n)| = +\infty \tag{2.19}$$

We will demonstrate that the assumption made implies a contradiction in what follows

• We consider now the solution h_n of the Cauchy problem

$$\begin{cases} (h')^2 = 4h^3 + 2x_n h + \widetilde{C}_n & \text{with } \widetilde{C}_n = C - 2\displaystyle\int_{x_0}^{x_n} u(s)\, ds \\ h(0) = u(x_n), & h'(0) = u'(x_n) \end{cases} \tag{2.20}$$

where C is the constant given in (2.14). Notice by (2.18) that

$$\lim_{n} |\widetilde{C}_n| = +\infty, \tag{2.21}$$

and by (2.16) then by (2.14) that

$$\begin{cases} |h_n(0)| = (2\gamma)^{-1/3} \big|\widetilde{C}_n\big|^{1/3} \big(1 + o(1)\big) \\ \\ |h_n'(0)| = \big|2\gamma^{-1} \mathrm{e}^{i\phi_n} + 1\big|^{1/2} \big|\widetilde{C}_n\big|^{1/2} \big(1 + o(1)\big), \qquad \phi_n \in \mathbb{R}. \end{cases} \tag{2.22}$$

Writing

$$h_n(t) = \widetilde{C}_n^{1/3} H_n(X),\, X = \widetilde{C}_n^{1/6} t, \tag{2.23}$$

one obtains that H_n is solution of the following *elliptic differential equation* (see (1.7)) with given initial data:

$$\begin{cases} (H')^2 = 4H^3 + 2\theta_n H + 1, \;\text{ with } \theta_n = x_n \widetilde{C}_n^{-2/3} \\ \\ H_n(0) = \widetilde{C}_n^{-1/3} u(x_n), \qquad |H_n(0)| = (2\gamma)^{-1/3}\big(1 + o(1)\big), \\ \\ H_n'(0) = \widetilde{C}_n^{-1/2} u'(x_n), \qquad |H_n'(0)| = \big|2\gamma^{-1}\mathrm{e}^{i\phi_n} + 1\big|^{1/2} \big(1 + o(1)\big) \end{cases} \tag{2.24}$$

From the properties of elliptic functions, we know that H_n can be analytically continued as a doubly periodic meromorphic function with double poles at the period lattice $a_n + m\omega_1(\theta_n) + n\omega_2(\theta_n)$, $(n,m) \in \mathbb{Z}^2$, for some $a_n \in \mathbb{C}$ and $\omega_{1,2}(\theta_n) = Cte_{1,2} + O(\theta_n)$.

• Next we consider the function U_n defined by

$$u(x) = \widetilde{C}_n^{1/3} U_n(X),\, X = \widetilde{C}_n^{1/6}(x - x_n), \tag{2.25}$$

so that (2.1) translates into the property that U_n is solution of the ODE

$$U'' = 6U^2 + \theta_n + \varepsilon_n X, \;\text{ with } \varepsilon_n = \widetilde{C}_n^{-5/6}, \tag{2.26}$$

and, more precisely from (2.14), that

$$\begin{cases} (U')^2 = 4U^3 + 2\theta_n U + 1 + 2\varepsilon_n \left(XU - \displaystyle\int_0^X U(S)\, dS \right) \\ U_n(0) = \widetilde{C}_n^{-1/3} u(x_n), \qquad U_n'(0) = \widetilde{C}_n^{-1/2} u'(x_n) \end{cases} \tag{2.27}$$

• We want to show that U_n and H_n are locally holomorphically equivalent: we look for a function G_n holomorphic near 0 such that

$$U_n = H_n \circ G_n \;\text{ with } G_n(X) = X + g_n(X), \qquad g_n(0) = 0,\; g_n'(0) = 0. \tag{2.28}$$

We know from (2.24) that $H_n'' = 6H_n^2 + \theta_n$, hence from (2.26) we deduce that

$$2g_n' H_n'' \circ G_n + (g_n')^2 H_n'' \circ G_n + g_n'' H_n' \circ G_n = \varepsilon_n X.$$

which is also

$$2g_n'(H_n' \circ G_n)' + g_n'' H_n' \circ G_n = \varepsilon_n X + (g_n')^2 H_n'' \circ G_n.$$

Multiplying both parts of this equality by $H_n' \circ G_n$ and integrating, one gets:

$$\begin{cases} w_n = (H_n' \circ G_n)^{-2} \displaystyle\int_0^X H_n' \circ G_n(S) \Big[\varepsilon_n S + w_n^2(S).H_n'' \circ G_n(S)\Big]\, dS = \mathcal{N}(w_n) \\ g_n(X) = \displaystyle\int_0^X w_n(S)\, dS, \qquad w_n(0) = 0, \qquad G_n(X) = X + g_n(X). \end{cases} \tag{2.29}$$

Let $D(0, \frac{|\varepsilon_n|^{-1/4}}{2})$ be the disc centred at 0 of diameter $|\varepsilon_n|^{-1/4}$. We note $\widetilde{D(0, \frac{|\varepsilon_n|^{-1/4}}{2})}$ the disc $D(0, \frac{|\varepsilon_n|^{-1/4}}{2})$ without the discs of diameter $d(\gamma)$ around the poles and the zeros of H_n'.

We consider a path $\lambda(0, X_0)$ in $\widetilde{D(0, \frac{|\varepsilon_n|^{-1/4}}{2})}$. In (2.29) the integrals $\int_0^X$ are considered along $\lambda(0, X) \subset \lambda(0, X_0)$. We can assume that the length of any subsegment $\lambda(0, X)$ of $\lambda(0, X_0)$ is less that $2|X|$.

Let $a \in]1/4, 1/2[$ and let $(\mathbb{B}, \|.\|)$ be the Banach space

$$\mathbb{B} = \{f \in \mathcal{C}^0(\lambda(0, X_0))\}, \qquad \|f\| = \sup_{x \in \lambda(0, X_0)} |f(x)|.$$

We consider also the ball $B = \{f \in \mathbb{B}, \|f\| \le |\varepsilon_n|^a\}$. If $w \in B$ and $g(X) = \int_0^X w(S)\, dS$, one has

$$\|g\| \le \sup_{X \in \lambda(0, X_0)} \left| \int_0^X w(S)\, dS \right| \le \|w\|.\mathrm{Length}(\lambda(0, X_0)) \le |\varepsilon_n|^{a-1/4}.$$

One can assume that $d(\gamma) \ge 3|\varepsilon_n|^{a-1/4}$ so that

$$\|\mathcal{N}(w)\| \le |\varepsilon_n| Cte_1(\gamma)|\varepsilon_n|^{-1/2} + Cte_2(\gamma)|\varepsilon_n|^{2a-1/4}.$$

Therefore $\|\mathcal{N}(w)\| \le |\varepsilon_n|^a$ for $|\varepsilon_n|$ small enough. Quite similarly, for $w_1, w_2 \in B$,

$$\|\mathcal{N}(w_1) - \mathcal{N}(w_2)\| = O(|\varepsilon_n|^{a-1/4})\|w_1 - w_2\|.$$

We conclude by the contraction mapping theorem that $\mathcal{N}$ has a unique fixed point in B, for $|\varepsilon_n|$ small enough.

• We have seen that, for $|\varepsilon_n|$ small enough and $a \in]1/4, 1/2[$, we have

$$U_n(X) = H_n\big(X + g_n(X)\big), \qquad |g(X)| \le |\varepsilon_n|^{a-1/4}, \qquad X \in \widetilde{D(0, \frac{|\varepsilon_n|^{-1/4}}{2})}.$$

Therefore,

$$\sup_{X \in \widetilde{D(0, \frac{|\varepsilon_n|^{-1/4}}{2})}} \left| \widetilde{C}_n^{-1/3} u(x_n + \widetilde{C}_n^{-1/6} X) - H_n(X) \right| = O(|\varepsilon_n|^{a-1/4}). \tag{2.30}$$

We remind that when $\underset{\sim}{x_n \to \widetilde{x}}$ one has $|\widetilde{C}_n| \to +\infty$ and $|\varepsilon_n| = |\widetilde{C}_n^{-5/6}| \to 0$.

Now when $X \in \widetilde{D(0, \frac{|\varepsilon_n|^{-1/4}}{2})}$ then $\widetilde{C}_n^{-1/6} X$ belong to a disc of radius $|\widetilde{C}_n|^{1/24}$ without some discs of radius $d(\gamma)|\widetilde{C}_n|^{-1/6}$. Consequently, for n large enough,

$$\forall\, x \in D_{x_0},\ \exists\, X \in D(0, \frac{\widetilde{|\varepsilon_n|^{-1/4}}}{2}),\ \Big| x - (x_n + \widetilde{C}_n^{-1/6} X) \Big| \leq \frac{d(\gamma)}{2} |\widetilde{C}_n|^{-1/6}.$$

Choosing $x = x_0$, we see from (2.30) that u is unbounded near x_0 which is a regular point for u: contradiction.

Therefore, $u^{-3}(x) \displaystyle\int_{x_0}^{x} u(s)\, ds \to 0$ when $x \to \widetilde{x}$ along $\lambda(x_0, \widetilde{x})$. It is now an easy exercice by lemma 2.5 and (2.14) to see that $\min\{|u|, |u'|\} \to +\infty$ necessarily when $x \to \widetilde{x}$. (Just assume that $u^{-1}(x) \to 0$ is false and see that there is a contradiction.) $\square$

End of the Proof of theorem 2.1. What remains to show is that $\widetilde{x}$ is a second order pole. The substitution $u = 1/v^2$ transforms (2.14) into

$$(v')^2 = 1 + \frac{x}{2} v^4 - \frac{v^6}{2} \int_{x_0}^{x} \frac{ds}{v^2(s)} + \frac{C}{4} v^6. \tag{2.31}$$

We know from lemma 2.6 that $\displaystyle\frac{v^6}{2} \int_{x_0}^{x} \frac{ds}{v^2(s)}\, ds \to 0$ and $v \to 0$ along a path $\lambda(x_0, \widetilde{x})$ which avoids the poles of u in D_{x_0}. Therefore

$$(v')^2 = 1 + o(1), \ \text{ then } \ v^2(x) = (x - \widetilde{x})^2 \big(1 + o(1)\big).$$

Using this last equality in (2.31) one gets

$$(v')^2(x) = 1 + \frac{\widetilde{x}}{2} (x - \widetilde{x})^4 + o((x - \widetilde{x})^4), \ \text{then}\ v^2(x) = (x - \widetilde{x})^2 + \frac{\widetilde{x}}{10} (x - \widetilde{x})^6 + o((x - \widetilde{x})^6).$$

One uses (2.31) again and one concludes with lemma 2.3.

References

1. I. Aniceto, R. Schiappa, M. Vonk, *The resurgence of instantons in string theory.* Commun. Number Theory Phys. **6** (2012), no. 2, 339-496.
2. C.M. Bender, S. A. Orszag, *Advanced mathematical methods for scientists and engineers. I. Asymptotic methods and perturbation theory.* Reprint of the 1978 original. Springer-Verlag, New York, 1999.
3. P. Boutroux, *Recherches sur les transcendantes de M. Painlevé et l'étude asymptotique des équations différentielles du second ordre.* Annales scientifiques de l'Ecole Normale Supérieure Sér. 3, **30** (1913), 255-375.
4. P. Boutroux, *Recherches sur les transcendantes de M. Painlevé et l'étude asymptotique des équations différentielles du second ordre (suite).* Annales scientifiques de l'Ecole Normale Supérieure Sér. 3, **31** (1914), 99-159.
5. G. Casale, *Une preuve galoisienne de l'irréductibilité au sens de Nishioka-Umemura de la première équation de Painlevé.* Asterisque **323** (2009), 83-100.
6. O. Costin, *Correlation between pole location and asymptotic behavior for Painlevé I solutions.* Comm. Pure Appl. Math. **52** (1999), no. 4, 461-478.
7. O. Costin, M. Huang, S. Tanveer, *Proof of the Dubrovin conjecture and analysis of the tritronquée solutions of P_I*, Duke Math. J. **163** (2014), no. 4, 665-704.
8. P. Di Francesco, P. Ginsparg, J. Zinn-Justin, *2D gravity and random matrices.* Phys. Rep. **254** (1995), no. 1-2, 133 pp.
9. B. Dubrovin, T. Grava, C. Klein, *On universality of critical behavior in the focusing nonlinear Schrödinger equation, elliptic umbilic catastrophe and the tritronquée solution to the Painlevé-I equation.* J. Nonlinear Sci. **19** (2009), no. 1, 57-94.
10. A.S. Fokas, A.R. Its, A.A. Kapaev, V.Y. Novokshenov, *Painlevé transcendents. The Riemann-Hilbert approach.* Mathematical Surveys and Monographs, 128. American Mathematical Society, Providence, RI, 2006.

11. R. Garnier, *Sur des équations différentielles du troisième ordre dont l'intégrale générale est uniforme et sur une classe d'équations nouvelles d'ordre supérieur dont l'intégrale générale a ses points critiques fixes.* Ann. Sci. Ecole Norm. Sup. (3) **29** (1912), 1-126.
12. S. Garoufalidis, A. Its, A. Kapaev, M. Mariño, *Asymptotics of the instantons of Painlevé I.* Int. Math. Res. Not. IMRN, 2012, no. 3, 561-606.
13. V. Gromak, I. Laine, S. Shimomura, *Painlevé differential equations in the complex plane.* de Gruyter Studies in Mathematics, 28. Walter de Gruyter & Co., Berlin, 2002.
14. E. Hille, *Ordinary differential equations in the complex domain.* Pure and Applied Mathematics. Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1976.
15. E.L. Ince, *Ordinary Differential Equations.* Dover Publications, New York, 1956.
16. M. Jimbo, T. Miwa, *Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. II.* Phys. D **2** (1981), no. 3, 407-448.
17. M. Jimbo, T. Miwa, K. Ueno, *Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. I. General theory and τ-function.* Phys. D **2** (1981), no. 2, 306-352.
18. N. Joshi, *Asymptotic studies of the Painlevé equations.* In The Painlevé property, 181-227, CRM Ser. Math. Phys., Springer, New York, 1999.
19. N. Joshi, A. V. Kitaev, *On Boutroux's tritronquée solutions of the first Painlevé equation.* Stud. Appl. Math. **107** (2001), no. 3, 253-291.
20. N. Joshi, A. V. Kitaev, P. A Treharne, *On the linearization of the first and second Painlevé equations.* J. Phys. A **42** (2009), no. 5, 055208, 18 pp.
21. N. Joshi, M. Kruskal, *The Painlevé connection problem: an asymptotic approach. I.* Stud. Appl. Math. **86** (1992), no. 4, 315-376.
22. A.A. Kapaev, *Quasi-linear stokes phenomenon for the Painlevé first equation.* J. Phys. A **37** (2004), no. 46, 11149-11167.
23. A.A. Kapaev, A. V. Kitaev, *Connection formulae for the first Painlevé transcendent in the complex domain.* Lett. Math. Phys. **27** (1993), no. 4, 243-252.
24. T. Kawai, Y. Takei, *WKB analysis of Painlevé transcendents with a large parameter. I.* Adv. Math. **118** (1996), no. 1, 1-33.
25. T. Kawai, Y. T. Koike, Y. Nishikawa, Y. Takei, *On the Stokes geometry of higher order Painlev equations.* Analyse complexe, systèmes dynamiques, sommabilité des séries divergentes et théories galoisiennes. II. Astérisque No. **297** (2004), 117-166.
26. I. Laine, *Nevanlinna theory and complex differential equations.* de Gruyter Studies in Mathematics, 15. Walter de Gruyter & Co., Berlin, 1993.
27. P. Lax *Integrals of nonlinear equations of evolution and solitary waves.* Comm. Pure Appl. Math. **21** (1968), 467-490.
28. M. Loday-Richaud, *Divergent series and differential equations.* To appear
29. M. Mariño, R. Schiappa, M. Weiss, *Nonperturbative effects and the large-order behavior of matrix models and topological strings.* Commun. Number Theory Phys. **2** (2008), no. 2, 349-419.
30. M. Mariño, R. Schiappa, M. Weiss, *Multi-instantons and multicuts.* J. Math. Phys. **50** (2009), no. 5, 052301, 31 pp.
31. D. Masoero, *Poles of intégrale tritronquée and anharmonic oscillators. A WKB approach.* J. Phys. A: Math. Theor. **43**, no. 9, 095201, (2010).
32. D. Masoero, *Poles of intégrale tritronquée and anharmonic oscillators. Asymptotic localization from WKB analysis.* Nonlinearity **23** , no. 10, 2501-2507, (2010).
33. C. Mitschi, *Monodromy-related problems for differential equations.* To appear.
34. K. Nishioka, *A note on the transcendency of Painlevé's first transcendent.* Nagoya Math. J. **109** (1988), 63-67.
35. K. Okamoto, *Polynomial Hamiltonians associated with Painlev equations. I.* Proc. Japan Acad. Ser. A Math. Sci. **56** (1980), no. 6, 264-268.
36. K. Okamoto, *Polynomial Hamiltonians associated with Painlev equations. II. Differential equations satisfied by polynomial Hamiltonians.* Proc. Japan Acad. Ser. A Math. Sci. **56** (1980), no. 8, 367-371.
37. P. Painlevé, *Mémoire sur les équations différentielles dont l'intégrale générale est uniforme.* Bull. Soc. Math. France **28** (1900), 201-261.
38. Y. Takei, *An explicit description of the connection formula for the first Painlevé equation.* In Toward the exact WKB analysis of differential equations, linear or non-linear (Kyoto, 1998), 204, 271-296, Kyoto Univ. Press, Kyoto, 2000.

Chapter 3
Tritruncated solutions for Painlevé I

Abstract This chapter is devoted to the construction of the tritruncated solutions for the first Painlevé equation, the existence of which being announced in Sect. 2.6. This example will introduce the reader to reasonings that are common in resurgence theory. We construct a prepared form associated with the first Painlevé equation (Sec 3.1). This prepared form ODE has a unique formal solution from which we deduce the existence of truncated solutions by application of the "'main asymptotic existence theorem" (Sect. 3.1.3). We then study the 1-summability property of the formal solution by various methods (Sect. 3.3 and Sect. 3.4). By Borel-Laplace summation, one deduces the existence of the tritruncated solutions for the first Painlevé equation (Sect. 3.5).

3.1 Normalization and formal series solution

Throughout this course, $\mathbb{C}[[z^{-1}]]$ stands for the differential algebra of formal power series of the form $\widetilde{g}(z) = \sum_{n\geq 0} a_n z^{-n}$, while $\mathbb{C}((z^{-1}))$ is the space of formal Laurent series. The space of formal Laurent series is a valuation field with the natural valuation

$$\mathrm{val}: \begin{array}{ccc} \mathbb{C}((z^{-1})) & \to & \mathbb{Z}\cup\infty \\ \sum_{n\in\mathbb{Z}} a_n z^{-n} & \mapsto & \mathrm{val}\,\widetilde{w} = \min\{n\in\mathbb{Z}\,/\,a_n\neq 0\}. \end{array} \tag{3.1}$$

3.1.1 Normalization, prepared form

We have seen in Sect. 2.6 that the first Painlevé equation is equivalent to the following differential equation,

$$v'' + \frac{v'}{z} = -\frac{1}{2} + \frac{4}{25}\frac{v}{z^2} + \frac{1}{2}v^2, \tag{3.2}$$

under the Boutroux's transformation $u(x) = \frac{\mathrm{e}^{\frac{\mathrm{i}\pi}{2}}}{\sqrt{6}} x^{\frac{1}{2}} v(z)$, $z = \mathrm{e}^{\frac{5\mathrm{i}\pi}{4}} \frac{24^{\frac{5}{4}}}{30} x^{\frac{5}{4}}$.

The variable z is the so-called **critical time**.

It is worth mentioning that the symmetries detailed in Sect. 2.5 translate into the fact that any solution v of (3.2) is mapped into another solution v_k through the transformation

$$v_k(z) = \mathrm{e}^{\mathrm{i}\pi k} v\big(\mathrm{e}^{\mathrm{i}\pi k/2} z\big), \qquad k = 0, \cdots, 3. \tag{3.3}$$

From what we have seen at the end of Sect. 2.6, it is only natural to look for a formal solution of (2.8) under the form: $\widetilde{v}(z) = \sum_{l=0}^{\infty} b_l z^{-l} \in \mathbb{C}[[z^{-1}]]$. Plugging this formal series expansion in (3.2) one gets that necessarily $b_0^2 = 1$, $b_1 = 0$ and $b_2 = -\dfrac{4}{25}$. Thanks to the symmetries (3.3), there is no restriction in assuming that $b_0 = 1$. Also, it will be convenient in the sequel to make a new transformation,

$$v(z) = 1 - \frac{4}{25}\frac{1}{z^2} + \frac{1}{z^2} w(z) \tag{3.4}$$

which has the virtue to bring (3.2) into the following differential equation :

$$w'' - \frac{3}{z} w' - w = \frac{392}{625}\frac{1}{z^2} - \frac{4}{z^2} w + \frac{1}{2z^2} w^2. \tag{3.5}$$

Definition 3.1. The differential equation (3.5), which reads

$$P(\partial) w + \frac{1}{z} Q(\partial) w = F(z, w), \tag{3.6}$$

with $P(\partial) = \partial^2 - 1$, $Q(\partial) = -3\partial$ and

$$F(z, w) = \frac{392}{625}\frac{1}{z^2} - \frac{4}{z^2} w + \frac{1}{2z^2} w^2 = f_0(z) + f_1(z) w + f_2(z) w^2,$$

is called the **prepared form** equation associated with the first Painlevé equation.

Remark 3.1. For general comments on normalization procedures see, e.g. [7] and exercise 3.1. Notice that the prepared form is not unique.

3.1.2 Formal series solution

When replacing the formal series expansion $\sum_{l=0}^{\infty} a_l z^{-l}$ into (3.6), then identifying the powers, one obtains the quadratic recursion relation:

$$\begin{cases} a_0 = a_1 = 0, \quad a_2 = -\dfrac{392}{625} \\[2ex] a_l = l^2 a_{l-2} - \dfrac{1}{2} \displaystyle\sum_{p=0}^{l-2} a_{(p)} a_{(l-2-p)}, \; l = 3, 4, \cdots \end{cases} \tag{3.7}$$

One easily deduces the following properties from (3.7).

Proposition 3.1. *There exists a unique formal series*

$$\widetilde{w}(z) = \sum_{l=0}^{\infty} a_l z^{-l} \in \mathbb{C}[[z^{-1}]]. \tag{3.8}$$

solution of (3.6). Moreover val $\widetilde{w} = 2$*, the series expansion*

$$\widetilde{w}(z) = -\frac{392}{625} z^{-2} - \frac{6272}{625} z^{-4} - \frac{141196832}{390625} z^{-6} + \cdots$$

is even and the coefficients a_l are all real negative.

Remark 3.2. 1. One infers from (3.7) that the series expansion $\widetilde{w}$ diverges since obviously $|a_{2m}| \geq (m!)^2 |a_2|$ for $m \geq 1$. We expect $\widetilde{w}$ to be 1-Gevrey and this is what we will see in a moment, by considering its formal Borel transform.

We recall (with, e.g. [16, 24]) that a series $\widetilde{g}(z) = \sum_{n\geq 0} a_n z^{-n} \in \mathbb{C}[[z^{-1}]]$ is 1-Gevrey when there exist constants $C > 0$, $A > 0$ so that $|a_n| \leq C(n!)A^n$ for all n. The space $\mathbb{C}[[z^{-1}]]_1$ of 1-Gevrey series expansions is a differential algebra.

2. The differential equation (3.6) can be written as a fixed point problem,

$$w = \mathcal{N}(w), \qquad \mathcal{N}(w) = -F(z,w) - \frac{3}{z} w' + w''.$$

We consider the differential operator $\mathcal{N}$ as acting on the ring $\mathbb{C}[[z^{-1}]]$, $\mathcal{N} : \mathbb{C}[[z^{-1}]] \to \mathbb{C}[[z^{-1}]]$. When $\mathbb{C}[[z^{-1}]]$ is seen as a complete metric space (for the so-called Krull topology [24]), then $\mathcal{N}$ is a contractive map and the formal solution $\widetilde{w}$ given by lemma 3.1 is the unique solution of the fixed point problem.
This way of demonstrating the existence of the formal solution $\widetilde{w}$ is also useful for pratical calculations. All the calculations in this course are produced that way under Maple 12.0 (released: 2008).

3.1.3 Towards truncated solutions

3.1.3.1 Main asymptotic existence theorem

We have previously seen that the ODE (3.6) is formally solved by a (unique) formal series $\widetilde{w} \in \mathbb{C}[[z^{-1}]]$.

Question 3.1. Can we associate to $\widetilde{w}$ a holomorphic solution whose asymptotics are governed by this formal series ?

This question is the matter of the "main asymptotic existence theorem". This theorem is detailed in [16] for linear ODEs and extends to nonlinear equations. We will here refer the reader to [26], theorems 12.1 and 14.1, see also [23] for extension to Gevrey asymptotics.

In what follows, we refer to definition 3.6 for our notations for arcs, and to definition 3.7 for sectors of type $\dot{\mathfrak{s}}^{\infty}(I)$.

Theorem 3.1 (main asymptotic existence theorem (M.A.E.T.)). *Let I be an open arc of $\mathbb{S}^1$ of aperture $|I| \leq \pi/(q+1)$ where q is a nonnegative integer. Let $\boldsymbol{F}(z,\boldsymbol{w})$ be a m-dimensional vector function subject to the following conditions:*

1. *$\boldsymbol{F}(z,\boldsymbol{w})$ is holomorphic in $(z,\boldsymbol{w})$ on the domain of $\overset{\bullet}{\mathfrak{s}}{}^{\infty}(I)\times B(0,r)$ with $B(0,r)=\{\boldsymbol{w}\in\mathbb{C}^m,\ \|\boldsymbol{w}\|\le r\}$ for some $r>0$;*
2. *$\boldsymbol{F}(z,\boldsymbol{w})$ admits an asymptotic expansion in powers of z^{-1} at infinity in $\overset{\bullet}{\mathfrak{s}}{}^{\infty}(I)$, uniformaly valid in $\boldsymbol{w}\in B(0,r)$;*
3. *the equation $z^{-q}\boldsymbol{w}'=\boldsymbol{F}(z,\boldsymbol{w})$ is formally satisfied by a formal power series solution $\widetilde{\boldsymbol{w}}(z)\in\mathbb{C}^m[[z^{-1}]]$;*
4. *if $F_j(z,\boldsymbol{w})$ denotes the components of $\boldsymbol{F}(z,\boldsymbol{w})$, the Jacobian matrix $\lim_{z\to\infty,z\in\overset{\bullet}{\mathfrak{s}}{}^{\infty}(I)}\begin{pmatrix}\frac{\partial F_1}{\partial w_1}(z,0) & \cdots & \frac{\partial F_1}{\partial w_m}(z,0)\\ \cdots & \cdots & \cdots\\ \frac{\partial F_m}{\partial w_1}(z,0) & \cdots & \frac{\partial F_m}{\partial w_m}(z,0)\end{pmatrix}$ has non zero eigenvalues.*

Then there exists a solution $\boldsymbol{w}$ of the equation $z^{-q}\boldsymbol{w}'=\boldsymbol{F}(z,\boldsymbol{w})$, holomorphic in a domain of the form $\overset{\bullet}{\mathfrak{s}}{}^{\infty}(I)$, whose (Poincaré) asymptotics at infinity in every proper subsector of $\overset{\bullet}{\mathfrak{s}}{}^{\infty}(I)$, is given by the formal solution $\widetilde{\boldsymbol{w}}$.

3.1.3.2 Application

Let us transform (3.6) into a one order ODE of dimension 2 : we introduce $\boldsymbol{w}=\begin{pmatrix}w_1\\ w_2\end{pmatrix}=\begin{pmatrix}w\\ w'\end{pmatrix}$ and we obtain the companion system:

$$\partial\boldsymbol{w}=\begin{pmatrix}0 & 1\\ 1 & \frac{3}{z}\end{pmatrix}\boldsymbol{w}+\begin{pmatrix}0\\ F(z,w_1)\end{pmatrix}=\begin{pmatrix}F_1(z,\boldsymbol{w})\\ F_2(z,\boldsymbol{w})\end{pmatrix}=\boldsymbol{F}(z,\boldsymbol{w})\in\mathbb{C}^2[z^{-1},\boldsymbol{w}]. \tag{3.9}$$

We now fix an open arc I of $\mathbb{S}^1$, arbitrary but of aperture $|I|\le\pi$. We also consider a domain of the form $\overset{\bullet}{\mathfrak{s}}{}^{\infty}(I)$. We observe that:

1. $\boldsymbol{F}(z,\boldsymbol{w})$ is polynomial with respect to $\boldsymbol{w}$, with coefficients belonging to $\mathbb{C}[z^{-1}]$. Therefore $\boldsymbol{F}(z,\boldsymbol{w})$ is holomorphic in $(z,\boldsymbol{w})$ on the domain $\overset{\bullet}{\mathfrak{s}}{}^{\infty}(I)\times B(0,r)$ with $B(0,r)=\{\boldsymbol{w}\in\mathbb{C}^2,\ \|\boldsymbol{w}\|\le r\}$ for some $r>0$;
2. again because $\boldsymbol{F}(z,\boldsymbol{w})\in\mathbb{C}^2[z^{-1},\boldsymbol{w}]$, $\boldsymbol{F}(z,\boldsymbol{w})$ admits an asymptotic expansion in powers of z^{-1} at infinity in $\overset{\bullet}{\mathfrak{s}}{}^{\infty}(I)$, uniformaly valid in $\boldsymbol{w}\in B(0,r)$;
3. the equation(3.9) is formally satisfied by a formal power series solution $\widetilde{\boldsymbol{w}}(z)=\begin{pmatrix}\widetilde{w}\\ \widetilde{w}'\end{pmatrix}\in\mathbb{C}^2[[z^{-1}]]$;
4. the Jacobian matrix $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}=\begin{pmatrix}\frac{\partial F_1}{\partial w_1}(\infty,0) & \frac{\partial F_1}{\partial w_2}(\infty,0)\\ \frac{\partial F_2}{\partial w_1}(\infty,0) & \frac{\partial F_2}{\partial w_2}(\infty,0)\end{pmatrix}$ has non zero eigenvalues $\mu_1=-1$ and $\mu_2=1$.

These properties allow to apply the (M.A.E.T.) and this shows the following proposition (see also [15]):

Proposition 3.2. *For any open arc I of $\mathbb{S}^1$ of aperture $|I|\le\pi$, there exists a solution w of (3.6), holomorphic in a domain of the form $\overset{\bullet}{\mathfrak{s}}{}^{\infty}(I)$, whose (Poincaré) asymptotics at infinity in every proper subsector of $\overset{\bullet}{\mathfrak{s}}{}^{\infty}(I)$, is given by the formal solution $\widetilde{w}$ given by proposition 3.1.*

Proposition 3.2 thus describes the minimal opening of sectors on which holomorphic solutions w asymptotic to $\widetilde{w}$ exist. Through the transformations (3.4), (2.6) and (2.7), these solutions w corresponds to holomorphic functions

u solutions of the first Painlevé equation, defined on opening sectors of aperture $4\pi/5$: we thus get a first insight towards the truncated solutions for the first Painlevé equation (theorem 2.4).

As a matter of fact, from the above informations and the property for any solution of the first Painlevé equation to be a meromorphic function, one can even show the existence of tritruncated solutions [15]. However, to get more precise informations, we decide in what follows to turn to the question of the Borel-Laplace summability of $\widetilde{w}$.

3.2 Formal series solution and Borel transform

We denote by $\tilde{\mathcal{B}}\widetilde{w}$ the transform of $\widetilde{w}$ through the formal Borel transform $\tilde{\mathcal{B}}(z \to \zeta)$.

We denote by $\tilde{\mathcal{B}}$ the formal Borel transform (instead of $\mathcal{B}$ like in [24, 16]). Given a formal series $\widetilde{g}(x) = \sum_{l=0}^{\infty} b_l x^l \in \mathbb{C}[[x]]$, its formal Borel transform $\tilde{\mathcal{B}}\widetilde{g}$ is defined by $\big(\tilde{\mathcal{B}}\widetilde{g}\big)(\zeta) = b_0\delta + \widehat{g}(\zeta)$ where $\widehat{g}(\zeta) = \sum_{l=1}^{\infty} b_l \dfrac{\zeta^{l-1}}{\Gamma(l)} \in \mathbb{C}[[\zeta]]$. The series expansion $\widehat{g}$ is the **minor** of $\widetilde{g}$. The inverse map $\tilde{\mathcal{L}} : \mathbb{C}\delta \oplus \mathbb{C}[[\zeta]] \to \mathbb{C}[[z^{-1}]]$ is the formal Laplace transform.

Since we know by proposition 3.1 that val $\widetilde{w} > 0$, the formal Borel transform of $\widetilde{w}$ just reduces to its minor $\widehat{w}$. We now use the fact that $\widetilde{w}(z)$ is the unique solution in $\mathbb{C}[[z^{-1}]]$ of the differential equation (3.6). One gets the following result from the general properties of the Borel transform.

Proposition 3.3. *The formal series $\widetilde{w}(z) \in \mathbb{C}[[z^{-1}]]$ is a formal solution of (3.6) if and only if its minor $\widehat{w}(\zeta) \in \mathbb{C}[[\zeta]]$ is solution of the following convolution equation:*

$$\begin{aligned} &P(\partial)\widehat{w} + 1 * \big[Q(\partial)\widehat{w}\big] = \widehat{f}_0 + \widehat{f}_1 * \widehat{w} + \widehat{f}_2 * \widehat{w} * \widehat{w}, \\ &P(\partial) = \zeta^2 - 1, \qquad Q(\partial) = 3\zeta, \\ &\widehat{f}_0(\zeta) = \frac{392}{625}\zeta, \quad \widehat{f}_1(\zeta) = -4\zeta, \quad \widehat{f}_2(\zeta) = \frac{1}{2}\zeta. \end{aligned} \tag{3.10}$$

For $g(\zeta) = \sum_n b_n\zeta^n$ and $h(\zeta) = \sum_n c_n\zeta^n \in \mathbb{C}[[\zeta]]$, the convolution product $g * h$ is given by the Hurwitz product, $g*h(\zeta) = \sum_k d_k\zeta^k$, $d_k = \sum_{n+m+1=k} \dfrac{n!m!}{(n+m+1)!} b_n c_m$, which reads also $g * h(\zeta) = \int_0^\zeta g(\eta)h(\zeta-\eta)d\eta$. This formula provides the convolution product on $\mathcal{O}_0$. By formal Borel transform $\tilde{\mathcal{B}}(z \to \zeta)$, the derivation $\partial = \dfrac{\mathrm{d}}{\mathrm{d}z}$ is transported into the operator ∂ of multiplication by $(-\zeta)$ while the usual product becomes the convolution product. See, e.g. [24, 16, 4, 5] and Chapt. 7.

3.3 Formal series solution and 1-summability : first approach

We want to analyse the 1-summability of the formal solution $\widetilde{w}$ of (3.6) defined in proposition 3.1, this we do in this section by analysing its formal Borel transform through a perturbation approach. This way of doing has the advantage to give a first insight into the resurgent structure.

3.3.1 A perturbation approach

We will consider equation (3.10) as a perturbation of the equation

$$P(\partial)\widehat{w} = \widehat{f}_0(\zeta) \tag{3.11}$$

which is quite easy to solve:

- either formally, in the space $\mathbb{C}[[\zeta]]$, since $P(\partial) = \zeta^2 - 1$ is invertible in that space;
- or analytically, in the space of analyic functions, since $P(\partial) = \zeta^2 - 1$ has an inverse which is a meromorphic function with two simple poles.

In this approach, it is convenient to transform equation (3.10) into the following one parameter family of convolution equations,

$$P(\partial)\widehat{h} = \widehat{f}_0 + \varepsilon\Big(-1 * \big[Q(\partial)\widehat{h}\big] + \widehat{f}_1 * \widehat{h} + \widehat{f}_2 * \widehat{h} * \widehat{h}\Big), \tag{3.12}$$

and to look for a solution under the form

$$\widehat{h}(\zeta, \varepsilon) = \sum_{l \geq 0} \widehat{h}_l(\zeta)\varepsilon^l. \tag{3.13}$$

When plugging (3.13) into (3.12) and identifying the same powers in ε, one obtains a recursive system of convolution equations, namely:

$$\begin{cases} P(\partial)\widehat{h}_0 = \widehat{f}_0, \\ P(\partial)\widehat{h}_1 = -1 * \big[Q(\partial)\widehat{h}_0\big] + \widehat{f}_1 * \widehat{h}_0 + \widehat{f}_2 * \widehat{h}_0 * \widehat{h}_0, \\ P(\partial)\widehat{h}_n = -1 * \big[Q(\partial)\widehat{h}_{n-1}\big] + \widehat{f}_1 * \widehat{h}_{n-1} + \displaystyle\sum_{n_1+n_2=n-1} \widehat{f}_2 * \widehat{h}_{n_1} * \widehat{h}_{n_2}, \quad n \geq 1. \end{cases} \tag{3.14}$$

3.3.1.1 Formal analysis

Lemma 3.1. *The system of equations (3.14) provides a unique sequence $(\widehat{h}_l)_{l\geq 0}$ of solutions in $\mathbb{C}[[\zeta]]$. Furthermore $\widehat{h}_l \in \zeta^{2l+1}\mathbb{C}[[\zeta]]$ for every $l \geq 0$.*

Proof. Use the fact that $P(\partial)$ is invertible in $\mathbb{C}[[\zeta]]$ and the general properties of the convolution product. □

The above lemma has the following consequence:

Proposition 3.4. *The series* $\sum_{l\geq 0} \widehat{h}_l(\zeta)$ *is well defined in* $\mathbb{C}[[\zeta]]$ *and is formally convergent to the unique formal solution* $\widehat{w}(\zeta) \in \mathbb{C}[[\zeta]]$ *of the convolution equation (3.10).*

We mention that proposition 3.4 has a counterpart by formal Laplace transform $\tilde{\mathcal{L}}(\zeta \to z)$. Introducing $\widetilde{h}_l = \tilde{\mathcal{L}}\widehat{h}_l$, one gets from lemma 3.1 that the sequence $(\widetilde{h}_l)_{l\geq 0}$ solves in $\mathbb{C}[[z^{-1}]]$ the following recursive system of linear nonhomogeneous ODEs:

$$\begin{cases} P(\partial)\widetilde{h}_0 = f_0(z) \\ P(\partial)\widetilde{h}_1 = -\dfrac{1}{z}Q(\partial)\widetilde{h}_0 + f_1(z)\widetilde{h}_0 + f_2(z)\widetilde{h}_0^2 \\ P(\partial)\widetilde{h}_n = -\dfrac{1}{z}Q(\partial)\widetilde{h}_{n-1} + f_1(z)\widetilde{h}_{n-1} + f_2(z)\displaystyle\sum_{n_1+n_2=n-1} \widetilde{h}_{n_1}\widetilde{h}_{n_2}, \quad n \geq 1. \end{cases} \tag{3.15}$$

One deduces from 3.1 again that $\widetilde{h}_l \in z^{-2l-2}\mathbb{C}[[z^{-1}]]$ for every $l \geq 0$, thus:

Proposition 3.5. *The series* $\sum_{l\geq 0} \widetilde{h}_l(z)$ *is well defined in* $\mathbb{C}[[z^{-1}]]$ *and is formally convergent to the unique formal solution* $\widetilde{w}(z) \in \mathbb{C}[[z^{-1}]]$ *of the differential equation (3.6).*

3.3.1.2 Analytic properties

Instead of working in the space $\mathbb{C}[[\zeta]]$, one can rather work in the space of analytic functions. What we get from (3.14) is the following result.

Proposition 3.6. *For every* $l \geq 0$*, the formal series* $\widehat{h}_l$ *converges to a holomorphic function in the open disc* $D(0,1)$ *(and still denoted by* $\widehat{h}_l$*). Moreover, for any* $l \geq 0$*, the holomorphic function* $\widehat{h}_l$ *can be analytically continued on the universal covering of* $\mathbb{C} \setminus \{0, \pm 1, \cdots, \pm l, \pm(l+1)\}$.

As a consequence, for every $l \in \mathbb{N}$, $\widehat{h}_l$ belongs to the space of functions $\hat{\mathscr{R}}_{\mathbb{Z}}$ that will be introduced later on, see definition 4.2.

Proof. We use (3.14) and the properties of the convolution product (see, e.g. [24], or Chapt. 7). From the fact that the open disc $D(0,1)$ is a star-shaped domain with respect to the origin, the space $\mathcal{O}\big(D(0,1)\big)$ is stable under convolution product. Since $P(\partial) = \zeta^2 - 1$ is invertible in $\mathcal{O}\big(D(0,1)\big)$ one easily infers by induction from (3.14) that $\widehat{h}_l \in \mathcal{O}\big(D(0,1)\big)$ for every $l \geq 0$.
We then use the fact that if $\widehat{\varphi}, \widehat{\psi} \in \mathbb{C}\{\zeta\}$ are such that $\widehat{\varphi}$, *resp.* $\widehat{\psi}$, can be analytically continued to the universal covering of $\mathbb{C} \setminus \{0, \pm 1, \cdots, \pm p\}$, *resp.* $\mathbb{C} \setminus \{0, \pm 1, \cdots, \pm q\}$, then $\widehat{\varphi} * \widehat{\psi} \in \mathbb{C}\{\zeta\}$ can be analytically continued to the universal covering of $\mathbb{C} \setminus \{0, \pm 1, \cdots, \pm(p+q)\}$. The result announced in the proposition is thus shown by induction from (3.14). □

The function $\widehat{h}_0$ is a meromorphic function with simples poles at $\zeta = \pm 1$, thus $\widehat{h}_0$ belongs to $\hat{\mathscr{R}}_{\mathbb{Z}}^{\mathrm{simp}}$, the space of resurgent functions with simple singularities (see definition 7.54 and [24]). Since this space is a convolution algebra, the function $P(\partial)\widehat{h}_1$ given by (3.14) belongs also to $\hat{\mathscr{R}}_{\mathbb{Z}}^{\mathrm{simp}}$, but this is no more the case for $\widehat{h}_l$, $l \geq 1$, which present other types of singularities.

3.3.2 Preparations

We have previously shown (proposition 3.4) that the formal Borel transform $\widehat{w}(\zeta)$ of the formal series $\widetilde{w}(z)$ solution of the prepared form equation (3.6), can be written in the space $\mathbb{C}[[\zeta]]$ as $\widehat{w}(\zeta) = \sum_{l \geq 0} \widehat{h}_l(\zeta)$, where the sequence $(\widehat{h}_l)_{l \geq 0}$ solves the recursive system of equations (3.14).
To investigate the Borel-Laplace summability of $\widehat{w}$ one thus have to show:

- that the series of functions $\sum_{l \geq 0} \widehat{h}_l(\zeta)$ converges to a holomorphic function near the origin and can be analytically continued in a convenient sector;
- that this holomorphic function has at most exponential growth of order 1 at infinity in this sector.

We know also by proposition 3.6 that each $\widehat{h}_l(\zeta)$ can be analytically continued to the universal covering of $\mathbb{C} \setminus \mathbb{Z}$. This is why we introduce the following definition.

Definition 3.2. For $0 < \rho < 1$ one defines the domain $\mathcal{D}_\rho^{(0)} = \bigcup_{\lambda = \pm 1} D(\lambda, \rho)$.
($D(a, r)$ is the open disc centred in a with radius r.) We denote by $\overset{\bullet}{\mathscr{R}}{}_\rho^{(0)}$ the star-shaped domain defined by:

$$\overset{\bullet}{\mathscr{R}}{}_\rho^{(0)} = \mathbb{C} \setminus \left\{ t\zeta \;\mid\; t \in [1, +\infty[, \, \zeta \in \overline{D(\pm 1, \rho)} \right\} \subset \mathbb{C} \setminus \overline{\mathcal{D}}_\rho^{(0)}.$$

(See Fig. 3.1). We note $\overset{\bullet}{\mathscr{R}}{}^{(0)} = \bigcup_{0 < \rho < 1} \overset{\bullet}{\mathscr{R}}{}_\rho^{(0)} = \mathbb{C} \setminus \pm[1, +\infty[$.

Definition 3.3. Assume that $f(\zeta) = \sum_{l \geq 0} a_l \zeta^l$ is an analytic function on the open disc $D(0, r)$ centred at 0. One defines the function $|f|$, analytic on $D(0, r)$, by $|f|(\xi) = \sum_{l \geq 0} |a_l| \xi^l$.

Lemma 3.2. *For every $\rho \in]0, 1[$, there exists $M_{\rho,(0)} > 0$ such that, for every $\zeta \in \mathbb{C} \setminus \mathcal{D}_\rho^{(0)}$ and for $p = 0, 1$, one has $\left| \dfrac{\zeta^p}{P(-\zeta)} \right| \leq M_{\rho,(0)}$. In particular, for every polynom $q \in \mathbb{C}[\zeta]$ of degree ≤ 1, $\left| \dfrac{q(\zeta)}{P(-\zeta)} \right| \leq M_{\rho,(0)} |q|(1)$, for every $\zeta \in \mathbb{C} \setminus \mathcal{D}_\rho^{(0)}$. Moreover, on can take $M_{\rho,(0)} = \dfrac{1}{\rho}$.*

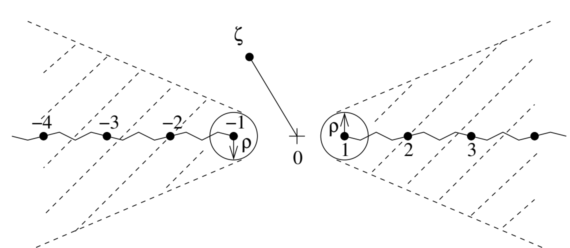


Fig. 3.1 The domain $\overset{\bullet}{\mathscr{R}}{}_\rho^{(0)}$.

Proof. We have $P(-\zeta) = (\zeta-1)(\zeta+1)$. By definition of $\mathcal{D}_\rho^{(0)}$ one observes that for every $\zeta \in \mathbb{C} \setminus \mathcal{D}_\rho^{(0)}$, $\dfrac{1}{|\zeta \pm 1|} \leq \dfrac{1}{\rho}$ and $\left|\dfrac{\zeta}{\zeta \pm 1}\right| \leq 1 + \dfrac{1}{\rho}$. Therefore, for $p = 0, 1, 2$, $\left|\dfrac{\zeta^p}{P(-\zeta)}\right| \leq \dfrac{1}{\rho^{2-p}}\left(1+\dfrac{1}{\rho}\right)^p \leq \dfrac{1}{\rho^2}(\rho+1)^p \leq \dfrac{2^p}{\rho^2}$, and this means that one can take $M_{\rho,(0)} = \dfrac{2}{\rho^2}$.
As a matter of fact, one can be more precise. Suppose for instance that $\Re(\zeta) \geq 0$. Then $|\zeta+1| \geq \max\{1, |\zeta|\}$, thus $\dfrac{\max\{1,|\zeta|\}}{|P(-\zeta)|} \leq \dfrac{1}{\rho}$. In a nutshell, one can take $M_{\rho,(0)} = \dfrac{1}{\rho}$. □

As a rule, we will combined lemma 3.2 with the following lemma:

Lemma 3.3. *Let U be a star-shaped domain from 0. Suppose that $\widehat{\varphi}$ and $\widehat{\psi}$ are two holomorphic functions in U and satisfy the conditions:*

$$\text{for every } \zeta \in U,\ \left|\widehat{\varphi}(\zeta)\right| \leq F\big(|\zeta|\big) \quad \text{and} \quad \left|\widehat{\psi}(\zeta)\right| \leq G\big(|\zeta|\big)$$

with F, G positive continuous functions on $\mathbb{R}^+$. Then, for every $\zeta \in U$,

$$\left|\widehat{\varphi} * \widehat{\psi}(\zeta)\right| \leq F * G\big(|\zeta|\big) \quad \text{and} \quad \left|\big(\zeta\widehat{\varphi}\big) * \widehat{\psi}(\zeta)\right| \leq |\zeta|\Big(F * G\big(|\zeta|\big)\Big).$$

Proof. If $\zeta = |\zeta|\exp(\mathrm{i}\theta)$, then (since U is a star-shaped domain)

$$\left|\widehat{\varphi}*\widehat{\psi}(\zeta)\right| \leq \int_0^{|\zeta|} \left|\widehat{\varphi}(r\mathrm{e}^{\mathrm{i}\theta})\right|\left|\widehat{\psi}\big((|\zeta|-r)\mathrm{e}^{\mathrm{i}\theta}\big)\right| dr \leq \int_0^{|\zeta|} F(r)G(|\zeta|-r)\, dr \leq F*G\big(|\zeta|\big).$$

The last statement is shown in a similar way. □

3.3.3 Majorant functions

We have in mind to show that the series $\sum_{l\geq 0} \widehat{h}_l(\zeta)$ converges uniformaly on any compact subset of $\dot{\mathscr{R}}^{(0)}$. To do that we will use majorant functions.

3.3.3.1 Definition of the majorant functions

We consider, for any $0 < \rho < 1$, the sequence of functions $(\widehat{h}_l)_{l\geq 0}$ recursively defined by:

$$\begin{cases} \dfrac{1}{M_{\rho,(0)}}\widehat{H}_0 = |\widehat{f}_0|(\xi), \\ \dfrac{1}{M_{\rho,(0)}}\widehat{H}_1 = \big(3+|\widehat{f}_1|\big) * \widehat{H}_0 + |\widehat{f}_2| * \widehat{H}_0 * \widehat{H}_0, \\ \dfrac{1}{M_{\rho,(0)}}\widehat{H}_n = \big(3+|\widehat{f}_1|\big) * \widehat{H}_{n-1} + \displaystyle\sum_{n_1+n_2=n-1} |\widehat{f}_2| * \widehat{H}_{n_1} * \widehat{H}_{n_2}, \quad n \geq 1. \end{cases}$$
$$\text{with } |\widehat{f}_0|(\xi) = \frac{392}{625}\xi, \ \ |\widehat{f}_1|(\xi) = 4\xi, \ \ |\widehat{f}_2|(\xi) = \frac{1}{2}\xi. \tag{3.16}$$

and $M_{\rho,(0)}$ given by lemma 3.2. (Compare this system with (3.14).) We claim that for every $l \in \mathbb{N}$, $\widehat{h}_l$ is a majorant function for $\widehat{h}_l$. Precisely:

Lemma 3.4. *We consider the sequence of functions $(\widehat{h}_l)_{l\geq 0}$ defined by (3.16). For every $0<\rho<1$ and for every $l \in \mathbb{N}$, the following properties are satisfied:*

1. *$\widehat{h}_l(\xi)$ is a polynomial function and belongs to $\xi^{l+1}\mathbb{C}[\xi]$;*
2. *furthermore,*

$$\textit{for every } \ \zeta \in \overset{\bullet}{\mathscr{R}}{}^{(0)}_{\rho}, \ \big|\widehat{h}_l(\zeta)\big| \leq \widehat{H}_l(\xi) \quad \textit{with} \quad \xi = |\zeta|. \tag{3.17}$$

Proof. The fact that $\widehat{H}_l(\xi) \in \xi^{l+1}\mathbb{C}[\xi]$ is proved by induction from (3.16) and from the properties of the convolution product. By (3.14) and lemma 3.2, for every $\zeta \in \overset{\bullet}{\mathscr{R}}{}^{(0)}_{\rho}$,

$$\big|\widehat{h}_0(\zeta)\big| \leq \left|\frac{1}{P(\partial)}\right| \big|\widehat{f}_0(\zeta)\big| \leq M_{\rho,(0)}|\widehat{f}_0|(\xi) \quad \text{with} \quad \xi = |\zeta|,$$

so that (3.17) is true for $l=0$. Assume now that (3.17) is true for $l = 0, \cdots, (n-1)$, for some $n \in \mathbb{N}^\star$. By lemma 3.3 and the induction hypothesis, for every $\zeta \in \overset{\bullet}{\mathscr{R}}{}^{(0)}_{\rho}$,

$$\left|\frac{1}{P(\partial)}\right| . \left|1 * \big[Q(\partial)\widehat{h}_{n-1}\big](\zeta)\right| \leq \left|\frac{1}{P(\partial)}\right| |Q|(|\zeta|)\left(1 * \widehat{H}_{n-1}(|\zeta|)\right),$$

where $|Q|(\xi) = 3\xi$. Therefore, by lemma 3.2,

$$\left|\frac{1}{P(\partial)}\right| . \left|1 * \big[Q(\partial)\widehat{h}_{n-1}\big](\zeta)\right| \leq M_{\rho,(0)}\, |Q|(1)\left(1 * \widehat{H}_{n-1}(\xi)\right)$$

with $\xi = |\zeta|$. More generally, for similar reasons, still writing $\xi = |\zeta|$,

$$\frac{1}{M_{\rho,(0)}}\big|\widehat{h}_n(\zeta)\big| \leq \left(3 * \widehat{H}_{n-1}(\xi)\right) + |\widehat{f}_1| * \widehat{H}_{n-1}(\xi) + \sum_{n_1+n_2=n-1} |\widehat{f}_2| * \widehat{H}_{n_1} * \widehat{H}_{n_2}(\xi).$$

Thus, for every $\zeta \in \overset{\bullet}{\mathscr{R}}{}^{(0)}_{\rho}$, $\big|\widehat{h}_n(\zeta)\big| \leq \widehat{H}_n(\xi)$. This ends the proof. □

3.3.3.2 Upper bounds for the majorant functions

Before keeping on studying the majorant functions, we state a property that will be useful in the sequel. We first recall two notations.

Definition 3.4. Let $U \subset \mathbb{C}$ be an open set. We denote by $\mathcal{O}(\overline{U})$ the space of functions continuous on the closure $\overline{U}$ of U, and holomorphic in U.
For $R_0 > 0$, we note $D(\infty, R_0) = \{z \in \mathbb{C},\ |z| > \dfrac{1}{R_0}\}$.

Lemma 3.5. *For $R_0 > 0$, we suppose $f \in \mathcal{O}\big(\overline{D(\infty, R_0)}\big)$ with $f(z) = O(z^{-m})$ at infinity, $m \in \mathbb{N}$, and we note $M = \sup\limits_{z \in D(\infty,R_0)} |f(z)|$. Then the formal Borel transform $\tilde{\mathcal{B}}f = f_0\delta + \widehat{f}$ of f satisfies the following properties:*

1. $\widehat{f} \in \mathcal{O}(\mathbb{C})$ and $|f_0| \leq \dfrac{M}{R_0}$.

2. for every $\zeta \in \mathbb{C}$, $|\widehat{f}(\zeta)| \leq \big|\widehat{f}\big|(\xi) \leq \dfrac{M}{R_0} \mathrm{e}^{\frac{\xi}{R_0}}$ with $\xi = |\zeta|$ and, when $m \geq 2$,

$$|\widehat{f}(\zeta)| \leq \frac{M}{R_0^m} \frac{\xi^{m-2}}{(m-2)!} * \mathrm{e}^{\frac{\xi}{R_0}}, \qquad \xi = |\zeta|.$$

Proof. We assume that $f \in \mathcal{O}(\overline{D(\infty, R_0)})$ with $R_0 > 0$. Its Taylor series expansion at infinity reads $f(z) = \sum\limits_{k \geq m} f_k z^{-k} = z^{-(m-1)} \sum\limits_{l \geq 1} f_{m+l-1} z^{-l}$, and by the Cauchy inequalities, $|f_k| \leq \dfrac{M}{R_0^k}$ for any $k \in \mathbb{N}$. The formal Borel transform of f reads $\tilde{\mathcal{B}}f = f_0\delta + \widehat{f}$ with :

1. $\widehat{f}(\zeta) = \sum\limits_{l \geq 1} f_l \dfrac{\zeta^{l-1}}{(l-1)!}$ as rule,
2. $\widehat{f}(\zeta) = \dfrac{\zeta^{m-2}}{(m-2)!} * \left(\sum\limits_{l \geq 1} f_{m+l-1} \dfrac{\zeta^{l-1}}{(l-1)!} \right)$ when $m \geq 2$.

Now, for any $m \geq 1$, for every $\zeta \in \mathbb{C}$, writing $\xi = |\zeta|$,

$$\sum_{l \geq 1} |f_{m+l-1}| \frac{|\zeta|^{l-1}}{(l-1)!} \leq \sum_{l \geq 1} \frac{M}{R_0^{m+l-1}} \frac{\xi^{l-1}}{(l-1)!} \leq \frac{M}{R_0^m} \mathrm{e}^{\frac{\xi}{R_0}}.$$

This ensures the uniform convergence on any compact set of $\mathbb{C}$, thus $\widehat{f} \in \mathcal{O}(\mathbb{C})$, and this provides the upper bounds. □

We now return to the majorant functions with the following lemma.

Lemma 3.6. *For every $l \in \mathbb{N}$, the majorant function $\widehat{H}_l(\xi)$ is the Borel transform of the function $\widetilde{H}_l(z)$ which has the following properties:*

- *$\widetilde{H}_l(z)$ belongs to $\mathbb{C}[z^{-1}]$;*
- *for every $0 < \rho < 1$, $\widetilde{H}_l(z)$ is bounded in the domain $|z| > \dfrac{8}{\rho}$, precisely $\sup\limits_{|z| > \frac{8}{\rho}} |\widetilde{H}_l(z)| \leq \dfrac{1}{2^l}$.*

Proof. To obtain more informations about the majorant functions, we consider their generating function, namely we introduce the series:

$$\widehat{H} = \sum_{l=0}^{\infty} \widehat{H}_l\, \varepsilon^l \ \in \mathbb{C}[\xi][[\varepsilon]]. \tag{3.18}$$

From (3.16) we observe that this generating function formally solves the convolution equation

$$\frac{1}{M_{\rho,(0)}}\widehat{H} = |\widehat{f}_0| + \varepsilon\Big[\big(3+|\widehat{f}_1|\big) * \widehat{H} + |\widehat{f}_2| * \widehat{H} * \widehat{H}\Big]. \tag{3.19}$$

This translates into the fact that the generating function $\widehat{H}$ is the transform, through the formal Borel transform $\tilde{\mathcal{B}}(z \to \xi)$, of the solution $\widetilde{h} = \sum_{l=0}^{\infty} \widetilde{H}_l\,\varepsilon^l \in \mathbb{C}[z^{-1}][[\varepsilon]]$ of the following second order algebraic equation:

$$\begin{aligned} &\frac{1}{M_{\rho,(0)}}\widetilde{H} = |f_0|(z) + \varepsilon\Big[\big(\frac{3}{z} + |f_1|\big)\widetilde{H} + |f_2|\widetilde{H}^2\Big] \\ &\text{with } |f_0|(z) = \frac{392}{625}\frac{1}{z^2}, \quad |f_1|(z) = \frac{4}{z^2}, \quad |f_2|(z) = \frac{1}{2z^2}. \end{aligned} \tag{3.20}$$

This equation has two branches solutions, one of which being asymptotic to the equation $\frac{1}{M_{\rho,(0)}}\widetilde{H} = |f_0|(z)$ when ε goes to zero. We are interested in that solution. Instead of using an explicit calculation, we rather use another method which can be generalized. In (3.20) we make the change of variable $t = \frac{1}{z}$. When writing $\widetilde{H}(z,\varepsilon) = H(t,\varepsilon)$, equation (3.20) becomes:

$$\begin{aligned} &\mathcal{F}(t,\varepsilon,H) = 0, \\ &\text{with } \mathcal{F}(t,\varepsilon,H) = \frac{1}{M_{\rho,(0)}}H - |f_0|(t^{-1}) - \varepsilon\Big[\big(3t + |f_1|(t^{-1})\big)H + |f_2|(t^{-1})H^2\Big]. \end{aligned} \tag{3.21}$$

Since

$$\mathcal{F}(0,0,0) = 0 \;\text{ and }\; \frac{\partial\mathcal{F}}{\partial H}(0,0,0) = \frac{1}{M_{\rho,(0)}} \neq 0,$$

the implicit function theorem provides a unique holomorphic solution $H(t,\varepsilon)$ to (3.21), for $|t|$ and $|\varepsilon|$ small enough : there exist $r_1 > 0$, $r_2 > 0$, $r_3 > 0$ and a holomorphic function $H : (t,\varepsilon) \in D(0,r_1) \times D(0,r_2) \mapsto H(t,\varepsilon) \in D(0,r_3)$ such that

$$\text{for every } (t,\varepsilon,H) \in D(0,r_1)\times D(0,r_2)\times D(0,r_3), \quad \Big[\mathcal{F}(t,\varepsilon,H) = 0 \Leftrightarrow H = H(t,\varepsilon)\Big].$$

To get more precise informations about that solution $H(t,\varepsilon)$ we are interested, we now view the implicit problem (3.21) as a fixed-point problem,

$$\begin{aligned} H &= \mathcal{N}(H), \qquad\qquad (3.22)\\ \mathcal{N}(H) &= M_{\rho,(0)}\left(|f_0|(t^{-1}) + \varepsilon\Big[\big(3t + |f_1|(t^{-1})\big)H + |f_2|(t^{-1})H^2\Big]\right) \\ &= M_{\rho,(0)}\left(\frac{392}{625}t^2 + \varepsilon\Big[\big(3t + 4t^2\big)H + \frac{1}{2}t^2H^2\Big]\right). \end{aligned}$$

We set $M_{\rho,(0)} = \frac{1}{\rho}$ (see lemma 3.2) and we introduce the space $O(\overline{U})$ of functions in (t,ε) which are holomorphic on the polydisc $U = D(0,\frac{\rho}{8}) \times D(0,2)$ and continuous on the closure $\overline{U}$ of U. We recall that $\big(O(\overline{U}), \|\ \|\big)$ is a Banach algebra where $\|\ \|$ stands for the maximum norm.

We have the following more general theorem : let U be a bounded open subset of $\mathbb{C}^n$, $n \geq 1$, E be a Banach space and $\mathcal{O}(\overline{U})$ be the space of functions $f : x \mapsto f(x) \in E$ which are continuous on $\overline{U}$ and holomorphic on U. With the the maximum norm $\|f\| = \sup_{z\in U} |f(z)|$, $(\mathcal{O}(\overline{U}), \|.\|)$ is a Banach algebra. See [25].

For a reason of homogeneity, we introduce the ball $B_\rho = \{H \in O(\overline{U}),\ \|H\| \leq \rho\}$. For any $H, H_1, H_2 \in B_\rho$,

$$\|\mathcal{N}(H)\| \leq \frac{1}{\rho}\left(\frac{392}{625}\frac{\rho^2}{64} + 2\left[\frac{7\rho}{16}\|H\| + \frac{\rho^2}{128}\|H\|^2\right]\right) \leq \rho$$

(remember that $\rho < 1$), while

$$\begin{aligned}\|\mathcal{N}(H_1) - \mathcal{N}(H_2)\| &\leq \frac{2}{\rho}\left(\frac{7\rho}{16}\|H_1 - H_2\| + \frac{\rho^2}{128}\|H_1 - H_2\|\Big(\|H_1\| + \|H_2\|\Big)\right)\\ &\leq \frac{29}{32}\|H_1 - H_2\|.\end{aligned}$$

The mapping $\mathcal{N}_{|B_\rho} : H \in B_\rho \mapsto \mathcal{N}(H) \in B_\rho$ is thus contractive. Since B_ρ is a close subset of a complete space, $(B_\rho, \|.\|)$ is complete and the contraction mapping theorem can be applied. We deduce the existence of a unique solution H in B_ρ of the fixed-point problem (3.22).

This solution $H(t,\varepsilon)$, thus holomorphic in $U = D(0, \frac{\rho}{8}) \times D(0,2)$, has a Taylor expansion with respect to ε at 0, $H(t,\varepsilon) = \sum_{l=0}^{\infty} H_l(t)\,\varepsilon^l$, where $(H_l)_{l\geq 0}$ is a sequence of holomorphic functions in the disc $D(0, \frac{\rho}{8})$. Moreover, by the Cauchy inequalities and using the fact that $\sup_{(t,\varepsilon)\in U} |H(t,\varepsilon)| \leq \rho$, one gets the property: for every $l \in \mathbb{N}$, $\sup_{t\in D(0,\frac{\rho}{8})} |H_l(t)| \leq \frac{\rho}{2^l}$. This ends the proof of lemma 3.6. $\square$

Lemma 3.7. *For every $0 < \rho < 1$ and every $l \in \mathbb{N}$, the majorant function $\widehat{H}_l(\xi)$ is a polynomial function and satisfies: for every $\xi \in \mathbb{C}$, $|\widehat{H}_l(\xi)| \leq \frac{8}{2^l} \mathrm{e}^{\frac{8}{\rho}|\xi|}$.*

Proof. Just a consequence of lemma 3.6 and lemma 3.5. $\square$

3.3.4 Formal series solution and Borel-Laplace summability

We are ready to show the following theorem.

Theorem 3.2. *The formal solution $\widetilde{w}(z)$ of the prepared equation (3.6) associated with the first Painlevé equation, belongs to the space $\mathbb{C}[[z^{-1}]]_1$ of 1-Gevrey series and satisfies the following properties:*

1. *its formal Borel transform $\widehat{w}(\zeta)$ belongs to the space $\zeta\mathbb{C}\{\zeta\}$, is odd and can be analytically continued on the cut plane $\overset{\bullet}{\mathscr{R}}{}^{(0)}$;*
2. *$\widehat{w}(\zeta)$ has at most exponential growth of order 1 at infinity along non-horizontal directions. More precisely, for every $0 < \rho < 1$, there exist $A > 0$ and $\tau > 0$ such that*

$$\textit{for every } \zeta \in \overset{\bullet}{\mathscr{R}}^{(0)}_{\rho},\ |\widehat{w}(\zeta)| \leq A\mathrm{e}^{\tau|\zeta|};$$

3. moreover in the above upper bounds one can choose $A = 16$ *and* $\tau = \dfrac{8}{\rho}$.

Proof. Combining lemma 3.4 and lemma 3.7, we know that, for every $0 < \rho < 1$, the functions $\widehat{h}_l(\zeta)$, $l \geq 0$ are all holomorphic on $\overset{\bullet}{\mathscr{R}}^{(0)}_{\rho}$ and satisfy: for every $R > 0$,

$$\sum_{l\geq 0} \sup_{\overline{U_R}} |\widehat{h}_l(\zeta)| \leq \sum_{l\geq 0} \widehat{H}_l(R) \leq \sum_{l\geq 0} \frac{8}{2^l}\mathrm{e}^{\frac{8}{\rho}R} \leq 16\mathrm{e}^{\frac{8}{\rho}R} \quad \text{with} \quad U_R = D(0,R)\cap\overset{\bullet}{\mathscr{R}}^{(0)}_{\rho}.$$

This normal convergence ensures the uniform convergence on any compact subset of $\overset{\bullet}{\mathscr{R}}^{(0)}$ of the series $\sum_{l\geq 0} \widehat{h}_l(\zeta)$, which thus defines a holomorphic function on $\overset{\bullet}{\mathscr{R}}^{(0)}$. We end the proof with proposition 3.4 from which we know that the series $\sum_{l\geq 0} \widehat{h}_l(\zeta)$ is formally convergent to the formal Borel transform $\widehat{w}(\zeta)$ of the formal solution $\widetilde{w}(z)$ of the ODE (3.6). □

Remark 3.3. Actually better estimates can be easily obtained, see corollary 3.1, see also exercise 3.3.

3.4 Formal series solution and 1-summability : second approach

In this second approach, however related to the first one, we introduce a Banach space (following [6, 7]) that will be convenient to demonstrate the analyticity of $\widehat{w}$, the formal Borel transform of the formal series $\widetilde{w}$ solution of the ODE (3.6). Then one introduces the reader to a "Grönwall-like lemma", which will give us the upper bounds we are looking for.

3.4.1 Convolution algebra and uniform norm

Definition 3.5. We assume that $U = U_R \subset \mathbb{C}$ is an open neighbourdhood of the origin, that U is a bounded star-shaped domain with $R = \sup_{\zeta\in U} |\zeta|$ the "radius" of U. We note $\big(\mathcal{O}(\overline{U}), +, ., *\big)$ the convolution $\mathbb{C}$-algebra (without unit) of functions which are continuous on $\overline{U}$ and holomorphic on U. We note $\mathcal{MO}(\overline{U})$ the maximal ideal of $\mathcal{O}(\overline{U})$ defined by $\mathcal{MO}(\overline{U}) = \{f \in \mathcal{O}(\overline{U}),\ f(0) = 0\}$. We define

$$\partial : f \in \mathcal{O}(\overline{U}) \mapsto \partial f(\zeta) = -\zeta f(\zeta) \in \mathcal{MO}(\overline{U}).$$

For $\nu \geq 0$ we introduce the norm $\|.\|_\nu$ defined by: for every $f \in \mathcal{O}(\overline{U})$,

$$\|f\|_\nu = R \sup_{\zeta\in U} \left|\mathrm{e}^{-\nu|\zeta|} f(\zeta)\right|.$$

We extend this norm to $\mathcal{O}(\overline{U}) \oplus \mathbb{C}\delta$ by defining, for every $f \in \mathcal{O}(\overline{U})$ and every $c \in \mathbb{C}$, $\|c\delta + f\|_\nu = |c| + \|f\|_\nu$, while $\partial\delta = 0$.

Proposition 3.7. *With the above definitions, $\big(\mathcal{O}(\overline{U})\oplus\mathbb{C}\delta, \|.\|_\nu\big)$ is a Banach algebra. In particular, for every $f,g\in\mathcal{O}(\overline{U})\oplus\mathbb{C}\delta$, $\|f*g\|_\nu\le\|f\|_\nu\|g\|_\nu$. Also $\mathcal{MO}(\overline{U})$ is closed in the norm space $\big(\mathcal{O}(\overline{U}),\|.\|_\nu\big)$. Moreover, for $\nu>0$:*

1. *for every $n\in\mathbb{N}$, for every $g\in\mathcal{O}(\overline{U})$, $\|\zeta^n*g\|_\nu\le\dfrac{n!}{\nu^{n+1}}\|g\|_\nu$, $\|\zeta^{n+1}\|_\nu\le\dfrac{n!}{\nu^{n+1}}R$ and $\|1\|_\nu=R$.*
2. *for every $f,g\in\mathcal{O}(\overline{U})$, $\|fg\|_\nu\le\dfrac{1}{R}\|f\|_\nu\|g\|_0$.*
3. *for every $f\in\mathcal{O}(\overline{U_R})$, $\nu\ge\nu_0\ge0\Rightarrow\|f\|_\nu\le\|f\|_{\nu_0}$.*
4. *for every $f\in\mathcal{MO}(\overline{U_R})$, $\lim\limits_{\nu\to\infty}\|f\|_\nu=0$.*
5. *the map $\partial_{|\mathcal{O}(\overline{U})}:f\in\mathcal{O}(\overline{U})\mapsto\partial f\in\mathcal{MO}(\overline{U})$ is a derivative in the convolution space $\mathcal{O}(\overline{U})$ and is invertible. Its inverse map ∂^{-1} satisfies: for every $f\in\mathcal{O}(\overline{U})$, for every $g\in\mathcal{MO}(\overline{U})$, $\partial^{-1}(f*g)\in\mathcal{MO}(\overline{U})$ and*
$$\|\partial^{-1}(f*g)\|_\nu\le\frac{1}{\nu R}\|f\|_\nu\|\partial^{-1}g\|_0.$$
For every $f\in\mathcal{O}(\overline{U})\oplus\mathbb{C}\delta$, for every $g\in\mathcal{MO}(\overline{U})$, $\partial^{-1}(f*g)\in\mathcal{O}(\overline{U})$ and
$$\|\partial^{-1}(f*g)\|\|_\nu\le\|f\|_\nu\|\partial^{-1}g\|_\nu.$$

Proof. Since $R\mathrm{e}^{-\nu R}\sup\limits_{\zeta\in U}\big|f(\zeta)\big|\le R\sup\limits_{\zeta\in U}\big|\mathrm{e}^{-\nu|\zeta|}f(\zeta)\big|\le R\sup\limits_{\zeta\in U}\big|f(\zeta)\big|$, we see that $\|.\|_\nu$ is equivalent to the usual maximum norm on the vector space $\mathcal{O}(\overline{U})$ and this normed vector space is complete. This shows the completeness of $\Big(\big(\mathcal{O}(\overline{U}),+,.\big),\|.\|_\nu\Big)$ and of $\big(\mathcal{O}(\overline{U})\oplus\mathbb{C}\delta,\|.\|_\nu\big)$ as well.
For $f,g\in\mathcal{O}(\overline{U})$ we have, writing $\zeta=|\zeta|\mathrm{e}^{\mathrm{i}\theta}\in U$,
$$\begin{aligned}R\mathrm{e}^{-\nu|\zeta|}f*g(\zeta)&=R\mathrm{e}^{-\nu|\zeta|}\int_0^{|\zeta|}f(s\mathrm{e}^{\mathrm{i}\theta})g\big((|\zeta|-s)\mathrm{e}^{\mathrm{i}\theta}\big)\,\mathrm{e}^{\mathrm{i}\theta}ds\\&=R\int_0^{|\zeta|}f(s\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{-\nu s}g\big((|\zeta|-s)\mathrm{e}^{\mathrm{i}\theta}\big)\mathrm{e}^{-\nu(|\zeta|-s)}\,\mathrm{e}^{\mathrm{i}\theta}ds.\end{aligned}$$

Therefore $R|\mathrm{e}^{-\nu|\zeta|}f*g(\zeta)|\le\|f\|_\nu\|g\|_\nu\displaystyle\int_0^{|\zeta|}\frac{1}{R}\,ds\le\|f\|_\nu\|g\|_\nu$. We conclude that for every $f,g\in\mathcal{O}(\overline{U})$, $\|f*g\|_\nu\le\|f\|_\nu\|g\|_\nu$, hence $\Big(\mathcal{O}(\overline{U}),\|.\|_\nu\Big)$ is a Banach algebra and $\big(\mathcal{O}(\overline{U})\oplus\mathbb{C}\delta,\|.\|_\nu\big)$ as well.
We now assume that $\nu>0$.

1. For the particular case $f=\zeta^n$, we write for $g\in\mathcal{O}(\overline{U})$,
$$\begin{aligned}R\mathrm{e}^{-\nu|\zeta|}\big|(\zeta^n*g)(\zeta)\big|&\le R\int_0^{|\zeta|}\mathrm{e}^{-\nu s}s^n\Big|g\big((|\zeta|-s)\mathrm{e}^{\mathrm{i}\theta}\big)\Big|\mathrm{e}^{-\nu(|\zeta|-s)}\,ds\\&\le\|g\|_\nu\int_0^{|\zeta|}\mathrm{e}^{-\nu s}s^n\,ds\\&\le\|g\|_\nu\int_0^{\infty}\mathrm{e}^{-\nu s}s^n\,ds\end{aligned}$$
This shows that $\|\zeta^n*g\|_\nu\le\dfrac{n!}{\nu^{n+1}}\|g\|_\nu$. The other properties follow.

2. It is obvious to show that $\|fg\|_\nu \leq \|f\|_\nu \sup_U |g| \leq \frac{1}{R}\|f\|_\nu \|g\|_0$, for every $f, g \in \mathcal{O}(\overline{U})$.
3. If $f \in \mathcal{O}(\overline{U_R})$, it is straightforward to see that $\nu \geq \nu_0 \geq 0 \Rightarrow \|f\|_\nu \leq \|f\|_{\nu_0}$.
4. If $f \in \mathcal{MO}(\overline{U_R})$, then $f = \zeta g$ with $g \in \mathcal{O}(\overline{U})$. From the previous property, $\|f\|_\nu \leq \frac{1}{R}\|\zeta\|_\nu \|g\|_0 \leq \frac{1}{\nu}\|g\|_0$. Thus $\lim_{\nu\to\infty} \|f\|_\nu = 0$.
5. If $f \in \mathcal{O}(\overline{U}) \oplus \mathbb{C}\delta$ and $g \in \mathcal{MO}(\overline{U})$ then obviously $f * g \in \mathcal{MO}(\overline{U})$. Assume now that $f \in \mathcal{O}(\overline{U})$ and $g \in \mathcal{MO}(\overline{U})$. Then $\partial^{-1}(f*g)(0) = 0$ and writing $\zeta = |\zeta|\mathrm{e}^{\mathrm{i}\theta} \in U$,

$$
\begin{aligned}
R\mathrm{e}^{-\nu|\zeta|} f * g(\zeta) &= R\mathrm{e}^{-\nu|\zeta|} \int_0^{|\zeta|} g(s\mathrm{e}^{\mathrm{i}\theta}) f\big((|\zeta| - s)\, \mathrm{e}^{\mathrm{i}\theta} ds \qquad (3.23)\\
&= R \int_0^{|\zeta|} s\mathrm{e}^{\mathrm{i}\theta} (\partial^{-1} g)(s\mathrm{e}^{\mathrm{i}\theta}) \mathrm{e}^{-\nu s} f\big((|\zeta| - s)\mathrm{e}^{\mathrm{i}\theta}\big) \mathrm{e}^{-\nu(|\zeta|-s)}\, \mathrm{e}^{\mathrm{i}\theta} ds.
\end{aligned}
$$

On the one hand, from (3.23),

$$
R|\mathrm{e}^{-\nu|\zeta|} f * g(\zeta)| \leq \frac{1}{R}\|f\|_\nu \|\partial^{-1} g\|_\nu \int_0^{|\zeta|} s\, ds \leq \frac{|\zeta|^2}{2R}\|f\|_\nu \|\partial^{-1} g\|_\nu,
$$

so that

$$
R|\mathrm{e}^{-\nu|\zeta|} \partial^{-1}(f * g)(\zeta)| \leq \frac{|\zeta|}{2R}\|f\|_\nu \|\partial^{-1} g\|_\nu \leq \|f\|_\nu \|\partial^{-1} g\|_\nu.
$$

Thus $\|\partial^{-1}(f*g)\|_\nu \leq \|f\|_\nu \|\partial^{-1} g\|_\nu$. One easily extends this formula when $f \in \mathcal{O}(\overline{U}) \oplus \mathbb{C}\delta$. On the other hand, from (3.23),

$$
R|\mathrm{e}^{-\nu|\zeta|} f * g(\zeta)| \leq \|f\|_\nu \sup_U |\partial^{-1} g| \int_0^{|\zeta|} s\mathrm{e}^{-\nu s}\, ds \leq \frac{|\zeta|}{\nu R}\|f\|_\nu \|\partial^{-1} g\|_0
$$

hence $R|\mathrm{e}^{-\nu|\zeta|} \partial^{-1}(f * g)(\zeta)| \leq \frac{1}{\nu R}\|f\|_\nu \|\partial^{-1} g\|_0$, and thus $\|\partial^{-1}(f * g)\|_\nu \leq \frac{1}{\nu R}\|f\|_\nu \|\partial^{-1} g\|_0$.
□

3.4.2 A Grönwall-like lemma

We start with the following observation.

Lemma 3.8. *Let be $N \in \mathbb{N}^\star$ and $(\widehat{F}_n)_{0\leq n\leq N}$ a sequence of entire functions, real and positive on $\mathbb{R}^+$, with at most exponential growth of order 1 at infinity. We suppose $a, b, c, d \geq 0$.*
Then, the convolution equation

$$
\widehat{\mathsf{w}}(\xi) = d + [a + b\xi] * \widehat{\mathsf{w}}(\xi) + c\left(\widehat{F}_0(\xi) + \sum_{n=1}^{N} \widehat{F}_n * \widehat{\mathsf{w}}^{*n}(\xi)\right) \qquad (3.24)
$$

has a unique solution in $\mathbb{C}[[\xi]]$, whose sum converges to an entire function $\widehat{\mathsf{w}}_d(\xi)$ with at most exponential growth of order 1 at infinity. The function

$\widehat{\mathsf{w}}_d(\xi)$ is real, positive and non-decreasing on $\mathbb{R}^+$ and, for every $\xi \in \mathbb{C}$, the mapping $d \mapsto \widehat{\mathsf{w}}_d(\xi)$ is continuous on $\mathbb{R}^+$.

Proof. Obviously, (3.24) has a unique solution $\widehat{\mathsf{w}}_d \in \mathbb{R}^+[[\xi]]$. Its formal Laplace transform, $\widetilde{\mathsf{w}}_d = \tilde{\mathcal{L}}(\widehat{\mathsf{w}}_d) \in \mathbb{R}^+[[z^{-1}]]$, solves the algebraic equation

$$\widetilde{\mathsf{w}}(z) = \frac{d}{z} + \left[\frac{a}{z} + \frac{b}{z^2}\right]\widetilde{\mathsf{w}}(z) + c\sum_{n=0}^{N} F_n(z)\widetilde{\mathsf{w}}^n(z), \tag{3.25}$$

where the F_n, $0 \le n \le N$, are holomorphic functions on a neighbourhood of infinity with $F_n(z) = O(z^{-1})$. This shows (by a reasoning already done) that $\widetilde{\mathsf{w}}_d = O(z^{-1})$ is a holomorphic function in (z,d) for $d \in \mathbb{C}$ and z on a neighbourhood of infinity (independent on d). Therefore, $\widehat{\mathsf{w}}$ defines a holomorphic function in $(\xi, d) \in \mathbb{C}^2$, with at most exponential growth of order 1 at infinity in ξ. The fact that, for $d \ge 0$, $\widehat{\mathsf{w}}_d$ is real, positive and non-decreasing on $\mathbb{R}^+$, is evident. $\square$

Lemma 3.9 (Grönwall lemma). *Let U be a star-shaped domain from 0 and $N \in \mathbb{N}^\star$. Let $(\widehat{f}_n)_{0\le n\le N}$ be a sequence of functions in $\mathcal{O}(U)$ so that there exists a sequence $(\widehat{F}_n)_{0\le n\le N}$ of entire functions, real and positive on $\mathbb{R}^+$, such that, for every $0 \le n \le N$,*

$$\text{for every } \zeta \in U,\ |\widehat{f}_n(\zeta)| \le \widehat{F}_n(\xi), \quad \xi = |\zeta|.$$

Let $p, q, r \in \mathbb{C}[\zeta]$ be polynomial functions so that p does not vanish on U and we assume that following upper bounds are valid:

$$a = \sup_{\zeta\in U}\frac{|q|(|\zeta|)}{|p(\zeta)|} < \infty, \qquad b = \sup_{\zeta\in U}\frac{|r|(|\zeta|)}{|p(\zeta)|} < \infty, \qquad c = \sup_{\zeta\in U}\frac{1}{|p(\zeta)|} < \infty.$$

We furthermore assume that $\widehat{w} \in \mathcal{O}(U)$ solves the following convolution equation:

$$p(\zeta)\widehat{w}(\zeta) + 1 * [q(\zeta)\widehat{w}](\zeta) = \zeta * [r(\zeta)\widehat{w}](\zeta) + \widehat{f}_0(\zeta) + \sum_{n=1}^{N}\widehat{f}_n * \widehat{w}^{*n}(\zeta). \tag{3.26}$$

Then for every $d \ge 0$, for every $\zeta \in U$,

$$|\widehat{w}(\zeta)| \le \widehat{\mathsf{w}}_d(\xi), \qquad \xi = |\zeta|,$$

where $\widehat{\mathsf{w}}_d$ is the holomorphic solution of the convolution equation (3.24).

Proof. (Adapted from [17]). We assume that $\widehat{w} \in \mathcal{O}(U)$ is a solution of the convolution equation (3.24). We thus have, for every $\zeta \in U$,

$$\begin{aligned} p(\zeta)\widehat{w}(\zeta) = \widehat{f}_0(\zeta) - \int_0^\zeta q(\eta)\widehat{w}(\eta)\,d\eta + \int_0^\zeta (\zeta-\eta)r(\eta)\widehat{w}(\eta)\,d\eta \\ + \sum_{n=1}^{N}\int_0^\zeta \widehat{f}_n(\zeta-\eta)\widehat{w}^{*n}(\eta)\,d\eta \end{aligned}$$

Thus, writing $\xi = |\zeta|$ and $\zeta = \xi e^{i\theta}$,

$$|\widehat{w}(\zeta)| \leq \frac{1}{|p(\zeta)|}\widehat{F}_0(\xi) + \int_0^{\xi} \left[\frac{|q|(\xi)}{|p(\zeta)|} + \frac{|r|(\xi)}{|p(\zeta)|}(\xi - r)\right] |\widehat{w}(r\mathrm{e}^{\mathrm{i}\theta})|\, dr$$
$$+ \sum_{n=1}^{N} \int_0^{\xi} \frac{1}{|p(\zeta)|}\widehat{F}_n(\xi - r)|\widehat{w}^{*n}(r\mathrm{e}^{\mathrm{i}\theta})|\, dr.$$

Therefore,

$$|\widehat{w}(\zeta)| \leq c\widehat{F}_0(\xi) + \int_0^{\xi} [a + b(\xi - r)]\, |\widehat{w}(r\mathrm{e}^{\mathrm{i}\theta})|\, dr + c\sum_{n=1}^{N} \int_0^{\xi} \widehat{F}_n(\xi - r)|\widehat{w}^{*n}(r\mathrm{e}^{\mathrm{i}\theta})|\, dr.$$

We notice from (3.26) that $|\widehat{w}(0)| = \left|\frac{\widehat{f}_0(0)}{p(0)}\right|$, while $\widehat{\mathsf{w}}_d(0) = c\widehat{F}_0(0) + d$, where $\widehat{\mathsf{w}}_d$ solves (3.24). Notice that $|\widehat{w}(0)| \leq c\widehat{F}_0(0)$ by definition of c and by hypothesis on $\widehat{F}_0$.

Case 3.1. We assume $\widehat{\mathsf{w}}_d(0) > |\widehat{w}(0)|$. We want to demonstrate that $|\widehat{w}(\zeta)| < \widehat{\mathsf{w}}_d(\xi)$ for ζ on the ray $\zeta = \xi\mathrm{e}^{\mathrm{i}\theta} \in U$.
Assume on the contrary that there exists $\zeta_1 = \xi_1\mathrm{e}^{\mathrm{i}\theta} \in U$ such that $|\widehat{w}(\zeta_1)| \geq \widehat{\mathsf{w}}_d(\xi_1)$. Define $\chi = \{\zeta \in [0, \zeta_1] \mid |\widehat{w}(\zeta)| \geq \widehat{\mathsf{w}}_d(|\zeta|)\}$. This is a non-empty closed set, bounded from below, and we note ζ_2 its infimum.

- If $|\widehat{w}(\zeta)| \geq \widehat{\mathsf{w}}_d(|\zeta|)$ for some $\zeta \in]0, \zeta_2[$, then $\zeta \in \chi$ and this contradicts the definition of ζ_2. Thus, for every $\zeta \in [0, \zeta_2[$, $|\widehat{w}(\zeta)| < \widehat{\mathsf{w}}_d(|\zeta|)$.
- If $|\widehat{w}(\zeta_2)| > \widehat{\mathsf{w}}_d(|\zeta_2|)$ then, by continuity of $\widehat{w}$ and $\widehat{\mathsf{w}}_d$, one can find $\alpha > 0$ such that $|\widehat{w}\big((|\zeta_2| - \alpha)\mathrm{e}^{\mathrm{i}\theta}\big)| > \widehat{\mathsf{w}}_d(|\zeta_2| - \alpha)$, but this this contradicts again the definition of ζ_2. Therefore $|\widehat{w}(\zeta_2)| = \widehat{\mathsf{w}}_d(|\zeta_2|)$.

Putting things together, one gets with $\xi_2 = |\zeta_2|$:

$$\begin{aligned}
&|\widehat{w}(\zeta_2)| \\
&\leq c\widehat{F}_0(\xi_2) + \int_0^{\xi_2} [a + b(\xi_2 - r)]\, |\widehat{w}(r\mathrm{e}^{\mathrm{i}\theta})|\, dr + c\sum_{n=1}^{N} \int_0^{\xi_2} \widehat{F}_n(\xi_2 - r)|\widehat{w}^{*n}(r\mathrm{e}^{\mathrm{i}\theta})|\, dr \\
&\leq c\widehat{F}_0(\xi_2) + \int_0^{\xi_2} [a + b(\xi_2 - r)]\, \widehat{\mathsf{w}}_d(r)\, dr + c\sum_{n=1}^{N} \int_0^{\xi_2} \widehat{F}_n(\xi_2 - r)\widehat{\mathsf{w}}_d^{*n}(r)\, dr \\
&\leq \widehat{\mathsf{w}}_d(\xi_2) - d
\end{aligned}$$

and we get a contradiction. As a conclusion, for every $d > 0$, for every $\zeta \in U$, $|\widehat{w}(\zeta)| \leq \widehat{\mathsf{w}}_d(\xi)$ with $\xi = |\zeta|$.

Case 3.2. The case $\widehat{\mathsf{w}}_d(0) = |\widehat{w}(0)|$ (thus, in particular, $d = 0$) is deduced from the above result. Indeed, for a given $\zeta \in U$, one has by $|\widehat{w}(\zeta)| \leq \widehat{\mathsf{w}}_d(\xi)$ for every $d > 0$. Since the mapping $d \mapsto \widehat{\mathsf{w}}_d(\xi)$ is continuous on $\mathbb{R}^+$ (cf. lemma 3.8), one gets the result by letting $d \to 0$.
□

3.4.3 Applications

We demontrate the theorem 3.2 with the tools introduced in this section.

For $R > 0$ and $\rho > 0$ we introduce the star-shaped domain $U_R = D(0,R) \cap \overset{\bullet}{\mathscr{R}}{}^{(0)}_{\rho}$. One defines $B_r = \{\widehat{v} \in \mathcal{O}(\overline{U_R}), \|\widehat{v}\|_\nu \leq r\}$, for any $r > 0$ and $\nu > 0$.
We now consider the convolution equation (3.10), viewed as a fixed-point problem. Precisely, we consider the mapping

$$\mathcal{N} : \widehat{v} \in B_r \mapsto \frac{1}{P(\partial)}\Big[-1 * \big[Q(\partial)\widehat{v}\big] + \widehat{f}_0(\zeta) + \widehat{f}_1 * \widehat{v}(\zeta) + \widehat{f}_2 * \widehat{v} * \widehat{v}(\zeta)\Big].$$

By lemmas 3.2 and proposition 3.7, one first gets:

$$\|\mathcal{N}(\widehat{v})\|_\nu \leq M_{\rho,(0)} \| -1 * \big[Q(\partial)\widehat{v}\big] + \widehat{f}_0 + \widehat{f}_1 * \widehat{v} + \widehat{f}_2 * \widehat{v} * \widehat{v}\|_\nu.$$

By proposition 3.7 again, one easily obtains, since $Q(\partial) = 3\zeta$:

$$\|1 * \big[Q(\partial)\widehat{v}\big]\|_\nu \leq \frac{1}{\nu}\|Q(\partial)\widehat{v}\|_\nu \leq \frac{1}{R\nu}\|Q(\partial)\|_0\|\widehat{v}\|_\nu \leq \frac{3}{\nu}\|\widehat{v}\|_\nu.$$

The functions $\widehat{f}_0, \widehat{f}_1, \widehat{f}_2$ belong to $\mathcal{MO}(\overline{U_R})$. This implies, by proposition 3.7, that $\lim\limits_{\nu\to\infty} \|\widehat{f}_i\|_\nu = 0$, $i = 0,1,2$. We then deduce that $\|\mathcal{N}(\widehat{v})\|_\nu \leq r$ by taking $\nu > 0$ large enough.
By the same arguments, one easily sees that $\|\mathcal{N}(\widehat{v}_1) - \mathcal{N}(\widehat{v}_2)\|_\nu \leq k\|\widehat{v}_1 - \widehat{v}_2\|_\nu$ with $k < 1$, for $\widehat{v}_1, \widehat{v}_2 \in B_r$ and for $\nu > 0$ large enough.
This means that $\mathcal{N}$ is contractive in the closed set B_r of the Banach space $\big(\mathcal{O}(\overline{U_R}), \|.\|_\nu\big)$, for $\nu > 0$ large enough. The contraction mapping theorem provides a unique solution $\widehat{w} \in B_r$ for the fixed-point problem $\widehat{v} = \mathcal{N}(\widehat{v})$. Since R and ρ can be arbitrarily chosen, we deduce (by uniqueness) that the formal Borel transform $\widehat{w}$ of the unique formal series $\widetilde{w}$ solution of (3.6), defines a holomorphic in $\overset{\bullet}{\mathscr{R}}{}^{(0)}$.

One now turns to the Grönwall lemma to get upper bounds. Working in the star-shaped domain $\overset{\bullet}{\mathscr{R}}{}^{(0)}_{\rho}$, for any $0 < \rho < 1$, one sees by lemma 3.2, lemma 3.3 and the Grönwall lemma 3.9, that for every $\zeta \in \overset{\bullet}{\mathscr{R}}{}^{(0)}_{\rho}$, $|\widehat{w}(\zeta)| \leq \widehat{\mathsf{w}}(\xi)$, $\xi = |\zeta|$, where $\widehat{\mathsf{w}}(\xi)$ solves the following convolution equation:

$$\frac{1}{M_{\rho,(0)}}\widehat{\mathsf{w}} = |\widehat{f}_0| + \big(3 + |\widehat{f}_1|\big) * \widehat{\mathsf{w}} + |\widehat{f}_2| * \widehat{\mathsf{w}} * \widehat{\mathsf{w}}.$$

This is nothing but (3.19) with $\varepsilon = 1$. We adopt the notations and reasoning made for the proof of lemma 3.6. We set $\widetilde{\mathsf{w}}(z)$ the inverse Borel transform of $\widehat{\mathsf{w}}$ and we note $\widetilde{\mathsf{w}}(z) = H(t)$, $t = z^{-1}$. The function H solves the fixed-point problem $H = \mathcal{N}(H)$ with

$$\mathcal{N}(H) = M_{\rho,(0)}\left(\frac{392}{625}t^2 + \big(3t + 4t^2\big)H + \frac{1}{2}t^2H^2\right). \tag{3.27}$$

We set $M_{\rho,(0)} = \dfrac{1}{\rho}$, $U = D(0, \dfrac{\rho}{4.22})$, and $B_\rho = \{H \in O(\overline{U}),\ \|H\| \leq \rho\}$. One easily shows that for any $H, H_1, H_2 \in B_1$,

$$\mathcal{N}(H) \in B_\rho \quad \text{ and } \quad \|\mathcal{N}(H_1) - \mathcal{N}(H_2)\| \leq \frac{44150}{44521}\|H_1 - H_2\|.$$

We conclude with the contraction mapping theorem that $\widetilde{\mathsf{w}}(z)$ is holomorphic on the domain $|z| > \dfrac{4.22}{\rho}$ and is bounded by ρ there. Therefore, by lemma

3.5, $\widehat{\mathrm{w}}$ is an entire function and satisfies: for every $\xi \in \mathbb{C}$, $|\widehat{\mathrm{w}}(\xi)| \leq 4.22\mathrm{e}^{\frac{4.22}{\rho}|\xi|}$. To sum up:

Corollary 3.1. *In theorem 3.2, one can take* $A = 4.22$ *and* $\tau = \dfrac{4.22}{\rho}$.

3.5 Tritruncated solutions for the first Painlevé equation

Theorem 3.2 shows that one can apply the Borel-Laplace summation scheme to the unique formal series expansion $\widetilde{w} \in \mathbb{C}[[z^{-1}]]$ which solves equation (3.6). This is what we do in this section.

3.5.1 Formal series solution and Borel-Laplace summation

3.5.1.1 Notations

We will use essentially common notations with [24, 16]. In particular:

Definition 3.6. We note $\mathbb{S}^1 \subset \mathbb{C}$ the circle of directions about 0 of half-lines on $\mathbb{C}$. We usually identify $\mathbb{S}^1$ with $\mathbb{R}/2\pi\mathbb{Z}$.
For a direction $\theta \in \mathbb{S}^1$ and an open arc I of $\mathbb{S}^1$, we note $\breve{\theta}$ the open arc of $\mathbb{S}^1$ defined by $\breve{\theta} =]-\frac{\pi}{2}-\theta, -\theta+\frac{\pi}{2}[$, and $\breve{I} = \bigcup_{\theta\in I}\breve{\theta}$.
For an open arc $I =]\alpha,\beta[$ of $\mathbb{S}^1$, we set $\bar{I} = [\alpha,\beta]$ its closure, and we note $I^\star =]-\beta,-\alpha[$ the complex conjugate open arc.
For an arc $I = (\alpha,\beta)$, we note $|I| = \beta-\alpha$ its aperture.

Definition 3.7. For I an open arc of $\mathbb{S}^1$ and for $0 \leq r < R \leq \infty$, we define the open domain $\dot{\mathfrak{s}}_r^R(I) = \{\zeta = \xi\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C} \mid \theta \in I, r < \xi < R\}$.
For $0 < r < R < \infty$, we set $\dot{\bar{\mathfrak{s}}}_0^R(I) = \{\zeta = \xi\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C} \mid \theta \in \bar{I}, 0 < \xi \leq R\}$ and $\dot{\bar{\mathfrak{s}}}_r^\infty(I) = \{\zeta = \xi\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C} \mid \theta \in \bar{I}, r \leq \xi < \infty\}$.
We note $\dot{\mathfrak{s}}_0(I)$, $\dot{\bar{\mathfrak{s}}}_0(I)$, $\dot{\mathfrak{s}}^\infty(I)$ and $\dot{\bar{\mathfrak{s}}}^\infty(I)$ when R or r is unspecified.
For a direction θ and $\tau \in \mathbb{R}$, we write $\dot{\Pi}_\tau^\theta$ for the following open half-plane, bisected by the half-line $\mathrm{e}^{-\mathrm{i}\theta}\mathbb{R}^+$: $\dot{\Pi}_\tau^\theta = \{z \in \mathbb{C}, \Re(z\mathrm{e}^{\mathrm{i}\theta}) > \tau\}$.
For I an open arc of $\mathbb{S}^1$ of length $|I| \leq \pi$ and $\gamma : I \to \mathbb{R}$ locally bounded, we note

$$\dot{\mathscr{D}}(I,\gamma) = \bigcup_{\theta\in I}\dot{\Pi}_{\gamma(\theta)}^\theta.$$

The domain is called a call **sectorial neighbourhood of infinity**.

3.5.1.2 Borel-Laplace summation

We start noticing that for any ρ such that $0 < \rho < 1$, we can define

$$\delta = \sin^{-1}(\rho) = \arcsin(\rho) \in]0, \frac{\pi}{2}[.$$

From this remark, theorem 3.2 and corollary 3.1 have the following obvious consequences:

Corollary 3.2. *The Borel transform* $\widehat{w} \in \mathcal{O}(\overset{\bullet}{\mathscr{R}}^{(0)})$ *of the formal solution* $\widetilde{w}$ *of equation (3.6) satisfies the following property.*
For every $\delta \in]0, \frac{\pi}{2}[$, *there exist* $A_\delta > 0$ *and* $\tau_\delta > 0$ *so that*

$$\text{for every } \zeta \in \overset{\bullet}{\mathfrak{s}}_0^\infty(]\delta, \pi-\delta[), \ \ |\widehat{w}(\zeta)| \leq A_\delta \mathrm{e}^{\tau_\delta |\zeta|}. \tag{3.28}$$

Moreover one can choose $A_\delta = 4.22$, $\tau_\delta = \dfrac{4.22}{\sin(\delta)}$.

From corollary 3.2 and the general properties of the Laplace transform $\mathcal{L}$ ([24, 16, 7] and Chapt. 7), we see that for every $\delta \in]0, \frac{\pi}{2}[$, the Borel-Laplace sum $\mathscr{S}^\theta \widetilde{w}$ of $\widetilde{w}$ in any direction $\theta \in]\delta, \pi-\delta[$,

$$\mathscr{S}^\theta \widetilde{w}(z) = (\mathcal{L}^\theta\, \widehat{w})(z) = \int_0^{\infty \mathrm{e}^{\mathrm{i}\theta}} \mathrm{e}^{-z\zeta} \widehat{w}(\zeta) d\zeta,$$

is well-defined and is holomorphic in a half-plane of type $\overset{\bullet}{\Pi}{}^\theta_{\tau_\delta}$ with $\tau_\delta = \dfrac{4.22}{\sin(\delta)}$. These holomorphic functions glue together to give the Borel-Laplace sum $\mathscr{S}^{]\delta,\pi-\delta[}\widetilde{w}$ which is holomorphic in the domain $\displaystyle\bigcup_{\theta \in]\delta,\pi-\delta[} \overset{\bullet}{\Pi}{}^\theta_{\tau_\delta}$, or even

$$\mathscr{S}^{]0,\pi[}\widetilde{w} \in \mathcal{O}\big(\overset{\bullet}{\mathscr{D}}(]0,\pi[,\tau)\big), \qquad \overset{\bullet}{\mathscr{D}}(]0,\pi[,\tau) = \bigcup_{\theta \in]0,\pi[} \overset{\bullet}{\Pi}{}^\theta_{\tau(\theta)}, \qquad \tau(\theta) = \frac{4.22}{\sin(\theta)}.$$

(See Fig. 3.2, see also exercise 3.4). Moreover, since $\widetilde{w}$ formally solves equation (3.6), then $\mathscr{S}^{]0,\pi[}\widetilde{w}$ is a holomorphic solution of equation (3.6) and is Gevrey asymptotic of order 1 at infinity to $\widetilde{w}(z) = \displaystyle\sum_{l=0}^{\infty} a_l z^{-l}$ on the sector $\overset{\bullet}{\mathscr{D}}(]0,\pi[,\tau)$: for every proper-subsector $\overset{\bullet}{\bar{\mathfrak{s}}}{}^\infty \Subset \overset{\bullet}{\mathscr{D}}(]0,\pi[,\tau)$, there exist constants $C > 0$ and $A > 0$ such that for every $N \in \mathbb{N}$ and every $z \in \overset{\bullet}{\bar{\mathfrak{s}}}{}^\infty$,

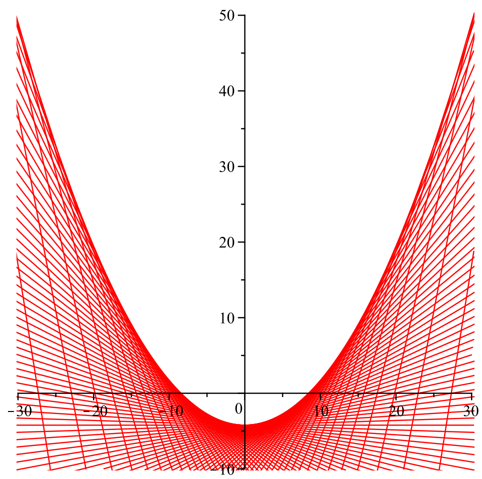
$$\left| \mathscr{S}^{]0,\pi[}\widetilde{w}(z) - \sum_{l=0}^{N-1} a_l z^{-l} \right| \leq C N! A^N |z|^{-N}. \tag{3.29}$$

Similarly, the formal series $\widetilde{w}$ is 1-summable in the directions of the interval $]\pi, 2\pi[$. This provides the Borel-Laplace sum $\mathscr{S}^{]\pi,2\pi[}\widetilde{w} \in \mathcal{O}\big(\overset{\bullet}{\mathscr{D}}(]\pi,2\pi[,\tau)\big)$ with $\tau(\theta) = \dfrac{4.22}{\sin(\theta)}$.

3.5.2 Fine Borel-Laplace summation

When using fine Borel-Laplace summation (see [16, 24]), it is possible to give more precise estimates than those given by (3.29). This is what we do in this subsection.

Fig. 3.2 The (shaded) domain $\overset{\bullet}{\mathscr{D}}(]0,\pi[,\tau)$ for $\tau(\theta)=\dfrac{4.22}{\sin(\theta)}$.



3.5.2.1 Fine Borel-Laplace summation and Nevanlinna theorem

Definition 3.8. We note $S_r(\theta)$ the open half-strip $S_r(\theta)=\bigcup_{s\in\mathbb{R}^+} D(s\mathrm{e}^{\mathrm{i}\theta},r)$, for $r>0$ and a direction θ.

The following lemma is the easy part of a theorem of Nevanlinna, [24, 16, 18, 12].

Proposition 3.8 (Nevanlinna). *We consider* $\widetilde{\varphi}(z)=\sum_{n=0}^{+\infty}\dfrac{a_n}{z^n}\in\mathbb{C}[[z^{-1}]]_1$ *and we note* $\tilde{\mathcal{B}}\widetilde{\varphi}=a_0\delta+\widehat{\varphi}$ *ist formal Borel transform through* $\tilde{\mathcal{B}}(z\to\zeta)$*. We assume* $\theta\in\mathbb{R}$ *and* $r>0$, $A>0$, $\tau>0$*. Then property (1) implies property (2) in what follows.*

1. *The minor* $\widehat{\varphi}$ *is analytically continuable on* $S_r(\theta)$ *and for every* $\zeta\in S_r(\theta)$, $|\widehat{\varphi}(\zeta)|\leq A\mathrm{e}^{\tau|\zeta|}$.
2. *The Borel-Laplace sum* $\mathscr{S}^\theta\widetilde{\varphi}(z)$ *is holomorphic in* $\overset{\bullet}{\Pi}{}^\theta_\tau$ *and for every* $p\geq 0$, $N\geq 0$ *and* $z\in\overset{\bullet}{\Pi}{}^\theta_\tau$ *:*

$$\Big|\frac{d^p\mathscr{S}^\theta\widetilde{\varphi}}{dz^p}(z)-\sum_{k=p}^{N}(-1)^p a_{k-p}\frac{(k-p)\cdots(k-1)}{z^k}\Big|\leq R_{as}(r,A,\tau,N,z\mathrm{e}^{\mathrm{i}\theta};p) \tag{3.30}$$

where

$$R_{as}(r,A,\tau,N,z;p)=A\frac{N!\mathrm{e}^{\tau r}}{r^N|z|^N}\frac{p!}{(\Re(z)-\tau)^{p+1}}\sum_{l=0}^{p}\frac{\big(r(\Re(z)-\tau)\big)^l}{l!} \tag{3.31}$$

Proof. One can assume that $\theta=0$. We note $\widehat{\varphi}_p(\zeta)=(-\zeta)^p\widehat{\varphi}(\zeta)$. Assuming that property (1) is true, one gets by the Cauchy formula:

$$\text{for every }\ \zeta\in\mathbb{R}^+,\ \big|\widehat{\varphi}_p^{(N)}(\zeta)\big|\leq\frac{N!}{r^N}\sup_{|\eta-\zeta|<r}\big|\widehat{\varphi}_p(\eta)\big|\leq A\frac{N!}{r^N}(\zeta+r)^p\mathrm{e}^{\tau(\zeta+r)}.$$

By integration by part we have, for $z \in \overset{\bullet}{\Pi}{}^0_\tau$ and $N \geq 0$,

$$\frac{d^p \mathscr{S}^0 \widetilde{\varphi}}{dz^p}(z) - \sum_{k=p}^{N} (-1)^p a_{k-p} \frac{(k-p)\cdots(k-1)}{z^k} = \frac{1}{z^N} \int_0^\infty \widehat{\varphi}_p^{(N)}(\zeta) \mathrm{e}^{-z\zeta}\, d\zeta$$

so that

$$\Big| \frac{d^p \mathscr{S}^0 \widetilde{\varphi}}{dz^p}(z) - \sum_{k=p}^{N} (-1)^p a_{k-p} \frac{(k-p)\cdots(k-1)}{z^k} \Big| \\ \leq A \frac{N! \mathrm{e}^{\tau r}}{r^N |z|^N} \int_0^\infty (\zeta + r)^p \mathrm{e}^{-\zeta(\Re(z) - \tau)}\, d\zeta.$$

We conclude with the identity:

$$\int_0^\infty (\zeta + r)^p \mathrm{e}^{-\zeta(\Re(z)-\tau)}\, d\zeta = \frac{p!}{(\Re(z)-\tau)^{p+1}} \sum_{l=0}^{p} \frac{\big(r(\Re(z)-\tau)\big)^l}{l!}.$$

□

3.5.2.2 Applications

We return to theorem 3.2 and corollary 3.1. We consider a direction $\theta \in]0, \pi[$ and we choose $r > 0$ and $0 < \rho < 1$ such that $\sin(\theta) = r + \rho$. This ensures that the half-strip $S_r(\theta)$ is a subset of the domain $\overset{\bullet}{\mathscr{R}}{}^{(0)}_\rho$ and, by theorem 3.2, there exist $A > 0$ and $\tau > 0$ such that

$$\text{for every } \ \zeta \in S_r(\theta),\ |\widehat{w}(\zeta)| \leq A \mathrm{e}^{\tau|\zeta|}, \quad \text{with} \quad \sin(\theta) = r + \rho.$$

Also, from corollary 3.1, one can choose $A = 4.22$, $\tau = \dfrac{4.22}{\rho}$. As a consequence, proposition 3.8 can be applied. The reader will easily adapt the previous considerations when the directions $\theta \in]\pi, 2\pi[$ are considered.

We now summarize what we have obtained.

Proposition 3.9. *The 1-Gevrey series $\widetilde{w} \in \mathbb{C}[[z^{-1}]]_1$, solution of the prepared equation (3.6) associated with the first Painlevé equation, is 1-summable in the directions of the arcs $I_0 =]0, \pi[$, resp. $I_1 =]\pi, 2\pi[$. This provides two Borel-Laplace sums,*

$$w_{tri,0} = \mathscr{S}^{]0,\pi[}\widetilde{w} \quad \underline{\text{resp.}} \quad w_{tri,1} = \mathscr{S}^{]\pi,2\pi[}\widetilde{w}.$$

These two sums $w_{tri,0}$ and $w_{tri,1}$ are holomorphic solutions of the differential equation (3.6) and satisfy the following properties.
For every $\theta \in I_0$, resp. $\theta \in I_1$, for every $r > 0$ and $\rho > 0$ so that $|\sin(\theta)| = r + \rho$, there exist $\tau > 0$ and $A > 0$ such that :

- *$w_{tri,j} \in \mathcal{O}(\overset{\bullet}{\Pi}{}^\theta_\tau)$, $j = 0$ resp. $j = 1$;*
- *for every $z \in \overset{\bullet}{\Pi}{}^\theta_\tau$, for every $N \in \mathbb{N}$, for $j = 0$ resp. $j = 1$,*

$$\Big| w_{tri,j}(z) - \sum_{k=0}^{N} \frac{a_k}{z^k} \Big| \leq A \frac{N! \mathrm{e}^{\tau r}}{r^N |z|^N} \frac{1}{\Re(z\mathrm{e}^{\mathrm{i}\theta}) - \tau}; \tag{3.32}$$

$$\left|\frac{dw_{tri,j}}{dz}(z)+\sum_{k=1}^{N}\frac{(k-1)a_{(k-1)}}{z^k}\right| \leq A\frac{N!\mathrm{e}^{\tau r}}{r^N|z|^N}\frac{1+r(\Re(z\mathrm{e}^{\mathrm{i}\theta})-\tau)}{\left(\Re(z\mathrm{e}^{\mathrm{i}\theta})-\tau\right)^2} \tag{3.33}$$

where the coefficients a_k are given by (3.7);

- *morover one can take $A = 4.22$, $\tau = \dfrac{4.22}{\rho}$. In particular $w_{tri,0}$, resp. $w_{tri,1}$, is holomorphic in $\overset{\bullet}{\mathscr{D}}(I_0,\tau)$, resp. in $\overset{\bullet}{\mathscr{D}}(I_1,\tau)$, with $\tau(\theta) = \dfrac{4.22}{|\sin(\theta)|}$.*

3.5.2.3 Remarks

1. We would like to make a link with 1-summability theory. We fix some notations (these are classical notations [16, 18] but for the fact that we consider asymptotics at infinity) and we recall the Borel-Ritt theorem for which we refer to [16].

   Definition 3.9. Let $I \subset \mathbb{S}^1$ be an open arc and $\overset{\bullet}{\mathfrak{s}}^\infty = \overset{\bullet}{\mathfrak{s}}^\infty(I)$ a sector.

   - $\overline{\mathcal{A}}(\overset{\bullet}{\mathfrak{s}}^\infty)$, resp. $\overline{\mathcal{A}}(I)$, is the differential algebra of holomorphic functions on the sector $\overset{\bullet}{\mathfrak{s}}^\infty$ admitting Poincaré asymptotics at infinity in this sector, resp. asymptotics germs at infinity over I.
   - $\overline{\mathcal{A}}_1(\overset{\bullet}{\mathfrak{s}}^\infty)$, resp. $\overline{\mathcal{A}}_1(I)$, is the differential algebra of holomorphic functions on the sector $\overset{\bullet}{\mathfrak{s}}^\infty$ with 1-Gevrey asymptotics at infinity in this sector, resp. 1-Gevrey asymptotics germs at infinity over I.
   - $\overline{\mathcal{A}}^{<0}(\overset{\bullet}{\mathfrak{s}}^\infty)$, resp. $\overline{\mathcal{A}}^{<0}(I)$, is the space of flat functions on $\overset{\bullet}{\mathfrak{s}}^\infty$, resp. flat germs at infinity over I.
   - $\overline{\mathcal{A}}^{\leq -1}(\overset{\bullet}{\mathfrak{s}}^\infty)$, resp. $\overline{\mathcal{A}}^{\leq -1}(I)$, is the space of 1-exponentially flat functions on $\overset{\bullet}{\mathfrak{s}}^\infty$, resp. 1-exponentially flat germs at infinity over I.

     We recall that $\overline{\mathcal{A}}^{<0}(\overset{\bullet}{\mathfrak{s}}^\infty)$ is a differential ideal of $\overline{\mathcal{A}}(\overset{\bullet}{\mathfrak{s}}^\infty)$ and that $\overline{\mathcal{A}}^{\leq -1}(\overset{\bullet}{\mathfrak{s}}^\infty)$ is a differential ideal of $\overline{\mathcal{A}}_1(\overset{\bullet}{\mathfrak{s}}^\infty)$.
   - $\mathcal{A}$ is the sheaf over $\mathbb{S}^1$ of asymptotic functions at infinity associated with the presheaf $\overline{\mathcal{A}}$. We denote by $\mathcal{A}_1$ the sheaf over $\mathbb{S}^1$ of 1-Gevrey asymptotic functions at infinity associated with the presheaf $\overline{\mathcal{A}}_1$. We denote by $\mathcal{A}^{<0}$ the sheaf over $\mathbb{S}^1$ of flat germs at infinity associated with the presheaf $\overline{\mathcal{A}}^{<0}$. Finally $\mathcal{A}^{\leq -1}$ stands for the sheaf over $\mathbb{S}^1$ of 1-Gevrey flat germs at infinity associated with the presheaf $\overline{\mathcal{A}}^{\leq -1}$.

   Theorem 3.3 (Borel-Ritt). *The quotient sheaf $\mathcal{A}/\mathcal{A}^{<0}$, resp. $\mathcal{A}_1/\mathcal{A}^{\leq -1}$, is isomorphic via the Taylor map T, resp. the 1-Gevrey Taylor map T_1, to the constant sheaf $\mathbb{C}[[z^{-1}]]$ and, resp. $\mathbb{C}[[z^{-1}]]_1$.*

   We go back to proposition 3.9. The domain $\overset{\bullet}{\mathscr{D}}(I_0,\tau)$ is a "sectorial neighbourhood of ∞" ([4] and [24]) with aperture $\breve{I}_0 =]-\frac{3}{2}\pi, +\frac{1}{2}\pi[$, while $\overset{\bullet}{\mathscr{D}}(I_1,\tau) = \mathrm{e}^{-\mathrm{i}\pi}\overset{\bullet}{\mathscr{D}}(I_0,\tau)$ is a "sectorial neighbourhood of ∞" with aperture $\breve{I}_1 =]-\frac{5}{2}\pi, -\frac{1}{2}\pi[$. These two open arcs provide a good covering $\{\breve{I}_0, \breve{I}_1\}$ of the circle of directions $\mathbb{S}^1$. We note $J_0 =]-\frac{1}{2}\pi, \frac{1}{2}\pi[$ and

$J_1 =]-\frac{3}{2}\pi, -\frac{1}{2}\pi[$ the two intersection arcs. Both $w_{tri,0}$ and $w_{tri,1}$ can be considered as defining sections of $\mathcal{A}_1$, namely $w_{tri,0} \in \Gamma(\check{I}_0, \mathcal{A}_1)$ and $w_{tri,1} \in \Gamma(\check{I}_1, \mathcal{A}_1)$, and are asymptotic to the same 1-Gevrey formal series $\widetilde{w}$. The pair $(w_{tri,0}, w_{tri,1})$ defines a 0-cochain in the sense of Čech cohomology, and the 1-coboundary $(w_{tri,0} - w_{tri,1}, w_{tri,1} - w_{tri,0})$ belongs to $\Gamma(J_0, \mathcal{A}^{\leq -1}) \times \Gamma(J_1, \mathcal{A}^{\leq -1})$.

2. For any $j \in \mathbb{Z}$ and $I_j = I_0 + j\pi =]0,\pi| + j\pi$, one can of course consider the Borel-Laplace sum $w_{tri,j} = \mathscr{S}^{I_j}\widetilde{w}$, which defines a holomorphic function on the domain $\dot{\mathscr{D}}(I_j, \tau)$, a sectorial neighbourhood of ∞ with aperture $\check{I}_j = \check{I}_0 - j\pi =]-\frac{3}{2}\pi, +\frac{1}{2}\pi[-j\pi$. Morever, for every $j \in \mathbb{Z}$,

$$w_{tri,j+2}(z) = w_{tri,j}(z) \text{ for } z \in \dot{\mathscr{D}}(I_j, \tau) \tag{3.34}$$

because $\widetilde{w} \in \mathbb{C}[[z^{-1}]]_1$.

3. We mentioned in proposition 3.1 that the formal series $\widetilde{w}(z)$ is even. One deduces that for any $\theta \in]0,\pi[$, for every $z \in \dot{\Pi}_\tau^{\pi-\theta}$

$$\mathscr{S}^{\pi-\theta}\,\widetilde{w}(z) = \mathscr{S}^{-\theta}\,\widetilde{w}(-z).$$

Therefore, for every $j \in \mathbb{Z}$,

$$\text{for every } z \in \dot{\mathscr{D}}(I_j, \tau),\ w_{tri,j}(z) = w_{tri,j+1}(-z). \tag{3.35}$$

4. We know by proposition 3.1 that $\widetilde{w}(z)$ actually belongs to $\mathbb{R}[[z^{-1}]]$. This has the following consequence : for any $\theta \in]0,\pi[$, for $z \in \dot{\Pi}_\tau^{\theta}$,

$$\overline{\mathscr{S}^{\theta}\,\widetilde{w}}(z) = \mathscr{S}^{-\theta}\,\widetilde{w}(\overline{z})$$

($\overline{a}$ stands for the complex conjugate of $a \in \mathbb{C}$). In other words, for any $j \in \mathbb{Z}$, the two functions $w_{tri,j}$ and $w_{tri,j+1}$ are complex conjugate,

$$\text{for every } z \in \dot{\mathscr{D}}(I_j, \tau),\ \overline{w_{tri,j}}(z) = w_{tri,j+1}(\overline{z}). \tag{3.36}$$

However, neither $w_{tri,0}$ nor $w_{tri,1}$ are real analytic functions, since this would mean that the 1-coboundary $w_{tri,0} - w_{tri,1}$ is zero which is not as we shall see later on.

5. The properties (3.35) and (3.36) have the following consequences: for every $j \in \mathbb{Z}$, $w_{tri,j}$ is " $\mathcal{PT}$-symmetric" [10, 11, 13], in the sense that for every $z \in \dot{\mathscr{D}}(I_j, \tau)$,

$$w_{tri,j}(z) = \overline{w_{tri,j}}(-\overline{z}). \tag{3.37}$$

In particular, for $r > 0$ large enough,

$$w_{tri,0}(r\mathrm{e}^{-\mathrm{i}\pi/2}) \in \mathbb{R}, \qquad w'_{tri,0}(r\mathrm{e}^{-\mathrm{i}\pi/2}) \in i\mathbb{R}. \tag{3.38}$$

6. By Stirling formula one has $N! \sim \sqrt{2\pi}N^{N+\frac{1}{2}}\mathrm{e}^{-N}$ for large N. Since for a given $z \neq 0$ the function $N \mapsto \dfrac{N^N \mathrm{e}^{-N}}{(r|z|)^N}$ reaches its minimal value at $n = r|z|$, it turns out from formula (3.32) that one can estimates the

value of $w_{tri,0}$ or $w_{tri,1}$ from the truncated series expansion $\displaystyle\sum_{k=0}^{N}\frac{a_k}{z^k}$ with $N=\big[r|z|\big]$ where $\big[.\big]$ is the entire part. This gives rise to the **summation to the least term**.

7. Along this state of mind, there are many ways of computing Borel-Laplace sums approximatively in practice (see, e.g., [14, 3]). Among them, one may quote the so-called **hyperasymptotic** methods [1] which have strong links with resurgence theory. These methods, originally arising from (and extending to) geometrical considerations on (multiple) singular integrals [22, 9, 8], can be applied to a wide class of problems stemming from applied mathematics and physics, see [19, 20, 21] and references therein. Other ways are available, for instance those based on the use of conformal mappings [2] with realistic upper bounds. It is also theoretically possible to calculate a 1-sum exactly by means of factorial series expansions [18, 12].

3.5.3 Tritruncated solutions

3.5.3.1 Tritruncated solutions

One can easily translate proposition 3.9 into properties for the first Painlevé equation (2.1). However, to cope with the Boutroux's transformations (2.6), (2.7), it is worth to work on the Riemann surface of the logarithm and we thus fix some notations.

Definition 3.10. We denote by $\underset{\bullet}{\mathbb{C}}$ the Riemann surface of the logarithm,

$$\underset{\bullet}{\mathbb{C}}=\{z=r\mathrm{e}^{\mathrm{i}\theta}\ |\ r>0,\ \theta\in\mathbb{R}\},\qquad \pi: z\in\underset{\bullet}{\mathbb{C}}\mapsto \overset{\bullet}{z}=r\mathrm{e}^{\mathrm{i}\theta}\in\mathbb{C}^{\star}.$$

For any $z=r\mathrm{e}^{\mathrm{i}\theta}\in\underset{\bullet}{\mathbb{C}}$, we refer to θ as to its argument, denoted by $\theta=\arg z$. We denote by $\underset{\bullet}{\mathbb{S}}^1$ (usually identified with $\mathbb{R}$) the set of directions of half-lines about 0 on $\underset{\bullet}{\mathbb{C}}$. We note $\dot{\pi}:\underset{\bullet}{\mathbb{S}}^1\to\mathbb{S}^1$ the natural projection ($\dot{\pi}=\pi|_{\underset{\bullet}{\mathbb{S}}^1}$), which makes $\underset{\bullet}{\mathbb{S}}^1$ an étalé space on $\mathbb{S}^1$ (and even a universal covering).

Definition 3.11. For a direction $\theta\in\underset{\bullet}{\mathbb{S}}^1$ and $\tau\in\mathbb{R}$, we define

$$\Pi_\tau^\theta=\{z=r\mathrm{e}^{\mathrm{i}\alpha}\in\underset{\bullet}{\mathbb{C}}\ |\ \alpha\in\check{\theta}\ \text{and}\ \pi(z)\in\overset{\bullet}{\Pi}{}_\tau^\theta\}.$$

For I an open arc of $\underset{\bullet}{\mathbb{S}}^1$ and $\gamma: I\to\mathbb{R}$ locally bounded, we note

$$\mathscr{D}(I,\gamma)=\bigcup_{\theta\in I}\Pi_{\gamma(\theta)}^\theta\subset\underset{\bullet}{\mathbb{C}}.$$

One calls $\mathscr{D}(I,\gamma)$ a **sectorial neighbourhood of** ∞ on $\underset{\bullet}{\mathbb{C}}$.

In order to define the transformations (2.6) and (2.7) properly, we introduce a conformal mapping:

Definition 3.12. The conformal mapping $\mathscr{T}$ is defined by:

$$\underset{\bullet}{\mathbb{C}} \xrightarrow{\mathscr{T}} \underset{\bullet}{\mathbb{C}}, \qquad z \mapsto x = \mathscr{T}(z) = \frac{30^{4/5}}{24}\mathrm{e}^{-\mathrm{i}\pi} z^{4/5}. \tag{3.39}$$

For I an open arc of $\underset{\bullet}{\mathbb{S}}^1$ and $\gamma : I \to \mathbb{R}$ locally bounded, the domain $\mathscr{D}(I,\gamma)$ is sent onto $\mathscr{T}\big(\mathscr{D}(I_j,\tau)\big) \subset \underset{\bullet}{\mathbb{C}}$ through the mapping $\mathscr{T}$, and we set

$$\mathcal{S}(I,\gamma) = \mathscr{T}\big(\mathscr{D}(I,\gamma)\big), \qquad \overset{\bullet}{\mathcal{S}}(I,\gamma) = \pi\Big(\mathcal{S}(I,\gamma)\Big). \tag{3.40}$$

We will consider the domains $\mathscr{D}(I_j,\tau)$, $j \in \mathbb{Z}$, for $I_j = I_0 + j\pi =]0,\pi| + j\pi$ and $\tau(\theta) = \dfrac{4.22}{|\sin(\theta)|}$. Notice that $\mathscr{D}(I_{j+1},\tau) = \mathrm{e}^{-\mathrm{i}\pi}\mathscr{D}(I_j,\tau)$ for any $j \in \mathbb{Z}$. The domain $\mathcal{S}(I_j,\tau)$ (see Fig. 3.3 and Fig. 3.4) is a sectorial neighbourhood of ∞ of aperture

$$K_j =]-\frac{11}{5}\pi, -\frac{3}{5}\pi[-\frac{4}{5}j\pi$$

and we may notice that, for any $j \in \mathbb{Z}$, $\mathcal{S}(I_{j+1},\tau) = \mathrm{e}^{-4\mathrm{i}\pi/5}\mathcal{S}(I_j,\tau)$. In particular, $\overset{\bullet}{\mathcal{S}}(I_{j+5},\tau) = \overset{\bullet}{\mathcal{S}}(I_j,\tau)$.
We now think of $w_{tri,j} = \mathscr{S}^{I_j}\widetilde{w}$ as a holomorphic function on $\mathscr{D}(I_j,\tau)$. By (3.35) and (3.37), these functions satisfy relationships: for any $j \in \mathbb{Z}$, for every $z \in \mathscr{D}(I_j,\tau)$,

$$\begin{aligned} w_{tri,j}(z) &= w_{tri,j+1}(z\mathrm{e}^{-\mathrm{i}\pi}), \\ \overline{w_{tri,j}}(z) &= w_{tri,j}(\overline{z}\mathrm{e}^{-(2j+1)\mathrm{i}\pi}), \end{aligned} \tag{3.41}$$

with the convention $\overline{z} = r\mathrm{e}^{-\mathrm{i}\alpha} \in \underset{\bullet}{\mathbb{C}}$ for $z = r\mathrm{e}^{\mathrm{i}\alpha} \in \underset{\bullet}{\mathbb{C}}$.

This gives sense without ambiguity to (3.4), (2.6) and (2.7) with the transformation

$$\begin{aligned} &z \in \mathscr{D}(I_j,\tau) \leftrightarrow x \in \mathcal{S}(I_j,\tau) \\ &w_{tri,j}(z) \leftrightarrow u_{tri,j}(x) = \frac{\mathrm{e}^{\mathrm{i}\pi/2}}{\sqrt{6}} x^{1/2}\left(1 - \frac{4}{25\big(\mathscr{T}^{-1}(x)\big)^2} + \frac{w_{tri,j}\big(\mathscr{T}^{-1}(x)\big)}{\big(\mathscr{T}^{-1}(x)\big)^2}\right). \end{aligned} \tag{3.42}$$

The functions $u_{tri,j}$ are solutions for the first Painlevé equation (2.1) and, by (3.41) and (3.42), they satisfy the following relationships: for any $j \in \mathbb{Z}$, for every $x \in \mathcal{S}(I_j,\tau)$,

$$\begin{aligned} u_{tri,j}(x) &= \mathrm{e}^{2\mathrm{i}\pi/5} u_{tri,j+1}(x\mathrm{e}^{-4\mathrm{i}\pi/5}), \\ \overline{u_{tri,j}}(x) &= \mathrm{e}^{\frac{2}{5}(2j+1)\mathrm{i}\pi} u_{tri,j}(\overline{x}\mathrm{e}^{-\frac{2}{5}(4j+7)\mathrm{i}\pi}), \end{aligned} \tag{3.43}$$

We recover here the symmetries discussed in Sect. 2.5.
By projection, $u_{tri,j}$ becomes a holomorphic function on the domain $\overset{\bullet}{\mathcal{S}}(I_j,\tau)$. This provides five distinct holomorphic functions $u_{tri,j}(x)$, $j = 0, \cdots, 4$, the so-called **tri-truncated solutions**.

Since $w_{tri,j}$ is a section on $\check{I}_j$ of $\mathcal{A}_1$, we deduce that the tritruncated solution $u_{tri,j}(x)$ belongs to the space of holomorphic functions with Gevrey asymptotic expansion of order $4/5$ (or equivalently of level $5/4$, see [16]) at infinity in $\overset{\bullet}{\mathcal{S}}_j$. One can thus recover $u_{tri,j}(x)$ by its asymptotics through 5/4-

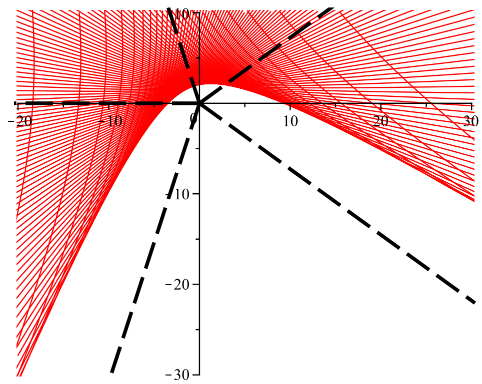


Fig. 3.3 The projection of the (shaded) domain $\mathcal{S}(I_0,\tau)$, image by the transformation (3.39), of the domain $\mathscr{D}(I_0,\tau)$ drawn on Fig. 3.2 for $\tau(\theta)=\dfrac{4.22}{|\sin(\theta)|}$. The dash lines recall the sectors (2.9).

summability.
It is also worth mentioning that, from property (3.43), $u_{tri,2}(x)$ is a real analytic function.

Proposition 3.10. *We note* $\overset{\bullet}{\mathcal{S}}(I_0,\tau)=\pi\Big(\mathscr{T}\big(\mathscr{D}(I_0,\tau)\big)\Big)$ *with* $\tau(\theta)=\dfrac{4.22}{|\sin(\theta)|}$ *and, for* $j=0,\cdots,4$,

$$\overset{\bullet}{\mathcal{S}}(I_j,\tau)=\omega_j^2\,\overset{\bullet}{\mathcal{S}}(I_0,\tau)\qquad \omega_j=\mathrm{e}^{-\frac{2i\pi}{5}j}.$$

The first Painlevé equation (2.1) has 5 tritruncated solutions $u_{tri,j}(x)$, $j=0,\cdots 4$.

The tri-truncated solution $u_{tri,j}(x)$ *is holomorphic in* $\overset{\bullet}{\mathcal{S}}(I_j,\tau)$, *a sectorial neighbourhood of* ∞ *of aperture* $K_j=]-\frac{11}{5}\pi,-\frac{3}{5}\pi[-\frac{4}{5}j\pi$, *and has in* $\overset{\bullet}{\mathcal{S}}(I_j,\tau)$ *a Gevrey asymptotic expansion of order 4/5 which determined* $u_{tri,j}(x)$ *uniquely. Moreover, for every* $x\in\overset{\bullet}{\mathcal{S}}(I_j,\tau)$,

$$u_{tri,j}(x)=\omega_j u_{tri,0}\big(\omega_j^{-2}x\big),\qquad \omega_j=\mathrm{e}^{-\frac{2i\pi}{5}j},\qquad j=0,\cdots,4,$$

and $u_{tri,2}$ *is a real analytic function.*

Remark 3.4. It is shown in exercise 3.3 that for any $j=0,\cdots,4$, the tri-truncated solution $u_{tri,j}$ can be analytically continued to the domain $\overset{\bullet}{\mathcal{S}}(I_j,\tau)$ with $\tau(\theta)=\dfrac{1.4}{|\sin(\theta)|}$. We will see later on that each tri-truncated solution $u_{tri,j}$ can be analytically continued to a wider domain than $\overset{\bullet}{\mathcal{S}}(I_j,\tau)$.

Exercices

3.1. We consider an ordinary differential equation of the form

$$P(\partial)w = G(z, w, w', ..., w^{(n-1)}) \tag{3.44}$$
$$P(\partial) = \sum_{m=0}^{n} \alpha_{n-m}\partial^m \in \mathbb{C}[\partial],\ \alpha_0 \neq 0,\ \alpha_n \neq 0$$

where $G(z, \boldsymbol{y})$ is holomorphic in a neighbourhood of $(z, \boldsymbol{y}) = (\infty, \boldsymbol{0}) \in \mathbb{C}\times\mathbb{C}^n$, $n \in \mathbb{N}^\star$. We furthermore suppose that

- $G(z, \boldsymbol{0}) = O(z^{-1})$,
- $\dfrac{\partial^{|\boldsymbol{l}|}G(z, \boldsymbol{0})}{\partial \boldsymbol{y}^{\boldsymbol{l}}} = O(z^{-1})$ when $|\boldsymbol{l}| = 1$.

1. Show that for every $M \in \mathbb{N}$ and up to making transformations of the type
$$w = \sum_{k=1}^{M} a_k z^{-k} + v, \tag{3.45}$$
one can instead assume that $G(z, \boldsymbol{0}) = O(z^{-M-1})$.
2. We thus suppose that for some $M \in \mathbb{N}^\star$, $G(z, \boldsymbol{y})$ is such that $G(z, \boldsymbol{0}) = O(z^{-1-M})$. Show that, up to making a (so called) shearing transformation of the form
$$w = z^{-M}v, \tag{3.46}$$
one can rather assume that
   - $G(z, \boldsymbol{0}) = O(z^{-1})$;
   - $\dfrac{\partial^{|\boldsymbol{l}|}G(z, \boldsymbol{0})}{\partial \boldsymbol{y}^{\boldsymbol{l}}} = O(z^{-1})$ when $|\boldsymbol{l}| = 1$;
   - $\dfrac{\partial^{|\boldsymbol{l}|}G(z, \boldsymbol{0})}{\partial \boldsymbol{y}^{\boldsymbol{l}}} = O(z^{-M(|\boldsymbol{l}|-1)})$ when $|\boldsymbol{l}| \geq 2$.
3. Deduce that, through transformations of the type (3.45) and (3.46), one can put equation (3.44) under the prepared form:
$$P(\partial)w + \frac{1}{z}Q(\partial)w = F(z, w, w', ..., w^{(n-1)}) \tag{3.47}$$
$$P(\partial) = \sum_{m=0}^{n} \alpha_{n-m}\partial^m \in \mathbb{C}[\partial]\ ,\ Q(\partial) = \sum_{m=0}^{n-1} \beta_{n-m}\partial^m \in \mathbb{C}[\partial]$$
where $F(z, \boldsymbol{y})$ is holomorphic in a neighbourhood of $(z, \boldsymbol{y}) = (\infty, \boldsymbol{0}) \in \mathbb{C} \times \mathbb{C}^n$ and such that
   - $F(z, \boldsymbol{0}) = O(z^{-2-M_0})$, $M_0 \in \mathbb{N}$;
   - $\dfrac{\partial^{|\mathbf{l}|}F(z, \boldsymbol{0})}{\partial \boldsymbol{y}^{\boldsymbol{l}}} = O(z^{-2})$ when $|\boldsymbol{l}| = 1$;
   - $\dfrac{\partial^{|\boldsymbol{l}|}F(z, \boldsymbol{0})}{\partial \boldsymbol{y}^{\boldsymbol{l}}} = O(z^{-2-M_{|\boldsymbol{l}|}})$, $M_{|\boldsymbol{l}|} \in \mathbb{N}$, when $|\boldsymbol{l}| \geq 2$.
4. Show that the shearing transform $w = z^{-M}v$, $M \in \mathbb{N}^\star$, transforms equation (3.47) into an equation of the form
$$P(\partial)v + \frac{1}{z}\left(Q(\partial) - MP'(\partial)\right)v = g(z, v, v', ..., v^{(n-1)}).$$

3.2. In this exercice we still consider the equation (3.10) and its unique solution $\widehat{w} \in \mathcal{O}(\overset{\bullet}{\mathscr{R}}^{(0)})$.

1. Show that, for every $\zeta \in \mathbb{C} \setminus \mathcal{D}_\rho^{(0)}$, one has $\dfrac{\max\{1, |\zeta|\}}{|P(-\zeta)|} \leq \dfrac{1}{\rho}$.
2. Show that, for any $0 < \rho < 1$, for any $\zeta = \xi e^{i\theta} \in \overset{\bullet}{\mathscr{R}}_\rho^{(0)}$, $\xi = |\zeta|$,
$$\rho|\widehat{w}(\zeta)| \leq \frac{392}{625} + 7\int_0^\xi |\widehat{w}(re^{i\theta})|\, dr + \frac{1}{2}\int_0^\xi |\widehat{w}^{*2}(re^{i\theta})|\, dr.$$
3. for any $0 < \rho < 1$, we consider the (unique) entire function $\widehat{\mathsf{w}}$ solution of the convolution equation
$$\rho\widehat{\mathsf{w}}(\xi) = \frac{392}{625} + 7 * \widehat{\mathsf{w}}(\xi) + \frac{1}{2} * \widehat{\mathsf{w}} * \widehat{\mathsf{w}}(\xi).$$
We note $\widetilde{\mathsf{w}}(z)$ the inverse Borel transform of $\widehat{\mathsf{w}}$.
Show that $\widetilde{\mathsf{w}}(z) \in \mathcal{O}\left(\overline{\left\{|z| > \dfrac{203}{25\rho}\right\}}\right)$. (Consider the discriminant locus).
Show that for $|z| > \dfrac{203}{25\rho}$, $\widetilde{\mathsf{w}}(z) = \dfrac{784}{625}\left((\rho z - 7) + \left((\rho z - 7)^2 - \dfrac{784}{625}\right)^{1/2}\right)^{-1}$,
$\widetilde{\mathsf{w}}(z) = O(z^{-1})$ at infinity, and that $|\widetilde{\mathsf{w}}(z)| \leq \dfrac{784}{625}\dfrac{1}{|\rho z - 7|} \leq \dfrac{28}{25}$.
4. Show that, for every $\xi \in \mathbb{C}$, $|\widehat{\mathsf{w}}(\xi)| \leq \dfrac{5684}{625\rho} e^{\frac{203}{25\rho}|\xi|}$.
5. Deduce that for every $0 < \rho < 1$ and for every $\zeta \in \overset{\bullet}{\mathscr{R}}_\rho^{(0)}$, $|\widehat{w}(\zeta)| \leq \dfrac{5684}{625\rho} e^{\frac{203}{25\rho}|\zeta|}$.

3.3. In this exercice we consider the ODE
$$y'' + \frac{y'}{z} - y = \frac{392}{625} z^{-4} + \frac{1}{2} y^2. \tag{3.48}$$
deduced from (3.2) by the transformation $v(z) = 1 - \dfrac{4}{25z^2} + y(z)$ or, from (3.6) through the transformation $y(z) = z^{-2}w(z)$. In particular there exists a unique formal series $\widetilde{y}(z) = z^{-2}\widetilde{w}(z) \in \mathbb{C}[[z^{-1}]]$ solution of (3.48). We thus know that the formal Borel transform $\widehat{y}$ belongs to $\mathcal{MO}(\overset{\bullet}{\mathscr{R}}^{(0)})$ and satisfies the convolution equation associated with (3.48) by formal Borel transformation:
$$(\zeta^2 - 1)\widehat{y} - 1 * (\zeta\widehat{y}) = \frac{392}{625}\frac{\zeta^3}{\Gamma(4)} + \frac{1}{2}\widehat{y} * \widehat{y}. \tag{3.49}$$

1. Assume that $f \in \mathbb{C}\{\zeta\}$ with $f(0) = 0$. Show that the solutions $g \in \mathbb{C}\{\zeta\}$ of the convolution equation
$$(\zeta^2 - 1)g - 1 * (\zeta g) = f$$
are given by
$$g(\zeta) = \frac{C}{(1-\zeta^2)^{1/2}} - \frac{f(\zeta)}{1-\zeta^2} + \frac{1}{(1-\zeta^2)^{1/2}}\int_0^\zeta \frac{\eta}{(1-\eta^2)^{3/2}} f(\eta)\, d\eta, \qquad C \in \mathbb{C}.$$

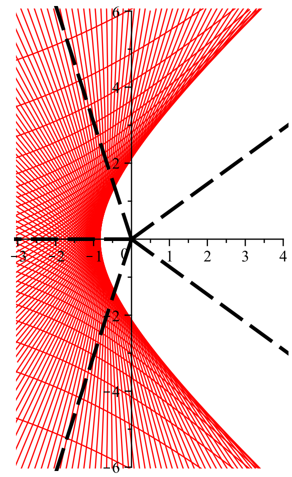


Fig. 3.4 The projection of the (shaded) domain $\mathcal{S}_2(I_2,\tau)$, image of the domain $\mathscr{D}(I_2,\tau)$ by the conformal mapping (3.39), for $\tau(\theta) = \dfrac{1.4}{|\sin(\theta)|}$. The dash lines recall the sectors (2.9).

(Hint : take $g(\zeta) = \frac{G(\zeta)}{1-\zeta^2}$, differentiate the convolution equation to obtain a nonhomogeneous linear differential equation of order 1, and solve this equation).

2. Show that $\widehat{y}$ satisfies the convolution equation (3.49) in $\mathcal{MO}(\overset{\bullet}{\mathscr{R}}^{(0)})$ if and only if $\widehat{y}$ satisfies the following fixed-point problem:

$$\widehat{y} = \mathscr{P}\left(\frac{392}{625}\frac{\zeta^3}{\Gamma(4)}\right) + \frac{1}{2}\mathscr{P}\big(\widehat{y} * \widehat{y}\big) \qquad \text{with}$$
$$\big(\mathscr{P}g\big)(\zeta) = -\frac{g(\zeta)}{1-\zeta^2} + \frac{1}{(1-\zeta^2)^{1/2}}\int_0^\zeta \frac{\eta}{(1-\eta^2)^{3/2}} g(\eta)\, d\eta, \tag{3.50}$$

3. Show that for any $0 < \rho < 1$ and for any $\zeta \in \overset{\bullet}{\mathscr{R}}_\rho^{(0)}$ one has $\dfrac{\max\{1, |\zeta|\}}{|P(-\zeta)|} \leq \dfrac{1}{\rho}$ and $\left|\dfrac{\zeta}{(1-\zeta^2)^{3/2}}\right| \leq \dfrac{1}{\rho^{3/2}}$.

4. Show that for any $0 < \rho < 1$ and for any $\zeta \in \overset{\bullet}{\mathscr{R}}_\rho^{(0)}$, $|\widehat{y}(\zeta)| \leq \widehat{Y}(\xi)$ with $\xi = |\zeta|$, where $\widehat{Y}$ is an entire function that solves the fixed-point problem:

$$\widehat{Y} = \mathscr{Q}\left(\frac{392}{625}\frac{\xi^3}{\Gamma(4)}\right) + \frac{1}{2}\mathscr{Q}\big(\widehat{Y} * \widehat{Y}\big) \tag{3.51}$$
$$\big(\mathscr{Q}G\big)(\xi) = \frac{G(\xi)}{\rho} + \frac{1}{\rho^2}\Big(1 * G\Big)(\xi)$$

5. For any $0 < \rho < 1$, we note $\widetilde{Y}(z)$ the inverse Borel transform of $\widehat{Y}$. Show that $\widetilde{Y}(z)$ satisfies the algebraic equation

$$\rho\widetilde{Y} = \left(\frac{392}{625}\frac{1}{z^4} + \frac{1}{2}\widetilde{Y}^2\right)\left(1 + \frac{1}{\rho z}\right), \qquad \widetilde{Y}(z) = \frac{392}{625}\frac{1}{\rho z^4} + O(z^{-5}). \tag{3.52}$$

6. Show that the fixed-point problem (3.52) has a unique solution in $B_{\rho^3/2} = \{H \in O(\overline{U}),\ \|H\| \leq \dfrac{\rho^3}{2}\}$, for $U = D(\infty, \dfrac{\rho}{1.4})$.

7. Deduce that the minor $\widehat{y}$ of the formal series $\widetilde{y}$ solution of equation (3.48) is holomorphic on $\overset{\bullet}{\mathscr{R}}^{(0)}$ and that, for any $0 < \rho < 1$, for every $\zeta \in \overset{\bullet}{\mathscr{R}}^{(0)}_{\rho}$, one has
$$|\widehat{y}(\zeta)| \leq 0.7\rho^2 \mathrm{e}^{\frac{1.4}{\rho}|\zeta|}. \tag{3.53}$$
8. Deduce that for any $j \in \mathbb{Z}$ and $I_j = I_0 + j\pi =]0,\pi| + j\pi$, the Borel-Laplace sum $y_{tri,j} = \mathscr{S}^{I_j}\widetilde{y}$ defines a holomorphic function on the domain $\overset{\bullet}{\mathscr{D}}(I_j,\tau)$ with $\tau(\theta) = \dfrac{1.4}{|\sin(\theta)|}$.
9. Deduce that the tri-truncated solution $u_{tri,j}$, $j \in \mathbb{Z}$, is holomorphic on the domain $\mathcal{S}(I_j,\tau) = \mathscr{T}\big(\mathscr{D}(I_j,\tau)\big)$ with $\tau(\theta) = \dfrac{1.4}{|\sin(\theta)|}$. See Fig. 3.4.

3.4. We consider the domain $\overset{\bullet}{\mathscr{D}}(]0,\pi[,\tau)$ for $\tau(\theta) = \dfrac{\lambda}{\sin(\theta)}$, $\lambda > 0$. We want to describe the boundary $\partial\overset{\bullet}{\mathscr{D}}(]0,\pi[,\tau)$ of this domain.

1. show that $\partial\overset{\bullet}{\mathscr{D}}(]0,\pi[,\tau)$ is the envelope of the following family of line curves:
$$z = x + \mathrm{i}y, \qquad x\cos(\theta) - y\sin(\theta) = \frac{\lambda}{\sin(\theta)}, \qquad \theta \in]0,\pi[.$$
2. Deduce that $\partial\overset{\bullet}{\mathscr{D}}(]0,\pi[,\tau)$ is the parabolic curve of equation $y = \dfrac{x^2}{4\lambda} - \lambda$.

References

1. M.V. Berry, C.J. Howls, *Hyperasymptotics for integrals with saddles.* Proc. Roy. Soc. Lond. A **434** (1991), 657-675.
2. W. Balser, D.A. Lutz, R. Schäfke, *On the convergence of Borel approximants.* J. Dynam. Control Systems **8** (2002), no. 1, 65-92
3. E. Caliceti, M. Meyer-Hermann, P. Ribeca, A. Surzhykov, U.D. Jentschura, U. D. *From useful algorithms for slowly convergent series to physical predictions based on divergent perturbative expansions.* Phys. Rep. **446** (2007), no. 1-3, 1-96.
4. B. Candelpergher, C. Nosmas, F. Pham, *Approche de la résurgence*, Actualités mathématiques, Hermann, Paris (1993).
5. B. Candelpergher, C. Nosmas, F. Pham, *Premiers pas en calcul étranger*, Ann. Inst. Fourier (Grenoble) **43** (1993) 201-224.
6. O. Costin, *On Borel summation and Stokes phenomena for rank-*1 *nonlinear systems of ordinary differential equations.* Duke Math. J. **93** (1998), no. 2, 289-344.
7. O. Costin, *Asymptotics and Borel summability*, Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, 141. CRC Press, Boca Raton, FL, 2009.
8. E. Delabaere, *Singular integrals and the stationary phase methods. Algebraic approach to differential equations*, 136-209, World Sci. Publ., Hackensack, NJ, 2010.
9. E. Delabaere, C. J. Howls, *Global asymptotics for multiple integrals with boundaries.* Duke Math. J. **112** (2002), 2, 199-264.
10. E. Delabaere, F. Pham, *Eigenvalues of complex Hamiltonians with PT-symmetry*, Phys. Lett. A **250** (1998), no. 1-3, 25-32.
11. E. Delabaere, F. Pham, *Resurgent methods in semi-classical asymptotics*, Ann. Inst. Henri Poincaré, Sect. A **71** (1999), no 1, 1-94.
12. E. Delabaere, J.-M. Rasoamanana, *Sommation effective d'une somme de Borel par séries de factorielles.* Ann. Inst. Fourier **57** (2007), no. 2, 421-456.
13. E. Delabaere, D.T. Trinh, *Spectral analysis of the complex cubic oscillator,* J.Phys. A: Math. Gen. **33** (2000), no. 48, 8771-8796.

14. U. Jentschura, Habilitation Thesis, Dresden University of Technology, 3rd edition (2004).
15. N. Joshi, A. V. Kitaev, *On Boutroux's tritronquée solutions of the first Painlevé equation.* Stud. Appl. Math. **107** (2001), no. 3, 253-291.
16. M. Loday-Richaud, *Divergent series and differential equations.* To appear.
17. M. Loday-Richaud, P. Remy, *Resurgence, Stokes phenomenon and alien derivatives for level-one linear differential systems.* J. Differential Equations **250** (2011), no. 3, 1591-1630.
18. B. Malgrange, *Sommation des séries divergentes.* Expositiones Mathematicae **13**, 163-222 (1995).
19. A.B. Olde Daalhuis, *Hyperterminants I.* J. Comput. Appl. Math. **76** (1996), 255-264.
20. A.B. Olde Daalhuis, *Hyperterminants II.* J. Comput. Appl. Math. **89** (1997), 87-95.
21. A.B. Olde Daalhuis, *Hyperasymptotics for nonlinear ODEs. II. The first Painlevé equation and a second-order Riccati equation.* Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **461** (2005), no. 2062, 3005-3021.
22. F. Pham, *Intégrales singulières.* Savoirs Actuels (Les Ulis). EDP Sciences, Les Ulis; CNRS Editions, Paris, 2005
23. J.-P. Ramis, Y. Sibuya, Y. *Hukuhara domains and fundamental existence and uniqueness theorems for asymptotic solutions of Gevrey type.* Asymptotic Anal. 2 (1989), no. 1, 39-94.
24. D. Sauzin, *Introduction to 1-summability and the resurgence theory.* To appear.
25. F. Trèves, *Basic linear partial differential equations.*, Pure and Applied Mathematics, Vol. 62. Academic Press, New York-London, 1975.
26. W. Wasow, *Asymptotic expansions for ODE.* Reprint of the 1965 edition. Robert E. Krieger Publishing Co., Huntington, N.Y., 1976.

Chapter 4
Beyond 1-summability

Abstract We have shown that the minor $\widehat{w}$ of the unique formal series solution $\widetilde{w}$ of the prepared ODE associated with the first Painlevé equation, defines a holomorphic function on a convenient star-shaped domain. We further analyze the analytic properties of $\widehat{w}$. We show in Sect. 4.4 how $\widehat{w}$ can be analytically continued onto a wider domain of a Riemann surface that we define in Sect. 4.1. This question is related to the problem of "mastering" the analytic continuations of a convolution product and, as a byproduct, of getting upper bounds for on any compact set. This is what we will (partly) do in Sect. 4.2 and Sect. 4.3, using only elementary geometric arguments.

4.1 Riemann surface and sheets

This section is devoted to defining the Riemann surface $\mathscr{R} = \mathscr{R}_{\mathbb{Z}}$ and some of its sheets. We do that in a way at first sight artificially complicated, but needed to state one of the main results of this chapter, namely proposition 4.2 and its consequences developed in Sect. 4.3.

4.1.1 Riemann surface

4.1.1.1 About paths

In what follows, a path λ in a topological space X is any continuous function $\lambda : [a, a+l] \to X$, where $[a, a+l] \subset \mathbb{R}$ is a (compact) interval possibly reduced to $\{a\}$. We often work with standard paths, that is paths defined on $[0,1]$. The path $\underline{\lambda} : t \in [0,1] \mapsto \lambda(a+tl)$ is the standardized path of λ. For two paths $\lambda_1 : [a, a+l] \to X$, $\lambda_2 : [b, b+k] \to X$ so that $\lambda_1(a+l) = \lambda_2(b)$, one defines their product (or concatenation) by

$$\lambda_1\lambda_2 : t \in [a, a+l+k] \mapsto \begin{cases} \lambda_1(t),\ t \in [a, a+l] \\ \lambda_2(t-a-l+b),\ t \in [a+l, a+l+k] \end{cases}$$

When the two paths λ_1, λ_2 have same extremities, they are homotopic when there exists a continuous map $H : [0,1] \times [0,1] \to X$ that realizes a homotopy between the standardized paths $\underline{\lambda}_1$ and $\underline{\lambda}_2$.

Sometimes one needs to use regular paths. We recall that any path can be uniformaly approached by $\mathcal{C}^\infty$-paths. For a piecewise $\mathcal{C}^1$-path λ, we denote its length by $\mathrm{length}(\lambda) = \displaystyle\int_0^1 |\underline{\lambda}'(t)|dt$.

4.1.1.2 Riemann surface

Definition 4.1. We note $\mathfrak{R} = \mathfrak{R}_0$ the set of paths λ that satisfies the condition: there exists $t_0 \in [0,1]$ so that $\underline{\lambda}([0,t_0]) = \{0\}$ and $\underline{\lambda}(]t_0,1]) \subset \mathbb{C} \setminus \mathbb{Z}$. For $\lambda \in \mathfrak{R}$, we note $\mathrm{cl}(\lambda)$ its equivalence class for the relation of homotopy $\sim_{\mathfrak{R}}$ of paths in $\mathfrak{R}$ with fixed extremities. We define

$$\mathscr{R} = \mathscr{R}_{\mathbb{Z}} = \{\zeta = \mathrm{cl}(\lambda) \mid \lambda \in \mathfrak{R}\} \text{ and } \mathfrak{p} : \zeta = \mathrm{cl}(\lambda) \mapsto \overset{\bullet}{\zeta} = \underline{\lambda}(1) \in \overset{\bullet}{\mathscr{R}}$$

where $\overset{\bullet}{\mathscr{R}} = \mathbb{C} \setminus \mathbb{Z}^\star$.

We precise the relation of homotopy $\sim_{\mathfrak{R}}$. For two paths λ_0, λ_1 ending at the same point, $\lambda_0 \sim_{\mathfrak{R}} \lambda_1$ if there is a homotopy $H : (s,t) \in [0,1] \times [0,1] \mapsto H_t(s) \in \mathbb{C}$ so that $H_0 = \underline{\lambda}_0$, $H_1 = \underline{\lambda}_1$ and $H_t \in \mathfrak{R}$ for every $t \in [0,1]$.

Notice that the origin plays a singular role in this definition. In particular, $\mathfrak{p}^{-1}(0)$ is reduced to a single point $0 = \mathrm{cl}(0)$. This is why one usually considers $(\mathscr{R}, 0)$ as a pointed space.

The space $\mathscr{R}$ shares many characteristics with $\widetilde{\mathbb{C} \setminus \mathbb{Z}}$, the universal covering of $\mathbb{C} \setminus \mathbb{Z}$ (see, e.g., [7, 3, 11]). In particular, by classical arguments, one can endow $\mathscr{R}$ with a separated topology, a basis $\mathscr{B} = \{\mathscr{U}\}$ of open sets defining this topology being given as follows. Let us consider $\zeta \in \mathscr{R}$:

- assume that $\zeta = 0$. For $\overset{\bullet}{\mathscr{U}} \subset \overset{\bullet}{\mathscr{R}}$ a connected and simply connected neighbourhood of 0, one defines $\mathscr{U} \subset \mathscr{R}$ as the set of all $\xi = \mathrm{cl}(\lambda)$ such that λ is any path of $\mathfrak{R}$ contained in $\overset{\bullet}{\mathscr{U}}$ and ending at $\overset{\bullet}{\xi}$. Notice that ξ is well-defined since $\overset{\bullet}{\mathscr{U}}$ is simply connected.
- assume that $\zeta \neq 0$. If $\overset{\bullet}{\mathscr{U}} \subset \overset{\bullet}{\mathscr{R}} \setminus \{0\}$ is a connected and simply connected neighbourhood of $\overset{\bullet}{\zeta}$, one defines $\mathscr{U} \subset \mathscr{R}$ as the set of all $\xi = \mathrm{cl}(\lambda_1\lambda_2)$ where $\lambda_1 \in \mathfrak{R}$ is so that $\zeta = \mathrm{cl}(\lambda_1)$, while λ_2 is a path starting from $\overset{\bullet}{\zeta}$, ending at $\overset{\bullet}{\xi}$ and contained in $\overset{\bullet}{\mathscr{U}}$.

With this topology, it is straightforward to see that the projection $\mathfrak{p}$ is a continuous mapping and, even, a local homeomorphism : for every $\mathscr{U} \in \mathscr{B}$, the mapping $\mathfrak{p}_{|\mathscr{U}} \to \overset{\bullet}{\mathscr{U}}$ is a homeomorphism. One eventually gets the following proposition.

Proposition 4.1. *The pointed space $(\mathscr{R}, 0)$ is a topologically separated space, arcconnected and simply connected. The projection $\mathfrak{p}$ makes $\mathscr{R}$ an étalé space on $\overset{\bullet}{\mathscr{R}}$. By pulling back by $\mathfrak{p}$ the complex structure of $\mathbb{C}$, the space $\mathscr{R}$ becomes a Riemann surface.*

By "étalé space", we mean that $\mathfrak{p} : \mathscr{R} \to \overset{\bullet}{\mathscr{R}}$ is a local homeomorphism. Notice that $\mathfrak{p}$ is not a covering map since the curve lifting property [7, 3] is not satisfied. For instance, as a rule, a path starting from and ending at 0 cannot be lifted from 0 on $\mathscr{R}$ with respect to $\mathfrak{p}$.

We precise the "pull back" of the complex structure. If $\mathscr{U}_1, \mathscr{U}_2, \mathscr{U}_1 \cap \mathscr{U}_2 \neq \emptyset$ are two open sets of $\mathscr{R}$ such that the mappings $\mathfrak{p}|_{\mathscr{U}_1} : \mathscr{U}_1 \to \mathfrak{p}(\mathscr{U}_1)$ and $\mathfrak{p}|_{\mathscr{U}_2} : \mathscr{U}_2 \to \mathfrak{p}(\mathscr{U}_2)$ are two homeomorphisms, then the chart transition $\mathfrak{p}|_{\mathscr{U}_2} \circ \mathfrak{p}|_{\mathscr{U}_1}^{-1} : \mathfrak{p}(\mathscr{U}_1 \cap \mathscr{U}_2) \to \mathfrak{p}(\mathscr{U}_1 \cap \mathscr{U}_2)$ is nothing but the identity map, thus is biholomorphic. This makes $\mathscr{R}$ a Riemann surface, that is a connected one-dimensional complex manifold [7, 3].

Definition 4.2. We note $\hat{\mathscr{R}} = \hat{\mathscr{R}}_{\mathbb{Z}}$ the space of germs $\widehat{\varphi}$ of analytic functions at the origin that satisfies the property : there exists a neighbourhood $\mathscr{U} \in \mathscr{B}$ of 0 such that the mapping $\Phi : \zeta \in \mathscr{U} \subset \mathscr{R} \mapsto \Phi(\zeta) = \widehat{\varphi}(\overset{\bullet}{\zeta}) \in \mathbb{C}$ can be analytically continued to $\mathscr{R}$.

4.1.2 Sheets of the Riemann surface

4.1.2.1 Principal sheet

By the very construction of the Riemann surface $\mathscr{R}$, there exists a unique open set $\mathscr{R}^{(0)}$ of $\mathscr{R}$ so that $\mathfrak{p}|_{\mathscr{R}^{(0)}}$ realises a homeomorphism between $\mathscr{R}^{(0)}$ and the simply connected domain $\overset{\bullet}{\mathscr{R}}{}^{(0)}$. This open set $\mathscr{R}^{(0)}$ is made of all the classes $\mathrm{cl}(\lambda)$ of paths λ that are homotopic to segments $[0, \overset{\bullet}{\zeta}]$, with $\overset{\bullet}{\zeta} \in \overset{\bullet}{\mathscr{R}}{}^{(0)}$.

Definition 4.3. One refers to $\mathscr{R}^{(0)}$ as to the **principal sheet** of the pointed Riemann surface $(\mathscr{R}, 0)$.
For every $0 < \rho < 1$, one defines $\mathscr{R}_\rho^{(0)}$ as the unique open subset of $\mathscr{R}^{(0)}$ such that $\mathfrak{p}(\mathscr{R}_\rho^{(0)}) = \overset{\bullet}{\mathscr{R}}{}_\rho^{(0)}$. (See Fig. 4.1).

4.1.2.2 Other sheets

Definition 4.4. Let be $m \in \mathbb{N}^\star$, $\boldsymbol{\varepsilon} = (\varepsilon_1, \cdots, \varepsilon_{m-1}) \in \{+, -\}^{m-1}$ a sequence of $m-1$ signs and $\boldsymbol{n} = (n_1, \cdots, n_{m-1}) \in (\mathbb{N}^\star)^{m-1}$ a sequence of positive integers. Let be $\theta_1 \in \{0, \pi\} \in \mathbb{S}^1$ a given direction.
When $m = 1$, one says that the path $\gamma \in \mathfrak{R}$ is of type $\gamma_{()}^{\theta_1}$ when γ closely follows the segment $\mathrm{e}^{\mathrm{i}\theta_1}]0, 1[=]0, \omega_1[$ toward $\omega_1 = \mathrm{e}^{\mathrm{i}\theta_1}$.
Otherwise, for $m \geq 2$, on says that the $\gamma \in \mathfrak{R}$ is of type type $\gamma_{\boldsymbol{\varepsilon}^{\boldsymbol{n}}}^{\theta_1}$ if γ connects the segment $]0, \omega_1[$ to the segment $]\omega_{m-1}, \omega_m[$, $\omega_m - \omega_{m-1} = \mathrm{e}^{\mathrm{i}\theta_m}$, through the following steps:

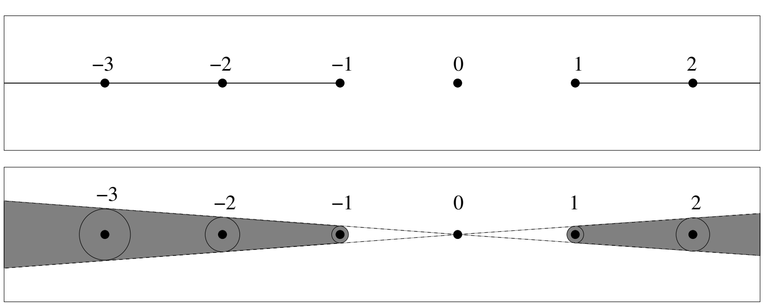


Fig. 4.1 Above, the domain $\overset{\bullet}{\mathscr{R}}{}^{(0)}$. Below, the domain $\overset{\bullet}{\mathscr{R}}{}_\rho^{(0)}$.

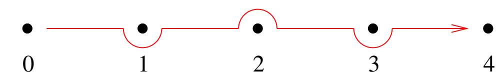


Fig. 4.2 A path of type $\gamma_{\boldsymbol{\varepsilon}}^{\theta}$ for $\boldsymbol{\varepsilon} = (+,-,+)$ and $\theta = 0$.

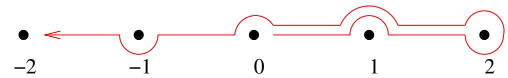


Fig. 4.3 A path of type $\gamma_{\boldsymbol{\varepsilon}^{\boldsymbol{n}}}^{\theta}$ for $\theta = 0$, $\boldsymbol{\varepsilon} = (-,+,+,+,-)$ and $\boldsymbol{n} = (1,2,1,1,1)$.

- γ closely follows the segment $]0,\omega_1[$ toward the direction θ_1, makes n_1 half-turns around the point ω_1, anti clockwise when $\varepsilon_1 = +$, clockwise when $\varepsilon_1 = -1$, and finally closely follows the segment $]\omega_1,\omega_2[$, $\omega_2 - \omega_1 = \mathrm{e}^{\mathrm{i}\theta_2}$, toward the direction $\theta_2 = \theta_1 + \varepsilon_1(n_1-1)\pi$;
- then, successively for $k = 2,\cdots,m-1$, γ makes n_k half-turns around the point ω_k, anti clockwise when $\varepsilon_k = +$, clockwise when $\varepsilon_k = -1$, and eventually closely follows the segment $]\omega_k,\omega_{k+1}[$, $\omega_{k+1} - \omega_k = \mathrm{e}^{\mathrm{i}\theta_{k+1}}$, toward the direction $\theta_{k+1} = \theta_k + \varepsilon_k(n_k-1)\pi$.

When $\boldsymbol{n} = (1,\cdots,1) \in \{1\}^{m-1}$, we simply say that $\gamma \in \mathfrak{R}$ is of type $\gamma_{\boldsymbol{\varepsilon}}^{\theta_1}$. (See Fig. 4.2 and Fig. 4.3).

For instance, if γ is of type $\gamma_{\boldsymbol{\varepsilon}}^{\theta}$, then someone standing at $0 \in \mathbb{C}$ and looking in the direction of the half-line $]0,\mathrm{e}^{\mathrm{i}\theta}\infty[$ will see the path γ avoiding the point $\omega_n = n\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C}^\star$ by swerving in the direction of his right hand when $\varepsilon_n = +$, of his left hand when $\varepsilon_n = -$.

Definition 4.5. For $m \in \mathbb{N}^\star$, $\boldsymbol{\varepsilon} \in \{+,-\}^m$, $\boldsymbol{n} \in (\mathbb{N}^\star)^m$ and a direction $\theta \in \{0,\pi\}$, we denote by $\mathscr{R}^{\boldsymbol{\varepsilon}^{\boldsymbol{n}},\theta}$ the sheet defined as the domain of $\mathscr{R}$ made of points $\zeta = \mathrm{cl}(\gamma\lambda)$, where γ is a path of type $\gamma_{\boldsymbol{\varepsilon}^{\boldsymbol{n}}}^{\theta}$ ending at $\overset{\bullet}{\xi} \in]p,(p+1)[=]\omega_m,\omega_{m+1}[$, and λ is a path starting from $\overset{\bullet}{\zeta}$, and contained in the simply connected domain $\mathbb{C} \setminus \{]-\infty,p] \cup [p+1,+\infty[\}$, star-shaped from $\overset{\bullet}{\xi}$. When $\boldsymbol{n} = (1,\cdots,1) \in \{1\}^m$, we simply write $\mathscr{R}^{\boldsymbol{\varepsilon},\theta} = \mathscr{R}^{\boldsymbol{\varepsilon}^{\boldsymbol{n}},\theta}$.

The set of sheets $\{\mathscr{R}^{(0)}, \mathscr{R}^{\boldsymbol{\varepsilon}^{\boldsymbol{n}},\theta}\}$ provides an open covering of $\mathscr{R}$, with the following property: the restriction $\mathfrak{p}|_{\mathscr{R}^{\boldsymbol{\varepsilon}^{\boldsymbol{n}},\theta}}$ realises a homeomorphism between $\mathscr{R}^{\boldsymbol{\varepsilon}^{\boldsymbol{n}},\theta}$ and the simply connected domain $\mathbb{C} \setminus \{]-\infty,p] \cup [p+1,+\infty[\}$ where $]p,(p+1)[=]\omega_m,\omega_{m+1}[$, with ω_m,ω_{m+1} as given by definition 4.4.

Remark that for every $\theta \in \{0,\pi\}$, for every $m \in \mathbb{N}^\star$ and for every $\boldsymbol{\varepsilon} \in \{+\}^m$ or $\boldsymbol{\varepsilon} \in \{-\}^m$, $\mathscr{R}^{(0)}$ and $\mathscr{R}^{\boldsymbol{\varepsilon},\theta}$ have a non-empty intersection (a half-plane on projection). This justifies the following definitions.

Definition 4.6. For $m \in \mathbb{N}^\star$, we define $(+)_{m-1} = (+,\cdots,+) \in \{+\}^{m-1}$ and $(-)_{m-1} = (-,\cdots,-) \in \{-\}^{m-1}$. We denote by $(\pm)_{m-1}$ any $(m-1)$-tuple of the form $(\pm,\cdots,\pm) \in \{+,-\}^{m-1}$. Also, $(+)_0 = (-)_0 = (\pm)_0 = ()$ is the 0-tuple.

Thus, the set of all $(\pm)_m$ is made of 2^m elements.

Definition 4.7. One says that $\mathscr{R}^{\boldsymbol{\varepsilon},\theta}$ is a $\mathscr{R}^{(0)}$**-nearby sheet** when $\boldsymbol{\varepsilon} \in \bigcup_{m\in\mathbb{N}^\star} \{(+)_m,(-)_m\}$. One denotes by $\mathscr{R}^{(1)}$ the domain of $\mathscr{R}$ defined as the union of the principal sheet and the collection of nearby sheets:

$$\mathscr{R}^{(1)} = \mathscr{R}^{(0)} \bigcup_{\theta\in\{0,\pi\},\, m\in\mathbb{N}^\star} \mathscr{R}^{(+)_m,\theta} \cup \mathscr{R}^{(-)_m,\theta}.$$

More generally, for any $k \in \mathbb{N}^\star$, one defines

$$\mathscr{R}^{(k+1)} = \mathscr{R}^{(k)} \bigcup_{\substack{\theta \in \{0,\pi\}, m \in \mathbb{N}^\star \\ \boldsymbol{n} \in (\mathbb{N}^\star)^k}} \mathscr{R}^{((\pm)_k^{\boldsymbol{n}},(+)_{m-1}),\theta} \cup \mathscr{R}^{((\pm)_k^{\boldsymbol{n}},(-)_{m-1}),\theta}.$$

Remark 4.1. Notice that

$$\mathfrak{p}(\mathscr{R}^{(+)_m,\theta}) = \mathfrak{p}(\mathscr{R}^{(-)_m,\theta}) = \mathbb{C} \setminus \mathrm{e}^{\mathrm{i}\theta}\{]-\infty, m] \cup [m+1, +\infty[\}$$

and that $\bigcup_k \mathscr{R}^{(k)} = \mathscr{R}$.

Remark also that the $\mathscr{R}^{(k)}$ are Riemann surfaces since they inherit from $\mathscr{R}$ the structure of complex manifold and since they are open connected. We will not be concerned by the Riemann surfaces $\mathscr{R}^{(k)}$, $k \geq 2$, until Chapt. 7.

4.1.3 Nearby sheets

Our aim in this subsection is to introduce other sheets of the Riemann surface, that will be convenient for our purpose.

We start with the following remark: for $0 < \rho < 1$ and $m \in \mathbb{N}^\star$, the closed discs $\overline{D}(m, m\rho)$ and $\overline{D}(m+1, (m+1)\rho)$ are disjoint as soon as $m < \dfrac{\rho^{-1}-1}{2}$. Thus, now assuming that $0 < \rho \leq \dfrac{1}{5}$ and introducing the integer part $\mathscr{M}(\rho) + 1 = \lfloor \frac{\rho^{-1}-1}{2} \rfloor \geq 2$ ($\lfloor . \rfloor$ is the floor function), one observes that the discs $\overline{D}(m, |m|\rho)$ do not overlap when with $|m| \leq \mathscr{M}(\rho) + 1$.

Definition 4.8. We assume that $0 < \rho \leq \dfrac{1}{5}$ and we note $\mathscr{M}(\rho) = \lfloor \dfrac{\rho^{-1}-1}{2} \rfloor - 1$, $\mathscr{M}(\rho) \in \mathbb{N}^\star$. For $|m| \leq \mathscr{M}(\rho) + 1$ and $m \neq 0$, we note $\overline{D}_m = \overline{D}(m, |m|\rho)$ the closed disc centered at m with radius $|m|\rho$. We note $\overline{D}_0 = \{0\}$.
For $\theta \in \{0, \pi\}$, we denote by $\overline{\mathcal{D}}_\rho^\theta \subset \mathbb{C}$ the closed subset defined by

$$\overline{\mathcal{D}}_\rho^\theta = \left\{ t\zeta \;\mid\; t \in [1, +\infty[,\, \zeta \in \overline{D}_{\mathrm{e}^{\mathrm{i}\theta}(M+1)} \right\} \bigcup_{0 \leq m \leq \mathscr{M}(\rho)} \overline{D}_{\mathrm{e}^{\mathrm{i}\theta} m}.$$

We define the domains $\overset{\bullet}{\mathscr{P}}{}_\rho^\theta = \mathbb{C} \setminus \overline{\mathcal{D}}_\rho^\theta$ and $\overset{\bullet}{\mathscr{R}}_\rho = \left(\overset{\bullet}{\mathscr{P}}{}_\rho^0 \cap \overset{\bullet}{\mathscr{P}}{}_\rho^\pi \right) \cup \{0\}$ (see Fig. 4.4).

Remark 4.2. Notice that $\overset{\bullet}{\mathscr{R}} = \bigcup_{0 < \rho \leq 1/5} \overset{\bullet}{\mathscr{R}}_\rho$.

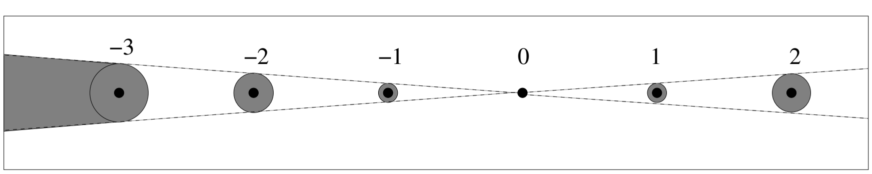


Fig. 4.4 The domain $\overset{\bullet}{\mathscr{R}}_\rho$ when $\dfrac{1}{9} < \rho \leq \dfrac{1}{7}$ (the scale is not correct).

The domains $\overset{\bullet}{\mathscr{P}}{}^{\theta}_{\rho}$ satisfy the following property :

Lemma 4.1. *We assume that* $\overset{\bullet}{\zeta}$ *belongs to* $\overset{\bullet}{\mathscr{P}}{}^{\theta}_{\rho}$. *Then for every* $n \in [1, \mathscr{M}(\rho)]$, *the set* $\overset{\bullet}{\zeta} - \overline{D}_{\mathrm{e}^{\mathrm{i}\theta} n}$ *is included in* $\overset{\bullet}{\mathscr{P}}{}^{\theta}_{\rho}$.

Proof. The proof is easy and is left as an exercise. □

Definition 4.9. With the hypotheses of definition 4.8, for $0 \leq m \leq \mathscr{M}(\rho)$ and $\theta \in \{0, \pi\}$, we define

$$\overline{\mathscr{E}}^{m,\theta}_{\rho} = \bigcup_{(\zeta,\xi) \in \overline{D}_{\mathrm{e}^{\mathrm{i}\theta} m} \times \overline{D}_{\mathrm{e}^{\mathrm{i}\theta}(m+1)}} \Big\{ \xi + t(\xi - \zeta), \zeta + t(\zeta - \xi) \ \mid \ t \in [0, +\infty[\Big\}$$

and $\overset{\bullet}{\mathscr{Q}}{}^{m,\theta}_{\rho} = \mathbb{C} \setminus \overline{\mathscr{E}}^{\theta}_{\rho,m}$. When $m > \mathscr{M}(\rho)$, we set $\overset{\bullet}{\mathscr{Q}}{}^{m,\theta}_{\rho} = \emptyset$.
For $m \geq 1$ and $\epsilon = \pm$, we set $\overset{\bullet}{\mathscr{Q}}{}^{(\epsilon)_m,\theta}_{\rho} = \overset{\bullet}{\mathscr{Q}}{}^{m,\theta}_{\rho} \cap \{\zeta \ \mid \ \epsilon \mathrm{e}^{\mathrm{i}\theta}(\Im \zeta) \leq 0\}$. See Fig. 4.5.

The domains $\overset{\bullet}{\mathscr{Q}}{}^{m,\theta}_{\rho}$ have been defined so as to enjoy the following property :

Lemma 4.2. *We assume that* $\overset{\bullet}{\zeta}$ *belongs to* $\overset{\bullet}{\mathscr{Q}}{}^{m,\theta}_{\rho}$ *for some* $m \in [1, \mathscr{M}(\rho)]$ *and some* $\theta \in \{0, \pi\}$. *Then, for every* $n \in [1, m]$, *the set* $\overset{\bullet}{\zeta} - \overline{D}_{\mathrm{e}^{\mathrm{i}\theta} n}$ *is included in* $\overset{\bullet}{\mathscr{Q}}{}^{m-n,\theta}_{\rho}$.

Proof. We only consider the case $\theta = 0$ and we suppose $\overset{\bullet}{\zeta} \in \overset{\bullet}{\mathscr{Q}}{}^{m,0}_{\rho}$.
For $n \in [1, m]$, we assume that there exists $\overset{\bullet}{\zeta}_n \in \overline{D}_n$ such that $\overset{\bullet}{\zeta} - \overset{\bullet}{\zeta}_n \notin \overset{\bullet}{\mathscr{Q}}{}^{m-n,0}_{\rho}$. This means that $\overset{\bullet}{\zeta} - \overset{\bullet}{\zeta}_n \in \overline{\mathscr{E}}^{m-n,0}_{\rho}$ (see definition 4.9). Thus, there exist $\overset{\bullet}{\zeta}_{m-n} \in \overline{D}_{m-n}$, $\overset{\bullet}{\zeta}_{m-n+1} \in \overline{D}_{m-n+1}$ and $t \in [0, +\infty[$ such that

$$\overset{\bullet}{\zeta} - \overset{\bullet}{\zeta}_n = \overset{\bullet}{\zeta}_{m-n} + t(\overset{\bullet}{\zeta}_{m-n} - \overset{\bullet}{\zeta}_{m-n+1}) \ \text{ or } \ \overset{\bullet}{\zeta} - \overset{\bullet}{\zeta}_n = \overset{\bullet}{\zeta}_{m-n+1} + t(\overset{\bullet}{\zeta}_{m-n+1} - \overset{\bullet}{\zeta}_{m-n}).$$

We look only at the first case, which we write as follows:

$$\overset{\bullet}{\zeta} = (\overset{\bullet}{\zeta}_{m-n} + \overset{\bullet}{\zeta}_n) + t\Big((\overset{\bullet}{\zeta}_{m-n} + \overset{\bullet}{\zeta}_n) - (\overset{\bullet}{\zeta}_{m-n+1} + \overset{\bullet}{\zeta}_n)\Big).$$

We observe that $\overset{\bullet}{\zeta}_{m-n} + \overset{\bullet}{\zeta}_n \in \overline{D}_m$ and that $\overset{\bullet}{\zeta}_{m-n+1} + \overset{\bullet}{\zeta}_n \in \overline{D}_{m+1}$. Therefore $\overset{\bullet}{\zeta} \in \overline{\mathscr{E}}^{m,0}_{\rho}$ and this contradicts the fact that $\overset{\bullet}{\zeta} \in \overset{\bullet}{\mathscr{Q}}{}^{m,0}_{\rho}$. □

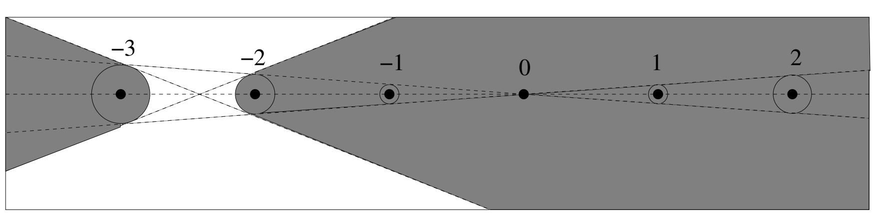


Fig. 4.5 The domain $\overset{\bullet}{\mathscr{Q}}{}^{2,\pi}_{\rho}$. The set $\overset{\bullet}{\mathscr{Q}}{}^{(-)_2,\pi}_{\rho}$ lies below the real axis, the domain $\overset{\bullet}{\mathscr{Q}}{}^{(+)_2,\pi}_{\rho}$ lies above the real axis.

Definition 4.10. With the hypotheses of definition 4.8, for $1 \leq m \leq \mathscr{M}(\rho)$ and $\theta \in \{0, \pi\}$, we denote by $\overline{\mathcal{D}}_\rho^{m,\theta} \subset \mathbb{C}$ the closed subset defined by

$$\overline{\mathcal{D}}_\rho^{m,\theta} = \left\{ t\zeta \mid t \in]-\infty, 1],\, \zeta \in \overline{D}_{\mathrm{e}^{\mathrm{i}\theta} m} \right\} \cup \left\{ t\zeta \mid t \in [1, +\infty[,\, \zeta \in \overline{D}_{\mathrm{e}^{\mathrm{i}\theta}(m+1)} \right\}.$$

We define the domain $\overset{\bullet}{\mathscr{P}}{}_\rho^{m,\theta} = \mathbb{C} \setminus \overline{\mathcal{D}}_\rho^{m,\theta}$. We set $\overset{\bullet}{\mathscr{P}}{}_\rho^{0,\theta} = \overset{\bullet}{\mathscr{Q}}{}_\rho^{0,0}$ while, for $m > \mathscr{M}(\rho)$, we set $\overset{\bullet}{\mathscr{P}}{}_\rho^{m,\theta} = \emptyset$.
For $\epsilon = \pm$ we note $\overset{\bullet}{\mathscr{P}}{}_\rho^{(\epsilon)_m,\theta} = \overset{\bullet}{\mathscr{P}}{}_\rho^{m,\theta} \cap \{\zeta \mid \epsilon \mathrm{e}^{\mathrm{i}\theta}(\Im\zeta) \leq 0\}$. (See Fig. 4.6).

Definition 4.11. Under the hypotheses of definition 4.8, for $\theta \in \{0, \pi\}$, $\epsilon = \pm$ and $m \in \mathbb{N}$, we define the domains

$$\overset{\bullet}{\mathscr{R}}{}_\rho^{(\epsilon)_m,\theta} = \overset{\bullet}{\mathscr{P}}{}_\rho^{(\epsilon)_m,\theta} \cup \overset{\bullet}{\mathscr{Q}}{}_\rho^{(-\epsilon)_m,\theta}$$

(see Fig. 4.7). We define as well the domains

$$\overset{\bullet}{\mathscr{R}}{}^{m,\theta} = \bigcup_{0<\rho\leq 1/5} \overset{\bullet}{\mathscr{R}}{}_\rho^{(+)_m,\theta} = \bigcup_{0<\rho\leq 1/5} \overset{\bullet}{\mathscr{R}}{}_\rho^{(-)_m,\theta} = \mathbb{C} \setminus \mathrm{e}^{\mathrm{i}\theta}\{]-\infty, m] \cup [m+1, +\infty[\}.$$

We have already noticed that for $\theta \in \{0, \pi\}$ and $m \in \mathbb{N}^\star$, the restriction $\mathfrak{p}|_{\mathscr{R}^{(+)_m,\theta}}$ and $\mathfrak{p}|_{\mathscr{R}^{(-)_m,\theta}}$ respectively, realises a homeomorphism between the nearby sheet $\mathscr{R}^{(+)_m,\theta}$ and $\mathscr{R}^{(-)_m,\theta}$ respectively, and the simply connected domain

$$\mathfrak{p}(\mathscr{R}^{(+)_m,\theta}) = \mathfrak{p}(\mathscr{R}^{(-)_m,\theta}) = \overset{\bullet}{\mathscr{R}}{}^{m,\theta}.$$

This justifies the following definition.

Definition 4.12. With the above notations, with $\epsilon = \pm$ and $1 \leq m \leq \mathscr{M}(\rho)$, one defines

$$\mathscr{R}_\rho^{(\epsilon)_m,\theta} = \mathfrak{p}|^{-1}_{\mathscr{R}^{(\epsilon)_m,\theta}}\left(\overset{\bullet}{\mathscr{R}}{}_\rho^{(\epsilon)_m,\theta}\right).$$

One says that the domains $\mathscr{R}_\rho^{(\epsilon)_m,\theta}$ are the $\mathscr{R}_\rho^{(0)}$**-nearby sheets** .
One defines the connected and simply connected domains $\mathscr{R}_\rho^{(1)} \subset \mathscr{R}^{(1)}$ by

$$\mathscr{R}_\rho^{(1)} = \mathscr{R}_\rho^{(0)} \bigcup_{\substack{1 \leq m \leq \mathscr{M}(\rho) \\ \theta \in \{0,\pi\}, \epsilon = \pm}} \mathscr{R}_\rho^{(\epsilon)_m,\theta},$$

and we denote by $\overline{\mathscr{R}}_\rho^{(1)}$ the closure of $\mathscr{R}_\rho^{(1)}$ in $\mathscr{R}^{(1)}$.

Observe that $\mathfrak{p}\left(\mathscr{R}_\rho^{(1)}\right) = \overset{\bullet}{\mathscr{R}}_\rho$. In the same line, one has also the following lemma which will be useful in a moment.

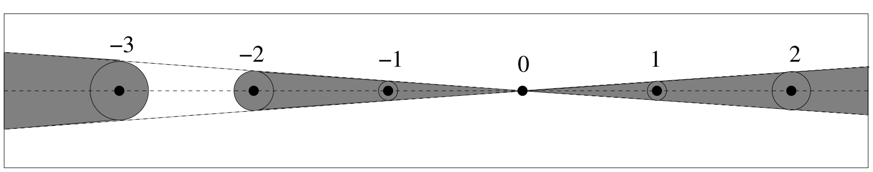


Fig. 4.6 The domain $\overset{\bullet}{\mathscr{P}}{}_\rho^{2,\pi}$. The set $\overset{\bullet}{\mathscr{P}}{}_\rho^{(-)_2,\pi}$ lies below the real axis, the set $\overset{\bullet}{\mathscr{P}}{}_\rho^{(+)_2,\pi}$ lies above the real axis.

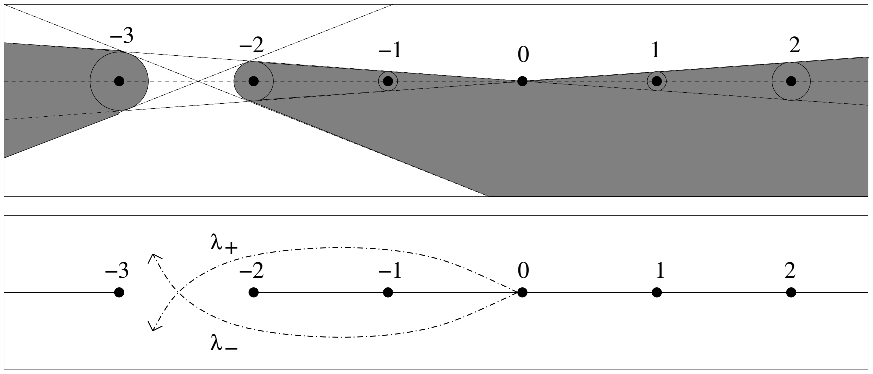


Fig. 4.7 Figure above, the domain $\overset{\bullet}{\mathscr{R}}_\rho^{(+)_2,\pi}$. Figure below, the domain $\overset{\bullet}{\mathscr{R}}^{2,\pi}$.

Lemma 4.3. *We assume that $\zeta \in \mathscr{R}_\rho^{(\epsilon)_m,\theta} \setminus \mathscr{R}_\rho^{(0)}$ for some $m \in [1, \mathscr{M}(\rho)]$, some $\theta \in \{0, \pi\}$, $\epsilon = \pm$. Then, for every $n \in [1, m]$, $\overset{\bullet}{\zeta} - \overline{D}_{e^{i\theta}n}$ is a subset of $\overset{\bullet}{\mathscr{R}}_\rho$ and there exists a closed set $\overline{\mathscr{U}}_{\zeta, e^{i\theta}n} \subset \mathscr{R}_\rho^{(\epsilon)_{m-n},\theta}$ such that $\overline{\mathscr{U}}_{\zeta, e^{i\theta}n}$ and $\overset{\bullet}{\zeta} - \overline{D}_{e^{i\theta}n}$ are $\mathfrak{p}$-homeomorphic.*

Proof. The lemma is a consequence of lemmas 4.1 and 4.2. □

4.2 Symmetrically contractile paths

4.2.1 Geodesics

With the hypotheses of definition 4.8, we consider the closure $\overline{\overset{\bullet}{\mathscr{R}}}_\rho$ of the domain $\overset{\bullet}{\mathscr{R}}_\rho$. This space $\overline{\overset{\bullet}{\mathscr{R}}}_\rho$ can be thought of as a complete real 2-dimensional Riemannian manifold with smooth $\mathcal{C}^1$-boundary embedded in the 2-dimensional euclidean space. For such a space, the following result takes place.

Lemma 4.4. *We note $X = \overline{\overset{\bullet}{\mathscr{R}}}_\rho$. For every two points $\overset{\bullet}{\zeta}_1, \overset{\bullet}{\zeta}_2 \in X$, there exists a geodesic in every homotopy class of curves from $\overset{\bullet}{\zeta}_1$ to $\overset{\bullet}{\zeta}_2$ in X, and this geodesic may be chosen as a shortest curve in the homotopy class.*

In this lemma, a geodesic is a locally shortest path for the metric. Lemma 4.4 can be seen as a consequence of the general Hopf-Rinow theorem [9] which can be applied. As a matter of fact, the situation is quite simple here : inside $\overset{\bullet}{\mathscr{R}}_\rho$, a geodesic is nothing but a straight line, otherwise one just follows the smooth boundary $\partial\, \overline{\overset{\bullet}{\mathscr{R}}}_\rho$ (see also [2]. See [1] and references therein for more general cases).

Lemma 4.5. *For every $\zeta \in \overline{\mathscr{R}}_\rho^{(1)}$, there exists $\lambda \in \mathfrak{R}$ ending at $\overset{\bullet}{\zeta}$ with the following properties:*

- *λ can be lifted uniquely with respect to p on $\overline{\mathscr{R}}_\rho^{(1)}$ into a path Λ starting from 0 and ending at ζ.*

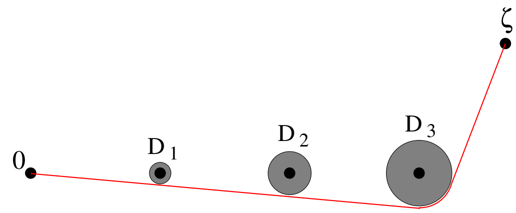


Fig. 4.8 The geodesic path λ for $\zeta = \mathrm{cl}(\lambda)$ in $\overline{\mathscr{R}}_\rho^{(+)_3,\theta} \setminus \overline{\mathscr{R}}_\rho^{(0)}$.

- *λ is of class $\mathcal{C}^1$ and is the shortest path in the homotopy class of paths in $\overset{\bullet}{\overline{\mathscr{R}}}_\rho$.*

Proof. we assume that $\zeta \in \overline{\mathscr{R}}_\rho^{(1)}$. Then:

1. **First case**: either ζ belongs to $\overline{\mathscr{R}}_\rho^{(0)}$. In that case we take for λ the segment $[0, \overset{\bullet}{\zeta}] \subset \overset{\bullet}{\overline{\mathscr{R}}}{}_\rho^{(0)}$.

2. **Second case**: or ζ belongs to $\overline{\mathscr{R}}_\rho^{(\epsilon)_m,\theta} \setminus \overline{\mathscr{R}}_\rho^{(0)}$ for some $\theta \in \{0, \pi\}$, $\epsilon = \pm$ and some $m \in [1, \mathscr{M}(\rho)]$. In that case we consider the path $\lambda = \gamma_0\delta\gamma_1$, made as the product of the following geodesics (see Fig. 4.8) :

   - γ_0 is the segment $[0, \overset{\bullet}{\zeta}_1] \subset \partial\big(\overset{\bullet}{\overline{\mathscr{P}}}{}_\rho^{(\epsilon)_m,\theta} \cap \overset{\bullet}{\overline{\mathscr{R}}}{}_\rho^{(0)} \big)$ that circumvents the segment $\mathrm{e}^{\mathrm{i}\theta}[1, m]$ to the right when $\epsilon = +$ and to the left when $\epsilon = -$;
   - δ is the arc-curve $\overset{\frown}{\overset{\bullet}{\zeta}_1, \overset{\bullet}{\zeta}_2}$ that follows in $\overset{\bullet}{\overline{\mathscr{R}}}{}_\rho^{(\epsilon)_m,\theta}$ the boundary $\partial D_{\mathrm{e}^{\mathrm{i}\theta} m}$;
   - γ_1 is the segment $[\overset{\bullet}{\zeta}_2, \overset{\bullet}{\zeta}]$ in $\overset{\bullet}{\overline{\mathscr{R}}}{}_\rho^{(\epsilon)_m,\theta}$ (possibly reduced to the point $\overset{\bullet}{\zeta}$).

In the two cases, the path λ can be lifted with respect to $\mathfrak{p}$ from 0 on $\overline{\mathscr{R}}_\rho^{(1)}$, by the very construction of $\overline{\mathscr{R}}_\rho^{(1)}$. This lifting is unique from the uniqueness of lifting (see [7]), because $\mathscr{R}$ is an étalé space on $\overset{\bullet}{\mathscr{R}}$. Obviously λ is $\mathcal{C}^1$ and is the shortest path in its homotopy class in $\overset{\bullet}{\overline{\mathscr{R}}}_\rho$. $\square$

4.2.2 Symmetric $\mathbb{Z}$-homotopy

We refer the reader to [12] for the definition of “symmetric Ω-homotopy”, see also [4, 11, 13, 14].

4.2.2.1 Shortest length symmetric $\mathbb{Z}$-homotopy

We are ready to show the following result.

Proposition 4.2 (Shortest length symmetric $\mathbb{Z}$-homotopy). *Let $\zeta = \mathrm{cl}(\lambda)$ be any point in $\overline{\mathscr{R}}_\rho^{(1)}$, with λ given by lemma 4.5.*

There exists a unique continuous map $H : (s,t) \in I \times J \mapsto H(s,t) = H_t(s) \in \overset{\bullet}{\overline{\mathscr{R}}}_\rho$ where $I = [0,1]$ and $J = [a,b]$ is a compact interval of $\mathbb{R}$, such that H satisfies the following properties:

1. *for each $t \in J$, the map $H_t : s \in I \mapsto H_t(s) = H(s,t)$ defines a path which satisfies:*
   a. *H_t belongs to $\mathfrak{R}$, is of class $\mathcal{C}^1$, and can be lifted uniquely from 0 with respect to $\mathfrak{p}$ on $\overline{\mathscr{R}}_\rho^{(1)}$ into a path $\mathcal{H}_t : s \in I \mapsto \mathcal{H}_t(s)$;*
   b. *the map $\mathcal{H} : (s,t) \in I \times J \mapsto \mathcal{H}(s,t) = \mathcal{H}_t(s) \in \overline{\mathscr{R}}_\rho^{(1)}$ thus defined is continuous, and the following diagram commutes:*

$$\begin{tikzcd} & \overline{\mathscr{R}}_\rho^{(1)} \arrow[d, "\mathfrak{p}"] \\ I \times J \arrow[ur, "\mathcal{H}"] \arrow[r, "H"'] & \overset{\bullet}{\overline{\mathscr{R}}}_\rho \end{tikzcd}$$

   c. *$H_t^{-1}(s) = H_t(1) - H_t(s)$ for every $s \in I$, where H_t^{-1} is the inverse path[1];*
   d. *H_t is the shortest path in the homotopy class of paths having the above properties 1a, 1b, 1c and H_t is homotopic to $\lambda|_{[0,T(t)]}$ in $\overset{\bullet}{\overline{\mathscr{R}}}_\rho$ for some $0 \leq T(t) \leq 1$;*
2. *the initial path H_a is a segment in $\overset{\bullet}{\overline{\mathscr{R}}}{}_\rho^{(0)}$;*
3. *the final path H_b is so that $\zeta = \mathrm{cl}(H_b)$. In other words, $\mathcal{H}_b(1) = \zeta$.*

Proof. We assume that $\zeta \in \overline{\mathscr{R}}_\rho^{(1)}$ and we consider the path λ given by lemma 4.5.

1. **First case**. When ζ belongs to $\overline{\mathscr{R}}_\rho^{(0)}$, we know that λ is the segment $[0, \overset{\bullet}{\zeta}] \subset \overset{\bullet}{\overline{\mathscr{R}}}{}_\rho^{(0)}$. In this case we set $J = \{0\}$ and define $H_0(s) = s\,\overset{\bullet}{\zeta}$ for every $s \in I$.
2. **Second case**: We now assume that ζ belongs to $\overline{\mathscr{R}}_\rho^{(\epsilon)_m,\theta} \setminus \overline{\mathscr{R}}_\rho^{(0)}$ for some $\theta \in \{0, \pi\}$, $\epsilon = \pm$ and some $m \in [1, \mathscr{M}(\rho)]$. For simplicity we will assume that $\theta = 0$ and $\epsilon = +$. In this case, the path $\lambda : [0,3] \to \overset{\bullet}{\overline{\mathscr{R}}}_\rho$ reads $\lambda = \gamma_0 \delta \gamma_1$,

$$\lambda(t) = \begin{cases} \gamma_0(t),\ t \in [0,1] \\ \delta(t-1),\ t \in [1,2] \\ \gamma_1(t-2),\ t \in [2,3] \end{cases},$$

   with $\gamma_0, \delta, \gamma_1 : [0,1] \to \mathbb{C}$ as described in the proof of lemma 4.5.
   We define H_1 to be the path γ_0, that is we define $H_1(s) = s\,\overset{\bullet}{\zeta}_1$, for $s \in I$. Since the point $\zeta_1 = \mathrm{cl}(H_1)$ belongs to $\overline{\mathscr{R}}_\rho^{(+)_m,0} \setminus \overline{\mathscr{R}}_\rho^{(0)}$, we can applied lemma 4.3. As a consequence, the path H_1 can be seen as a geodesic path (of shortest length) in $X = \overset{\bullet}{\overline{\mathscr{R}}}_\rho \setminus \bigcup_{1 \leq n \leq m} \{\overset{\bullet}{\zeta}_1 - D_n\}$, by application of lemma 4.4.
   Still according to lemma 4.3, the space $\overset{\bullet}{\overline{\mathscr{R}}}_\rho \setminus \bigcup_{1 \leq n \leq m} \{\overset{\bullet}{\xi} - D_n\}$ remains in the field of application of lemma 4.4, for every $\overset{\bullet}{\xi} = \mathfrak{p}(\xi)$ with $\xi \in \overline{\mathscr{R}}_\rho^{(+)_m,0} \setminus \overline{\mathscr{R}}_\rho^{(0)}$. In that way, one gets a local system $\left(\overset{\bullet}{\overline{\mathscr{R}}}_\rho \setminus \bigcup_{1 \leq n \leq m} \{\overset{\bullet}{\xi} - D_n\}\right)_\xi$ of Rieman-

[1] That is $H_t^{-1}(s) = H_t(1-s)$

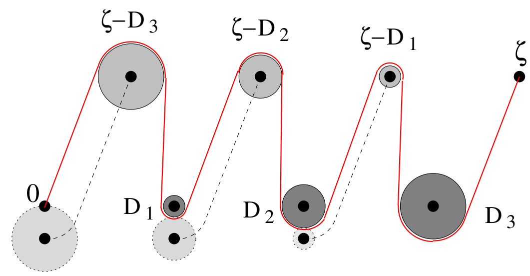


Fig. 4.9 The geodesic path H_b for $\zeta = \mathrm{cl}(H_b)$ in $\overline{\mathscr{R}}^{+}_{\rho,(3,4)} \setminus \overline{\mathscr{R}}^{(0)}_{\rho}$.

nian manifolds with smooth boundary.
For $t \in [1,3]$, we consider the restriction $\lambda_{|[0,t]}$ of λ to $[0,t]$. For $t \in [1,3]$ we note $\overset{\bullet}{\xi}_t = \mathfrak{p}(\mathrm{cl}(\lambda_{|[0,t]})$ and we continuously follow the class of H_1 in $\overset{\bullet}{\overline{\mathscr{R}}}_\rho \setminus \bigcup\limits_{1\leq n\leq m} \{\overset{\bullet}{\xi}_t - D_n\}$ when t moves from 1 to 3. In this class $\mathrm{cl}(H_1)$, we note K_t the geodesic path of shortest length. Obviously, when each K_t is viewed as a continuous functions in $\overset{\bullet}{\overline{\mathscr{R}}}_\rho$, this gives rise to a continuous map $K : t \in [1,3] \to \mathcal{C}^0\big([0,1], \overset{\bullet}{\overline{\mathscr{R}}}_\rho\,\big)$.

For $t \in [1,3]$, we finally define $H_t : s \in I \to \overset{\bullet}{\overline{\mathscr{R}}}_\rho \setminus \bigcup_{1\leq n\leq m}\{\overset{\bullet}{\xi}_t - D_n\}$ to be the path deduced from the product path $K_t(\delta\gamma_1|_{[1,t]})$ by standardization. This defines the homotopy H we had in mind,

$$H : (s,t) \in I \times J \mapsto H_t(s) = H(s,t) \in \overset{\bullet}{\overline{\mathscr{R}}}_\rho \setminus \bigcup_{1\leq n\leq m} \{\overset{\bullet}{\xi}_t - D_n\} \subset \overset{\bullet}{\overline{\mathscr{R}}}_\rho,$$

with $J = [a,b] = [1,3]$. See Fig. 4.9.
By its very construction, for every $t \in [1,3]$, the path H_t is symmetric with respect to its midpoint. Thus, up to making a change of parametrisation (arc-length parametrisation and standardization), one has $H_t^{-1}(s) = H_t(1) - H_t(s)$ for every $s \in I$.
Also, as a consequence of lemma 4.3, for every $t \in [1,3]$, the path H_t can be uniquely lifted from 0 with respect to $\mathfrak{p}$ on $\overline{\mathscr{R}}^{(1)}_\rho$ into a path $\mathcal{H}_t : s \in I \mapsto \mathcal{H}_t(s)$. This induces a continuous map $\mathcal{H} : (s,t) \in I \times J \mapsto \mathcal{H}(s,t) = \mathcal{H}_t(s) \in \overline{\mathscr{R}}^{(1)}_\rho$ as it follows from the lifting theorem for homotopies [7, 3].
The fact that $\zeta = \mathrm{cl}(H_3)$ is obvious. The uniqueness of H comes from the fact that any H_t is chosen as the shortest path.
□

With the notation of [12], the map H given by proposition 4.2 is an example of a “symmetric $\mathbb{Z}$-homotopy” and the final path H_b has the property of being a “symmetrically contractile path”, according to a terminology of Ecalle [4].

4.2.2.2 Length and L-points

Definition 4.13. For every $\zeta \in \overline{\mathscr{R}}_\rho^{(1)}$, we note $\mathrm{leng}(\zeta) = \mathrm{length}(H_b)$ the length of the associated symmetrically contractile path H_b given by proposition 4.2.

When returning to the very construction of the shortest length symmetric $\mathbb{Z}$-homotopy H, one sees that the mapping $\zeta \in \overline{\mathscr{R}}_\rho^{(1)} \mapsto \mathrm{leng}(\zeta)$ is continuous. This justifies the following definition:

Definition 4.14. For any $L > 0$, we note $\mathscr{U}_{\rho,L}$ the open subset of $\mathscr{R}_\rho^{(1)}$:

$$\mathscr{U}_{\rho,L} = \{\zeta \in \mathscr{R}_\rho^{(1)} \;\mid\; \mathrm{leng}(\zeta) < L\}.$$

An element of $\mathscr{U}_{\rho,L}$ is called a L-point.

Lemma 4.6. *We assume $\zeta \in \mathscr{U}_{\rho,L}$. We consider the shortest length symmetric $\mathbb{Z}$-homotopy H of proposition 4.2 associated with ζ, and its lifting $\mathcal{H} : (s,t) \in [0,1] \times [a,b] \mapsto \mathcal{H}_t(s) \in \overline{\mathscr{R}}_\rho^{(1)}$. Then:*

- *$\mathcal{H}_t(s)$ belongs to $\mathscr{U}_{\rho,L}$, for every $(s,t) \in [0,1] \times [a,b]$;*
- *equipping H_b with ist arc-length parametrisation, $H_b : \ell \in [0, \mathrm{leng}(\zeta)] \mapsto H_b(\ell)$, one has*
  - *$\mathrm{leng}(\mathcal{H}_b(\ell)) \leq \ell$.*
  - *$\mathrm{leng}(\mathcal{H}_b^{-1}(\ell)) \leq \mathrm{leng}(\zeta) - \ell$.*

Proof. For every $(s,t) \in [0,1] \times [a,b]$, to the point $\mathcal{H}_t(s) \in \overline{\mathscr{R}}_\rho^{(1)}$ is associated a (shortest length) symmetrically contractile path given by proposition 4.2, whose length, $\mathrm{leng}(\mathcal{H}_t(s))$, is obviously less than the length of $H_t|_{[0,s]}$. (Just look at Fig. 4.9). Thus $\mathrm{leng}(\mathcal{H}_t(s)) \leq \mathrm{length}(H_t|_{[0,s]})$.
Similarly, to the point $\mathcal{H}_t^{-1}(s) \in \overline{\mathscr{R}}_\rho^{(1)}$ is associated a (shortest length) symmetrically contractile path and evidence shows that $\mathrm{leng}(\mathcal{H}_t^{-1}(s))$ is less that the length of $H_t^{-1}|_{[0,s]}$. Therefore $\mathrm{leng}(\mathcal{H}_t^{-1}(s)) \leq \mathrm{length}(H_t) - \mathrm{length}(H_t|_{[0,s]})$.
□

We finally provide a result from [8], which gives an upper bound for the length of the symmetrically contractile path we work with.

Lemma 4.7. *For every $\zeta \in \overline{\mathscr{R}}_\rho^{(1)}$,*

- *either $\zeta \in \overline{\mathscr{R}}_\rho^{(0)}$ and then $\mathrm{leng}(\zeta) = |\,\dot{\zeta}\,|$;*
- *or $|\,\dot{\zeta}\,| \leq \mathrm{leng}(\zeta) \leq \dfrac{1}{\rho}|\,\dot{\zeta}\,| + \dfrac{1}{\rho}\left(\dfrac{1}{\rho} - 2\right)$.*

Proof. The first case is obvious. The second case means that $\zeta \in \overline{\mathscr{R}}_\rho^{(\epsilon)_m,\theta} \backslash \overline{\mathscr{R}}_\rho^{(0)}$ for some $\theta \in \{0,\pi\}$, $\epsilon = \pm$ and some $m \in [1, \mathscr{M}(\rho)]$. Let us assume that $\theta = 0$ and $\epsilon = +$ for simplicity. We return to the construction of the final path H_b of proposition 4.2, see also Fig. 4.9. The geodesic path H_b is made of :

- $m+1$ segments between $\partial \overline{D}_n$ and $\partial\big(\dot{\zeta} - \overline{D}_{m-n}\big)$, $n \in [0,m]$. Each of these segments has length less than $|\,\dot{\zeta} - m| + m\rho$.
- m segments between $\partial\big(\dot{\zeta} - \overline{D}_{m-n}\big)$ and $\partial \overline{D}_{n+1}$, $n \in [1,m]$. Each of these segments has length less than $|\,\dot{\zeta} - (m+1)| + (m+1)\rho$.

- $2m$ arcs of circle, the total length of which being less than $2(1+\cdots+m)2\pi\rho$.

Putting things together, one gets

$$\mathrm{leng}(\zeta) \le (2m+1)|\overset{\bullet}{\zeta}| + 2m(m+1)(1+\rho) + 2m(m+1)\pi\rho.$$

Since $\rho \le \dfrac{1}{5}$, one has $|\overset{\bullet}{\zeta}| \le \mathrm{leng}(\zeta) \le (2m+1)|\overset{\bullet}{\zeta}| + 4m(m+1)$. But since $\mathscr{M}(\rho)+1 = \lfloor \frac{\rho^{-1}-1}{2} \rfloor$, one has $m \le \mathscr{M}(\rho) \le \dfrac{1}{2\rho} - 1$ and we easily get the result. □

4.3 Convolution product and related properties

It is known that the space $\hat{\mathscr{R}} = \hat{\mathscr{R}}_{\mathbb{Z}}$ (definition 4.2) is a convolution algebra without unit [12, 10]. Also, bounds for the convolution product can be obtained [2, 11, 12], as demonstrated in [14].

Stability by convolution product is also valid for the space of germs of analytic functions that are "endlessly continuable" [2, 11]

In this subsection, we will show that these properties remains true for a larger space $\hat{\mathscr{R}}^{(1)}$, with more precise statements for the bounds.

4.3.1 Riemann surface and convolution

Definition 4.15. For $k \in \mathbb{N}^\star$, we denote by $\hat{\mathscr{R}}^{(k)}$ the space of germs of analytic functions at the origin $\widehat{\varphi}$ that can be analytically continued to $\mathscr{R}^{(k)}$: there exists a neighbourhood $\mathscr{U} \in \mathscr{B}$ of 0 such that the mapping $\Phi : \zeta \in \mathscr{U} \mapsto \Phi(\zeta) = \widehat{\varphi}(\overset{\bullet}{\zeta}) \in \mathbb{C}$ can be analytically continued to $\mathscr{R}^{(k)}$.

The proposition 4.2 allows to show the following property:

Proposition 4.3. *The space $\hat{\mathscr{R}}^{(1)}$ is a convolution algebra (without unit). Moreover, suppose $\widehat{\Phi}, \widehat{\Psi}$ in $\hat{\mathscr{R}}^{(1)}$ with the property:*

$$\textit{for every } \zeta \in \overline{\mathscr{R}}_\rho^{(1)},\ \big|\widehat{\Phi}(\zeta)\big| \le F\big(\mathrm{leng}(\zeta)\big),\ \big|\widehat{\Psi}(\zeta)\big| \le G\big(\mathrm{leng}(\zeta)\big)$$

with F, G two positive, non-decreasing and continuous functions on $\mathbb{R}^+$. Then the convolution product $\widehat{\Phi} * \widehat{\Psi}$ satisfies the following property:

$$\textit{for every } \zeta \in \overline{\mathscr{R}}_\rho^{(1)},\ \big|\widehat{\Phi} * \widehat{\Psi}(\zeta)\big| \le F * G\big(\mathrm{leng}(\zeta)\big).$$

Also,

$$\textit{for every } \zeta \in \overline{\mathscr{R}}_\rho^{(1)},\ \big|(\zeta\widehat{\Phi}) * \widehat{\Psi}(\zeta)\big| \le \mathrm{leng}(\zeta)\left(F * G\big(\mathrm{leng}(\zeta)\big)\right).$$

Proof. The standard proof for proving that $\hat{\mathscr{R}}$ is a convolution algebra [12, 11] can be copy as it stands for $\hat{\mathscr{R}}^{(1)}$. We sketch it here, essentially so as to fix notations that will be used later on. Assume that $\widehat{\varphi}$ and $\widehat{\psi}$ are two functions in $\mathcal{O}(\overset{\bullet}{\mathscr{R}}^{(0)})$ and that they can be analytically continued to the Riemann surface

$\mathscr{R}^{(1)}$. This means that there exist two fonctions $\widehat{\Phi}$ and $\widehat{\Psi}$ in $\hat{\mathscr{R}}^{(1)}$ such that $\widehat{\Phi}(\zeta) = \widehat{\varphi}(\overset{\bullet}{\zeta})$ and $\widehat{\Psi}(\zeta) = \widehat{\psi}(\overset{\bullet}{\zeta})$, for every $\zeta \in \mathscr{R}^{(0)}$. For $\zeta \in \mathscr{R}^{(0)}$, we note

$$\widehat{\chi}(\zeta) = \widehat{\Phi} * \widehat{\Psi}(\zeta) = \widehat{\varphi} * \widehat{\psi}(\overset{\bullet}{\zeta}).$$

We assume that $0 < \rho \leq \dfrac{1}{5}$. For every $\overset{\bullet}{\zeta}_a \in \overline{\overset{\bullet}{\mathscr{R}}}{}^{(0)}_{\rho}$ and every $\overset{\bullet}{\xi} \in \mathbb{C}$ with $|\overset{\bullet}{\xi}| < \dfrac{\rho}{2}$, one has $\overset{\bullet}{\zeta}_a + \overset{\bullet}{\xi} \in \overset{\bullet}{\mathscr{R}}{}^{(0)}$, thus there exists a unique point $\zeta_a + \overset{\bullet}{\xi} \in \mathscr{R}^{(0)}$ so that $\mathfrak{p}(\zeta_a + \overset{\bullet}{\xi}) = \overset{\bullet}{\zeta} + \overset{\bullet}{\xi}$, and the convolution product $\widehat{\chi}(\zeta_a + \overset{\bullet}{\xi}) = \widehat{\varphi} * \widehat{\psi}(\overset{\bullet}{\zeta}_a + \overset{\bullet}{\xi})$ reads :

$$\begin{aligned}\widehat{\chi}(\zeta_a + \overset{\bullet}{\xi}) &= \int_0^{\overset{\bullet}{\zeta}_a + \overset{\bullet}{\xi}} \widehat{\varphi}(\eta)\widehat{\psi}(\overset{\bullet}{\zeta}_a + \overset{\bullet}{\xi} - \eta)d\eta \\ &= \int_0^{\overset{\bullet}{\zeta}_a} \widehat{\varphi}(\eta)\widehat{\psi}(\overset{\bullet}{\zeta}_a + \overset{\bullet}{\xi} - \eta)d\eta + \int_0^{\overset{\bullet}{\xi}} \widehat{\varphi}(\overset{\bullet}{\zeta}_a + \eta)\widehat{\psi}(\overset{\bullet}{\xi} - \eta)d\eta.\end{aligned}$$

Now assume that $\overset{\bullet}{\zeta}_a$ is the endpoint $H_a(1)$ of the path H_a given by proposition 4.2. The above equality reads :

$$\begin{aligned}\widehat{\chi}\big(\mathcal{H}_a(1) + \overset{\bullet}{\xi}\,\big) = \int_0^1 \widehat{\varphi}\big(H_a(s)\big)\widehat{\psi}\big(H_a^{-1}(s) + \overset{\bullet}{\xi}\,\big)H_a'(s)ds \\ + \overset{\bullet}{\xi} \int_0^1 \widehat{\varphi}(H_a(1) + \overset{\bullet}{\xi}\, s)\widehat{\psi}\big(\overset{\bullet}{\xi}\,(1-s)\big)ds,\end{aligned}$$

that is also

$$\begin{aligned}\widehat{\chi}\big(\mathcal{H}_a(1) + \overset{\bullet}{\xi}\,\big) = \int_0^1 \widehat{\Phi}(\mathcal{H}_a(s))\widehat{\Psi}(\mathcal{H}_a^{-1}(s) + \overset{\bullet}{\xi})H_a'(s)ds \\ + \overset{\bullet}{\xi} \int_0^1 \widehat{\Phi}(\mathcal{H}_a(1) + \overset{\bullet}{\xi}\, s)\widehat{\psi}\big(\overset{\bullet}{\xi}\,(1-s)\big)ds.\end{aligned}$$

The analytic continuation of $\widehat{\chi}$ from $\mathcal{H}_a(1)$ along the path $t \in [a,b] \mapsto \mathcal{H}_t(1) \in \overline{\mathscr{R}}{}^{(1)}_{\rho}$ is thus given by

$$\begin{aligned}\widehat{\chi}\big(\mathcal{H}_t(1) + \overset{\bullet}{\xi}\,\big) = \int_0^1 \widehat{\Phi}(\mathcal{H}_t(s))\widehat{\Psi}(\mathcal{H}_t^{-1}(s) + \overset{\bullet}{\xi})H_t'(s)ds \\ + \overset{\bullet}{\xi} \int_0^1 \widehat{\Phi}(\mathcal{H}_t(1) + \overset{\bullet}{\xi}\, s)\widehat{\psi}\big(\overset{\bullet}{\xi}\,(1-s)\big)ds.\end{aligned}$$

(See the arguments given, e.g. in [12]). In particular, if $\zeta = \mathcal{H}_b(1)$,

$$\widehat{\Phi} * \widehat{\Psi}(\zeta) = \int_0^1 \widehat{\Phi}\big(\mathcal{H}_b(s)\big)\widehat{\Psi}\big(\mathcal{H}_b^{-1}(s)\big)\, H_b'(s)ds. \tag{4.1}$$

We remark that $\widehat{\Phi} * \widehat{\Psi}(\zeta)$ does not depend on the chosen path $\mathcal{H}_b$ since $\mathscr{R}^{(1)}$ is simply connected. We now turn to bounds and we follow reasoning from [8]. We take $\zeta \in \overline{\mathscr{R}}{}^{(1)}_{\rho}$ and we note H_b its associated symmetrically contractile path provided by proposition 4.2. We equip this path with its arc-length parametrisation,

$$H_b : s \in [0, \mathrm{leng}(\zeta)] \mapsto H_b(s)$$

so that for $s \in [0, \mathrm{leng}(\zeta)]$, the length of the restricted path $H_b|_{[0,s]}$ is s, while the length of $H_b^{-1}|_{[0,s]}$ is $\mathrm{leng}(\zeta) - s$. Therefore, (4.1) reads

$$\widehat{\Phi} * \widehat{\Psi}(\zeta) = \int_0^{\mathrm{leng}(\zeta)} \widehat{\Phi}\big(\mathcal{H}_b(s)\big)\widehat{\Psi}\big(\mathcal{H}_b^{-1}(s)\big)ds. \tag{4.2}$$

Using lemma 4.6, we then get :

$$\begin{aligned}
\big|\widehat{\Phi} * \widehat{\Psi}(\zeta)\big| &\leq \int_0^{\mathrm{leng}(\zeta)} |\widehat{\Phi}\big(\mathcal{H}_b(s)\big)|.|\widehat{\Psi}\big(\mathcal{H}_b^{-1}(s)\big)|ds. \\
&\leq \int_0^{\mathrm{leng}(\zeta)} F\big(s\big)G\big(\mathrm{leng}(\zeta) - s\big)ds \\
&\leq F * G\big(\mathrm{leng}(\zeta)\big).
\end{aligned}$$

The proof is complete. □

4.3.2 Convolution space and uniform norm

We start this subsection with the following remark:

Proposition 4.4. *For every $\rho > 0$ and $L > 0$, the space $\mathcal{O}(\overline{\mathscr{U}_{\rho,L}})$ of holomorphic functions on $\overline{\mathscr{U}_{\rho,L}}$ is a convolution algebra.*

Proof. Just adapt the proof of proposition 4.3 by using lemma 4.6. □

We introduce the following definition, analogous to definition 3.5.

Definition 4.16. We note $\mathscr{U} = \mathscr{U}_{\rho,L}$ the open set of L-points, for $L > 0$. We note $\big(\mathcal{O}(\overline{\mathscr{U}}), *\big)$ the convolution $\mathbb{C}$-algebra (without unit) of functions which are continuous on $\overline{\mathscr{U}}$ and holomorphic on $\mathscr{U}$. We note $\mathcal{MO}(\overline{\mathscr{U}})$ the maximal ideal of $\mathcal{O}(\overline{\mathscr{U}}))$ defined by $\mathcal{MO}(\overline{\mathscr{U}}) = \{f \in \mathcal{O}(\overline{\mathscr{U}}),\ f(0) = 0\}^2$.
For $\nu \geq 0$ we introduce the norm $\|.\|_\nu$ defined by: for every $f \in \mathcal{O}(\overline{\mathscr{U}})$,

$$\|f\|_\nu = L \sup_{\zeta \in \mathscr{U}} \big|\mathrm{e}^{-\nu \mathrm{leng}(\zeta)} f(\zeta)\big|.$$

We extend this norm to $\mathcal{O}(\overline{\mathscr{U}}) \oplus \mathbb{C}\delta$ by defining, for every $f \in \mathcal{O}(\overline{\mathscr{U}})$ and every $c \in \mathbb{C}$, $\|c\delta + f\|_\nu = |c| + \|f\|_\nu$.

We now state an analogous to proposition 3.7.

Proposition 4.5. *With the above definitions, $\big(\mathcal{O}(\overline{\mathscr{U}}) \oplus \mathbb{C}\delta, \|.\|_\nu\big)$ is a Banach algebra. In particular, for every $f, g \in \mathcal{O}(\overline{\mathscr{U}}) \oplus \mathbb{C}\delta$, $\|f * g\|_\nu \leq \|f\|_\nu \|g\|_\nu$. The space $\mathcal{MO}(\overline{\mathscr{U}})$ is closed in the norm space $\big(\mathcal{O}(\overline{\mathscr{U}}), \|.\|_\nu\big)$. Moreover, for $\nu > 0$:*

1. *for every $n \in \mathbb{N}$, for every $g \in \mathcal{O}(\overline{\mathscr{U}})$, $\|\zeta^n * g\|_\nu \leq \dfrac{n!}{\nu^{n+1}}\|g\|_\nu$, $\|\zeta^{n+1}\|_\nu \leq \dfrac{n!}{\nu^{n+1}} L$ and $\|1\|_\nu = L$.*

---

[2] When writing $f(0)$, we of course make reference to the origin of the pointed Riemann surface $(\mathscr{R}^{(1)}, 0)$.

2. for every $f,g \in \mathcal{O}(\overline{\mathscr{U}})$, $\|fg\|_\nu \le \dfrac{1}{L}\|f\|_\nu \|g\|_0$.
3. for every $f \in \mathcal{O}(\overline{\mathscr{U}})$, $\nu \ge \nu_0 \ge 0 \Rightarrow \|f\|_\nu \le \|f\|_{\nu_0}$.
4. for every $f \in \mathcal{MO}(\overline{\mathscr{U}})$, $\lim\limits_{\nu\to\infty} \|f\|_\nu = 0$.
5. the map $\partial_{|\mathcal{O}(\overline{\mathscr{U}})} : f \in \mathcal{O}(\overline{\mathscr{U}}) \mapsto \partial f \in \mathcal{MO}(\overline{\mathscr{U}})$ *is a derivative in the convolution space* $\mathcal{O}(\overline{\mathscr{U}})$ *and is invertible. Its inverse map* ∂^{-1} *satisfies: for every* $f \in \mathcal{O}(\overline{\mathscr{U}})$, *for every* $g \in \mathcal{MO}(\overline{\mathscr{U}})$, $\partial^{-1}(f*g) \in \mathcal{MO}(\overline{\mathscr{U}})$ *and*

$$\|\partial^{-1}(f*g)\|_\nu \le \frac{1}{\nu L}\|f\|_\nu \|\partial^{-1} g\|_0.$$

For every $\mathcal{O}(\overline{\mathscr{U}}) \oplus \mathbb{C}\delta$, *for every* $g \in \mathcal{MO}(\overline{\mathscr{U}})$, $\partial^{-1}(f*g) \in \mathcal{O}(\overline{\mathscr{U}})$ *and*

$$\|\partial^{-1}(f*g)\|\|_\nu \le \|f\|_\nu \|\partial^{-1} g\|_\nu.$$

Proof. Obviously, the norm $\|.\|_\nu$ is equivalent to the maximum norm on the vector space $\mathcal{O}(\overline{\mathscr{U}})$. This shows the completeness of $\big(\mathcal{O}(\overline{\mathscr{U}}), \|.\|_\nu\big)$ and of $\big(\mathcal{O}(\overline{\mathscr{U}}) \oplus \mathbb{C}\delta, \|.\|_\nu\big)$ as well.
We take $\zeta \in \mathscr{U}$ and H_b its associated symmetrically contractile path (proposition 4.2) equipped with its arc-length parametrisation. For $f,g \in \mathcal{O}(\overline{\mathscr{U}})$,

$$f*g(\zeta) = \int_0^{\mathrm{leng}(\zeta)} \mathrm{e}^{\nu[\mathrm{leng}(\mathcal{H}_b(s)) + \mathrm{leng}(\mathcal{H}_b^{-1}(s))]} f\big(\mathcal{H}_b(s)\big)\mathrm{e}^{-\nu \mathrm{leng}(\mathcal{H}_b(s))} g\big(\mathcal{H}_b^{-1}(s)\big)\mathrm{e}^{-\nu\mathrm{leng}(\mathcal{H}_b^{-1}(s))} ds.$$

We know from lemma 4.6 that $\mathrm{leng}(\mathcal{H}_b(s)) + \mathrm{leng}(\mathcal{H}_b^{-1}(s)) \le \mathrm{leng}(\zeta)$. Therefore

$$L\mathrm{e}^{-\nu\mathrm{leng}(\zeta)} |f*g(\zeta)| \le \|f\|_\nu \|g\|_\nu \int_0^{\mathrm{leng}(\zeta)} \frac{1}{L}\, ds \le \|f\|_\nu \|g\|_\nu.$$

This shows that for every $f,g \in \mathcal{O}(\overline{\mathscr{U}})$, $\|f*g\|_\nu \le \|f\|_\nu\|g\|_\nu$, hence $\Big(\mathcal{O}(\overline{\mathscr{U}}), \|.\|_\nu\Big)$ is a Banach algebra and $\big(\mathcal{O}(\overline{\mathscr{U}}) \oplus \mathbb{C}\delta, \|.\|_\nu\big)$ as well.
The other properties are shown in quite similar way than in the proof of proposition 3.7. □

4.3.3 An extended Grönwall-like lemma

We want demonstrate in this subsection a Grönwall-like lemma, similar to lemma 3.9.

Lemma 4.8 (Extended Grönwall lemma). *Let be* $N \in \mathbb{N}^\star$ *and* $(\widehat{f}_n)_{0\le n\le N}$ *be a sequence of functions in* $\hat{\mathscr{R}}^{(1)}$ *so that there exists a a sequence* $(\widehat{F}_n)_{0\le n\le N}$ *of entire functions, real, positive and non-decreasing on* $\mathbb{R}^+$, *with at mots exponential growth of order 1 at infinity and such that, for every* $0 \le n \le N$,

$$\textit{for every } \zeta \in \overline{\mathscr{R}}^{(1)}_\rho,\ |\widehat{f}_n(\zeta)| \le \widehat{F}_n\big(\mathrm{leng}(\zeta)\big).$$

Let p,q,r *be polynomial functions so that* p *does not vanish on* $\overline{\mathscr{R}}^{(1)}_\rho$ *and we suppose that the following upper bounds are valid:*

$$a = \sup_{\zeta\in\overline{\mathscr{R}}^{(1)}_\rho} \frac{|q|(\mathrm{leng}(\zeta))}{|p(\overset{\bullet}{\zeta})|} < \infty,\ b = \sup_{\zeta\in\overline{\mathscr{R}}^{(1)}_\rho} \frac{|r|(\mathrm{leng}(\zeta))}{|p(\overset{\bullet}{\zeta})|} < \infty,\ c = \sup_{\zeta\in\overline{\mathscr{R}}^{(1)}_\rho} \frac{1}{|p(\overset{\bullet}{\zeta})|} < \infty.$$

We furthermore assume that $\widehat{w} \in \mathcal{O}(\overline{\mathscr{R}}_\rho^{(1)})$ *solves the following convolution equation*

$$p(\zeta)\widehat{w}(\zeta) + 1 * [q(\zeta)\widehat{w}](\zeta) = \zeta * [r(\zeta)\widehat{w}](\zeta) + \widehat{f}_0(\zeta) + \sum_{n=1}^{N} \widehat{f}_n * \widehat{w}^{*n}(\zeta). \quad (4.3)$$

Then for every $d \geq 0$*, for every* $\zeta \in \overline{\mathscr{R}}_\rho^{(1)}$*,*

$$|\widehat{w}(\zeta)| \leq \widehat{\mathsf{w}}_d(\xi), \qquad \xi = \mathrm{leng}(\zeta),$$

where $\widehat{\mathsf{w}}_d$ *is the holomorphic solution of the convolution equation:*

$$\widehat{\mathsf{w}}(\xi) = d + [a + b\xi] * \widehat{\mathsf{w}}(\xi) + c\left(\widehat{F}_0(\xi) + \sum_{n=0}^{N} \widehat{F}_n * \widehat{\mathsf{w}}^{*n}(\xi)\right). \quad (4.4)$$

Proof. If $\widehat{w} \in \mathcal{O}(\overline{\mathscr{R}}_\rho^{(1)})$ solves the convolution equation (3.24), then for every $\zeta \in \overline{\mathscr{R}}_\rho^{(1)}$:

$$p(\zeta)\widehat{w}(\zeta) = \widehat{f}_0(\zeta) - \int_0^{\mathrm{leng}(\zeta)} q\big(\mathcal{H}_b(s)\big)\widehat{w}\big(\mathcal{H}_b(s)\big)\, ds$$
$$+ \int_0^{\mathrm{leng}(\zeta)} \mathcal{H}_b^{-1}(s) r\big(\mathcal{H}_b(s)\big)\widehat{w}\big(\mathcal{H}_b(s)\big)\, ds + \sum_{n=1}^{N}\int_0^{\mathrm{leng}(\zeta)} \widehat{f}_n\big(\mathcal{H}_b^{-1}(s)\big)\widehat{w}^{*n}\big(\mathcal{H}_b(s)\big)\, ds.$$

where H_b stands for the symmetrically contractile path associated with ζ, equipped with its arc-length parametrisation (proposition 4.2). We know by lemma 4.6 that $\mathrm{leng}(\mathcal{H}_b(s)) \leq s$ and $\mathrm{leng}(\mathcal{H}_b^{-1}(s)) \leq \xi - s$ with $\xi = \mathrm{leng}(\zeta)$. Using the hypotheses, one obtains:

$$|\widehat{w}(\zeta)| \leq \frac{1}{|\overset{\bullet}{p(\zeta)}|}\widehat{F}_0(\xi) + \int_0^{\xi}\left[\frac{|q|(s)}{|\overset{\bullet}{p(\zeta)}|} + \frac{|r|(s)}{|\overset{\bullet}{p(\zeta)}|}(\xi - s)\right]|\widehat{w}\big(\mathcal{H}_b(s)\big)|\, ds$$
$$+ \sum_{n=1}^{N}\int_0^{\xi}\frac{1}{|\overset{\bullet}{p(\zeta)}|}\widehat{F}_n(\xi - s)|\widehat{w}^{*n}\big(\mathcal{H}_b(s)\big)|\, ds.$$

Therefore

$$\begin{aligned}|\widehat{w}(\zeta)| \leq c\widehat{F}_0(\xi) + \int_0^{\xi}[a + b(\xi - s)]\,|\widehat{w}\big(\mathcal{H}_b(s)\big)|\, ds \\ + c\sum_{n=1}^{N}\int_0^{\xi}\widehat{F}_n(\xi - s)|\widehat{w}^{*n}\big(\mathcal{H}_b(s)\big)|\, ds.\end{aligned} \quad (4.5)$$

We now remind the reader that the existence and properties of $\widehat{\mathsf{w}}_d$ are given by lemma 3.8. We adapt the proof of lemma 3.9. We first notice that $|\widehat{w}(0)| \leq c\widehat{F}_0(0)$ by definition of c and by hypothesis on $\widehat{F}_0$. Since $\widehat{\mathsf{w}}(0) = d + c\widehat{F}_0(0)$, we have $|\widehat{w}(0)| \leq \widehat{\mathsf{w}}(0)$.

Case 4.1. We first assume $|\widehat{w}(0)| < \widehat{\mathsf{w}}(0)$.
One considers, for $L > 0$, the open set $\mathscr{U}_{\rho,L}$ of L-points. We remark that, once $L_0 > 0$ is chosen small enough, then for every $0 < L \leq L_0$, for very

$d > 0$, for every $\zeta \in \overline{\mathscr{U}}_{\rho,L}$, $|\widehat{w}(\zeta)| < \widehat{\mathsf{w}}_d(\xi)$ with $\xi = \mathrm{leng}(\zeta)$. This is just a consequence of lemma 3.9. (For $L > 0$ small enough, $\mathrm{leng}(\zeta) = |\zeta|$).

We now assume that there exist $L_1 > 0$ and $\zeta_1 \in \overline{\mathscr{U}}_{\rho,L_1}$ such that $|\widehat{w}(\zeta_1)| \geq \widehat{\mathsf{w}}_d(\xi_1)$, $\xi_1 = \mathrm{leng}(\zeta_1)$. Define

$$\chi = \{L \in [L_0, L_1] \mid \text{there exists } \zeta \in \overline{\mathscr{U}}_{\rho,L},\ |\widehat{w}(\zeta)| \geq \widehat{\mathsf{w}}_d(\mathrm{leng}(\zeta))\}.$$

This closed set χ has an infimum L_2, $L_0 < L_2 < L_1$. We now recall that the mapping $\zeta \in \overline{\mathscr{R}}_\rho^{(1)} \mapsto \mathrm{leng}(\zeta)$ is continuous. This implies that:

- for every $\zeta \in \overline{\mathscr{U}}_{\rho,L_2}$, $|\widehat{w}(\zeta)| \leq \widehat{\mathsf{w}}_d(\mathrm{leng}(\zeta))$;
- there exists $\zeta_2 \in \overline{\mathscr{U}}_{\rho,L_2}$ such that $|\widehat{w}(\zeta_2)| = \widehat{\mathsf{w}}_d(\xi_2)$, $\xi_2 = \mathrm{leng}(\zeta_2) = L_2$.

We take such a $\zeta_2 \in \overline{\mathscr{U}}_{\rho,L_2}$. By lemma (4.5),

$$|\widehat{w}(\zeta_2)| \leq c\widehat{F}_0(\xi_2) + \int_0^{\xi_2} [a + b(\xi_2 - s)]\, |\widehat{w}\big(\mathcal{H}_b(s)\big)|\, ds + c\sum_{n=1}^{N} \int_0^{\xi_2} \widehat{F}_n(\xi_2 - s) |\widehat{w}^{*n}\big(\mathcal{H}_b(s)\big)|\, ds$$

where H_b is the symmetrically contractile path associated with ζ_2, equipped with its arc-length parametrisation (proposition 4.2). We know by lemma 4.6 that $\mathrm{leng}(\mathcal{H}_b(s)) \leq s \leq \xi_2$, thus $\mathcal{H}_b(s) \in \overline{\mathscr{U}}_{\rho,L_2}$, and that $\widehat{\mathsf{w}}_d$ is real, positive and non-decreasing on $\mathbb{R}^+$. Therefore,

$$|\widehat{w}(\zeta_2)| \leq c\widehat{F}_0(\xi_2) + \int_0^{\xi_2} [a + b(\xi_2 - s)]\, \widehat{\mathsf{w}}_d(s)\, ds + c\sum_{n=1}^{N} \int_0^{\xi_2} \widehat{F}_n(\xi_2 - s) \widehat{\mathsf{w}}_d^{*n}(s)\, ds.$$

This shows that $|\widehat{w}(\zeta_2)| \leq \widehat{\mathsf{w}}_d(\xi_2) - d$ and we get a contradiction.

Case 4.2. The case $|\widehat{w}(0)| = \widehat{\mathsf{w}}(0)$ (thus $d = 0$) is done by an argument of continuity already used in the proof of proof of lemma 3.9.
□

4.4 Applications to the first Painlevé equation

This section is essentially devoted to proving theorem 4.1, which completes theorem 3.2.

4.4.1 Analytic continuation - statement

Theorem 4.1. *The formal solution $\widetilde{w}$ of the prepared equation (3.6) associated with the first Painlevé equation satisfies the following properties:*

1. *its formal Borel transform $\widehat{w}$ can be analytically continued to the Riemann surface $\mathscr{R}^{(1)}$;*
2. *$\widehat{w}$ has at most exponential growth of order 1 at infinity on $\mathscr{R}^{(1)}$. More precisely, for every $0 < \rho \leq \frac{1}{5}$, there exist $A = A(\rho) > 0$ and $\tau = \tau(\rho) > 0$ such that for every $\zeta \in \overline{\mathscr{R}}_\rho^{(1)}$, $|\widehat{w}(\zeta)| \leq A\mathrm{e}^{\tau\xi}$ with $\xi = \mathrm{leng}(\zeta)$.*

3. moreover $\operatorname{leng}(\zeta) \leq \frac{1}{\rho}|\overset{\bullet}{\zeta}| + \frac{1}{\rho}\left(\frac{1}{\rho} - 2\right)$ *and one can choose* $A = 4$ *and* $\tau = \frac{4}{\rho^3}$ *in the above upper bounds.*

4.4.2 Analytic continuation - proof

To prove this theorem, we will essentially copy what we have done in Sect. 3.4.3.

4.4.2.1 A lemma

We first state the following result which should be compared with lemma 3.2.

Lemma 4.9. *There exists* $M_{\rho,(1)} > 0$ *such that for every* $\zeta \in \overline{\mathscr{R}}_\rho^{(1)}$,

$$\text{for every } \zeta \in \overline{\mathscr{R}}_\rho^{(1)}, \ \frac{\operatorname{leng}(\zeta)^p}{|P(-\overset{\bullet}{\zeta})|} \leq M_{\rho,(1)}, \quad p = 0, 1.$$

In particular, for every polynom q *of degree* ≤ 1, $\frac{|q|(\operatorname{leng}(\zeta))}{|P(-\overset{\bullet}{\zeta})|}| \leq M_{\rho,(1)}|q|(1)$, *for every* $\zeta \in \overline{\mathscr{R}}_\rho^{(1)}$*. Moreover one can choose* $M_{\rho,(1)} = \dfrac{6}{5\rho^3}$

Proof. From lemma 3.2 we know that $\left|\dfrac{\overset{\bullet}{\zeta}{}^p}{P(-\overset{\bullet}{\zeta})}\right| \leq M_{\rho,(0)}$ with $M_{\rho,(0)} = \dfrac{1}{\rho}$, for $\overset{\bullet}{\zeta} \in \mathcal{D}_\rho^{(0)}$ and $p = 0, 1$. Using lemma 4.7 and since $0 < \rho \leq \dfrac{1}{5}$, one deduces that

$$\frac{\operatorname{leng}(\zeta)}{|P(-\overset{\bullet}{\zeta})|} \leq \left[\frac{1}{\rho} + \frac{1}{\rho}\left(\frac{1}{\rho} - 2\right)\right] M_{\rho,(0)} \leq \frac{6}{5\rho^3}.$$

This ends the proof. □

4.4.2.2 Analyticity of $\widehat{w}$ on $\mathscr{R}^{(1)}$

For $L > 0$ and $0 < \rho \leq 1/5$, we introduce the domain $\mathscr{U} = \mathscr{U}_{\rho,L}$. We note $B_r = \{\widehat{v} \in \mathcal{O}(\overline{\mathscr{U}}), \|\widehat{v}\|_\nu \leq r\}$, for any $r > 0$ and $\nu > 0$.
The convolution equation (3.10) is viewed as a fixed-point problem and we set

$$\mathcal{N} : \widehat{v} \in B_r \mapsto \frac{1}{P(\partial)}\Big[-1 * \big[Q(\partial)\widehat{v}\big] + \widehat{f}_0(\zeta) + \widehat{f}_1 * \widehat{v}(\zeta) + \widehat{f}_2 * \widehat{v} * \widehat{v}(\zeta)\Big].$$

By lemmas 4.9 and proposition 4.5,

$$\|\mathcal{N}(\widehat{v})\|_\nu \leq M_{\rho,(1)}\| - 1 * \big[Q(\partial)\widehat{v}\big] + \widehat{f}_0 + \widehat{f}_1 * \widehat{v} + \widehat{f}_2 * \widehat{v} * \widehat{v}\|_\nu.$$

By proposition 4.5, since $Q(\partial) = 3\zeta$:

$$\|1 * \left[Q(\partial)\widehat{v}\right]\|_\nu \leq \frac{1}{\nu}\|Q(\partial)\widehat{v}\|_\nu \leq \frac{1}{L\nu}\|Q(\partial)\|_0\|\widehat{v}\|_\nu \leq \frac{3L}{\nu}\|\widehat{w}\|_\nu.$$

The functions $\widehat{f}_0, \widehat{f}_1, \widehat{f}_2$ belong to $\mathcal{MO}(\overline{\mathscr{U}})$ and, by proposition 4.5, this implies $\lim\limits_{\nu\to\infty} \|\widehat{f}_i\|_\nu = 0$, $i = 0,1,2$. Therefore, $\|\mathcal{N}(\widehat{v})\|_\nu \leq r$ for $\nu > 0$ large enough.
The same arguments shows that $\|\mathcal{N}(\widehat{v}_1)-\mathcal{N}(\widehat{v}_2)\|_\nu \leq k\|\widehat{v}_1-\widehat{v}_2\|_\nu$ with $k<1$, for $\widehat{v}_1, \widehat{v}_2 \in B_r$ and for $\nu > 0$ large enough.
Thus, $\mathcal{N}$ is contractive in the closed set B_r of the Banach space $\big(\mathcal{O}(\overline{\mathscr{U}}), \|.\|_\nu\big)$, for $\nu > 0$ large enough. The contraction mapping theorem gives a unique solution $\widehat{w} \in B_r$ for the fixed-point problem $\widehat{v} = \mathcal{N}(\widehat{v})$. Since L and ρ can be arbitrarily chosen, we deduce (by uniqueness) that the formal Borel transform $\widehat{w}$ of the unique formal series $\widetilde{w}$ solution of (3.6), defines a holomorphic in $\mathscr{R}^{(1)}$.

4.4.2.3 Upper bounds

We use the Grönwall lemma 4.8 (with $d = 0$), which tells us that for every $\zeta \in \overline{\mathscr{R}}^{(1)}_\rho$,

$$|\widehat{w}(\zeta)| \leq \widehat{\mathsf{w}}_d(\xi), \qquad \xi = \mathrm{leng}(\zeta),$$

where $\widehat{\mathsf{w}}(\xi)$ solves the following convolution equation

$$\frac{1}{M_{\rho,(1)}}\widehat{\mathsf{w}} = |\widehat{f}_0| + \big(3 + |\widehat{f}_1|\big) * \widehat{\mathsf{w}} + |\widehat{f}_2| * \widehat{\mathsf{w}} * \widehat{\mathsf{w}},$$

(just use lemma 4.9) where we can take $M_{\rho,(1)} = \dfrac{6}{5\rho^3}$. Let us get explicit upper bounds. We consider $\widehat{\mathsf{w}}$ as the Borel transform of the holomorphic function $\widetilde{\mathsf{w}}$ solution of the second order algebraic equation,

$$\frac{1}{M_{\rho,(1)}}\widetilde{\mathsf{w}} = |f_0|(z) + \big(\frac{3}{z} + |f_1|\big)\widetilde{\mathsf{w}} + |f_2|\widetilde{\mathsf{w}}^2, \tag{4.6}$$

and asymptotic to $|f_0|(z)$ at infinity. Remember that $|f_0|(z) = \dfrac{392}{625}\dfrac{1}{z^2}$, $|f_1|(z) = \dfrac{4}{z^2}$, $|f_2|(z) = \dfrac{1}{2z^2}$. Setting $\widetilde{\mathsf{w}}(z) = H(t)$, $t = z^{-1}$, the above problem reads as a fixed-point problem,

$$H = \mathcal{N}(H), \qquad \mathcal{N}(H) = M_{\rho,(1)}\Big(|f_0|(t^{-1}) + \big(3t + |f_1|(t^{-1})\big)H + |f_2|(t^{-1})H^2\Big). \tag{4.7}$$

From homogeneity reasons, we introduce $U = D(0, \rho^3/4)$, the Banach algebra $\big(O(\overline{U}), \|\,\|\big)$ where $\|\,\|$ stands for the maximum norm, and we note $B_{\rho^3} = \{H \in O(\overline{U}),\ \|H\| \leq \rho^3\}$. It is easy to show that the mapping $\mathcal{N}_{|B_{\rho^3}} : H \in B_{rho^3} \mapsto \mathcal{N}(H) \in B_{\rho^3}$ is contractive (recall: $0 < \rho \leq 1/5$). Therefore, the contraction mapping theorem implies the existence of a unique solution H in B_{ρ^3} of the fixed-point problem (4.7). In return we deduce that $\widehat{\mathsf{w}}$ is an entire function and $|\widehat{\mathsf{w}}(\xi)| \leq 4\mathrm{e}^{\frac{4}{\rho^3}|\xi|}$, for every $\xi \in \mathbb{C}$. (See lemma 3.5). One ends with lemma 4.7.

4.5 Supplements to the convolution product

We end this chapter with some supplements to the convolution product that will be used later on.

Definition 4.17. For a direction $\theta \in \{0,\pi\}$, for $\alpha \in]0,\pi/2]$, for $L > 0$, we denote by $\mathfrak{R}^{(\theta,\alpha)}(L) \subset \mathfrak{R}$ the set of paths λ with the condition:

- either λ is a path on the open disc $D(0,1)$, thus homotopic (in the sense of $\sim_{\mathfrak{R}}$) to a segment $[0,\zeta]$, $\zeta \in D(0,1)$;
- or λ is a piecewise $\mathcal{C}^1$ paths λ with the following properties:
  1. for every $t \in [0,1]$, the right and left derivatives $\underline{\lambda}'(t)$ do not vanish and $\arg \underline{\lambda}'(t) \in]-\alpha+\theta, \theta+\alpha[$,
  2. the length of λ satisfies : $\mathrm{length}(\lambda) < L+1$.

We define $\mathscr{R}^{(\theta,\alpha)}(L) = \{\zeta = \mathrm{cl}(\lambda) \mid \lambda \in \mathfrak{R}^{(\theta,\alpha)}(L)\} \subset \mathscr{R}$.

The proof of the following lemma is left as exercise.

Lemma 4.10. *For any $L > 0$, $\mathscr{R}^{(\theta,\alpha)}(L)$ is an open and connected subset of $\mathscr{R}$ and satisfied $\mathscr{R}^{(\theta,\alpha)}(L) \subset \mathscr{R}^{(0)} \bigcup_{1\le j\le m} \mathscr{R}^{(\pm)_j,\theta}$ with $m = \lceil L \rceil$. Also, for any $m \in \mathbb{N}^\star$ and any path $\gamma \in \mathfrak{R}$ of type $\gamma^{\theta}_{\boldsymbol{\varepsilon}}$ with $\boldsymbol{\varepsilon} \in \{+,-\}^j$, $1 \le j \le m$, there exists $\lambda \in \mathscr{R}^{(\theta,\alpha)}(m)$ so that $\mathrm{cl}(\lambda) = \mathrm{cl}(\gamma)$.*

In the above lemma, $\lceil . \rceil$ is the ceiling function.

Remark 4.3. Notice that $\mathscr{R}^{(\theta,\alpha)}(L_1) \subset \mathscr{R}^{(\theta,\alpha)}(L_2)$ when $L_1 < L_2$. Also, since $\mathscr{R}^{(\theta,\alpha)}(L)$ is open and connected in $\mathscr{R}$, $\mathscr{R}^{(\theta,\alpha)}(L)$ inherits from $\mathscr{R}$ the structure of complex manifold, thus is a Riemann surface.

Definition 4.18. We denote $\hat{\mathscr{R}}^{(\theta,\alpha)}(L) \supset \hat{\mathscr{R}}$ the space of germs of analytic functions at the origin that can be analytically continued to $\mathscr{R}^{(\theta,\alpha)}(L)$.

Example 4.1. The minor $\widehat{w}_{(0,0)}$ associated with the formal solution $\widetilde{w}_{(0,0)}$ of the prepared equation (3.6), belongs to $\hat{\mathscr{R}}^{(\theta,\pi/2)}(L)$, for any direction $\theta \in \{0,\pi\}$ and any $L \in]0,1]$. This is a consequence of theorem 4.1.

Proposition 4.6. *The space $\hat{\mathscr{R}}^{(\theta,\alpha)}(L)$ is a (non unitary) convolution algebra.*

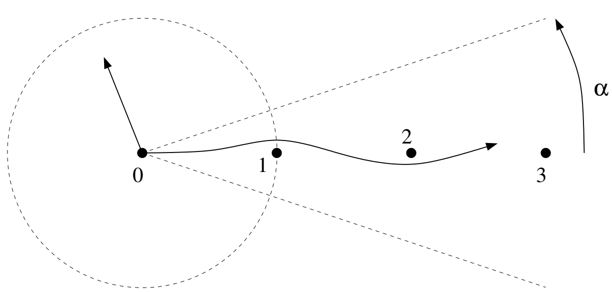


Fig. 4.10 Two paths belonging to $\mathfrak{R}^{(\theta,\alpha)}(L)$ for $\theta = 0$ and $L \ge 2$.

Proof. We just have to show the stability by convolution product and the proof can be done in the same manner than this for $\hat{\mathscr{R}}$. Here we follow ideas from [12]. The master piece is the existence, for any path $\gamma : I = [0,1] \to \mathbb{C}\backslash\mathbb{Z}$ with $|\gamma(0)| < 1$, of a symmetric $\mathbb{Z}$-homotopy

$$H : (s,t) \in I \times I \mapsto H(s,t) = H_t(s) \in \overset{\bullet}{\mathscr{R}}$$

so that

1. for each $t \in I$, the path H_t belongs to $\mathfrak{R}$, thus can be lifted uniquely from 0 with respect to $\mathfrak{p}$ on $\mathscr{R}$ into a path $\mathcal{H}_t$;
2. $H_t^{-1}(s) = H_t(1) - H_t(s)$ for every $s \in I$, where H_t^{-1} is the inverse path ;
3. the initial path H_0 is $H_0(s) = s\gamma(0)$;
4. for every $t \in I$, $H_t(1) = \gamma(t)$.

In particular, from the lifting theorem for homotopies, the map $\mathcal{H} : (s,t) \in I \times I \mapsto \mathcal{H}(s,t) = \mathcal{H}_t(s) \in \mathscr{R}$ is continuous, and the following diagram commutes:

$$\begin{tikzcd} & \mathscr{R} \arrow[d, "\mathfrak{p}"] \\ I \times I \arrow[ur, "\mathcal{H}"] \arrow[r, "H"'] & \overset{\bullet}{\mathscr{R}}. \end{tikzcd} \tag{4.8}$$

The symmetric $\mathbb{Z}$-homotopy can be constructed as follows: one takes a $\mathcal{C}^1$ function $\eta : \mathbb{C} \to [0,1]$ satisfying $\{\zeta \in \mathbb{C} \mid \eta(\zeta) = 0\} = \mathbb{Z}$ (the existence of which is given in [12]), and one considers the non-autonomous vector field $X(\zeta,t) = \dfrac{\eta(\zeta)}{\eta(\zeta) + \eta\big(\gamma(t) - \zeta\big)}\gamma'(t)$. The path H_t is obtained by deformation of the initial path H_0 through the flow of the vector field $g : (t_0, t, \zeta) \in [0,1]^2 \times \mathbb{C} \mapsto g^{t_0,t}(\zeta) \in \mathbb{C}$ of X, precisely $H_t(s) = g^{0,t}\big(H_0(s)\big)$. We now take a point $\zeta \in \mathscr{R}^{(\theta,\alpha)}(L)$. We can assume that $\zeta = \mathrm{cl}(\lambda)$ where $\lambda \in \mathfrak{R}^{(\theta,\alpha)}(L)$ is the product $\lambda = \lambda_0\gamma$ with $\lambda_0(s) = s\gamma(0)$. Let us analyze the above symmetric $\mathbb{Z}$-homotopy constructed from γ and $H_0 = \lambda_0$. The path $H^s : t \in I \mapsto H^s(t) = H(s,t) \in \mathbb{C} \setminus \mathbb{Z}$ satisfies $H^0 \equiv 0$ while for any $s \in]0,1]$:

1. $H^s(0) = \lambda_0(s)$,
2. $\frac{\mathrm{d}H^s(t)}{\mathrm{d}t} = X\big(H^s(t), t\big)$, thus $0 < \left|\frac{\mathrm{d}H^s(t)}{\mathrm{d}t}\right| \leq |\gamma'(t)|$ and $\arg \frac{\mathrm{d}H^s(t)}{\mathrm{d}t} \in]-\alpha+\theta, \theta+\alpha[$.

Denoting by $\lambda_0|_{[0,s]} : s' \in I \mapsto \lambda_0(s's)$ the restriction path, we see that the product of paths $F^s = \lambda_0|_{[0,s]}H^s$ has the following properties, for any $s \in]0,1]$:

1. the path F^s belongs to $\mathfrak{R}$,
2. $\mathrm{length}(F^s) \leq \mathrm{length}(\lambda_0|_{[0,s]}) + \mathrm{length}(H^s) \leq \mathrm{length}(\lambda) \leq L+1$,
3. for every $t \in [0,1]$, the right and left derivatives $(F^s)'(t)$ do not vanish and $\arg(F^s)'(t) \in]-\alpha+\theta, \theta+\alpha[$.

Therefore, F^s belongs $\mathfrak{R}^{(\theta,\alpha)}(L)$ and this means that the lifted map $\mathcal{H}$ given by (4.8) sends $I \times I$ into the space $\mathscr{R}^{(\theta,\alpha)}(L)$. We end the proof with the arguments recalled in the proof of proposition 4.3. $\square$

4.6 Comments

In resurgence theory, one has to deal as a rule with **endlessly continuable** functions. This notion is defined in [2], a more general definition of which being given by Ecalle in [5, 6]. The key point is the construction of **endless Riemann surfaces** and this is done in [11] for a slightly weaker version of endless continuablity. For such an endless Riemann surface, one can define "nearby sheets" in the way we did in Sect. 4.1 and analogues of propositions 4.3 and 4.6 can be stated.

References

1. F. Albrecht, I.D. Berg, *Geodesics in Euclidean space with analytic obstacle.* Proc. Amer. Math. Soc. **113** (1991), no. 1, 201-207.
2. B. Candelpergher, C. Nosmas, F. Pham, *Approche de la résurgence*, Actualités mathématiques, Hermann, Paris (1993).
3. W. Ebeling, *Functions of several complex variables and their singularities.* Translated from the 2001 German original by Philip G. Spain. Graduate Studies in Mathematics, 83. American Mathematical Society, Providence, RI, 2007.
4. J. Écalle, *Les algèbres de fonctions résurgentes*, Publ. Math. d'Orsay, Université Paris-Sud, 1981.05 (1981).
5. J. Ecalle, *Les fonctions résurgentes. Tome III : l'équation du pont et la classification analytique des objets locaux.* Publ. Math. d'Orsay, Université Paris-Sud, 1985.05 (1985).
6. J. Écalle, *Six lectures on transseries, Analysable functions and the Constructive proof of Dulac's conjecture*, Bifurcations and periodic orbits of vector fields (Montreal, PQ, 1992), 75-184, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 408, Kluwer Acad. Publ., Dordrecht, 1993.
7. O. Forster, *Lectures on Riemann Surfaces*, Graduate texts in mathematics; 81, Springer, New York (1981).
8. V. Gelfreich, D. Sauzin, *Borel summation and splitting of separatrices for the Hénon map*, Ann. Inst. Fourier (Grenoble) **51** (2001), no 2, 513-567.
9. J. Jost, *Riemannian geometry and geometric analysis.* Universitext. Springer-Verlag, Berlin, 1995.
10. Y. Ou, *On the stability by convolution product of a resurgent algebra.* Ann. Fac. Sci. Toulouse Math. (6) **19** (2010), no. 3-4, 687-705.
11. Y. Ou, *Sur la stabilité par produit de convolution d'algèbres de résurgence.* PhD thesis, Université d'Angers (2012).
12. D. Sauzin, *Introduction to 1-summability and the resurgence theory.* To appear.
13. D. Sauzin, *On the stability under convolution of resurgent functions.* Funkcial. Ekvac. **56** (2013), no. 3, 397-413.
14. D. Sauzin, *Nonlinear analysis with resurgent functions.* Preprint 2013, 35 pages. http://hal.archives-ouvertes.fr/hal-00766749

Chapter 5
Transseries and formal integral for Painlevé I

Abstract This chapter has two purposes. Our first goal is to construct the so-called "formal transseries solutions" for the prepared form associated with the first Painlevé equation, that will be used later on to get its truncated solutions : this is done in Sect. 5.3, after some preliminaries in Sect. 5.1 and some general nonsense on the "formal integrals" of Ecalle that we introduce in Sect. 5.2. Our second goal is to build the formal integral for the first Painlevé equation and this is what we do in Sect. 5.4. These informations will be used in a next chapter to investigate the resurgent structure for the first Painlevé equation.

5.1 Introduction

We return to the prepared form equation (3.6) associated with the first Painlevé equation, that we recall here:

$$P(\partial)w + \frac{1}{z}Q(\partial)w = F(z,w), \qquad P(\partial) = \partial^2 - 1, \quad Q(\partial) = -3\partial$$

and

$$F(z,w) = \frac{392}{625}\frac{1}{z^2} - \frac{4}{z^2}w + \frac{1}{2z^2}w^2 = f_0(z) + f_1(z)w + f_2(z)w^2,$$

We have seen in chapter 3 that the equation (3.6) has a unique formal solution, from now on denoted by $\widetilde{w}_{(0,0)} \in \mathbb{C}[[z^{-1}]]$, that $\widetilde{w}_{(0,0)}$ is 1-summable in every directions apart from the directions $k\pi$, $k \in \mathbb{Z}$ (theorem 3.2 and proposition 3.9). To the intervals $I_j =]0,\pi[+j\pi,\ j \in \mathbb{Z}$, one associates the Borel-Laplace sums, $w_{tri,j}(z) = \mathscr{S}^{I_j}\widetilde{w}_{(0,0)}(z)$ for z in $\overset{\bullet}{\mathscr{D}}(I_j,\tau)$. The domain $\overset{\bullet}{\mathscr{D}}(I_j,\tau)$ is a sectorial neighbourhoods of ∞ with aperture $\check{I}_j =]-\frac{3}{2}\pi, +\frac{1}{2}\pi[-j\pi$. As said in Remarks 3.5.2.3, $w_{tri,j} \in \Gamma(\check{I}_j, \mathcal{A}_1)$, $j = 0,1,2$ are sections of $\mathcal{A}_1$ that are asymptotic to the same 1-Gevrey series $\widetilde{w}_{(0,0)}$. Therefore the 1-coboundary $w_{tri,1} - w_{tri,0}$ belongs to $\Gamma(\check{I}_1 \cap \check{I}_0, \mathcal{A}^{\leq -1})$, while $w_{tri,2} - w_{tri,1}$ belongs to $\Gamma(\check{I}_2 \cap \check{I}_1, \mathcal{A}^{\leq -1})$. In other words, the 1-coboundaries

$$\begin{aligned} \mathcal{W}_{1,0}(z) &= w_{tri,1}(z) - w_{tri,0}(z), \quad -\frac{3}{2}\pi < \arg(z) < -\frac{1}{2}\pi, \quad |z| \text{ large enough} \\ \mathcal{W}_{2,1}(z) &= w_{tri,2}(z) - w_{tri,1}(z), \quad -\frac{5}{2}\pi < \arg(z) < -\frac{3}{2}\pi, \quad |z| \text{ large enough}, \end{aligned} \tag{5.1}$$

are exponentially flat functions of order 1 at infinity, and we deduce from equation (3.6) that $\mathcal{W}_{(j+1),j}$, $j = 0, 1$, satisfies the linear ODE:

$$P(\partial)\mathcal{W}_{(j+1),j} + \frac{1}{z}Q(\partial)\mathcal{W}_{(j+1),j} = f_1(z)\mathcal{W}_{(j+1),j} + f_2(z)(w_{tri,j+1} + w_{tri,j})\mathcal{W}_{(j+1),j}. \tag{5.2}$$

Question 5.1. Can we get more informations about $\mathcal{W}_{(j+1),j}$? In other words, can we analyze the Stokes phenonemon ?

Let us turn to the asymptotics. Denoting by T_1 the 1-Gevrey Taylor map, we set $\widetilde{\mathcal{W}}_{(j+1),j} = T_1(\breve{I}_{j+1} \cap \breve{I}_j)\mathcal{W}_{(j+1),j}$. We have $\widetilde{\mathcal{W}}_{(j+1),j} = 0$ by construction but, more interestingly for our purpose and since T_1 is a morphism of differential algebras, we deduce from (5.2) that $\widetilde{\mathcal{W}}_{(j+1),j}$ solves the problem $\mathfrak{P}_{\mathbf{0}}\widetilde{\mathcal{W}} = 0$, where $\mathfrak{P}_{\mathbf{0}}$ stands for the second order linear differential operator deduced from the operator $P(\partial) + \frac{1}{z}Q(\partial) - F(z, \cdot)$ by linearisation at $\widetilde{w}_{(0,0)}$:

$$\begin{aligned} \mathfrak{P}_{\mathbf{0}} &= P(\partial) + \frac{1}{z}Q(\partial) - \frac{\partial F(z, \widetilde{w}_{(0,0)})}{\partial w} \\ &= P(\partial) + \frac{1}{z}Q(\partial) - f_1(z) - 2\widetilde{w}_{(0,0)}(z) f_2(z) \\ &= (\partial^2 - 1) - \frac{3}{z}\partial + O(z^{-2}). \end{aligned} \tag{5.3}$$

The formal invariants of this equation is be governed by its Newton polygon at infinity $\mathcal{N}_\infty(\mathfrak{P}_{\mathbf{0}})$, drawn on Fig. 5.1.

See, e.g. [31]. We mention that the valuation v_∞ defined there is the opposite of our valuation val defined by (3.1).

The polygon $\mathcal{N}_\infty(\mathfrak{P}_{\mathbf{0}})$ has a single non-vertical side of slope -1: this corresponds to the fact that $\mathcal{W}_{(j+1),j}$, $j = 0, 1$, are exponentially flat functions of order 1 at infinity. The characteristic equation associated with this side is nothing but the equation

$$P(\mu) = 0, \qquad P(\mu) = \mu^2 - 1.$$

The polynomial $P(\mu)$ has two simple roots, $\mu_1 = -1$ and $\mu_2 = 1$. Therefore, from the theory of linear ODE [37, 31], we expect for $\mathcal{W}_{1,0}$ to behave like

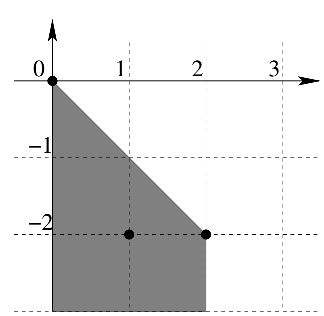


Fig. 5.1 The Newton polygon at infinity $\mathcal{N}_\infty(\mathfrak{P}_{\mathbf{0}})$ associated with the linear operator (5.3).

$e^{\mu_2 z} z^{-\tau_2} O(1)$ at infinity, and for $\mathcal{W}_{2,1}$ to behave like $e^{\mu_1 z} z^{-\tau_1} O(1)$ at infinity. Pursuing in that direction, the coefficients τ_1, τ_2 can be easily found : the $\widetilde{\mathcal{W}} = e^{\mu z} z^{-\tau} \widetilde{w}_\mu(z)$ solves the ODE (5.3) with $P(\mu) = 0$ and $\widetilde{w}_\lambda \in \mathbb{C}[[z^{-1}]]$, only under the condition

$$\tau = \frac{Q(\mu)}{P'(\mu)} = -\frac{3}{2}.$$

As a matter of fact, these behaviours are direct consequences of the analytic properties of the minor $\widehat{w}_{(0,0)}$ of $\widetilde{w}_{(0,0)}$. In particular, $\lambda_1 = -\mu_1$ and $\lambda_2 = -\mu_2$ are precisely the seen singularities of $\widehat{w}_{(0,0)}$.

The differential equation $\mathfrak{P}_{\mathbf{0}} \widetilde{\mathcal{W}} = 0$ has thus its general formal solution under the form $\widetilde{\mathcal{W}} = U_1 e^{\mu_1 z} z^{-\tau_1} \widetilde{w}_{\mu_1} + U_2 e^{\mu_2 z} z^{-\tau_2} \widetilde{w}_{\mu_2}$ and, as we will see later on, both $\widetilde{w}_{\mu_1}$ and $\widetilde{w}_{\mu_2}$ are 1-Gevery series whose minors have the same properties than $\widehat{w}_{(0,0)}$.

However, the expectation that $\mathcal{W}_{1,0}$ could be obtained from $U_1 e^{\mu_1 z} z^{-\tau_1} \widetilde{w}_{\mu_1}$ by Laplace-Borel summation for some well-chosen $U_1 \in \mathbb{C}$ is wrong. Indeed, this would mean that $w_{tri,1} = \mathscr{S}^{I_1}\left(\widetilde{w}_{(0,0)} + U_1 e^{\mu_1 z} z^{-\tau_1} \widetilde{w}_{\mu_1}\right)$, thus $\widetilde{w}_{(0,0)} + U_1 e^{\mu_1 z} z^{-\tau_1} \widetilde{w}_{\mu_1}$ is a formal solution of (3.6). This is not the case because of the nonlinearity of (3.6) and to the very nature of the Riemann surface $\mathscr{R}^{(1)}$ on which $\widehat{w}_{(0,0)}$ can be analytically continued (theorem 4.1). This raises the question : can we define an analogue of the general formal solution for the non-linear equation (3.6) ? The answer is given by the notion of "formal integral".

5.2 Formal integral : setting

5.2.1 Notations

It will be useful in the sequel to fix customary notations.

Definition 5.1. We suppose $n \in \mathbb{N}^\star$, $\boldsymbol{k}, \boldsymbol{h} \in \mathbb{N}^n$, $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{C}^n$.

- If $\boldsymbol{k} = (k_1, \cdots, k_n)$, we note $|\boldsymbol{k}| = k_1 + \cdots + k_n$.
- If $\boldsymbol{a} = (a_1, \cdots, a_n)$ or $\boldsymbol{a} = {}^t(a_1, \cdots, a_n)$, we note $\boldsymbol{a}^{\boldsymbol{k}} = a_1^{k_1} \cdots a_n^{k_n}$.
- If $\boldsymbol{b} = (b_1, \cdots, b_n)$, we note $\boldsymbol{a}.\boldsymbol{b} = a_1 b_1 + \cdots + a_n b_n$.
- We note $\boldsymbol{e}_j$ the j^{th} unit vector of $\mathbb{C}^n$.

5.2.2 Setting

5.2.2.1 Single level 1 ODE

To introduce the reader to the notion of Ecalle's **formal integral** [18], it will be useful to skip a moment from the ODE (3.6) to a more general one[1] with the same kind of properties. Namely we introduce

---

[1] Though far from the more general. For instance in (5.4) one could replace $F(z,w)$ by $F(z, w, \partial w, \cdots, \partial^{n-1} w)$, with F holomorphic in a neighbourhood of $(\infty, \mathbf{0})$ in $\mathbb{C} \times \mathbb{C}^{n-1}$, see exercise 3.1. We refrain of doing that only for a matter of simplicity. See [18] for more general results.

$$P(\partial)w + \frac{1}{z}Q(\partial)w = F(z,w) \tag{5.4}$$

$$P(\partial) = \sum_{m=0}^{n} \alpha_{n-m}\partial^m \in \mathbb{C}[\partial] \ , \ Q(\partial) = \sum_{m=0}^{n-1} \beta_{n-m}\partial^m \in \mathbb{C}[\partial]$$

with $n \in \mathbb{N}^\star$. We assume that P is a polynomial of degree n, that is $\alpha_0 \neq 0$, and that $F(z,w)$ is holomorphic in a neighbourhood of $(\infty, 0)$ in $\mathbb{C}^2$ with the condition $\frac{\partial^m F}{\partial w^m}(z,0) = O(z^{-2})$, $m \in \mathbb{N}$. (See exercise 3.1). We will add other assumptions to guarantee that the ODE (5.4) has a single level 1 at infinity.

When assuming furthermore that $\alpha_n \neq 0$, what have been said in Sect. 5.1 can be applied as well for (5.4). The equation (5.4) has a unique formal solution $\widetilde{w}_{\mathbf{0}} \in \mathbb{C}[[z^{-1}]]$ and $\operatorname{val} \widetilde{w}_{\mathbf{0}} \geq 2$. The Newton polygon at infinity $\mathcal{N}_\infty(\mathfrak{P}_{\mathbf{0}})$ associated with the linear differential operator $\mathfrak{P}_{\mathbf{0}} = P(\partial) + \frac{1}{z}Q(\partial) - \frac{\partial F}{\partial w}(z, \widetilde{w}_{\mathbf{0}})$ deduced from the operator $P(\partial) + \frac{1}{z}Q(\partial) - F(z,\cdot)$ by linearisation at $\widetilde{w}_{\mathbf{0}}$, has still a single non-vertical side of slope -1 and the characteristic equation associated with this single side remains the equation $P(\mu) = 0$.

Since $\alpha_n \neq 0$, the roots of the characteristic equation do not vanish. We will also assume that the polynomial

$$\mu \mapsto P(\mu) = \sum_{m=0}^{n} \alpha_{n-m}\mu^m = \alpha_0(\mu - \mu_1)\cdots(\mu - \mu_n)$$

has only simple roots $\mu = \mu_i$, $i = 1, \cdots, n$. In what follows, we adapt the following definitions from [2, 18].

Definition 5.2. Let $\{\mu_i\}$ be the set of the roots of the polynomial $P(\mu) = 0$, and we note $\lambda_i = -\mu_i$, $i = 1, \cdots, n$. The complex numbers $\lambda_1, \cdots, \lambda_n$ are called the **multipliers** of the ODE (5.4).
The ODE (5.4) is said to have a **single level** 1 **at infinity** when the multipliers are all nonzero.
One says that the multipliers are **non-resonant** when they are rationally independent, that is linearly independent over $\mathbb{Z}$. The multipliers are **positively resonant** when there exists $\boldsymbol{k}_{\text{reson}} = (k_1, \cdots, k_n) \in \mathbb{N}^n \setminus \{\mathbf{0}\}$ so that $\boldsymbol{\lambda}.\boldsymbol{k}_{\text{reson}} = 0$, where $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_n) \in (\mathbb{C}^\star)^n$ The number $|\boldsymbol{k}_{\text{reson}}| + 1$ is the order of the resonance, since the positive resonance brings **semi-positively resonances**, that is relationships of the type $\boldsymbol{\lambda}.(\boldsymbol{k}_{\text{reson}} + \boldsymbol{e}_j) = \lambda_j$ for any $j \in [1, n]$.

We mention that the following constants are properly defined, since P has only simple roots:

$$\tau_i = \frac{Q(-\lambda_i)}{P'(-\lambda_i)}, \qquad i = 1, \cdots, n. \tag{5.5}$$

From the theory of linear differential equations (see [31, 3, 37]), we notice that the linear equation $P(\partial)w + \frac{1}{z}Q(\partial)w = 0$ has a formal general solution under the form

$$w(z) = \sum_{i=1}^{n} v_i(z) y_i(z). \tag{5.6}$$

In (5.6), $y_i(z) = U_i \mathrm{e}^{-\lambda_i z} z^{-\tau_i}$, $U_i \in \mathbb{C}$, stands for the general solution of the differential equation $y_i' + \left(\lambda_i + \dfrac{\tau_i}{z}\right) y_i = 0$, while $v_i \in \mathbb{C}[[z^{-1}]]$ is invertible and is determined uniquely up to normalization.

5.2.2.2 Companion system, prepared form

Formal integrals have more natural foundations when differential equations of order one are considered. We therefore translate the ODE (5.4) into a one order ODE of dimension n by introducing $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} w \\ w' \\ \vdots \\ w^{(n-1)} \end{pmatrix}$. We get the companion system

$$\partial \mathbf{w} + A\mathbf{w} = \mathbf{f}(z, \mathbf{w}), \tag{5.7}$$

with $A = \begin{pmatrix} 0 & -1 \cdots & & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & 0 & -1 \\ \frac{\alpha_n}{\alpha_0} + \frac{\beta_n}{z\alpha_0} & \cdots & \cdots & \frac{\alpha_1}{\alpha_0} + \frac{\beta_1}{z\alpha_0} \end{pmatrix}$ and $\mathbf{f}(z, \mathbf{w}) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ F(z, w_1)/\alpha_0 \end{pmatrix}$.

Since (5.4) has a unique formal solution $\widetilde{w}_{\mathbf{0}} \in \mathbb{C}[[z^{-1}]]$, $\operatorname{val} \widetilde{w}_{\mathbf{0}} \geq 2$, we may remark that (5.7) has a unique formal solution $\widetilde{\mathbf{w}}_{\mathbf{0}} \in \mathbb{C}^n[[z^{-1}]]$ as well, and in fact $\widetilde{\mathbf{w}}_{\mathbf{0}} \in z^{-2}\mathbb{C}^n[[z^{-1}]]$.

Lemma 5.1. *There exists $T_0(z) \in GL_n(\mathbb{C}\{z^{-1}\}[z])$ so that the meromorphic gauge transform $\mathbf{w} = T_0(z)\mathbf{v}$ brings (5.7) into the **prepared form***

$$\partial \mathbf{v} + B_0 \mathbf{v} = \mathbf{g}(z, \mathbf{v}), \quad B_0 = \begin{pmatrix} \lambda_1 + \frac{\tau_1}{z} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n + \frac{\tau_n}{z} \end{pmatrix}, \tag{5.8}$$

with $\mathbf{g}$ a $\mathbb{C}^n$-valued function, holomorphic in a neighbourhood of $(\infty, \mathbf{0})$ and $\mathbf{g}(z, \mathbf{v}) = O(z^{-2}) + O(\|\mathbf{v}\|^2)$ when $z \to \infty$ and $\mathbf{v} \to \mathbf{0}$.
The prepared form (5.8) has a unique formal solution $\widetilde{\mathbf{v}}_{\mathbf{0}} \in \mathbb{C}^n[[z^{-1}]]$ and $\widetilde{\mathbf{v}}_{\mathbf{0}} \in z^{-2}\mathbb{C}^n[[z^{-1}]]$.

Proof. The proof is based on classical ideas for linear ODEs ([31, 3, 37], see also [14]). Looking at (5.6), we compare (5.7) with the linear equation

$$\partial \mathbf{u} + B_0 \mathbf{u} = 0, \quad B_0 = \begin{pmatrix} \lambda_1 + \frac{\tau_1}{z} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n + \frac{\tau_n}{z} \end{pmatrix} = \Lambda + \frac{1}{z} L, \tag{5.9}$$

whose general solution (holomorphic on $\underset{\bullet}{\mathbb{C}}$) is given in term of the fundamental matrix solution $z^{-L}\mathrm{e}^{-z\Lambda}$,

$$\mathbf{u}(z) = z^{-L}\mathrm{e}^{-z\Lambda}\boldsymbol{U} = \oplus_{i=1}^{n} z^{-\tau_i}\mathrm{e}^{-z\lambda_i}\boldsymbol{U}, \qquad \boldsymbol{U} \in \mathbb{C}^n. \tag{5.10}$$

We remark that

$$\left(\mathrm{e}^{-\lambda z} z^{-\tau}\right)^{(m)} = \mathrm{e}^{-\lambda z} z^{-\tau} \sum_{j=0}^{m} \binom{m}{j} (-\lambda)^{m-j} \frac{(-\tau)_j}{z^j} \tag{5.11}$$

for $(\lambda, \tau) \in \mathbb{C}^2$ and $m \in \mathbb{N}$, where $(-\tau)_j = j! \binom{-\tau}{j}$ mimics the Pochhammer symbol:

$$(-\tau)_0 = 1 \text{ and } (-\tau)_j = (-1)^j \tau(\tau+1)\cdots\big(\tau+j-1\big) \text{ for } j \geq 1. \tag{5.12}$$

We thus set the meromorphic gauge transform $\mathbf{w} = T_0(z)\mathbf{v}$ with $T_0(z) \in GL_n(\mathbb{C}\{z^{-1}\}[z])$ of the form:

$$T_0(z) = \begin{pmatrix} 1 & \cdots & 1 \\ -\lambda_1 - \frac{\tau_1}{z} & \cdots & -\lambda_n - \frac{\tau_n}{z} \\ \vdots & & \vdots \\ \sum_{j=0}^{n-1} \binom{n-1}{j} (-\lambda_1)^{n-1-j} \frac{(-\tau_1)_j}{z^j} & \cdots & \sum_{j=0}^{n-1} \binom{n-1}{j} (-\lambda_n)^{n-1-j} \frac{(-\tau_n)_j}{z^j} \end{pmatrix}. \tag{5.13}$$

By its very definition, this gauge transform brings (5.7) into the differential equation:

$$\begin{aligned} \partial \mathbf{v} &= -\left[T_0^{-1}(\partial T_0) + T_0^{-1} A T_0\right] \mathbf{v} + T_0^{-1} \mathbf{f}(z, T_0 \mathbf{v}) \\ &= -B_0 \mathbf{v} + \mathbf{g}(z, \mathbf{v}) \end{aligned} \tag{5.14}$$

where $\mathbf{g}$ has the properties described in the lemma. The fact that (5.8) has a unique formal solution $\widetilde{\mathbf{v}}_{\mathbf{0}} \in \mathbb{C}^n[[z^{-1}]]$ is obvious. □

Example 5.1. We have already seen that the companion system associated with (3.6) is (3.9). The gauge transform $\mathbf{w} = T_0(z)\mathbf{v}$, $T_0(z) = \begin{pmatrix} 1 & 1 \\ -1 + \frac{3}{2z} & 1 + \frac{3}{2z} \end{pmatrix}$, brings (3.9) into the prepared form:

$$\partial \mathbf{v} + \begin{pmatrix} 1 - \frac{3}{2z} & 0 \\ 0 & -1 - \frac{3}{2z} \end{pmatrix} \mathbf{v} = \frac{15}{8z^2} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \mathbf{v} + \frac{1}{2} \begin{pmatrix} -F(z, v_1 + v_2) \\ F(z, v_1 + v_2) \end{pmatrix}. \tag{5.15}$$

Remark 5.1. Let us consider the action of the gauge transform $\mathbf{y} = T_0(z)\mathbf{u}$ on the differential equation $\partial \mathbf{u} + B_0 \mathbf{u} = 0$. This differential equation is transformed into the system $\partial \mathbf{y} + A_0 \mathbf{y} = 0$ with $A_0 = T_0 B_0 T_0^{-1} - (\partial T_0) T_0^{-1}$ of the form $A_0 = \begin{pmatrix} 0 & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & 0 & -1 \\ p_n(z) & \cdots & \cdots & p_1(z) \end{pmatrix}$ with $p_n, \cdots, p_1 \in \mathbb{C}\{z^{-1}\}$ satisfying $p_n(z) = \frac{\alpha_n}{\alpha_0} + \frac{\beta_n}{z\alpha_0} + O(z^{-2}), \cdots, p_1(z) = \frac{\alpha_1}{\alpha_0} + \frac{\beta_1}{z\alpha_0} + O(z^{-2})$. The system $\partial \mathbf{y} + A_0 \mathbf{y} = 0$ is the companion system for the one-dimensional homogeneous ODE of order n,

$$\partial^n y + p_1(z) \partial^{n-1} y + \cdots p_n(z) y = 0, \tag{5.16}$$

whose general solution is $y(z) = \sum_{i=1}^{n} U_i \mathrm{e}^{-\lambda_i z} z^{-\tau_i}$, $(U_1, \cdots, U_n) \in \mathbb{C}^n$.

5.2.2.3 Normal form, formal reduction

We have previously reduced the companion system (5.7) to a prepared form through a meromorphic gauge transform. Under some conditions, one can go further, but through formal transformations, in the spirit of the Poincaré-Dulac theorem for vector fields [2].

Proposition 5.1. *We consider the ODE (5.8) and we assume that the multipliers $\lambda_1, \cdots, \lambda_n$ are non-resonant. Then there exists a formal transformation $\mathbf{v} = \widetilde{T}(z, \mathbf{u})$,*

$$\widetilde{T}(z, \mathbf{u}) = \sum_{\boldsymbol{k} \in \mathbb{N}^n} \mathbf{u}^{\boldsymbol{k}} \widetilde{\mathbf{v}}_{\boldsymbol{k}}(z), \qquad \widetilde{\mathbf{v}}_{\boldsymbol{k}}(z) \in \mathbb{C}^n[[z^{-1}]], \tag{5.17}$$

which formally transforms (5.8) into the linear **normal form** *equation $\partial \mathbf{u} + B_0 \mathbf{u} = 0$. In (5.17), $\widetilde{\mathbf{v}}_{\mathbf{0}}$ stands for the unique formal solution of (5.8); for $j = 1, \ldots, n$, $\widetilde{\mathbf{v}}_{\mathbf{e}_j}$ is uniquely determined when one prescribes its constant term to be equal to $\mathbf{e}_j$; then the formal series $\widetilde{\mathbf{v}}_{\boldsymbol{k}}$ are unique for $|\boldsymbol{k}| > 1$.*

We will see in the sequel how this proposition can be shown. Here, we rather concentrate on its consequences.

One can refer to, e.g., [30, 4] for a proof that extend to possibly nilpotent cases (but with no resonances), and to [18] for a very general frame.

We know that the general solution of the normal form $\partial \mathbf{u} + B_0 \mathbf{u} = 0$ is $\mathbf{u}(z) = \oplus_{i=1}^n z^{-\tau_i} \mathrm{e}^{-z\lambda_i} ({}^t\boldsymbol{U})$, $\boldsymbol{U} = (U_1, \cdots, U_n) \in \mathbb{C}^n$. Through the action of the formal transformation $\mathbf{v} = \widetilde{T}(z, \mathbf{u})$, this provides the following general formal solution for the ODE (5.8):

$$\widetilde{\mathbf{v}}(z, \boldsymbol{U}) = \sum_{\boldsymbol{k} = (k_1, \cdots, k_n) \in \mathbb{N}^n} \prod_{i=1}^n (U_i z^{-\tau_i} \mathrm{e}^{-z\lambda_i})^{k_i} \widetilde{\mathbf{v}}_{\boldsymbol{k}}(z) = \sum_{\boldsymbol{k} \in \mathbb{N}^n} \boldsymbol{U}^{\boldsymbol{k}} \mathrm{e}^{-\boldsymbol{\lambda}.\boldsymbol{k} z} z^{-\boldsymbol{\tau}.\boldsymbol{k}} \widetilde{\mathbf{v}}_{\boldsymbol{k}}(z) \tag{5.18}$$

with $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_n) \in (\mathbb{C}^\star)^n$ and $\boldsymbol{\tau} = (\tau_1, \cdots, \tau_n) \in \mathbb{C}^n$.

Definition 5.3. The formal series (5.18) is called the **formal integral** of (5.8).

Of course, one can obtain the formal integral $\widetilde{\mathbf{w}}(z, \mathbf{U})$ of (5.7) as well, by the gauge transform $\widetilde{\mathbf{w}} = T_0(z) \widetilde{\mathbf{v}}$, with $T_0(z)$ given by (5.13). When finally returning to the n-th order ODE (5.4) of dimension 1 we started with, one gets its formal integral.

Definition 5.4. We assume that the multipliers are non-resonant. The **formal integral** $\widetilde{w}(z, \boldsymbol{U})$ of the ODE (5.4) is defined by:

$$\begin{aligned} \widetilde{w}(z, \boldsymbol{U}) &= \sum_{\boldsymbol{k} \in \mathbb{N}^n} \boldsymbol{U}^{\boldsymbol{k}} \mathrm{e}^{-\boldsymbol{\lambda}.\boldsymbol{k} z} z^{-\boldsymbol{\tau}.\boldsymbol{k}} \widetilde{w}_{\boldsymbol{k}}(z), \quad \widetilde{w}_{\boldsymbol{k}}(z) = \widetilde{\mathbf{v}}_{\boldsymbol{k}}(z) \boldsymbol{.} (1, \cdots, 1) \in \mathbb{C}[[z^{-1}]], \\ &= \widetilde{\Phi}(z, U_1 \mathrm{e}^{-\lambda_1 z} z^{-\tau_1}, \cdots, U_n \mathrm{e}^{-\lambda_n z} z^{-\tau_n}) \end{aligned} \tag{5.19}$$

with $\widetilde{\Phi}(z, \mathbf{u}) = \sum_{\boldsymbol{k} \in \mathbb{N}^n} \mathbf{u}^{\boldsymbol{k}} \widetilde{w}_{\boldsymbol{k}}(z) \in \mathbb{C}[[z^{-1}, \mathbf{u}]]$. The formal transformation $w = \widetilde{\Phi}(z, \mathbf{u})$ formally transforms (5.4) into the normal form equation $\partial \mathbf{u} + B_0 \mathbf{u} = 0$.

The formal integral (5.19), thus depending on the maximal n free parameters $\boldsymbol{U} = (U_1, \cdots, U_n) \in \mathbb{C}^n$, plays the role of the general formal solution for the ODE (5.4) of order n. Formal integrals can be defined as well for difference and differential-difference equations, see, e.g. [18, 30]. This notion has been enlarged for nonlinear partial differential equations in [33].

Remark 5.2. Although working at the formal level, one may wonder what is the chosen branch when we write $z^{-\boldsymbol{\tau}.\boldsymbol{k}}$. As a matter of fact, this is not relevant at this stage since moving from a determination to another one just translates into rescaling the free parameter $\mathbf{U}$.

Remark 5.3. Introducing $\boldsymbol{V}^{\boldsymbol{k}} = \boldsymbol{U}^{\boldsymbol{k}} \mathrm{e}^{-\boldsymbol{\lambda}.\boldsymbol{k}z} z^{-\boldsymbol{\tau}.\boldsymbol{k}}$, we remark the identity:

$$\partial_z\big(\boldsymbol{V}^{\boldsymbol{k}}\widetilde{w}_{\boldsymbol{k}}\big) = \left[\Big(\partial_z - \sum_{i=1}^{n}(\lambda_i + \frac{\tau_i}{z})u_i\partial_{u_i}\Big)\big(\mathbf{u}^{\boldsymbol{k}}\widetilde{w}_{\boldsymbol{k}}\big)\right]|_{\mathbf{u}=\boldsymbol{V}}.$$

Looking at the equality

$$\widetilde{w}(z,\boldsymbol{U}) = \widetilde{\Phi}(z, U_1\mathrm{e}^{-\lambda_1 z}z^{-\tau_1}, \cdots, U_n\mathrm{e}^{-\lambda_n z}z^{-\tau_n}) \tag{5.20}$$

and since the formal integral (5.19) solves the differential equation (5.4), one deduces that $\widetilde{\Phi}$ satisfies:

$$P\Big(\partial_z - \sum_{i=1}^{n}(\lambda_i + \frac{\tau_i}{z})u_i\partial_{u_i}\Big)\widetilde{\Phi} + \frac{1}{z}Q\Big(\partial_z - \sum_{i=1}^{n}(\lambda_i + \frac{\tau_i}{z})u_i\partial_{u_i}\Big)\widetilde{\Phi} = F(z,\widetilde{\Phi}). \tag{5.21}$$

5.2.3 Formal integral, general considerations

Under convenient hypotheses, we have previously introduced the formal integral for the ODE (5.4), that is a n-parameters formal expansion of the form

$$w(z,\boldsymbol{U}) = \sum_{\boldsymbol{k}\in\mathbb{N}^n} \boldsymbol{U}^{\boldsymbol{k}} \mathrm{e}^{-\boldsymbol{\lambda}.\boldsymbol{k}z} z^{-\boldsymbol{\tau}.\boldsymbol{k}} w_{\boldsymbol{k}}(z), \qquad \boldsymbol{\lambda}, \boldsymbol{\tau} \in \mathbb{C}^n, \tag{5.22}$$

Let us start with (5.22) and investigate the conditions to impose on the $w_{\boldsymbol{k}}$'s in order for (5.22) to be formally solution of (5.4).

We could start with (5.21) as well.

Using the identity (5.11) for $m \in \mathbb{N}$, one obtains from (5.22):

$$\begin{aligned} w^{(m)} &= \sum_{|\boldsymbol{k}|\geq 0} \boldsymbol{U}^{\boldsymbol{k}} \sum_{p=0}^{m} \binom{m}{p} (\mathrm{e}^{-\boldsymbol{\lambda}.\boldsymbol{k}z} z^{-\boldsymbol{\tau}.\boldsymbol{k}})^{(p)} w_{\boldsymbol{k}}^{(m-p)} \\ &= \sum_{|\boldsymbol{k}|\geq 0} \boldsymbol{U}^{\boldsymbol{k}} \mathrm{e}^{-\boldsymbol{\lambda}.\boldsymbol{k}z} z^{-\boldsymbol{\tau}.\boldsymbol{k}} T_{\boldsymbol{k},m+1}(w_{\boldsymbol{k}}) \end{aligned}$$

where $T_{(0,0),m+1}(w_{\boldsymbol{0}}) = w_{\boldsymbol{0}}^{(m)}$ and, more generally for $\boldsymbol{k} \in \mathbb{N}^2$,

$$T_{\boldsymbol{k},m+1}(w_{\boldsymbol{k}}) = \sum_{p=0}^{m} \binom{m}{p} \Big[\sum_{j=0}^{p} \binom{p}{j} (-\boldsymbol{\lambda}.\boldsymbol{k})^{p-j} \frac{(-\boldsymbol{\tau}.\boldsymbol{k})_j}{z^j}\Big] w_{\boldsymbol{k}}^{(m-p)}$$
$$= \sum_{j=0}^{m} \binom{m}{j} \frac{(-\boldsymbol{\tau}.\boldsymbol{k})_j}{z^j} \Big[\sum_{q=0}^{m-j} \binom{m-j}{q} (-\boldsymbol{\lambda}.\boldsymbol{k})^{m-j-q} w_{\boldsymbol{k}}^{(q)}\Big],$$

that is also

$$T_{\boldsymbol{k},m+1}(w_{\boldsymbol{k}}) = \sum_{j=0}^{m} \binom{m}{j} \frac{(-\boldsymbol{\tau}.\boldsymbol{k})_j}{z^j} \Big[(-\boldsymbol{\lambda}.\boldsymbol{k} + \partial)^{m-j} w_{\boldsymbol{k}}\Big]. \tag{5.23}$$

In what follows we will simply write $T_{\boldsymbol{k},m+1}$ instead of $T_{\boldsymbol{k},m+1}(w_{\boldsymbol{k}})$. We introduce the notation

$$\boldsymbol{V}^{\boldsymbol{k}} = \boldsymbol{U}^{\boldsymbol{k}} \mathrm{e}^{-\boldsymbol{\lambda}.\boldsymbol{k}z} z^{-\boldsymbol{\tau}.\boldsymbol{k}}.$$

and we notice that for every $\boldsymbol{k}_1, \boldsymbol{k}_2 \in \mathbb{N}^n$, $\boldsymbol{V}^{\boldsymbol{k}_1}\boldsymbol{V}^{\boldsymbol{k}_2} = \boldsymbol{V}^{\boldsymbol{k}_1+\boldsymbol{k}_1}$.
On the one hand,

$$P(\partial)w = \sum_{k=0}^{\infty} \boldsymbol{V}^{\boldsymbol{k}} \Big[\sum_{m=0}^{n} \alpha_{n-m} T_{\boldsymbol{k},m+1}\Big] = \sum_{|\boldsymbol{k}|\geq 0} \boldsymbol{V}^{\boldsymbol{k}} p_{\boldsymbol{k}}(\partial) w_{\boldsymbol{k}} \tag{5.24}$$

where for $|\boldsymbol{k}| \geq 0$,

$$p_{\boldsymbol{k}}(\partial) = \sum_{m=0}^{n} \alpha_{n-m}(-\boldsymbol{\lambda}.\boldsymbol{k} + \partial)^m$$
$$+ \sum_{m=1}^{n} \alpha_{n-m}\Big\{\sum_{j=1}^{m} \binom{m}{j} \frac{(-\boldsymbol{\tau}.\boldsymbol{k})_j}{z^j} (-\boldsymbol{\lambda}.\boldsymbol{k} + \partial)^{m-j}\Big\}.$$

In other words, for $|\boldsymbol{k}| \geq 0$,

$$p_{\boldsymbol{k}}(\partial) = P(-\boldsymbol{\lambda}.\boldsymbol{k} + \partial) + \sum_{j=1}^{n} \frac{1}{z^j} \binom{-\boldsymbol{\tau}.\boldsymbol{k}}{j} P^{(j)}(-\boldsymbol{\lambda}.\boldsymbol{k} + \partial). \tag{5.25}$$

Similarly

$$Q(\partial)w = \sum_{|\boldsymbol{k}|\geq 0} \boldsymbol{V}^{\boldsymbol{k}} q_{\boldsymbol{k}}(\partial) w_{\boldsymbol{k}} \tag{5.26}$$

with

$$q_{\boldsymbol{k}}(\partial) = Q(-\boldsymbol{\lambda}.\boldsymbol{k} + \partial) + \sum_{j=1}^{n-1} \frac{1}{z^j} \binom{-\boldsymbol{\tau}.\boldsymbol{k}}{j} Q^{(j)}(-\boldsymbol{\lambda}.\boldsymbol{k} + \partial). \tag{5.27}$$

On the other hand we consider the Taylor expansion of $F(z, w(z, \boldsymbol{U}))$ at $w_{\boldsymbol{0}}$, namely

$$F(z,w) = F(z,w_{\boldsymbol{0}}) + \sum_{\ell\geq 1} \frac{\Big(\sum_{|\boldsymbol{k}|\geq 1} \boldsymbol{V}^{\boldsymbol{k}} w_{\boldsymbol{k}}\Big)^{\ell}}{\ell!} \frac{\partial^{\ell} F(z,w_{\boldsymbol{0}})}{\partial w^{\ell}}. \tag{5.28}$$

We observe that for every $\ell \in \mathbb{N}^{\star}$,

$$\Big(\sum_{|\boldsymbol{k}|\geq 1} \boldsymbol{V}^{\boldsymbol{k}} w_{\boldsymbol{k}}\Big)^{\ell} = \sum_{|\boldsymbol{p}|\geq \ell} \boldsymbol{V}^{\boldsymbol{p}} \sum_{\substack{\boldsymbol{p}_1+\cdots+\boldsymbol{p}_\ell=\boldsymbol{p} \\ |\boldsymbol{p}_i|\geq 1,\, 1\leq i\leq \ell}} w_{\boldsymbol{p}_1}\cdots w_{\boldsymbol{p}_\ell}. \tag{5.29}$$

As a result, equation (5.28) reads

$$F(z,w) = F(z,w_{\boldsymbol{0}}) + \sum_{\substack{\ell\geq 1 \\ |\boldsymbol{p}|\geq \ell}} \boldsymbol{V}^{\boldsymbol{p}} \sum_{\substack{\boldsymbol{p}_1+\cdots+\boldsymbol{p}_\ell=\boldsymbol{p} \\ |\boldsymbol{p}_i|\geq 1,\, 1\leq i\leq \ell}} \frac{w_{\boldsymbol{p}_1}\cdots w_{\boldsymbol{p}_\ell}}{\ell!} \frac{\partial^\ell F(z,w_{\boldsymbol{0}})}{\partial w^\ell} \tag{5.30}$$

Finally, plugging the formal expansion (5.22) into the differential equation (5.4), using the identities (5.24), (5.26), (5.30) and identifying the powers $\boldsymbol{V}^{\boldsymbol{k}}$, one gets the next lemma 5.2 which justifies the following definition.

Definition 5.5. For $\boldsymbol{k}\in\mathbb{N}^n$, we define

$$\begin{aligned} P_{\boldsymbol{k}}(\partial) &= P(-\boldsymbol{\lambda}.\boldsymbol{k}+\partial), \\ Q_{\boldsymbol{k}}(\partial) &= -\boldsymbol{\tau}.\boldsymbol{k}P'(-\boldsymbol{\lambda}.\boldsymbol{k}+\partial) + Q(-\boldsymbol{\lambda}.\boldsymbol{k}+\partial) \end{aligned} \tag{5.31}$$

$$R_{\boldsymbol{k}}(\partial) = \sum_{j=0}^{n-2} \frac{1}{z^j}\left[\binom{-\boldsymbol{\tau}.\boldsymbol{k}}{j+2} P^{(j+2)}(-\boldsymbol{\lambda}.\boldsymbol{k}+\partial) + \binom{-\boldsymbol{\tau}.\boldsymbol{k}}{j+1} Q^{(j+1)}(-\boldsymbol{\lambda}.\boldsymbol{k}+\partial)\right]. \tag{5.32}$$

For $\boldsymbol{k}\in\mathbb{N}^n$, we denote by $\mathfrak{D}_{\boldsymbol{k}} = \mathfrak{D}_{\boldsymbol{k}}(w_{\boldsymbol{0}})$ the linear differential operator

$$\mathfrak{D}_{\boldsymbol{k}} = P_{\boldsymbol{k}}(\partial) + \frac{1}{z}Q_{\boldsymbol{k}}(\partial) + \frac{1}{z^2}R_{\boldsymbol{k}} - \frac{\partial F(z,w_{\boldsymbol{0}})}{\partial w}$$

where $w_{\boldsymbol{0}}$ satisfies $P(\partial)w_{\boldsymbol{0}} + \frac{1}{z}Q(\partial)w_{\boldsymbol{0}} = F(z,w_{\boldsymbol{0}})$.
For $\boldsymbol{k}\in\mathbb{N}^n$, we denote by $\mathfrak{P}_{\boldsymbol{k}} = \mathfrak{P}_{\boldsymbol{k}}(w_{\boldsymbol{0}})$ the linear differential operator

$$\mathfrak{P}_{\boldsymbol{k}} = P(-\boldsymbol{\lambda}.\boldsymbol{k}+\partial) + \frac{1}{z}Q(-\boldsymbol{\lambda}.\boldsymbol{k}+\partial) - \frac{\partial F(z,w_{\boldsymbol{0}})}{\partial w}. \tag{5.33}$$

Lemma 5.2. *The n-parameters formal expansion*

$$w(z,\boldsymbol{U}) = \sum_{\boldsymbol{k}\in\mathbb{N}^n} \boldsymbol{U}^{\boldsymbol{k}} \mathrm{e}^{-\boldsymbol{\lambda}.\boldsymbol{k}z} z^{-\boldsymbol{\tau}.\boldsymbol{k}} w_{\boldsymbol{k}}(z) \tag{5.34}$$

solves (5.4) if and only if :

$$P(\partial)w_{\boldsymbol{0}} + \frac{1}{z}Q(\partial)w_{\boldsymbol{0}} = F(z,w_{\boldsymbol{0}}), \tag{5.35}$$

$$\mathfrak{D}_{\boldsymbol{e}_i} w_{\boldsymbol{e}_i} = 0 \tag{5.36}$$

with $\boldsymbol{e}_i$ the i-th vector of the canonical base of $\mathbb{C}^n$, and for $|\boldsymbol{k}|\geq 2$,

$$\mathfrak{D}_{\boldsymbol{k}} w_{\boldsymbol{k}} = \sum_{\substack{\boldsymbol{k}_1+\cdots+\boldsymbol{k}_\ell=\boldsymbol{k} \\ |\boldsymbol{k}_i|\geq 1,\, \ell\geq 2}} \frac{w_{\boldsymbol{k}_1}\cdots w_{\boldsymbol{k}_\ell}}{\ell!} \frac{\partial^\ell F(z,w_{\boldsymbol{0}})}{\partial w^\ell}. \tag{5.37}$$

Remark 5.4. Notice that in lemma 5.2 we have neither supposed that $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_n)$ are the multipliers, nor that $\boldsymbol{\tau} = (\tau_1, \cdots, \tau_n)$ are such that $\tau_i = \dfrac{Q(-\lambda_i)}{P'(-\lambda_i)}$, $i = 1, \cdots, n$. However, these conditions will come in the next section.

Example 5.2. We take equation (3.6) where $n = 2$, $P(\partial) = \partial^2 - 1$, $Q(\partial) = -3\partial$. Then, for every $\boldsymbol{k} \in \mathbb{N}^2$,

$$\begin{aligned} P_{\boldsymbol{k}}(\partial) &= \partial^2 - 2\boldsymbol{\lambda}.\boldsymbol{k}\partial + (\boldsymbol{\lambda}.\boldsymbol{k})^2 - 1, \\ Q_{\boldsymbol{k}}(\partial) &= (3 + 2\boldsymbol{\tau}.\boldsymbol{k})(-\partial + \boldsymbol{\lambda}.\boldsymbol{k}), \\ R_{\boldsymbol{k}}(\partial) &= \boldsymbol{\tau}.\boldsymbol{k}(\boldsymbol{\tau}.\boldsymbol{k} + 4). \end{aligned} \tag{5.38}$$

In particular, taking $\boldsymbol{\lambda} = (1, -1)$ (the zeros of $\zeta \mapsto P(-\zeta)$) and $\boldsymbol{\tau} = \left(-\dfrac{3}{2}, -\dfrac{3}{2}\right)$ (we take the values given by (5.5)), then writing $\boldsymbol{k} = (k_1, k_2)$:

$$\begin{aligned} P_{\boldsymbol{k}}(\partial) &= \partial^2 - 2(k_1 - k_2)\partial + (k_1 - k_2)^2 - 1, \\ Q_{\boldsymbol{k}}(\partial) &= 3(1 - k_1 - k_2)(-\partial + k_1 - k_2), \\ R_{\boldsymbol{k}}(\partial) &= \frac{9}{4}(k_1 + k_2)\left(k_1 + k_2 - \frac{8}{3}\right). \end{aligned} \tag{5.39}$$

We eventually mention some identities for later purposes, the proof of which being left as an exercise.

Lemma 5.3. *The operators $\mathfrak{P}_{\boldsymbol{k}}$ and $\mathfrak{D}_{\boldsymbol{k}}$ given by definition 5.5 satisfy the identities: for any $\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2 \in \mathbb{N}^n$, $\mathrm{e}^{-\boldsymbol{\lambda}.\boldsymbol{k}_1 z}\mathfrak{P}_{\boldsymbol{k}_1}\mathrm{e}^{\boldsymbol{\lambda}.\boldsymbol{k}_1 z} = \mathrm{e}^{-\boldsymbol{\lambda}.\boldsymbol{k}_2 z}\mathfrak{P}_{\boldsymbol{k}_2}\mathrm{e}^{\boldsymbol{\lambda}.\boldsymbol{k}_2 z}$, $z^{-\boldsymbol{\tau}.\boldsymbol{k}}\mathfrak{D}_{\boldsymbol{k}} = \mathfrak{P}_{\boldsymbol{k}} z^{-\boldsymbol{\tau}.\boldsymbol{k}}$ and*

$$(\mathrm{e}^{-\boldsymbol{\lambda}.\boldsymbol{k}_1 z} z^{-\boldsymbol{\tau}.\boldsymbol{k}_1})\mathfrak{D}_{\boldsymbol{k}_1}(\mathrm{e}^{-\boldsymbol{\lambda}.\boldsymbol{k}_1 z} z^{-\boldsymbol{\tau}.\boldsymbol{k}_1})^{-1} = (\mathrm{e}^{-\boldsymbol{\lambda}.\boldsymbol{k}_2 z} z^{-\boldsymbol{\tau}.\boldsymbol{k}_2})\mathfrak{D}_{\boldsymbol{k}_2}(\mathrm{e}^{-\boldsymbol{\lambda}.\boldsymbol{k}_2 z} z^{-\boldsymbol{\tau}.\boldsymbol{k}_2})^{-1}.$$

Setting $W_{\boldsymbol{k}} = z^{-\boldsymbol{\tau}.\boldsymbol{k}} w_{\boldsymbol{k}}$ for $\boldsymbol{k} \in \mathbb{N}^n$ and the $w_{\boldsymbol{k}}$ given by lemma 5.2, one has $\mathfrak{P}_{\boldsymbol{e}_i} W_{\boldsymbol{e}_i} = 0$, $i = 1, 2$ while and for $|\boldsymbol{k}| \geq 2$,

$$\mathfrak{P}_{\boldsymbol{k}} W_{\boldsymbol{k}} = \sum_{\substack{\boldsymbol{k}_1 + \cdots + \boldsymbol{k}_\ell = \boldsymbol{k} \\ |\boldsymbol{k}_i| \geq 1,\, \ell \geq 2}} \frac{W_{\boldsymbol{k}_1} \cdots W_{\boldsymbol{k}_\ell}}{\ell!} \frac{\partial^\ell F(z, w_{\boldsymbol{0}})}{\partial w^\ell}. \tag{5.40}$$

5.3 Formal transseries for the first Painlevé equation

We partly describe in this section the contains of lemma 5.2 for the prepared form equation (3.6) associated with the first Painlevé equation. Thus $n = 2$, $P(\partial) = \partial^2 - 1$, $Q(\partial) = -3\partial$ and $F(z, w) = f_0(z) + f_1(z)w + f_2(z)w^2$. Also, we will for the moment specialise our study to only one-parameter formal expansions, that is we will assume that either $U_1 = 0$ or $U_2 = 0$ in (5.34). This study will be enough to get the truncated solutions. We will keep on our study of the formal integral associated with (3.6) in Sect. 5.4 where will we see the effects of resonances.

5.3.1 Transseries solution - statement

This section will be devoted to prove the following proposition.

Proposition 5.2. *We consider the prepared ODE (3.6). We note* $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) = (1, -1)$ *where the* λ_i*'s are the multipliers, that is the roots of the polynomial* $\zeta \mapsto P(-\zeta)$. *We introduce* $\boldsymbol{\tau} = (\tau_1, \tau_2) = \left(-\dfrac{3}{2}, -\dfrac{3}{2}\right)$, *where* $\tau_i = \dfrac{Q(-\lambda_i)}{P'(-\lambda_i)}$, $i = 1, 2$. *We note* $\boldsymbol{e}_i$ *the* i*-th vector of the canonical base of* $\mathbb{C}^2$ *Then for each* $i = 1, 2$, *there exists a formal one-parameter solution of (3.6) in the graded algebra* $\bigoplus_{k \in \mathbb{N}} z^{-\tau_i k} \mathrm{e}^{-\lambda_i k z} \mathbb{C}[[z^{-1}]]$ *of the form*

$$\widetilde{w}(z, U\boldsymbol{e}_i) = \sum_{k=0}^{\infty} U^k \mathrm{e}^{-\lambda_i k z} z^{-\tau_i k} \widetilde{w}_{k\boldsymbol{e}_i}(z), \qquad \widetilde{w}_{k\boldsymbol{e}_i} \in \mathbb{C}[[z^{-1}]]. \tag{5.41}$$

This formal expansion is unique once one fixes the normalization of $\widetilde{w}_{\boldsymbol{e}_i}$ *to be* $\widetilde{w}_{\boldsymbol{e}_i}(z) = 1 + O(z^{-1})$. *Morever* $\widetilde{w}_{k\boldsymbol{e}_i} \in \mathbb{R}[[z^{-1}]]$ *and* $\mathrm{val}\, \widetilde{w}_{k\boldsymbol{e}_i} = 2(k-1)$ *with* $\widetilde{w}_{k\boldsymbol{e}_i}(z) = \dfrac{k}{12^{k-1}} z^{-2(k-1)}(1 + O(z^{-1}))$ *for every* $k \geq 1$. *Furthermore changing the normalization of* $\widetilde{w}_{\boldsymbol{e}_i}$ *is equivalent in rescaling the parameter* $U \in \mathbb{C}$. *Eventually,* $\widetilde{w}_{k\boldsymbol{e}_1}(z) = \widetilde{w}_{k\boldsymbol{e}_2}(-z)$ *for every* $k \geq 0$.

Definition 5.6. The expansion (5.41) is called a formal **transseries**. The terms $\mathrm{e}^{-\lambda_i k z} z^{-\tau_i k}$ are (log-free) **transmonomials**. The formal series $\widetilde{w}_{k\boldsymbol{e}_i}$ are called the k**-th series** of the transseries. We denote $\widetilde{W}_{k\boldsymbol{e}_i} = z^{-\tau_i k} \widetilde{w}_{k\boldsymbol{e}_i}$.

Remark 5.5. The term "transseries" is due to Ecalle [19]. These are objects that are widely used in resurgence theory, see, e.g. [8, 34, 27, 28]. More details on transseries can be founded in [18, 19], see also [6, 7]. Transseries are also common objects in theoretical physics : these are the so-called "multi-instanton expansions", see, e.g. [38, 24, 25, 26, 32, 23, 1, 15, 16, 17].

In quantum mechanics or quantum field theory, an **instanton action** (the terminology of which is due to Gerard 't Hooft) is a classical solution of the equations of motion, with a finite and non-zero action. A well-known instanton effect in quantum mechanics is given by a particle in a double well potential. The tunneling effect provides a non-zero probability that the particle crosses the potential barrier. This gives rise to a tunneling amplitude proportionnal to the **instanton** $\mathrm{e}^{-S/\hbar}$ where S is the instanton action, $\hbar$ being the Planck constant or the coupling constant. For the bound states, this translates into the fact that they can be described at a formal level by a multi-instanton expansion, that is a transseries of the form $\sum_{k \geq 0} \widetilde{E}_k(\hbar) \mathrm{e}^{-kS/\hbar}$ where the perturbative fluctuations $\widetilde{E}_k(\hbar)$ are formal expansions with respect to $\hbar$. The bound states are deduced from the multi-instanton expansion by (median) Laplace-Borel summation, see [36, 9, 10, 11, 12, 13, 21, 22].

For later use, we mention a lemma that result from proposition 5.2 and lemma 5.3.

Lemma 5.4. *Under the conditions of proposition 5.2 and for any* $\boldsymbol{k} \in \mathbb{N}^2$, *the (so-called) general formal solution of the linear differential equation* $\mathfrak{P}_{\boldsymbol{k}}(\widetilde{w}_{\boldsymbol{0}})\widetilde{W} = 0$ *is* $\widetilde{W} = \mathrm{e}^{\boldsymbol{\lambda}.\boldsymbol{k} z}\big(C_1 \mathrm{e}^{-\lambda_1 z} \widetilde{W}_{\boldsymbol{e}_1} + C_2 \mathrm{e}^{-\lambda_2 z} \widetilde{W}_{\boldsymbol{e}_2}\big)$, $C_1, C_2 \in \mathbb{C}$. *For any* $\boldsymbol{k} \in \mathbb{N}^2$ *the (so-called) general formal solution of the linear differential equation* $\mathfrak{D}_{\boldsymbol{k}}(\widetilde{w}_{\boldsymbol{0}})\widetilde{w} = 0$ *is* $\widetilde{w}(z) = \mathrm{e}^{\boldsymbol{\lambda}.\boldsymbol{k} z} z^{\boldsymbol{\tau}.\boldsymbol{k}}\big(C_1 \mathrm{e}^{-\lambda_1 z} \widetilde{W}_{\boldsymbol{e}_1} + C_2 \mathrm{e}^{-\lambda_2 z} \widetilde{W}_{\boldsymbol{e}_2}\big)$, $C_1, C_2 \in \mathbb{C}$.

5.3.2 Transseries solutions - proof

5.3.2.1 A useful lemma

We start with the following lemma which will be useful in the sequel.

Lemma 5.5. *We suppose $n, N \in \mathbb{N}^\star$. We consider the ordinary differential equation*

$$P(\partial)w + \frac{1}{z}R(\partial)w = \widetilde{f}(z),\ \widetilde{f}(z) = f_N z^{-N}(1+O(z^{-1})) \in z^{-N}\mathbb{C}[[z^{-1}]],\ f_N \neq 0$$

with

$$P(\partial) = \sum_{m=0}^{n} \alpha_{n-m}\partial^m \in \mathbb{C}[\partial],\ \alpha_n \neq 0,\ R(\partial) = \textstyle\sum_{m=0}^{n-1} \gamma_{n-m}(z)\partial^m \in \mathbb{C}[[z^{-1}]][\partial]$$

This ODE has a unique solution $\widetilde{w}$ in $\mathbb{C}[[z^{-1}]]$, moreover $\operatorname{val}\widetilde{w} = \operatorname{val}\widetilde{f}$ and $\widetilde{w}(z) = \frac{f_N}{P(0)} z^{-N}(1+O(z^{-1}))$.

Proof. In the valuation ring $\mathbb{C}[[z^{-1}]]$ we consider the following map :

$$\begin{aligned}\mathcal{N} : \mathbb{C}[[z^{-1}]] &\to \mathbb{C}[[z^{-1}]]\\ w &\to \frac{1}{P(0)}\Big[\widetilde{f}(z) - \Big(P(\partial) - P(0)\Big)w - \frac{1}{z}R(\partial)w\Big].\end{aligned}$$

(Remember that $P(0) = \alpha_n$ is nonzero). From the hypotheses made one easily observes that $\mathcal{N}(\mathbb{C}[[z^{-1}]]) \subset z^{-1}\mathbb{C}[[z^{-1}]]$ while, for every $p \in \mathbb{N}^\star$,

$$\text{if } u, v \in z^{-p}\mathbb{C}[[z^{-1}]],\ \text{ then } \mathcal{N}(u) - \mathcal{N}(v) \in z^{-p-1}\mathbb{C}[[z^{-1}]].$$

This means that $\mathcal{N}$ is contractive in $\mathbb{C}[[z^{-1}]]$, thus the fixed point problem $w = \mathcal{N}(w)$ has a unique solution $\widetilde{w} = \lim\limits_{p\to\infty} \mathcal{N}^p(0)$ in $\mathbb{C}[[z^{-1}]]$. Since $\mathcal{N}(0) = \widetilde{f}(z)/P(0)$ one gets $\widetilde{w}(z) = \frac{f_N}{P(0)} z^{-N}(1+O(z^{-1}))$. $\square$

5.3.2.2 Proof of proposition 5.2

We precise at an introduction that the fact that $\widetilde{w}_{k\boldsymbol{e}_i} \in \mathbb{R}[[z^{-1}]]$ is just a consequence of the realness of equation (3.6). The relationships $\widetilde{w}_{(0,k)}(z) = \widetilde{w}_{(k,0)}(-z)$ for every $k \geq 0$, come from the property of equation (3.6) to be invariant under the change of variable $z \mapsto -z$ and to the chosen normalization of $\widetilde{w}_{\boldsymbol{e}_i}$, $i = 1, 2$.

5.3.2.3 The return of the formal solution

We remark that $w_{\mathbf{0}} = w_{(0,0)}$ has to solve (5.35) which is nothing but the equation (3.6) one started with. In particular we know that this equation has a unique formal solution $\widetilde{w}_{\mathbf{0}} \in \mathbb{C}[[z^{-1}]]$ that has been investigated in the previous chapters.
In what follows, one will always replace $w_{\mathbf{0}}$ by this formal solution $\widetilde{w}_{\mathbf{0}}$. We mention the following obvious fact, essentially due to the property that

$\operatorname{val}\widetilde{w}_{\mathbf{0}} \geq 2$ and that for every $\ell = 0,1,2$, $\dfrac{\partial^\ell F(z,0)}{\partial w^\ell} \in z^{-2}\mathbb{C}\{z^{-1}\}$. (This is one place where it is interesting to work with a "well-prepared" equation, see what we have done in Sect. 3.1 to get (3.6) and exercise 3.1):

Lemma 5.6. *If $\widetilde{w}_{\mathbf{0}}(z) = \sum_{l\geq 2} a_{\mathbf{0},l} z^{-l} \in \mathbb{C}[[z^{-1}]]$ is the formal solution of (3.6), then for every $\ell = 0,1,2$, $\dfrac{\partial^\ell F(z,\widetilde{w}_{\mathbf{0}})}{\partial w^\ell} \in \mathbb{C}[[z^{-1}]]$ has valuation 2, and vanishes identically for every $\ell \geq 3$. Also:*

1. $\dfrac{\partial F(z,\widetilde{w}_{\mathbf{0}})}{\partial w} = -4z^{-2} + z^{-2}\widetilde{w}_{\mathbf{0}}$ *is even and its coefficients are all real negative;*
2. $\dfrac{\partial^2 F(z,\widetilde{w}_{\mathbf{0}})}{\partial w^2} = z^{-2}$.

5.3.2.4 The cases $|\boldsymbol{k}\boldsymbol{e_i}| = 1$

Formula (5.36) with $\boldsymbol{k} = \boldsymbol{e}_1$ provides

$$\mathfrak{D}_{\boldsymbol{e}_1} w_{\boldsymbol{e}_1} = 0 \tag{5.42}$$

where $\mathfrak{D}_{\boldsymbol{e}_1} = P_{\boldsymbol{e}_1}(\partial) + \dfrac{1}{z}Q_{\boldsymbol{e}_1}(\partial) + \dfrac{1}{z^2}R_{\boldsymbol{e}_1} - \dfrac{\partial F(z,\widetilde{w}_{\mathbf{0}})}{\partial w}$ with

$$\begin{aligned} P_{\boldsymbol{e}_1}(\partial) &= P(-\lambda_1+\partial) = P(-\lambda_1) + P'(-\lambda_1)\partial + \frac{P''(-\lambda_1)}{2!}\partial^2 \\ Q_{\boldsymbol{e}_1}(\partial) &= -\tau_1 P'(-\lambda_1+\partial) + Q(-\lambda_1+\partial) \\ R_{\boldsymbol{e}_1} &= \tau_1(\tau_1+4) \end{aligned}$$

Assuming that $w_{\boldsymbol{e}_1} \in \mathbb{C}[[z^{-1}]]$, one observes that the right-hand side of (5.42) has a valuation less or equal to $(\operatorname{val} w_{\boldsymbol{e}_1})-2$, because of lemma 5.6. In order to get a non identically vanishing solution, one thus has to impose the condition $P(-\lambda_1) = 0$. Following our conventions, we take $\lambda_1 = 1$.
The same reasoning leads to impose furthermore that $-\tau_1 P'(-\lambda_1) + Q(-\lambda_1) = 0$, thus $\tau_1 = -\dfrac{3}{2}$. Therefore,

$$P_{\boldsymbol{e}_1}(\partial) = \partial^2 - 2\partial, \qquad Q_{\boldsymbol{e}_1}(\partial) = 0, \qquad R_{\boldsymbol{e}_1}(\partial) = -\frac{15}{4}.$$

Symmetrically for $\boldsymbol{k} = \boldsymbol{e}_2$, one gets $\lambda_2 = -1$, $\tau_2 = -\dfrac{3}{2}$ as a necessary condition and

$$\mathfrak{D}_{\boldsymbol{e}_2} w_{\boldsymbol{e}_2} = 0 \tag{5.43}$$

where $\mathfrak{D}_{\boldsymbol{e}_2} = P_{\boldsymbol{e}_2}(\partial) + \dfrac{1}{z}Q_{\boldsymbol{e}_2}(\partial) + \dfrac{1}{z^2}R_{\boldsymbol{e}_2} - \dfrac{\partial F(z,\widetilde{w}_{\mathbf{0}})}{\partial w}$ whereas $P_{\boldsymbol{e}_2}(\partial) = \partial^2 + 2\partial$, $Q_{\boldsymbol{e}_2}(\partial) = 0$, $R_{\boldsymbol{e}_2}(\partial) = -\dfrac{15}{4}$.

Lemma 5.7. *The linear homogeneous equations (5.42), (5.43) have both a one-parameter family of formal solutions $w_{\boldsymbol{e}_1} = U_1\widetilde{w}_{\boldsymbol{e}_1}$ and $w_{\boldsymbol{e}_2} = U_2\widetilde{w}_{\boldsymbol{e}_2}$ in $\mathbb{C}[[z^{-1}]]$, where $\widetilde{w}_{\boldsymbol{e}_1}$ and $\widetilde{w}_{\boldsymbol{e}_2}$ are uniquely determined when normalized so that $\widetilde{w}_{\boldsymbol{e}_1} = 1 + O(z^{-1})$, $\widetilde{w}_{\boldsymbol{e}_2} = 1 + O(z^{-1})$. Moreover $\widetilde{w}_{\boldsymbol{e}_i} \in \mathbb{R}[[z^{-1}]]$ and $\widetilde{w}_{\boldsymbol{e}_2}(z) =$*

$\widetilde{w}_{\boldsymbol{e}_1}(-z)$. *Furthemore, if* $\widetilde{w}_{\boldsymbol{0}}(z) = \sum_{l\geq 0} a_{\boldsymbol{0},l} z^{-l}$ *and* $\widetilde{w}_{\boldsymbol{e}_1}(z) = \sum_{l\geq 0} a_{\boldsymbol{e}_1,l} z^{-l}$, *the following quadratic recursion relation is valid:*

$$\begin{cases} a_{\boldsymbol{e}_1,0} = 1, \\ a_{\boldsymbol{e}_1,l} = \dfrac{1}{8l}\left(-(2l-1)^2 a_{\boldsymbol{e}_1,l-1} + 4\displaystyle\sum_{p=0}^{l-1} a_{\boldsymbol{e}_1,p} a_{\boldsymbol{0},l-p-1}\right), \; l = 1, 2, \cdots \end{cases} \tag{5.44}$$

Proof. We only examine (5.42). We look at this equation in the space of normalized formal series $\mathbb{C}[[z^{-1}]]$, namely

$$\begin{cases} (\partial - 2)\partial w_{\boldsymbol{e}_1} = \left(\dfrac{15}{4}\dfrac{1}{z^2} + \dfrac{\partial F(z, \widetilde{w}_{\boldsymbol{0}})}{\partial w}\right) w_{\boldsymbol{e}_1} \\ w_{\boldsymbol{e}_1} \in \mathbb{C}[[z^{-1}]], \quad w_{\boldsymbol{e}_1} = 1 + O(z^{-1}). \end{cases} \tag{5.45}$$

We remark that the restriction of the derivation operator ∂ to the maximal ideal $z^{-1}\mathbb{C}[[z^{-1}]]$ is a bijective operator between $z^{-1}\mathbb{C}[[z^{-1}]]$ and $z^{-2}\mathbb{C}[[z^{-1}]]$; we note ∂^{-1} the inverse operator,

$$z^{-1}\mathbb{C}[[z^{-1}]] \underset{\partial^{-1}}{\overset{\partial}{\rightleftarrows}} z^{-2}\mathbb{C}[[z^{-1}]].$$

We transform (5.45) into the equation

$$-2\partial w_{\boldsymbol{e}_1} = \left(-\partial^2 + \frac{15}{4}\frac{1}{z^2} + \frac{\partial F(z, \widetilde{w}_{\boldsymbol{0}})}{\partial w}\right) w_{\boldsymbol{e}_1}$$

and we see that the right-hand side of this equation belongs to $z^{-2}\mathbb{C}[[z^{-1}]]$ once $w_{\boldsymbol{e}_1}$ belongs to $\mathbb{C}[[z^{-1}]]$, because of lemma 5.6 and to the choice of the coefficient τ_1. This means that the map

$$\begin{aligned} \mathcal{N} : \mathbb{C}[[z^{-1}]] &\to \mathbb{C}[[z^{-1}]] \\ w_{\boldsymbol{e}_1} &\to 1 - \frac{1}{2}\partial^{-1}\left(-\partial^2 + \frac{15}{4}\frac{1}{z^2} + \frac{\partial F(z, \widetilde{w}_{\boldsymbol{0}})}{\partial w}\right) w_{\boldsymbol{e}_1} \end{aligned}$$

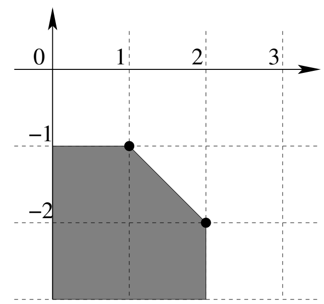
is well defined and the problem (5.45) is equivalent to the fixed-point problem $w_{\boldsymbol{e}_1} = \mathcal{N}(w_{\boldsymbol{e}_1})$. One easily sees that the map $\mathcal{N}$ is contractive in $\mathbb{C}[[z^{-1}]]$ so that the fixed point problem $w_{\boldsymbol{e}_1} = \mathcal{N}(w_{\boldsymbol{e}_1})$ has a unique solution $\widetilde{w}_{\boldsymbol{e}_1}$ in $\mathbb{C}[[z^{-1}]]$.

From the fact that (5.42) is a homogeneous equation, one immediately concludes that $U_1\widetilde{w}_{\boldsymbol{e}_1}$, $U_1 \in \mathbb{C}$, provides a one-parameter family of formal solutions.

The proof for the quadratic recursion relation (5.44) is left to the reader (see also [23, 1]). □

Remark 5.6. 1. The Newton polygon at infinity $\mathcal{N}_\infty(\mathfrak{D}_{\boldsymbol{e}_1})$ drawn on Fig. 5.2, has one horizontal side that corresponds to the operator -2∂. General nonsense in asymptotic theory ([31], or [5, 29]) provides the existence of the formal (normalized) series solution $\widetilde{w}_{\boldsymbol{e}_1}$. The other (normalized) formal solution associated with the side of slope -1 is $\mathrm{e}^{2z}\widetilde{w}_{\boldsymbol{e}_2}$ (see lemma 5.4) which, in our frame, is already incorporated in the other transseries solution.

Fig. 5.2 The Newton polygon at infinity $\mathcal{N}_\infty(\mathfrak{D}_{\boldsymbol{e}_1})$ associated with the linear operator (5.45).



2. From lemma 5.6 or (5.44), one easily shows that

$$\widetilde{w}_{\boldsymbol{e}_1}(z) = 1 - \frac{1}{8}z^{-1} + \frac{9}{128}z^{-2} - \frac{341329}{1920000}z^{-3} + \cdots$$

is a real formal expansion, with coefficients that alternate in sign.

5.3.2.5 The cases $|k\boldsymbol{e_i}| \geq 2$

Lemma 5.8. *For any $\boldsymbol{k} = k\boldsymbol{e}_i$, $i = 1,2$ and $k \geq 2$, equation (5.37) has a unique formal solution $w_{k\boldsymbol{e}_i} = \widetilde{w}_{k\boldsymbol{e}_i}$ in $\mathbb{C}[[z^{-1}]]$. Moreover $\operatorname{val} \widetilde{w}_{k\boldsymbol{e}_i} = 2(k-1)$. Furthermore, when considering $U\widetilde{w}_{\boldsymbol{e}_i}$ instead of $\widetilde{w}_{\boldsymbol{e}_i}$ for the solution of (5.36), then the unique solution of (5.37) at rank $\boldsymbol{k} = k\boldsymbol{e}_i$, $k \geq 2$, is $U^k\widetilde{w}_{k\boldsymbol{e}_i}$. Also, $\widetilde{w}_{k\boldsymbol{e}_i} \in \mathbb{R}[[z^{-1}]]$, $\widetilde{w}_{k\boldsymbol{e}_i}(z) = \dfrac{k}{12^{k-1}}z^{-2(k-1)}(1 + O(z^{-1}))$ and $\widetilde{w}_{(0,k)}(z) = \widetilde{w}_{(k,0)}(-z)$ for every $k \geq 2$.*
Eventually, writing $\widetilde{w}_{k\boldsymbol{e}_1}(z) = \sum_{l\geq 0} a_{k\boldsymbol{e}_1,l}z^{-l}$, one has the following quadratic recursion relations, for every $k \geq 2$:

$$\begin{cases} a_{k\boldsymbol{e}_1,0} = a_{k\boldsymbol{e}_1,1} = 0, \\ (k^2-1)a_{k\boldsymbol{e}_1,l} = k(3k-2l-1)a_{k\boldsymbol{e}_1,l-1} - \frac{1}{4}(3k-2l)^2 a_{k\boldsymbol{e}_1,l-2} \\ + \sum_{p=0}^{l-2}\left(a_{k\boldsymbol{e}_1,p}a_{\boldsymbol{0},l-p-2} + \frac{1}{2}\displaystyle\sum_{\substack{k_1+k_2=k \\ k_1\geq 1,\, k_2\geq 1}} a_{k_1\boldsymbol{e}_1,p}a_{k_2\boldsymbol{e}_1,l-p-2}\right), \; l = 2,3,\cdots \end{cases} \tag{5.46}$$

Proof. We only examine the case $\boldsymbol{k} = k\boldsymbol{e}_1$, $k \geq 2$.
The proof is done by induction on k. We first consider equation (5.37) for $k = 2$:

$$\mathfrak{D}_{2\boldsymbol{e}_1}w_{2\boldsymbol{e}_1} = \frac{\widetilde{w}^2_{\boldsymbol{e}_1}}{2!}\frac{\partial^2 F(z,\widetilde{w}_{\boldsymbol{0}})}{\partial w^2}, \tag{5.47}$$

with $\mathfrak{D}_{2\boldsymbol{e}_1} = P_{2\boldsymbol{e}_1}(\partial) + \dfrac{1}{z}Q_{2\boldsymbol{e}_1}(\partial) + \dfrac{1}{z^2}R_{2\boldsymbol{e}_1} - \dfrac{\partial F(z,\widetilde{w}_{\boldsymbol{0}})}{\partial w}$. By (5.31) one has $P_{2\boldsymbol{e}_1}(\partial) = P(-2\lambda_1 + \partial) = \partial^2 - 4\partial + 3$, thus $P_{2\boldsymbol{e}_1}(0) = 3$ is non zero. Using lemma 5.6, one sees that lemma 5.5 can be applied to (5.47) and this provides a unique solution $\widetilde{w}_{2\boldsymbol{e}_1} \in \mathbb{C}[[z^{-1}]]$. Its valuation is 2, explicit calculation giving, for instance:

$$\widetilde{w}_{2\boldsymbol{e}_1}(z) = \frac{1}{6}z^{-2} - \frac{11}{72}z^{-3} + \frac{53}{192}z^{-4} + \cdots, \qquad \widetilde{w}_{2\boldsymbol{e}_2}(z) = \widetilde{w}_{2\boldsymbol{e}_1}(-z).$$

One easily checks that replacing $\widetilde{w}_{\boldsymbol{e}_1}$ by $U\widetilde{w}_{\boldsymbol{e}_1}$ implies changing $\widetilde{w}_{2\boldsymbol{e}_1}$ into $U^2\widetilde{w}_{2\boldsymbol{e}_1}$.
We now assume that the properties of lemma 5.8 are true for every $2 \leq k \leq K-1$. When considering equation (5.37) for K one gets :

$$\mathfrak{D}_{K\boldsymbol{e}_1} w_{K\boldsymbol{e}_1} = \sum_{\substack{k_1+k_2=K \\ k_1 \geq 1,\, k_2 \geq 1}} \frac{\widetilde{w}_{k_1\boldsymbol{e}_1}\widetilde{w}_{k_2\boldsymbol{e}_1}}{2!} \frac{\partial^2 F(z,\widetilde{w}_{\mathbf{0}})}{\partial w^2}, \tag{5.48}$$

with $\mathfrak{D}_{K\boldsymbol{e}_1} = P_{K\boldsymbol{e}_1}(\partial) + \dfrac{1}{z}Q_{K\boldsymbol{e}_1}(\partial) + \dfrac{1}{z^2}R_{K\boldsymbol{e}_1} - \dfrac{\partial F(z,\widetilde{w}_{\mathbf{0}})}{\partial w}$ and $P_{K\boldsymbol{e}_1}(\partial) = \partial^2 - 2K\partial + (K^2-1)$. One deduces the conclusion of lemma 5.8 at the rank K by the arguments used previously. For what concerns the valuation, observe that when $k_1+k_2 = K$, $\operatorname{val}\widetilde{w}_{k_1\boldsymbol{e}_1}\widetilde{w}_{k_2\boldsymbol{e}_1} \geq 2(k_1-1)+2(k_2-1)$, thus $\operatorname{val}\widetilde{w}_{k_1\boldsymbol{e}_1}\widetilde{w}_{k_2\boldsymbol{e}_1} \geq 2(K-2)$. As a matter of fact, for every $k \geq 2$, $\widetilde{w}_{(k,0)}(z) = b_k z^{-2(K-1)}(1+O(z^{-1}))$ with

$$\begin{cases} b_1 = 1, \\ b_k = \frac{1}{2(k-1)(k+1)} \sum_{p=1}^{k-1} b_p b_{k-p}, \quad k \geq 2, \end{cases}$$

which easily provides $b_k = \frac{k}{12^{k-1}}$ by induction. The reader will easily check that the recursive relations (5.46) are true. (See also [23, 1]). □

Remark 5.7. Here again, we are not interested in the whole formal fundamental solutions of equations (5.47), (5.48), that incorporate the general solutions $(\mathrm{e}^{-\lambda_1 k z} z^{-\tau_1 k})^{-1}\big(C_1\mathrm{e}^{-\lambda_1 z}z^{-\tau_1}\widetilde{w}_{\boldsymbol{e}_1} + C_2\mathrm{e}^{-\lambda_2 z}z^{-\tau_2}\widetilde{w}_{\boldsymbol{e}_2}\big)$ of the associated homogeneous linear ODEs $\mathfrak{D}_{(k,0)}w = 0$ (cf. lemma 5.4). Taking into account the term $(\cdots)\widetilde{w}_{\boldsymbol{e}_1}$ would imply a rescaling on U_1. The other term $(\cdots)\widetilde{w}_{\boldsymbol{e}_2}$ concerns the other transseries.

5.4 Formal integral for the first Painlevé equation

We made general considerations on formal integrals in Sect. 5.2. We started the study of the formal integral for the prepared equation (3.6) associated with the first Painlevé equation in Sect. 5.3 : this gave us the transseries described by proposition 5.2. When no resonances occur, one gets with quite similar arguments the formal integral. However, this is not that simple for the first Painlevé equation where we have to cope with resonances.

5.4.1 Notations and preliminary results

5.4.1.1 Notations

It will be useful for our purpose to introduce the following notations:

Definition 5.7. For any $n \in \mathbb{N}^\star$, we note $\mathbf{n} = n(1,1)$ and

$$\boldsymbol{\Xi}_{n,0} = \{\boldsymbol{k} = (k_1, k_2) \in \mathbb{N}^2 \setminus \{\mathbf{0}\} \mid k_1 < n \text{ or } k_2 < n\} \cup \{\mathbf{n}\}.$$

We also set $\boldsymbol{\Xi}_{0,0} = \{(0,0)\}$.

Example 5.3. • $\boldsymbol{\Xi}_{1,0} = (\mathbb{N}^\star \times \{0\}) \cup (\{0\} \times \mathbb{N}^\star) \cup \{(1,1)\}$,
• $\boldsymbol{\Xi}_{2,0} = (\mathbb{N}^\star \times \{0,1\}) \cup (\{0,1\} \times \mathbb{N}^\star) \cup \{(2,2)\}$.
Notice that for every $n \in \mathbb{N}$, $\boldsymbol{\Xi}_{n+1,0} \setminus \boldsymbol{\Xi}_{n,0} = \mathbf{n} + \boldsymbol{\Xi}_{1,0}$.

5.4.1.2 Resonances : first consequences

Equation (3.6) has the feature to have *positively resonant* multipliers $\lambda_1 = 1$, $\lambda_2 = -1$ because $\boldsymbol{\lambda}.\mathbf{n} = 0$, for every $n \in \mathbb{N}^\star$ (see definition 5.2). This brings semi-positively resonances, the cases of semi-positive resonances being all described by $\lambda_1 = \boldsymbol{\lambda}.(\mathbf{n} + \boldsymbol{e_1})$ and $\lambda_2 = \boldsymbol{\lambda}.(\mathbf{n} + \boldsymbol{e_2})$, for every $n \in \mathbb{N}^\star$.

We have already seen (proposition 5.2) that these properties have no consequence for the transseries but, as we shall see, this produces new phenomena when the formal integral is concerned, these being essentially consequences of the following fact. (This derives from lemma 5.3).

Lemma 5.9. *For every $n \in \mathbb{N}$, $\boldsymbol{k} \in \mathbb{N}^2$, the following identities are satisfied:*

$$\mathfrak{P}_{\mathbf{n}+\boldsymbol{k}} = \mathfrak{P}_{\boldsymbol{k}}, \qquad \mathfrak{D}_{\mathbf{n}+\boldsymbol{k}} = z^{\boldsymbol{\tau}.\mathbf{n}} \mathfrak{D}_{\boldsymbol{k}} z^{-\boldsymbol{\tau}.\mathbf{n}}, \qquad \boldsymbol{\tau}.\mathbf{n} = -3n.$$

5.4.1.3 Preliminary lemmas

In a moment, we will have to deal with formal expansions of the type $\displaystyle\sum_{l=0}^{p} \log^l(z) \widetilde{f}_l(z)$, $p \in \mathbb{N}$, with the $\widetilde{f}_l$'s in $\mathbb{C}[[z^{-1}]]$.

Definition 5.8. We equip the graded algebra $\displaystyle\bigoplus_{l \in \mathbb{N}} \log^l(z)\mathbb{C}[[z^{-1}]]$ with the valuation val defined by: $\displaystyle \operatorname{val}\left(\sum_l \log^l(z)\widetilde{f}_l\right) = \min_l \{\operatorname{val} \widetilde{f}_l\}$.

Lemma 5.10. *We suppose $n, N \in \mathbb{N}^\star$ and $p \in \mathbb{N}$. We consider the ordinary differential equation*

$$P(\partial)w + \frac{1}{z}R(\partial)w = \widetilde{f}(z), \qquad \widetilde{f}(z) \in \bigoplus_{l=0}^{p} \log^l(z)\mathbb{C}[[z^{-1}]], \tag{5.49}$$

$$P(\partial) = \sum_{m=0}^{n} \alpha_{n-m}\partial^m \in \mathbb{C}[\partial],\ \alpha_n \neq 0, \qquad R(\partial) = \sum_{m=0}^{n-1} \gamma_{n-m}(z)\partial^m \in \mathbb{C}[[z^{-1}]][\partial]$$

Then (5.49) has a unique solution $\displaystyle\widetilde{w} \in \bigoplus_{l=0}^{p} \log^l(z)\mathbb{C}[[z^{-1}]]$ *and* $\operatorname{val}\widetilde{w} = \operatorname{val}\widetilde{f}$.
Moreover, if $\widetilde{f} = \sum_{l=0}^{p} \log^l(z)\widetilde{f}_l$ *and* $\widetilde{w} = \sum_{l=0}^{p} \log^l(z)\widetilde{w}_l$*, then:*

1. $\widetilde{w}_p$ *solves the ODE:* $P(\partial)w + \frac{1}{z}R(\partial)w = \widetilde{f}_p$*;*
2. *if* $\operatorname{val}\widetilde{f}_p < \operatorname{val} \sum_{l=0}^{p-1} \log^l(z)\widetilde{f}_l$ *then* $\operatorname{val}\widetilde{w}_p < \operatorname{val} \sum_{l=0}^{p-1} \log^l(z)\widetilde{w}_l$.

Proof. One easily sees that the arguments used for the proof of lemma 5.5 can be extended, when observing that $\operatorname{val} \partial \left(\sum_l \log^l(z)\widetilde{f}_l\right) \leq \operatorname{val}\left(\sum_l \log^l(z)\widetilde{f}_l\right) + 1$.
□

We have seen in lemma 5.7 that the operators $\mathfrak{D}_{\boldsymbol{e}_i}$, $i = 1, 2$, have specific behaviours. This is the purpose of the following lemma.

Lemma 5.11. *We suppose $p \in \mathbb{N}$ and $i \in \{1, 2\}$. We assume that $\widetilde{f} = \sum_{l=0}^{p} \log^l(z)\widetilde{f}_l \in \bigoplus_{l=0}^{p} \log^l(z)\mathbb{C}[[z^{-1}]]$ satisfies the conditions:*

1. $\operatorname{val}\widetilde{f}_p = 1$, $\widetilde{f}_p = f_{p1}z^{-1}(1 + 0(z^{-1}))$, $f_{p1} \neq 0$
2. $\operatorname{val}\left(\sum_{l=0}^{p-1} \log^l(z)\widetilde{f}_l\right) \geq 2$.

Then the equation $\mathfrak{D}_{\boldsymbol{e}_i} w = \widetilde{f}$ has a unique solution $\widetilde{w} = \sum_{l=0}^{p+1} \log^l(z)\widetilde{w}_l$ in $\bigoplus_{l=0}^{p+1} \log^l(z)\mathbb{C}[[z^{-1}]]$ that satisfies the condition $\operatorname{val}\left(\sum_{l=0}^{p} \log^l(z)\widetilde{w}_l\right) \geq 1$. Moreover $\widetilde{w}_{p+1} = \dfrac{f_{p1}}{(p+1)P'(-\lambda_i)}\widetilde{w}_{\boldsymbol{e}_i}$.
Otherwise, the general solution of the ODE $\mathfrak{D}_{\boldsymbol{e}_i} w = \widetilde{f}$ in $\bigoplus_{l=0}^{p+1} \log^l(z)\mathbb{C}[[z^{-1}]]$ is of the form $w = \widetilde{w} + U\widetilde{w}_{\boldsymbol{e}_i}$ where $U \in \mathbb{C}$.

Proof. We examine the case $i = 1$ only. The ODE $\mathfrak{D}_{\boldsymbol{e}_1} w = \widetilde{f}$ is equivalent to the equation :

$$P'(-\lambda_1)\partial w = \widetilde{f} + \left(-\partial^2 + \frac{15}{4}\frac{1}{z^2} + \frac{\partial F(z, \widetilde{w}_{\mathbf{0}})}{\partial w}\right) w, \quad P'(-\lambda_1) = -2.$$

By arguments already used in the proof of lemma 5.7, this problem amounts to looking for a formal solution that satisfies the fixed-point problem

$$w = U(z) + \frac{1}{P'(-\lambda_1)}\partial^{-1}\left(-\partial^2 + \frac{15}{4}\frac{1}{z^2} + \frac{\partial F(z, \widetilde{w}_{\mathbf{0}})}{\partial w}\right) w$$

where $U(z) = \partial^{-1}\left(\dfrac{\widetilde{f}}{P'(-\lambda_1)}\right) = \dfrac{f_{p1}}{(p+1)P'(-\lambda_1)} \log^{p+1}(z) + \displaystyle\sum_{l=0}^{p} \log^l(z)O(z^{-1})$.
Notice that we take the primitive with no constant term. This fixed-point problem has a unique formal solution under the form

$$\widetilde{w} = \frac{f_{p1}}{(p+1)P'(-\lambda_1)}\widetilde{w}_{\boldsymbol{e}_i} \log^{p+1}(z) + \sum_{l=0}^{p} \log^l(z)\widetilde{w}_l$$

and $\operatorname{val}\left(\sum_{l=0}^{p} \log^l(z)\widetilde{w}_l\right) \geq 1$. Eventually one can add to this particular solution any solution of the homogeneous equation $\mathfrak{D}_{\boldsymbol{e}_i} w = 0$, that is any term of the form $U\widetilde{w}_{\boldsymbol{e}_i}$ with $U \in \mathbb{C}$. □

5.4.2 Painlevé I, formal integral

We are now in position to detail the formal integral associated with the first Painlevé equation.

Theorem 5.1. *We consider the ODE (3.6). Let be* $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) = (1, -1)$ *where the* λ_i*'s are the multipliers, and* $\boldsymbol{\tau} = (\tau_1, \tau_2) = \left(-\frac{3}{2}, -\frac{3}{2}\right)$, $\tau_i = \frac{Q(-\lambda_i)}{P'(-\lambda_i)}$, $i = 1, 2$. *We set* $\boldsymbol{V}^{\boldsymbol{k}} = \boldsymbol{U}^{\boldsymbol{k}} \mathrm{e}^{-\boldsymbol{\lambda}.\boldsymbol{k} z} z^{-\boldsymbol{\tau}.\boldsymbol{k}}$ *for any* $\boldsymbol{k} \in \mathbb{N}^2$ *and any* $\boldsymbol{U} = (U_1, U_2) \in \mathbb{C}^2$. *We write* $\mathbf{n} = n(1,1)$ *for any* $n \in \mathbb{N}$.
There exists a two-parameter formal solution of (3.6), freely depending on $\boldsymbol{U} \in \mathbb{C}^2$, *of the form*

$$\widetilde{w}(z, \boldsymbol{U}) = \widetilde{w}_{\mathbf{0}}(z) + \sum_{n=0}^{\infty} \sum_{\boldsymbol{k} \in \Xi_{n+1,0} \setminus \Xi_{n,0}} \boldsymbol{V}^{\boldsymbol{k}} \widetilde{w}_{\boldsymbol{k}}(z), \tag{5.50}$$

and uniquely determined by the following conditions:

1. $\widetilde{w}_{\mathbf{0}} \in \mathbb{C}[[z^{-1}]]$;
2. $\widetilde{w}_{\boldsymbol{k}} = \sum_{l=0}^{n} \log^l(z) \widetilde{w}_{\boldsymbol{k}}^{[l]} \in \bigoplus_{l=0}^{n} \log^l(z) \mathbb{C}[[z^{-1}]]$, *for every* $\boldsymbol{k} \in \Xi_{n+1,0} \setminus \Xi_{n,0}$, $n \in \mathbb{N}$;
3. *for* $i = 1, 2$, $\widetilde{w}_{\boldsymbol{e}_i}$ *satisfies* $\widetilde{w}_{\boldsymbol{e}_i}(z) = 1 + O(z^{-1})$;
4. *for every* $n \in \mathbb{N}^\star$ *and* $i = 1, 2$, $\widetilde{w}_{\mathbf{n}+\boldsymbol{e}_i} = \sum_{l=0}^{n} \log^l(z) \widetilde{w}_{\mathbf{n}+\boldsymbol{e}_i}^{[l]}$ *satisfies* $\operatorname{val} \widetilde{w}_{\mathbf{n}+\boldsymbol{e}_1}^{[n]} < \operatorname{val} \left(\sum_{l=0}^{n-1} \log^l(z) \widetilde{w}_{\mathbf{n}+\boldsymbol{e}_i}^{[l]}\right)$.

Moreover, the following properties are satisfied:

5. *changing the normalization of* $\widetilde{w}_{\boldsymbol{e}_i}$, $i = 1, 2$, *is equivalent to rescaling the parameter* $\boldsymbol{U} \in \mathbb{C}^2$;
6. *for every* $n \in \mathbb{N}$ *and every* $\boldsymbol{k} \in \Xi_{n+1,0} \setminus \Xi_{n,0}$, $\widetilde{w}_{\boldsymbol{k}} \in \bigoplus_{l=0}^{n} \log^l(z) \mathbb{R}[[z^{-1}]]$.
   Furthermore $\widetilde{w}_{(k_1,k_2)}^{[l]}(z) = \widetilde{w}_{(k_2,k_1)}^{[l]}(-z)$ *for every* $l \in [0, n]$;
7. *for every* $n \in \mathbb{N}^\star$ *and every* $\boldsymbol{k} \in \Xi_{n+1,0} \setminus \Xi_{n,0}$,

$$\widetilde{w}_{\boldsymbol{k}} = \sum_{l=0}^{n} \frac{1}{l!} (\boldsymbol{\varkappa}.\boldsymbol{k})^l z^{\boldsymbol{\tau}.\mathbf{1}} \log^l(z) \widetilde{w}_{\boldsymbol{k}-\mathbf{1}}^{[0]} \tag{5.51}$$

   where $\boldsymbol{\varkappa} = (\varkappa_1, \varkappa_2) = (\frac{5}{12}, -\frac{5}{12})$ *is defined by:*

$$\varkappa_i = \frac{a^2}{P'(-\lambda_i)} \left(\frac{1}{P(0)} + \frac{1}{2!} \frac{1}{P(-2\lambda_i)}\right) = \frac{5}{12} \lambda_i, \quad i = 1, 2, \tag{5.52}$$

   whereas a *is given by* $\frac{\partial^2 F(z,0)}{\partial w^2} = a z^{-2} + o(z^{-2})$. *As a consequence, for every* $n \in \mathbb{N}^\star$, $\widetilde{w}_{\mathbf{n}} \in \mathbb{R}[[z^{-1}]]$;
8. *for every* $\boldsymbol{k} \in \mathbb{N}^2 \setminus \{\mathbf{0}\}$, $\operatorname{val} \widetilde{w}_{\boldsymbol{k}}^{[0]} = 2(|\boldsymbol{k}| - 1)$.

Proof. Once for all:

- the property 5. is easily derived by an argument of homogeneity;
- the realness and eveness in property 6. are just consequences of the realness of equation (3.6) and its property of being be invariant under the change of variable $z \mapsto -z$, and to the chosen normalizations.

In what follows, we investigate the terms under the form $\widetilde{w}_{\boldsymbol{k}}$ with $\boldsymbol{k} \in \Xi_{n+1,0} \setminus \Xi_{n,0}$ and $n \in \mathbb{N}$. We first look at what happens when $n = 0$ and $n = 1$, step by step so as to draw some conclusions, then we complete the proof by induction on n.

Case $\boldsymbol{n = 0}$ and $\boldsymbol{k = 1}$ This is the first case where a resonance appears. However, this case yields no surprise. Indeed, equation (5.37) for $\boldsymbol{k} = \mathbf{1}$ reads

$$\begin{aligned} P_{\mathbf{1}}(\partial)w_{\mathbf{1}} + \frac{1}{z}Q_{\mathbf{1}}(\partial)w_{\mathbf{1}} &= \left(-\frac{1}{z^2}R_{\mathbf{1}} + \frac{\partial F(z,\widetilde{w}_{\mathbf{0}})}{\partial w}\right)w_{\mathbf{1}} \\ &\quad + \widetilde{w}_{\boldsymbol{e}_1}\widetilde{w}_{\boldsymbol{e}_2}\frac{\partial^2 F(z,\widetilde{w}_{\mathbf{0}})}{\partial w^2} \end{aligned} \tag{5.53}$$

with $P_{\mathbf{1}}(\partial) = P_{\mathbf{0}}(\partial) = \partial^2 - 1$. Therefore lemma 5.5 can be applied and one gets a unique solution $\widetilde{w}_{\mathbf{1}} \in \mathbb{C}[[z^{-1}]]$ with, moreover, $\operatorname{val}\widetilde{w}_{\mathbf{1}} = 2$ and $\widetilde{w}_{\mathbf{1}}(z) = \dfrac{a}{P(0)}z^{-2} + o(z^{-2})$ where $a = 1$ is given by: $\frac{\partial^2 F(z,0)}{\partial w^2} = az^{-2} + o(z^{-2})$.

Explicit calculation yields: $\widetilde{w}_{\mathbf{1}}(z) = -z^{-2} - \dfrac{9}{8}z^{-4} - \dfrac{902139}{80000}z^{-6} - \cdots$.

Cases $\boldsymbol{n = 1}$ and $\boldsymbol{k \in \Xi_{2,0} \setminus \Xi_{1,0}}$

Cases $\boldsymbol{k} = \mathbf{1} + \boldsymbol{e}_i$, $i = 1, 2$ These are the first cases of semi-positive resonances and are more serious.

Let us concentrate on the case $\boldsymbol{k} = \mathbf{1} + \boldsymbol{e}_1$ for which equation (5.37) is

$$\mathfrak{D}_{\mathbf{1}+\boldsymbol{e}_1} w_{\mathbf{1}+\boldsymbol{e}_1} = \left(\widetilde{w}_{\mathbf{1}}\widetilde{w}_{\boldsymbol{e}_1} + \widetilde{w}_{2\boldsymbol{e}_1}\widetilde{w}_{\boldsymbol{e}_2}\right)\frac{\partial^2 F(z,\widetilde{w}_{\mathbf{0}})}{\partial w^2},$$

that is also, from lemma 5.9 and proposition 5.2,

$$\begin{aligned} \mathfrak{D}_{\boldsymbol{e}_1}(z^3 w_{\mathbf{1}+\boldsymbol{e}_1}) &= \widetilde{g}_{\mathbf{1}+\boldsymbol{e}_1}, \\ \widetilde{g}_{\mathbf{1}+\boldsymbol{e}_1} &= z^3\left(\widetilde{w}_{\mathbf{1}}\widetilde{w}_{\boldsymbol{e}_1} + \widetilde{w}_{2\boldsymbol{e}_1}\widetilde{w}_{\boldsymbol{e}_2}\right)\frac{\partial^2 F(z,\widetilde{w}_{\mathbf{0}})}{\partial w^2} \\ &= \left(\frac{1}{P(0)} + \frac{1}{2!}\frac{1}{P(-2\lambda_1)}\right)a^2 z^{-1} + O(z^{-2}) \\ &= -\frac{5}{6}z^{-1} + O(z^{-2}). \end{aligned} \tag{5.54}$$

The conditions of application of lemma 5.11 are fulfilled: equation (5.54) has a one-parameter family of formal solutions, depending on $U_{[1],1} \in \mathbb{C}$, of the form

$$\begin{aligned} &w_{\mathbf{1}+\boldsymbol{e}_1} = \widetilde{w}_{\mathbf{1}+\boldsymbol{e}_1} + U_{[1],1}z^{-3}\widetilde{w}_{\boldsymbol{e}_1}, \qquad \widetilde{w}_{\mathbf{1}+\boldsymbol{e}_1} = \widetilde{w}^{[1]}_{\mathbf{1}+\boldsymbol{e}_1}\log(z) + \widetilde{w}^{[0]}_{\mathbf{1}+\boldsymbol{e}_1}, \\ &\widetilde{w}^{[1]}_{\mathbf{1}+\boldsymbol{e}_1} = \varkappa_1 z^{-3}\widetilde{w}_{\boldsymbol{e}_1}, \qquad \operatorname{val}\widetilde{w}^{[0]}_{\mathbf{1}+\boldsymbol{e}_1} \geq 4. \\ &\varkappa_1 = \tfrac{a^2}{P'(-\lambda_1)}\left(\tfrac{1}{P(0)} + \tfrac{1}{2!}\tfrac{1}{P(-2\lambda_1)}\right) = \tfrac{5}{12}. \end{aligned} \tag{5.55}$$

Explicitly,

$$\widetilde{w}^{[0]}_{\mathbf{1}+\boldsymbol{e}_1}(z) = \frac{11}{72}z^{-4} - \frac{197}{576}z^{-5} + \frac{23903}{82944}z^{-6} - \cdots$$

Also remark that the property $\operatorname{val}\widetilde{w}^{[0]}_{\mathbf{1}+\boldsymbol{e}_1} \geq 4$ characterizes the particular solution $\widetilde{w}_{\mathbf{1}+\boldsymbol{e}_1}$ among the one-parameter family of solutions.

The case $\boldsymbol{k} = \mathbf{1}+\boldsymbol{e}_i$ is deduced from the above result from the invariance of (3.6) under the change of variable $z \mapsto -z$. One gets a one-parameter family of formal solutions, depending on $U_{[1],2} \in \mathbb{C}$, of the form

$$\begin{aligned}
&w_{\mathbf{1}+\boldsymbol{e}_2} = \widetilde{w}_{\mathbf{1}+\boldsymbol{e}_2} + U_{[1],2}\widetilde{w}_{\boldsymbol{e}_2}, \qquad \widetilde{w}_{\mathbf{1}+\boldsymbol{e}_2} = \widetilde{w}^{[1]}_{\mathbf{1}+\boldsymbol{e}_2}\log(z) + \widetilde{w}^{[0]}_{\mathbf{1}+\boldsymbol{e}_2},\\
&\widetilde{w}^{[1]}_{\mathbf{1}+\boldsymbol{e}_2}(z) = \widetilde{w}^{[1]}_{\mathbf{1}+\boldsymbol{e}_1}(-z) = \varkappa_2 z^{-3}\widetilde{w}_{\boldsymbol{e}_2}(z), \qquad \widetilde{w}^{[0]}_{\mathbf{1}+\boldsymbol{e}_2}(z) = \widetilde{w}^{[0]}_{\mathbf{1}+\boldsymbol{e}_1}(-z)\\
&\varkappa_2 = \tfrac{a^2}{P'(-\lambda_2)}\left(\tfrac{1}{P(0)} + \tfrac{1}{2!}\tfrac{1}{P(-2\lambda_2)}\right) = -\tfrac{5}{12}.
\end{aligned} \tag{5.56}$$

In the sequel, we fix $U_{[1],1} = U_{[1],2} = 0$, that is we only consider the (well and uniquely defined) particular solutions $\widetilde{w}_{\mathbf{1}+\boldsymbol{e}_i}$, $i = 1, 2$.

We stress that adding terms of the form $U_{[1],1}\widetilde{w}_{\boldsymbol{e}_1}$ and $U_{[1],2}\widetilde{w}_{\boldsymbol{e}_2}$ has the effect to rescaling the parameter (U_1, U_2). In particular, changing the branch of the log has non consequence for the formal integral.

Cases $\boldsymbol{k} = \mathbf{1} + k\boldsymbol{e}_i$ One step further, we consider the case $\boldsymbol{k} = \mathbf{1} + 2\boldsymbol{e}_i$. We take $i = 1$ only for simplicity. From (5.37) and lemma 5.9, we get:

$$\mathfrak{D}_{2\boldsymbol{e}_1}(z^3 w_{\mathbf{1}+2\boldsymbol{e}_1}) = z^3\left(\widetilde{w}_{\mathbf{1}+\boldsymbol{e}_1}\widetilde{w}_{\boldsymbol{e}_1} + \widetilde{w}_{2\boldsymbol{e}_1}\widetilde{w}_{\mathbf{1}} + \widetilde{w}_{3\boldsymbol{e}_1}\widetilde{w}_{\boldsymbol{e}_2}\right)\frac{\partial^2 F(z,\widetilde{w}_{\mathbf{0}})}{\partial w^2}. \tag{5.57}$$

By proposition 5.2 and the above result, the right-hand side of equation (5.57) is a formal series expansion of the type $\widetilde{f} = \widetilde{f}^{[1]}\log(z) + \widetilde{f}^{[0]}$ with $\operatorname{val}\widetilde{f}^{[1]} = 2$ and $\operatorname{val}\widetilde{f}^{[0]} = 3$. Applying lemma 5.10, we get for (5.57) a unique formal solution of the form $\widetilde{w}_{\mathbf{1}+2\boldsymbol{e}_1} = \widetilde{w}^{[1]}_{\mathbf{1}+2\boldsymbol{e}_1}\log(z) + \widetilde{w}^{[0]}_{\mathbf{1}+2\boldsymbol{e}_1} \in \bigoplus_{l=0}^{1}\log^l(z)\mathbb{C}[[z^{-1}]]$ with $\operatorname{val}\widetilde{w}^{[1]}_{\mathbf{1}+2\boldsymbol{e}_1} = 5$ and $\operatorname{val}\widetilde{w}^{[0]}_{\mathbf{1}+2\boldsymbol{e}_1} = 6$. Moreover, $\widetilde{w}^{[1]}_{\mathbf{1}+2\boldsymbol{e}_1}$ solves the ODE

$$\begin{aligned}
\mathfrak{D}_{2\boldsymbol{e}_1}(z^3 w^{[1]}_{\mathbf{1}+2\boldsymbol{e}_1}) &= z^3\widetilde{w}^{[1]}_{\mathbf{1}+\boldsymbol{e}_1}\widetilde{w}_{\boldsymbol{e}_1}\frac{\partial^2 F(z,\widetilde{w}_{\mathbf{0}})}{\partial w^2}\\
&= \varkappa_1\widetilde{w}^2_{\boldsymbol{e}_1}\frac{\partial^2 F(z,\widetilde{w}_{\mathbf{0}})}{\partial w^2}.
\end{aligned}$$

Comparing to (5.47), one concludes that

$$\begin{aligned}
&\widetilde{w}_{\mathbf{1}+2\boldsymbol{e}_1} = \widetilde{w}^{[1]}_{\mathbf{1}+2\boldsymbol{e}_1}\log(z) + \widetilde{w}^{[0]}_{\mathbf{1}+2\boldsymbol{e}_1},\\
&\widetilde{w}^{[1]}_{\mathbf{1}+2\boldsymbol{e}_1} = 2\varkappa_1 z^{-3}\widetilde{w}_{2\boldsymbol{e}_1}, \qquad \operatorname{val}\widetilde{w}^{[0]}_{\mathbf{1}+2\boldsymbol{e}_1} = 6.
\end{aligned}$$

We now reason by induction, assuming that for every $k \in [2, K-1]$ with $K \geq 3$, one has

$$\begin{aligned}
&\widetilde{w}_{\mathbf{1}+k\boldsymbol{e}_1} = \widetilde{w}^{[1]}_{\mathbf{1}+k\boldsymbol{e}_1}\log(z) + \widetilde{w}^{[0]}_{\mathbf{1}+k\boldsymbol{e}_1},\\
&\widetilde{w}^{[1]}_{\mathbf{1}+k\boldsymbol{e}_1} = k\varkappa_1 z^{-3}\widetilde{w}_{k\boldsymbol{e}_1}, \qquad \operatorname{val}\widetilde{w}^{[0]}_{\mathbf{1}+k\boldsymbol{e}_1} = 2(k+1).
\end{aligned}$$

Then, by (5.37) and lemma 5.9,

$$\begin{aligned}\mathfrak{D}_{K\boldsymbol{e}_1}(z^3\widetilde{w}_{\mathbf{1}+K\boldsymbol{e}_1}) &= z^3 \sum_{\substack{\boldsymbol{k}_1+\boldsymbol{k}_2=\mathbf{1}+K\boldsymbol{e}_1\\ |\boldsymbol{k}_1|\geq 1,\,|\boldsymbol{k}_2|\geq 1}} \frac{\widetilde{w}_{\boldsymbol{k}_1}\widetilde{w}_{\boldsymbol{k}_2}}{2}\frac{\partial^2 F(z,w_{\mathbf{0}})}{\partial w^2} \\ &= z^3 \sum_{\substack{k_1+k_2=K\\ k_1\geq 1,\,k_2\geq 1}} \widetilde{w}_{\mathbf{1}+k_1\boldsymbol{e}_1}\widetilde{w}_{k_2\boldsymbol{e}_1}\frac{\partial^2 F(z,w_{\mathbf{0}})}{\partial w^2} \\ &\quad + z^3\left(\widetilde{w}_{\mathbf{1}}\widetilde{w}_{K\boldsymbol{e}_1} + \widetilde{w}_{(1+K)\boldsymbol{e}_1}\widetilde{w}_{\boldsymbol{e}_2}\right)\frac{\partial^2 F(z,w_{\mathbf{0}})}{\partial w^2}\end{aligned} \tag{5.58}$$

With the above reasoning, one gets a unique solution $\widetilde{w}_{\mathbf{1}+K\boldsymbol{e}_1} = \widetilde{w}^{[1]}_{\mathbf{1}+K\boldsymbol{e}_1}\log(z) + \widetilde{w}^{[0]}_{\mathbf{1}+K\boldsymbol{e}_1} \in \bigoplus_{l=0}^{1}\log^l(z)\mathbb{C}[[z^{-1}]]$ where $\widetilde{w}^{[1]}_{\mathbf{1}+K\boldsymbol{e}_1}$ solves the ODE

$$\begin{aligned}\mathfrak{D}_{K\boldsymbol{e}_1}(z^3\widetilde{w}^{[1]}_{\mathbf{1}+K\boldsymbol{e}_1}) &= \varkappa_1 \sum_{\substack{k_1+k_2=K\\ k_1\geq 1,\,k_2\geq 1}} k_1\widetilde{w}_{k_1\boldsymbol{e}_1}\widetilde{w}_{k_2\boldsymbol{e}_1}\frac{\partial^2 F(z,w_{\mathbf{0}})}{\partial w^2} \\ &= K\varkappa_1 \sum_{\substack{k_1+k_2=K\\ k_1\geq 1,\,k_2\geq 1}} \frac{\widetilde{w}_{k_1\boldsymbol{e}_1}\widetilde{w}_{k_2\boldsymbol{e}_1}}{2}\frac{\partial^2 F(z,w_{\mathbf{0}})}{\partial w^2}\end{aligned}$$

Comparing to (5.48), one concludes that

$$\begin{aligned}\widetilde{w}_{\mathbf{1}+K\boldsymbol{e}_1} &= \widetilde{w}^{[1]}_{\mathbf{1}+K\boldsymbol{e}_1}\log(z) + \widetilde{w}^{[0]}_{\mathbf{1}+K\boldsymbol{e}_1}, \\ \widetilde{w}^{[1]}_{\mathbf{1}+K\boldsymbol{e}_1} &= K\varkappa_1 z^{-3}\widetilde{w}_{K\boldsymbol{e}_1}, \qquad \operatorname{val}\widetilde{w}^{[0]}_{\mathbf{1}+K\boldsymbol{e}_1} = 2(K+1).\end{aligned}$$

Case $\boldsymbol{k}=(2,2)$ What remains to do when $\boldsymbol{k}\in\Xi_{2,0}\setminus\Xi_{1,0}$ is to examine the case $\boldsymbol{k}=(2,2)$. By (5.37) and lemma 5.9,

$$\begin{aligned}&\mathfrak{D}_{\mathbf{1}}(z^3 w_{\mathbf{2}}) = \\ &z^3\left(\widetilde{w}_{\mathbf{1}+\boldsymbol{e}_1}\widetilde{w}_{\boldsymbol{e}_2} + \widetilde{w}_{\mathbf{1}+\boldsymbol{e}_2}\widetilde{w}_{\boldsymbol{e}_1} + \widetilde{w}_{2\boldsymbol{e}_1}\widetilde{w}_{2\boldsymbol{e}_2} + \tfrac{1}{2}\widetilde{w}_{\mathbf{1}}\widetilde{w}_{\mathbf{1}}\right)\tfrac{\partial^2 F(z,\widetilde{w}_{\mathbf{0}})}{\partial w^2}.\end{aligned} \tag{5.59}$$

We observe from (5.55) and (5.56) that

$$\widetilde{w}^{[1]}_{\mathbf{1}+\boldsymbol{e}_1}\widetilde{w}_{\boldsymbol{e}_2} + \widetilde{w}^{[1]}_{\mathbf{1}+\boldsymbol{e}_2}\widetilde{w}_{\boldsymbol{e}_1} = \varkappa_1 z^{-3}\widetilde{w}_{\boldsymbol{e}_1}\widetilde{w}_{\boldsymbol{e}_2} + \varkappa_2 z^{-3}\widetilde{w}_{\boldsymbol{e}_2}\widetilde{w}_{\boldsymbol{e}_1} = 0.$$

Therefore the log-term disappears in the right-hand side of (5.59) as a consequence of the symmetries of the problem. Moreover

$$\operatorname{val}\left(\widetilde{w}^{[0]}_{\mathbf{1}+\boldsymbol{e}_1}\widetilde{w}_{\boldsymbol{e}_2} + \widetilde{w}^{[0]}_{\mathbf{1}+\boldsymbol{e}_2}\widetilde{w}_{\boldsymbol{e}_1} + \widetilde{w}_{2\boldsymbol{e}_1}\widetilde{w}_{2\boldsymbol{e}_2} + \frac{1}{2}\widetilde{w}_{\mathbf{1}}\widetilde{w}_{\mathbf{1}}\right) \geq 4.$$

By lemma 5.10, we get $\widetilde{w}_{\mathbf{2}}\in\mathbb{C}[[z^{-1}]]$ with $\operatorname{val}\widetilde{w}_{\mathbf{2}}=6$. Explicit calculation provides: $\widetilde{w}_{\mathbf{2}}(z) = -\dfrac{5}{6}\dfrac{1}{z^6} - \dfrac{2177}{432}\dfrac{1}{z^8} - \dfrac{5288521}{54000}\dfrac{1}{z^{10}} + \cdots$.

Induction We assume that N is an integer ≥ 2 and we suppose that the properties announced in theorem 5.1 are true for any integer $n\in[0,N-1]$ and any $\boldsymbol{k}\in\Xi_{n+1,0}\setminus\Xi_{n,0}$.

We notice on the one hand that $\boldsymbol{\Xi}_{N+1,0}\setminus\boldsymbol{\Xi}_{N,0} = \mathbf{1}+\boldsymbol{\Xi}_{N,0}\setminus\boldsymbol{\Xi}_{N-1,0}$. On the other hand, for every $\boldsymbol{k}\in\Xi_{N,0}\setminus\Xi_{N-1,0}$,

$$\mathfrak{D}_{\mathbf{1}+\boldsymbol{k}}(\widetilde{w}_{\mathbf{1}+\boldsymbol{k}}) = \sum_{\substack{\boldsymbol{k}_1+\boldsymbol{k}_2=\mathbf{1}+\boldsymbol{k}\\ |\boldsymbol{k}_1|\geq 1,\,|\boldsymbol{k}_2|\geq 1}} \frac{\widetilde{w}_{\boldsymbol{k}_1}\widetilde{w}_{\boldsymbol{k}_2}}{2}\frac{\partial^2 F(z,w_{\mathbf{0}})}{\partial w^2} \tag{5.60}$$

We set $X = \log(z)$ and we consider X as an indeterminate. The right-hand side of (5.60) is of the form $\widetilde{f} = \sum \widetilde{f}^{[l]} X^l$ with

$$\begin{aligned}\partial_X \widetilde{f} &= \partial_X \sum_{\substack{\boldsymbol{k}_1+\boldsymbol{k}_2=\boldsymbol{1}+\boldsymbol{k}\\ |\boldsymbol{k}_1|\geq 1,\, |\boldsymbol{k}_2|\geq 1}} \frac{\widetilde{w}_{\boldsymbol{k}_1}\widetilde{w}_{\boldsymbol{k}_2}}{2}\frac{\partial^2 F(z,w_{\boldsymbol{0}})}{\partial w^2}\\ &= \sum_{\substack{\boldsymbol{k}_1+\boldsymbol{k}_2=\boldsymbol{1}+\boldsymbol{k}\\ |\boldsymbol{k}_1|\geq 1,\, |\boldsymbol{k}_2|\geq 1}} \frac{(\partial_X\widetilde{w}_{\boldsymbol{k}_1})\widetilde{w}_{\boldsymbol{k}_2}+\widetilde{w}_{\boldsymbol{k}_1}(\partial_X\widetilde{w}_{\boldsymbol{k}_2})}{2}\frac{\partial^2 F(z,w_{\boldsymbol{0}})}{\partial w^2}.\end{aligned}$$

Using the induction hypothesis, when $\boldsymbol{1}+\boldsymbol{k}_1 \in \Xi_{n+1,0}\setminus\Xi_{n,0}$, for any $n\in[0,N-1]$,

$$\partial_X\left(\sum_{l=1}^{n}\widetilde{w}^{[l]}_{\boldsymbol{1}+\boldsymbol{k}_1}X^l\right) = (\boldsymbol{\varkappa}.\boldsymbol{k}_1)z^{-3}\sum_{l=0}^{n-1}\widetilde{w}^{[l]}_{\boldsymbol{k}_1}X^l,$$

that is $\partial_X\widetilde{w}_{\boldsymbol{1}+\boldsymbol{k}_1} = (\boldsymbol{\varkappa}.\boldsymbol{k}_1)z^{-3}\widetilde{w}_{\boldsymbol{k}_1}$. Therefore:

$$\begin{aligned}\partial_X \widetilde{f} &= z^{-3}\sum_{\substack{\boldsymbol{k}_1+\boldsymbol{k}_2=\boldsymbol{k}\\ |\boldsymbol{k}_1|\geq 1,\, |\boldsymbol{k}_2|\geq 1}} (\boldsymbol{\varkappa}.\boldsymbol{k}_1)\widetilde{w}_{\boldsymbol{k}_1}\widetilde{w}_{\boldsymbol{k}_2}\frac{\partial^2 F(z,w_{\boldsymbol{0}})}{\partial w^2}\\ &= (\boldsymbol{\varkappa}.\boldsymbol{k})z^{-3}\sum_{\substack{\boldsymbol{k}_1+\boldsymbol{k}_2=\boldsymbol{k}\\ |\boldsymbol{k}_1|\geq 1,\, |\boldsymbol{k}_2|\geq 1}} \frac{\widetilde{w}_{\boldsymbol{k}_1}\widetilde{w}_{\boldsymbol{k}_2}}{2}\frac{\partial^2 F(z,w_{\boldsymbol{0}})}{\partial w^2}.\end{aligned}$$

Thus $\partial_X\widetilde{f} = (\boldsymbol{\varkappa}.\boldsymbol{k})z^{-3}\mathfrak{D}_{\boldsymbol{k}}(\widetilde{w}_{\boldsymbol{k}})$ and (5.60) provides:

$$\partial_X\Big(\mathfrak{D}_{\boldsymbol{k}}(z^3\widetilde{w}_{\boldsymbol{1}+\boldsymbol{k}})\Big) = (\boldsymbol{\varkappa}.\boldsymbol{k})\mathfrak{D}_{\boldsymbol{k}}(\widetilde{w}_{\boldsymbol{k}}).$$

Observing that $\partial_X\mathfrak{D}_{\boldsymbol{k}}\partial_X^{-1} = \mathfrak{D}_{\boldsymbol{k}}$, one easily gets $\widetilde{w}_{\boldsymbol{1}+\boldsymbol{k}}$ either from lemma 5.11 or lemma 5.10, with $\widetilde{w}_{\boldsymbol{1}+\boldsymbol{k}} = (\boldsymbol{\varkappa}.\boldsymbol{k})z^{-3}\partial_X^{-1}\widetilde{w}_{\boldsymbol{k}}$.
The property for $\widetilde{w}_{\boldsymbol{n}+\boldsymbol{1}}$ is easy and is left to the reader. This ends the proof of theorem 5.1. □

Definition 5.9. The two-parameter formal solution defined by theorem 5.1 is the **formal integral** of the prepared ODE (3.6) associated with the first Painlevé equation. The coefficients λ_i, τ_i and $\varkappa_i$, $i=1,2$, are the **formal invariants**.
The formal series $\widetilde{w}^{[0]}_{\boldsymbol{k}}$ are called the **$\boldsymbol{k}$-th series** of the formal integral. We denote $\widetilde{W}^{[0]}_{\boldsymbol{k}} = z^{-\boldsymbol{\tau}.\boldsymbol{k}}\widetilde{w}^{[0]}_{\boldsymbol{k}}$ and $\widetilde{W}_{\boldsymbol{k}} = z^{-\boldsymbol{\tau}.\boldsymbol{k}}\widetilde{w}_{\boldsymbol{k}}$ for any $\boldsymbol{k}\in\mathbb{N}^2$.

Remark 5.8. Theorem 5.1 can be compared to [23] and specially to [1], where the calculations made there translate into ours up to renormalization.

Definition 5.10. For any $\boldsymbol{k}\in\mathbb{N}^2$, one denotes by $\mathfrak{E}_{\boldsymbol{k}}$ and $\mathfrak{F}_{\boldsymbol{k}}$ the following operators:

$$\begin{aligned}\mathfrak{E}_{\boldsymbol{k}} &= \frac{\boldsymbol{\varkappa}.\boldsymbol{k}}{z^4}P'(\partial-\boldsymbol{\lambda}.\boldsymbol{k}) + \frac{\boldsymbol{\varkappa}.\boldsymbol{k}}{z^5}\left(Q'(\partial-\boldsymbol{\lambda}.\boldsymbol{k}) - \frac{\boldsymbol{\tau}.(\boldsymbol{2k}-\boldsymbol{1})+1}{2!}P''(\partial-\boldsymbol{\lambda}.\boldsymbol{k})\right)\\ &= 2\frac{\boldsymbol{\varkappa}.\boldsymbol{k}}{z^4}(\partial-\boldsymbol{\lambda}.\boldsymbol{k}) - \frac{\boldsymbol{\varkappa}.\boldsymbol{k}}{z^5}\left(\boldsymbol{\tau}.(\boldsymbol{2k}-\boldsymbol{1})+4\right),\\ \mathfrak{F}_{\boldsymbol{k}} &= \frac{1}{2!}\frac{(\boldsymbol{\varkappa}.\boldsymbol{k})^2}{z^8}P''(\partial-\boldsymbol{\lambda}.\boldsymbol{k}) = \frac{(\boldsymbol{\varkappa}.\boldsymbol{k})^2}{z^8}.\end{aligned}$$

We need hardly mention the analogue of lemma 5.9.

Lemma 5.12. *For every* $n \in \mathbb{N}$, $\boldsymbol{k} \in \mathbb{N}^2$,

$$\mathfrak{E}_{\mathbf{n}+\boldsymbol{k}} = z^{\boldsymbol{\tau}.\mathbf{n}} \mathfrak{E}_{\boldsymbol{k}} z^{-\boldsymbol{\tau}.\mathbf{n}}, \qquad \mathfrak{F}_{\mathbf{n}+\boldsymbol{k}} = z^{\boldsymbol{\tau}.\mathbf{n}} \mathfrak{F}_{\boldsymbol{k}} z^{-\boldsymbol{\tau}.\mathbf{n}}.$$

We finally give a corollary stemming from theorem 5.1.

Corollary 5.1. *The formal integral (5.50) associated with the prepared ODE (3.6) can be written under the form:*

$$\widetilde{w}(z, \boldsymbol{U}) = \sum_{\boldsymbol{k} \in \mathbb{N}^2} \boldsymbol{V}^{\boldsymbol{k}} \widetilde{w}_{\boldsymbol{k}}^{[0]}, \qquad \boldsymbol{V}^{\boldsymbol{k}} = \boldsymbol{U}^{\boldsymbol{k}} \mathrm{e}^{-(\boldsymbol{\lambda}.\boldsymbol{k})z + (\boldsymbol{\varkappa}.\boldsymbol{k})\boldsymbol{U}^{\mathbf{1}} \log(z)} z^{-\boldsymbol{\tau}.\boldsymbol{k}}. \tag{5.61}$$

Equivalently,

$$\widetilde{w}(z, \boldsymbol{U}) = \widetilde{\Phi}(z, U_1 \mathrm{e}^{-\lambda_1 z - (\tau_1 - \varkappa_1 \boldsymbol{U}^{\mathbf{1}}) \log(z)}, U_2 \mathrm{e}^{-\lambda_2 z - (\tau_2 - \varkappa_2 \boldsymbol{U}^{\mathbf{1}}) \log(z)})$$

where $\widetilde{\Phi}(z, \mathbf{u}) = \sum_{\boldsymbol{k} \in \mathbb{N}^2} \mathbf{u}^{\boldsymbol{k}} \widetilde{w}_{\boldsymbol{k}}^{[0]}(z) \in \mathbb{C}[[z^{-1}, \mathbf{u}]]$ *is solution of the equation:*

$$P\Big(\partial_z - \sum_{i=1}^{2} (\lambda_i + \frac{\tau_i - \varkappa_i \mathbf{u}^{\mathbf{1}}}{z}) u_i \partial_{u_i}\Big)\widetilde{\Phi} + \frac{1}{z} Q\Big(\partial_z - \sum_{i=1}^{2} (\lambda_i + \frac{\tau_i - \varkappa_i \mathbf{u}^{\mathbf{1}}}{z}) u_i \partial_{u_i}\Big)\widetilde{\Phi} = F(z, \widetilde{\Phi}). \tag{5.62}$$

The formal series $\widetilde{w}_{\boldsymbol{k}}^{[0]} \in z^{-2|\boldsymbol{k}|+2} \mathbb{R}[[z^{-1}]]$ *satisfy:*

- *for any* $\boldsymbol{k} \in \Xi_{1,0} \setminus \Xi_{0,0}$, $\mathfrak{D}_{\boldsymbol{k}} \widetilde{w}_{\boldsymbol{k}}^{[0]} = \displaystyle\sum_{\substack{\boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{k} \\ |\boldsymbol{k}_i| \geq 1}} \frac{w_{\boldsymbol{k}_1}^{[0]} w_{\boldsymbol{k}_2}^{[0]}}{2!} \frac{\partial^2 F(z, \widetilde{w}_{\mathbf{0}})}{\partial w^2}$;
- *for any* $\boldsymbol{k} \in \Xi_{2,0} \setminus \Xi_{1,0}$, $\mathfrak{D}_{\boldsymbol{k}} \widetilde{w}_{\boldsymbol{k}}^{[0]} + \mathfrak{E}_{\boldsymbol{k}} \widetilde{w}_{\boldsymbol{k}-\mathbf{1}}^{[0]} = \displaystyle\sum_{\substack{\boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{k} \\ |\boldsymbol{k}_i| \geq 1}} \frac{w_{\boldsymbol{k}_1}^{[0]} w_{\boldsymbol{k}_2}^{[0]}}{2!} \frac{\partial^2 F(z, \widetilde{w}_{\mathbf{0}})}{\partial w^2}$;
- *otherwise,* $\mathfrak{D}_{\boldsymbol{k}} \widetilde{w}_{\boldsymbol{k}}^{[0]} + \mathfrak{E}_{\boldsymbol{k}} \widetilde{w}_{\boldsymbol{k}-\mathbf{1}}^{[0]} + \mathfrak{F}_{\boldsymbol{k}} \widetilde{w}_{\boldsymbol{k}-\mathbf{2}}^{[0]} = \displaystyle\sum_{\substack{\boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{k} \\ |\boldsymbol{k}_i| \geq 1}} \frac{w_{\boldsymbol{k}_1}^{[0]} w_{\boldsymbol{k}_2}^{[0]}}{2!} \frac{\partial^2 F(z, \widetilde{w}_{\mathbf{0}})}{\partial w^2}$;

Proof. Let us examine (5.50) more closely. The formal integral can be written as follows:

$$\widetilde{w}(z, \boldsymbol{U}) = \sum_{n=0}^{\infty} \boldsymbol{V}^{\mathbf{n}} \widetilde{w}_{\mathbf{n}}(z) + \sum_{i=1,2} \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} \boldsymbol{V}^{\mathbf{n}+k\boldsymbol{e}_i} \widetilde{w}_{\mathbf{n}+k\boldsymbol{e}_i}(z), \tag{5.63}$$

that is we consider the sums along the direction given by the vector $(1, 1)$ that determines the resonance. We set $\boldsymbol{T}^{\boldsymbol{k}} = \boldsymbol{U}^{\boldsymbol{k}} \mathrm{e}^{-(\boldsymbol{\lambda}.\boldsymbol{k})z + (\boldsymbol{\varkappa}.\boldsymbol{k})\boldsymbol{U}^{\mathbf{1}} \log(z)} z^{-\boldsymbol{\tau}.\boldsymbol{k}}$. For the first sum we know that each $\widetilde{w}_{\mathbf{n}}(z)$ belongs to $\mathbb{C}[[z^{-1}]]$ and $\sum_{n=0}^{\infty} \boldsymbol{V}^{\mathbf{n}} \widetilde{w}_{\mathbf{n}} = \sum_{n=0}^{\infty} \boldsymbol{T}^{\mathbf{n}} \widetilde{w}_{\mathbf{n}}$ because $\boldsymbol{\varkappa}.\mathbf{n} = 0$.
We now look at the other sums and we use the relations given by (5.51). We get for $i = 1, 2$,

$$\begin{aligned}\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\boldsymbol{V}^{\mathbf{n}+k\boldsymbol{e}_i}\widetilde{w}_{\mathbf{n}+k\boldsymbol{e}_i} &= \sum_{k=1}^{\infty}\boldsymbol{V}^{k\boldsymbol{e}_i}\sum_{n=0}^{\infty}\boldsymbol{V}^{\mathbf{n}}\sum_{l=0}^{n}\frac{1}{l!}\left(\varkappa_i k z^{-3}\log(z)\right)^l \widetilde{w}^{[0]}_{\mathbf{n}-\mathbf{l}+k\boldsymbol{e}_i}\\ &= \sum_{n=0}^{\infty}\boldsymbol{V}^{\mathbf{n}}\sum_{k=1}^{\infty}\boldsymbol{V}^{k\boldsymbol{e}_i}\mathrm{e}^{\left(\varkappa_i k\boldsymbol{U}^{\mathbf{1}}\log(z)\right)}\widetilde{w}^{[0]}_{\mathbf{n}+k\boldsymbol{e}_i}.\\ &= \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\boldsymbol{T}^{\mathbf{n}+k\boldsymbol{e}_i}\widetilde{w}^{[0]}_{\mathbf{n}+k\boldsymbol{e}_i}.\end{aligned}$$

The equation (5.62) is obtained by the arguments developed in remark 5.3. The reader will check that equation (5.62) is equivalent to the given hierarchy of equations. □

Let us write $u_1(z) = U_1\mathrm{e}^{-\lambda_1 z-(\tau_1-\varkappa_1\boldsymbol{U}^{\mathbf{1}})\log(z)}$, $u_2(z) = U_2\mathrm{e}^{-\lambda_2 z-(\tau_2-\varkappa_2\boldsymbol{U}^{\mathbf{1}})\log(z)}$ and observe that ${}^t(u_1,u_2)$ provides the general analytic solution for a non linear differential equation that only depends on the formal invariants:

$$\partial\begin{pmatrix}u_1\\u_2\end{pmatrix}+\begin{pmatrix}\lambda_1+\frac{\tau_1}{z} & 0\\ 0 & \lambda_2+\frac{\tau_2}{z}\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}=\begin{pmatrix}\frac{\varkappa_1}{z^4}u_1u_2 & 0\\ 0 & \frac{\varkappa_2}{z^4}u_1u_2\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}. \quad (5.64)$$

This means that corollary 5.1 can be written another way.

Corollary 5.2. *There exists a formal transformation* $w = \widetilde{\Phi}(z,\mathbf{u})$ *of the form*

$$\widetilde{\Phi}(z,\mathbf{u}) = \sum_{\boldsymbol{k}\in\mathbb{N}^2}\mathbf{u}^{\boldsymbol{k}}\widetilde{w}^{[0]}_{\boldsymbol{k}}(z), \qquad \widetilde{w}^{[0]}_{\boldsymbol{k}} \in \mathbb{C}[[z^{-1}]], \quad (5.65)$$

that formally transforms the prepared ODE (3.6) into the **normal form** *equation:*

$$\partial\mathbf{u}+B_0(z)\mathbf{u} = B_1\big(z,\mathbf{u}\big)\mathbf{u} \quad (5.66)$$

$$B_0 = \begin{pmatrix}\lambda_1+\frac{\tau_1}{z} & 0\\ 0 & \lambda_2+\frac{\tau_2}{z}\end{pmatrix}, \quad B_1(z,\mathbf{u}) = \frac{\mathbf{u}^{\mathbf{1}}}{z^4}\begin{pmatrix}\varkappa_1 & 0\\ 0 & \varkappa_2\end{pmatrix}, \quad \mathbf{u}^{\mathbf{1}} = u_1u_2.$$

5.5 Comments

Analogues of proposition 5.1 can be stated for differential equations, resp. difference equations, of of order 1 and dimension n, with one level and no resonance, given in prepared form :

$$\partial\mathbf{v}+B_0(z)\mathbf{v} = \boldsymbol{g}(z,\mathbf{v}) \quad (5.67)$$

with $B_0(z) = \bigoplus_j\left(\lambda_j I_{n_j}+z^{-1}M_j\right)$, $\sum_j n_j = n$, resp.

$$\mathbf{v}(z+1) = B_0(z)\mathbf{v}(z)+\boldsymbol{g}(z,\mathbf{v}) \quad (5.68)$$

with $B_0(z) = \bigoplus_j \mathrm{e}^{-\lambda_j z}(1+z^{-1})^{M_j}$. In each case, there exists a formal transformation of the type $\mathbf{v} = \widetilde{T}(z,\mathbf{u})$, $\widetilde{T}(z,\mathbf{u}) = \sum_{\boldsymbol{k}\in\mathbb{N}^n}\mathbf{u}^{\boldsymbol{k}}\widetilde{\mathbf{v}}_{\boldsymbol{k}}(z)$, $\widetilde{\mathbf{v}}_{\boldsymbol{k}}(z)\in\mathbb{C}^n[[z^{-1}]]$ that brings the equation to the linear normal form $\partial\mathbf{u}+B_0(z)\mathbf{u} = 0$, resp. $\mathbf{u}(z+1) = B_0(z)\mathbf{u}(z)$.

To be correct, the upshot for difference equations is more subtle.

This property is still valid for differential equations with more than one level, see [30, 4, 7] and references therein. In particular, the whole set of formal invariants are alreday given by the linear part (in Jordan form) of the equation.

When resonances occur and as we saw with the first Painlevé equation, the normal form equation is nonlinear and new formal invariants appear. This is essentially a consequence of the Poincaré-Dulac theorem [2]; for instance in (5.66), one recognizes the effect of the positively resonance of order 3 with the resonances monomials $u_1^2 u_2$ and $u_1 u_2^2$. The classification is detailed in [18], see also [20] where the notion of (so-called) moulds and arborification are used (a good introduction of which is [35]).

Acknowledgements I am indebted to my student Julie Belpaume for helping me to working out this chapter. I thank Jean Ecalle for interesting discussions on phenomena induced by resonances.

References

1. I. Aniceto, R. Schiappa, M. Vonk, *The resurgence of instantons in string theory.* Commun. Number Theory Phys. **6** (2012), no. 2, 339-496.
2. V. I. Arnol'd, *Geometrical methods in the theory of ordinary differential equations.* Grundlehren der Mathematischen Wissenschaften, 250. Springer-Verlag, New York-Berlin, 1983.
3. W. Balser, *Formal power series and linear systems of meromorphic ordinary differential equations.* Universitext. Springer-Verlag, New York, 2000.
4. B.L.J. Braaksma, *Multisummability of formal power series solutions of nonlinear meromorphic differential equations.* Ann. Inst. Fourier (Grenoble) **42** (1992), no. 3, 517-540.
5. E.A. Coddington, N. Levinson, *Theory of ordinary differential equations.* McGraw-Hill Book Company 1955.
6. O. Costin, *Topological construction of transseries and introduction to generalized Borel summability.* Analyzable functions and applications, 137-175, Contemp. Math., **373**, Amer. Math. Soc., Providence, RI, 2005.
7. O. Costin, *Asymptotics and Borel summability*, Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, 141. CRC Press, Boca Raton, FL, 2009.
8. E. Delabaere, H. Dillinger, F. Pham, *Résurgence de Voros et périodes des courbes hyperelliptiques*, Annales de l'Institut Fourier **43** (1993), no. 1, 163-199.
9. E. Delabaere, H. Dillinger, F. Pham, *Exact semi-classical expansions for one dimensional quantum oscillators*, Journal Math. Phys. **38** (1997), 12, 6126-6184.
10. E. Delabaere, F. Pham, *Unfolding the quartic oscillator*, Ann. Physics **261** (1997), no. 2, 180-218.
11. E. Delabaere, F. Pham, *Eigenvalues of complex Hamiltonians with PT-symmetry*, Phys. Lett. A **250** (1998), no. 1-3, 25-32.
12. E. Delabaere, F. Pham, *Resurgent methods in semi-classical asymptotics*, Ann. Inst. Henri Poincaré, Sect. A **71** (1999), no 1, 1-94.
13. E. Delabaere, D.T. Trinh, *Spectral analysis of the complex cubic oscillator,* J.Phys. A: Math. Gen. **33** (2000), no. 48, 8771-8796.
14. P. Deligne, *Equations différentielles à points singuliers réguliers.* Lecture Notes in Mathematics, Vol. 163. Springer-Verlag, Berlin-New York, 1970.
15. G. Dunne, M. Ünsal, *Resurgence and trans-series in quantum field theory: the* $\mathbb{CP}^{N-1}$ *model.* J. High Energy Phys. 2012, no. 11, 170, front matter + 84 pp.
16. G. Başar, G. Dunne, M. Ünsal, *Resurgence theory, ghost-instantons, and analytic continuation of path integrals.* J. High Energy Phys. 2013, no. 10, 041, front matter+34 pp.

17. G. Dunne, M. Ünsal, *Uniform WKB, multi-instantons, and resurgent trans-series.* Phys. Rev. D **89**, 105009 (2014)
18. J. Ecalle, *Les fonctions résurgentes. Tome III : l'équation du pont et la classification analytique des objets locaux.* Publ. Math. d'Orsay, Université Paris-Sud, 1985.05 (1985).
19. J. Écalle, *Fonctions analysables et preuve constructive de la conjecture de Dulac.* Actualités mathématiques, Hermann, Paris (1992).
20. J. Écalle, *Singularités non abordables par la géométrie.* Ann. Inst. Fourier (Grenoble) **42** (1992), no. 1-2, 73-164.
21. A. Getmanenko, *Resurgent analysis of the Witten Laplacian in one dimension.* Funkcial. Ekvac. **54** (2011), no. 3, 383-438.
22. A. Getmanenko, *Resurgent analysis of the Witten Laplacian in one dimension-II.* Funkcial. Ekvac. **56** (2013), no. 1, 121-176.
23. S. Garoufalidis, A. Its, A. Kapaev, M. Mariño, *Asymptotics of the instantons of Painlevé I.* Int. Math. Res. Not. IMRN, 2012, no. 3, 561-606.
24. U. Jentschura, J. Zinn-Justin, *Instantons in quantum mechanics and resurgent expansions.* Phys. Lett. B **596** (2004), no. 1-2, 138-144.
25. U. Jentschura, J. Zinn-Justin, *Multi-instantons and exact results. II. Specific cases, higher-order effects, and numerical calculations.* Ann. Physics **313** (2004), no. 2, 269-325.
26. U. Jentschura, J. Zinn-Justin, *Multi-instantons and exact results. III. Unification of even and odd anharmonic oscillators.* Ann. Physics **325** (2010), no. 5, 1135-1172.
27. T. Kawai, Y. Takei, *WKB analysis of Painlevé transcendents with a large parameter. I.* Adv. Math. **118** (1996), no. 1, 1-33.
28. T. Kawai, Y. Takei, *WKB analysis of higher order Painlevé equations with a large parameter. II. Structure theorem for instanton-type solutions of* $(PJ)_m$ $(J = I, 34, II{-}2$ *or* $IV)$ *near a simple* P*-turning point of the first kind.* Publ. Res. Inst. Math. Sci. **47** (2011), no. 1, 153-219.
29. M. Kohno, *Global analysis in linear differential equations.* Mathematics and its Applications, 471. Kluwer Academic Publishers, Dordrecht, 1999.
30. G.R. Kuik, *Transseries in Difference and Differential Equations.* PhD thesis, Rijksuniversiteit Groningen (2003).
31. M. Loday-Richaud, *Divergent series and differential equations.* To appear.
32. M. Mariño, R. Schiappa, M. Weiss, *Multi-instantons and multicuts.* J. Math. Phys. **50** (2009), no. 5, 052301, 31 pp.
33. C. Olivé, D. Sauzin, T. Seara, *Resurgence in a Hamilton-Jacobi equation.* Proceedings of the International Conference in Honor of Frédéric Pham (Nice, 2002). Ann. Inst. Fourier (Grenoble) **53** (2003), no. 4, 1185-1235.
34. D. Sauzin, *Résurgence paramétrique et exponentielle petitesse de l'écart des séparatrices du pendule rapidement forcé.* Ann. Inst. Fourier (Grenoble) **45** (1995), no. 2, 453-511.
35. D. Sauzin, *Mould expansions for the saddle-node and resurgence monomials.* In Renormalization and Galois theories, 83-163, IRMA Lect. Math. Theor. Phys., 15, Eur. Math. Soc., Zürich, 2009.
36. A. Voros, *The return of the quartic oscillator: the complex WKB method.* Ann. Inst. H. Poincar Sect. A (N.S.) **39** (1983), no. 3, 211-338.
37. W. Wasow, *Asymptotic expansions for ODE.* Reprint of the 1965 edition. Robert E. Krieger Publishing Co., Huntington, N.Y., 1976.
38. J. Zinn-Justin, *Quantum field theory and critical phenomena.* Oxford Univ. Press (1989).

Chapter 6
Truncated solutions for Painlevé I

Abstract In the previous chapters, we studied the unique formal solution of the first Painlevé equation then we introduced its formal integral. We show that its formal series components are 1-Gevrey and their minors have analytic properties quite similar to those for the minor of the formal series solution we started with (Sect. 6.1). We then make a focus on the transseries solution of the first Painlevé equation and show their Borel-Laplace summability (Sect. 6.2). This provides by Borel-Laplace summation the truncated solutions (Sect. 6.4).

6.1 Formal integral : 1-summability of the $\boldsymbol{k}$-th series and beyond

We have described with theorem 5.1 and its corollary 5.1 the formal integral $\widetilde{w}(z,\boldsymbol{U}) = \sum_{\boldsymbol{k}\in\mathbb{N}^2} \boldsymbol{V}^{\boldsymbol{k}} \widetilde{w}_{\boldsymbol{k}}^{[0]}$ associated with the first Painlevé equation. Our goal in this section is mainly to show that the following theorem.

Theorem 6.1. *For every $\boldsymbol{k}\in\mathbb{N}^2$, the $\boldsymbol{k}$-th series $\widetilde{w}_{\boldsymbol{k}}^{[0]}$ is 1-Gevrey, its minor $\widehat{w}_{\boldsymbol{k}}^{[0]}$ defines a holomorphic function on $\overset{\bullet}{\mathscr{R}}{}^{(0)}$ with at most exponential growth of order 1 at infinity. Moreover, $\widehat{w}_{\boldsymbol{k}}^{[0]}$ can be analytically continued to the Riemann surface $\mathscr{R}^{(1)}$, with at most exponential growth of order 1 at infinity on $\mathscr{R}^{(1)}$.*

We already know by theorem 3.2 and theorem 4.1 that $\widehat{w}_{\boldsymbol{0}} = \widehat{w}_{\boldsymbol{0}}^{[0]}$ enjoyes the above properties. Our task comes down to studying the other $\boldsymbol{k}$-th series. This is what we do in what follows and we start with some preliminaries.

6.1.1 Preliminary results

In what follows we use a notation introduced in definition 5.5.

Lemma 6.1. *We set $P(\partial) = \partial^2 - 1$ and for every $\boldsymbol{k}\in\mathbb{N}^2$, $P_{\boldsymbol{k}}(\partial) = P(-\boldsymbol{\lambda}.\boldsymbol{k}+\partial)$ with $\lambda = (\lambda_1,\lambda_2) = (1,-1)$. For $i=1,2$, we define $\widetilde{P}_{\boldsymbol{e}_i}(\partial)$ by $P_{\boldsymbol{e}_i}(\partial) = \widetilde{P}_{\boldsymbol{e}_i}(\partial)\partial$ so that $\widetilde{P}_{\boldsymbol{e}_i}(-\lambda_i) \neq 0$.*

Then, for any $\rho \in]0,1[$, *there exists* $M_{\rho,(0)} > 0$ *such that, for every* $\zeta \in \mathbb{C} \setminus \bigcup_{m\in\mathbb{Z}^\star} \overline{D}(m, m\rho)$ *:*

1. for $i = 1, 2$, $\left|\dfrac{1}{\widetilde{P}_{\boldsymbol{e}_i}(-\zeta)}\right| \leq M_{\rho,(0)}$*;*

2. for every $\boldsymbol{k} \in \boldsymbol{\Xi}_{1,0}$ *with* $|\boldsymbol{k}| \geq 2$*, for* $m = 0, 1$, $\left|\dfrac{(\zeta + \boldsymbol{\lambda}.\boldsymbol{k})^m}{P_{\boldsymbol{k}}(-\zeta)}\right| \leq \dfrac{M_{\rho,(0)}}{|\boldsymbol{k}| - 1}$

and, for $\boldsymbol{k} \neq (1,1)$, $\left|\dfrac{1}{P_{\boldsymbol{k}}(-\zeta)}\right| \leq \dfrac{M^2_{\rho,(0)}}{|\boldsymbol{k}|^2 - 1}$.

Moreover one can take $M_{\rho,(0)} = \dfrac{1}{\rho}$.

Proof. We only examine the case $\boldsymbol{k} \in \boldsymbol{\Xi}_{1,0} \setminus \{(1,1)\}$ with $|\boldsymbol{k}| > 1$. With no loss of generality, we can assume that $\boldsymbol{k} = (k, 0)$ with $k \geq 2$. Thus $P_{\boldsymbol{k}}(-\zeta) = (\zeta + k - 1)(\zeta + k + 1)$, $\zeta + \boldsymbol{\lambda}.\boldsymbol{k} = \zeta + k$ and we notice that $|\zeta + k - 1| \geq (k-1)\rho$ and $|\zeta + k + 1| \geq (k+1)\rho$ for $\zeta \in \mathbb{C} \setminus \bigcup_{m\in\mathbb{Z}^\star} \overline{D}(m, m\rho)$.

Therefore, $\dfrac{1}{|P_{\boldsymbol{k}}(-\zeta)|} \leq \dfrac{1}{(k^2-1)\rho^2}$ for $\zeta \in \mathbb{C} \setminus \bigcup_{m\in\mathbb{Z}^\star} \overline{D}(m, m\rho)$. Now:

- either $\Re(\zeta + k) \geq 0$, then $|\zeta + k + 1| \geq \max\{1, |\zeta + k|\}$. This implies that

$$\frac{\max\{1, |\zeta + \boldsymbol{\lambda}.\boldsymbol{k}|\}}{|P_{\boldsymbol{k}}(-\zeta)|} \leq \frac{1}{(k-1)\rho}.$$

- or $\Re(\zeta + k) \leq 0$, then $|\zeta + k - 1| \geq \max\{1, |\zeta + k|\}$. This implies that

$$\frac{\max\{1, |\zeta + \boldsymbol{\lambda}.\boldsymbol{k}|\}}{|P_{\boldsymbol{k}}(-\zeta)|} \leq \frac{1}{(k+1)\rho}.$$

□

Lemma 6.2. *Under the conditions of lemma 6.1, we note* $Q(\partial) = -3\partial$ *and* $Q_{\boldsymbol{k}}(\partial)$, $R_{\boldsymbol{k}}(\partial)$ *given by (5.31), (5.32) with* $\boldsymbol{\tau} = \left(-\dfrac{3}{2}, -\dfrac{3}{2}\right)$.

Then, for every $\boldsymbol{k} \in \boldsymbol{\Xi}_{1,0} \setminus \{(1,1)\}$ *with* $|\boldsymbol{k}| > 1$*, for every* $\zeta \in \overset{\bullet}{\mathscr{R}}{}^{(0)}_{\rho}$,

$$\frac{|Q_{\boldsymbol{k}}|(|\zeta|)}{|P_{\boldsymbol{k}}(-\zeta)|} \leq 3M_{\rho,(0)}, \qquad \frac{|R_{\boldsymbol{k}}|(|\zeta|)}{|P_{\boldsymbol{k}}(-\zeta)|} \leq \frac{9}{4}M^2_{\rho,(0)}.$$

Proof. We note that lemma 6.1 can be applied for $\zeta \in \overset{\bullet}{\mathscr{R}}{}^{(0)}_{\rho}$.
We have $|Q_{\boldsymbol{k}}|(\xi) = 3(|\boldsymbol{k}| - 1)\big|\xi + \boldsymbol{\lambda}.\boldsymbol{k}\big|$ (see (5.39)), Therefore, by lemma 6.1, $\dfrac{|Q_{\boldsymbol{k}}|(|\zeta|)}{|P_{\boldsymbol{k}}(-\zeta)|} \leq 3M_{\rho,(0)}$. In the same way, one easily sees that $|R_{\boldsymbol{k}}(\partial)| \leq \dfrac{9}{4}|\boldsymbol{k}|(|\boldsymbol{k}| - 1)$ (cf. (5.39)), thus the result by lemma 6.1. □

We eventually introduce the following notation that complements definition 3.3.

Definition 6.1. Assume that $G(\zeta, \boldsymbol{w}) = \sum_{|\boldsymbol{l}|\geq 0} c_{\boldsymbol{l}}(\zeta)\boldsymbol{w}^{\boldsymbol{l}}$ is an analytic function on the open polydisc $\Delta_{\boldsymbol{r}} = \prod_{i=0}^{n} D(0, r_i)$. One defines the function $|G|$, analytic on $\Delta_{\boldsymbol{r}}$, by $|G|(\xi, \boldsymbol{w}) = \sum_{l\geq 0} |c_{\boldsymbol{l}}|(\xi)\boldsymbol{w}^{\boldsymbol{l}}$.

6.1.2 The 1-st series

We start our proof of theorem 6.1 by paying special attention to $\widetilde{w}_{\boldsymbol{e}_i} = \widetilde{w}^{[0]}_{\boldsymbol{e}_i}$.

Lemma 6.3. *The* 1*-st series* $\widetilde{w}_{\boldsymbol{e}_i}$ *is 1-Gevrey. Its formal Borel transform reads* $\tilde{\mathcal{B}}(\widetilde{w}_{\boldsymbol{e}_i})(\zeta) = \delta + \widehat{w}_{\boldsymbol{e}_i}(\zeta)$ *and* $\widehat{w}_{\boldsymbol{e}_i}$ *is holomorphic on* $\overset{\bullet}{\mathscr{R}}{}^{(0)}$ *with at most exponential growth of order 1 at infinity. More precisely, for every* $\rho \in]0,1[$*, there exist* $A > 0$ *and* $\tau > 0$ *such that*

$$\text{for every } \zeta \in \overset{\bullet}{\mathscr{R}}{}^{(0)}_{\rho},\ |\widehat{w}_{\boldsymbol{e}_i}(\zeta)| \leq A\mathrm{e}^{\tau|\zeta|}.$$

In the above upper bounds one can choose $A = \tau = \dfrac{5.81}{\rho}$.
Moreover, $\widehat{w}_{\boldsymbol{e}_i}$ *can be analytically continued to the Riemann surface* $\mathscr{R}^{(1)}$*, with at most exponential growth of order 1 at infinity on* $\mathscr{R}^{(1)}$*.*

Proof. It is enough to study $\widetilde{w}_{\boldsymbol{e}_1}$ since $\widetilde{w}_{\boldsymbol{e}_2}(z) = \widetilde{w}_{\boldsymbol{e}_1}(-z)$. We know that $\widetilde{w}_{\boldsymbol{e}_1}$ solves the equation (5.45), namely:

$$\partial \widetilde{P}_{\boldsymbol{e}_1}(\partial)\widetilde{w}_{\boldsymbol{e}_1} = \left(\frac{15}{4}\frac{1}{z^2} + \frac{\partial F(z,\widetilde{w}_{\mathbf{0}})}{\partial w}\right)\widetilde{w}_{\boldsymbol{e}_1}, \qquad \widetilde{P}_{\boldsymbol{e}_i} = \partial - 2. \tag{6.1}$$

The formal Borel transform of $\widetilde{w}_{\boldsymbol{e}_1}$ reads $\tilde{\mathcal{B}}(\widetilde{w}_{\boldsymbol{e}_1})(\zeta) = \delta + \widehat{w}_{\boldsymbol{e}_1}(\zeta)$, where the minor $\widehat{w}_{\boldsymbol{e}_1}(\zeta) \in \mathbb{C}[[\zeta]]$ satisfies the following convolution equation deduced from (6.1):

$$\partial \widetilde{P}_{\boldsymbol{e}_1}(\partial)\widehat{w}_{\boldsymbol{e}_1} = \left(\frac{15}{4}\zeta + \frac{\partial \widehat{F}(\zeta,\widehat{w}_{\mathbf{0}})}{\partial w}\right) * (\delta + \widehat{w}_{\boldsymbol{e}_1}). \tag{6.2}$$

In this equation, we use the notation:

$$\frac{\partial \widehat{F}(\zeta,\widehat{w}_{\mathbf{0}})}{\partial w} = \widehat{f}_1(\zeta) + 2\widehat{f}_2 * \widehat{w}_{\mathbf{0}}(\zeta) = -4\zeta + \zeta * \widehat{w}_{\mathbf{0}}(\zeta). \tag{6.3}$$

The equation (6.2) can be thought of as a linear differential equation with a *regular singular* point at 0.

Instead of (6.2), consider the convolution equation $\partial \widetilde{P}_{\boldsymbol{e}_1}(\partial)\widehat{w} = \left(a_1\zeta + a_2\dfrac{\zeta^2}{2!}\right) * (\delta + \widehat{w})$. Set $\widehat{g} = \partial \widetilde{P}_{\boldsymbol{e}_1}(\partial)\widehat{w} = \zeta(\zeta+2)\widehat{w}$. For $\zeta \neq 0$, one gets $\widehat{g} = \left(a_1\zeta + a_2\dfrac{\zeta^2}{2!}\right) * \left(\delta + \dfrac{\widehat{g}}{\zeta(\zeta+2)}\right)$. This implies by differentiation that $\widehat{g}^{(4)} = a_1\left(\dfrac{\widehat{g}}{\zeta(\zeta+2)}\right)^{(2)} + a_2\left(\dfrac{\widehat{g}}{\zeta(\zeta+2)}\right)^{(1)}$ where $\widehat{g}^{(i)} = \dfrac{\mathrm{d}^i\widehat{g}}{\mathrm{d}\zeta^i}$. The last ODE has a regular singular point at 0. One can apply the same trick to (6.2) but for the fact of getting an infinite order differential operator.

The equation (6.2) can be analyzed with the tools developed in Sect. 3.4. We introduce $\widehat{G}(\zeta) = \dfrac{15}{4}\zeta + \dfrac{\partial \widehat{F}(\zeta,\widehat{w}_{\mathbf{0}})}{\partial w} = -\dfrac{\zeta}{4} + \zeta * \widehat{w}_{\mathbf{0}}(\zeta)$ and we remark that $\widehat{G}$ belongs to the maximal ideal $\mathcal{MO}(\overset{\bullet}{\mathscr{R}}{}^{(0)}_{\rho})$ of $\mathcal{O}(\overset{\bullet}{\mathscr{R}}{}^{(0)}_{\rho})$ for any $0 < \rho < 1$, thus $\partial^{-1}\widehat{G} \in \mathcal{O}(\overset{\bullet}{\mathscr{R}}{}^{(0)}_{\rho})$ is well-defined. We set $\widehat{w}_{\boldsymbol{e}_1} = \widetilde{P}^{-1}_{\boldsymbol{e}_1}(\partial)\partial^{-1}\widehat{G} + \widehat{v}_{\boldsymbol{e}_1}$ and (6.2) becomes

$$\partial \widetilde{P}_{\boldsymbol{e}_1}(\partial)\widehat{v}_{\boldsymbol{e}_1} = \widehat{G} * \left(\widetilde{P}^{-1}_{\boldsymbol{e}_1}(\partial)\partial^{-1}\widehat{G}\right) + \widehat{G} * \widehat{v}_{\boldsymbol{e}_1}. \tag{6.4}$$

Observe that $\widehat{G} * \left(\widetilde{P}_{\boldsymbol{e}_1}^{-1}(\partial)\partial^{-1}\widehat{G}\right)$ belongs to $\mathcal{MO}(\overset{\bullet}{\mathscr{R}}_{\rho}^{(0)})$. For $R > 0$, we consider the star-shaped domain $U_R = D(0,R) \cap \overset{\bullet}{\mathscr{R}}_{\rho}^{(0)}$ and we defines $B_r = \{\widehat{v} \in \mathcal{O}(\overline{U_R}), \|\widehat{v}\|_\nu \leq r\}$, for $r > 0$ and $\nu > 0$. By proposition 3.7 and lemma 6.1, $\|\widetilde{P}_{\boldsymbol{e}_1}^{-1}(\partial)\partial^{-1}\left(\widehat{G} * \left(\widetilde{P}_{\boldsymbol{e}_1}^{-1}(\partial)\partial^{-1}\widehat{G}\right)\right)\|_\nu \to 0$ when $\nu \to \infty$.

Explicitly

$$\begin{aligned}\|\widetilde{P}_{\boldsymbol{e}_1}^{-1}(\partial)\partial^{-1}\left(\widehat{G} * \left(\widetilde{P}_{\boldsymbol{e}_1}^{-1}(\partial)\partial^{-1}\widehat{G}\right)\right)\|_\nu &\leq \frac{M_{\rho,(0)}}{R}\|\partial^{-1}\left(\widehat{G} * \left(\widetilde{P}_{\boldsymbol{e}_1}^{-1}(\partial)\partial^{-1}\widehat{G}\right)\right)\|_\nu \\ &\leq \frac{M_{\rho,(0)}}{\nu R^2}\|\partial^{-1}\widehat{G}\|_0\|\widetilde{P}_{\boldsymbol{e}_1}^{-1}(\partial)\partial^{-1}\widehat{G}\|_\nu.\end{aligned}$$

Also, $\|\widetilde{P}_{\boldsymbol{e}_1}^{-1}(\partial)\partial^{-1}\left(\widehat{G} * \widehat{v}_{\boldsymbol{e}_1}\right)\|_\nu \leq \frac{M_{\rho,(0)}}{\nu R^2}\|\partial^{-1}\widehat{G}\|_0\|\widehat{v}_{\boldsymbol{e}_1}\|_\nu$, Thus equation (6.4) translates into a fixed point problem $\widehat{v}_{\boldsymbol{e}_1} = \mathcal{N}(\widehat{v}_{\boldsymbol{e}_1})$ where $\mathcal{N} : B_r \to B_r$ is a contractive mapping for ν large enough. This ensures the existence and uniquess of $\widehat{w}_{\boldsymbol{e}_1} \in \mathcal{O}(\overset{\bullet}{\mathscr{R}}^{(0)})$. The same reasoning can be applied for showing that $\widehat{w}_{\boldsymbol{e}_1}$ can be analytically continued to $\overset{\bullet}{\mathscr{R}}^{(1)}$, in application of lemma 4.5 and theorem 4.1.

To get upper bounds, we notice by (6.3) and lemma 3.3 that for every $\zeta \in \overset{\bullet}{\mathscr{R}}_{\rho}^{(0)}$, $\left|\partial^{-1}\widehat{G}(\zeta)\right| \leq \frac{1}{4} + 1 * \widehat{\mathsf{w}}_{\mathbf{0}}(|\zeta|)$ where $\widehat{\mathsf{w}}_{\mathbf{0}}(\xi) = A\mathrm{e}^{\tau\xi}$ stands for the majorant function of $\widehat{w}_{\mathbf{0}}$ given by theorem 3.2 and corollary 3.1, thus with $A = 4.22$ and $\tau = \dfrac{4.22}{\rho}$. Viewing the Grönwall-like lemma 3.9, one sees that for every $\zeta \in \overset{\bullet}{\mathscr{R}}_{\rho}^{(0)}$, $|\widehat{w}_{\boldsymbol{e}_1}(\zeta)| \leq \widehat{\mathsf{w}}_{\boldsymbol{e}_1}(|\zeta|)$ where $\widehat{\mathsf{w}}_{\boldsymbol{e}_1}$ solves the convolution equation:

$$\frac{1}{M_{\rho,(0)}}\widehat{\mathsf{w}}_{\boldsymbol{e}_1} = \left(\frac{1}{4} + 1 * \widehat{\mathsf{w}}_{\mathbf{0}}\right) * (\delta + \widehat{\mathsf{w}}_{\boldsymbol{e}_1}). \tag{6.5}$$

This means that $\widehat{\mathsf{w}}_{\boldsymbol{e}_1}$ has an analytic Laplace transform under the form[1]:

$$\widetilde{\mathsf{w}}_{\boldsymbol{e}_1}(z) = \sum_{n\geq 1}\frac{1}{\rho^n}\left(\frac{1}{4z} + \frac{1}{z}\frac{A}{z-\tau}\right)^n, \quad A = 4.22,\ \tau = \frac{4.22}{\rho}.$$

When assuming $|z| \geq \dfrac{5.81}{\rho}$, for instance, one gets $\left|\dfrac{1}{\rho}\left(\dfrac{1}{4z} + \dfrac{1}{z}\dfrac{A}{z-\tau}\right)\right| \leq 0.5$ (since $\rho < 1$), thus $|\widetilde{\mathsf{w}}_{\boldsymbol{e}_1}(z)| \leq 1$. Therefore by lemma 3.5, for any $0 < \rho < 1$, for every $\zeta \in \overset{\bullet}{\mathscr{R}}_{\rho}^{(0)}$, $|\widehat{w}_{\boldsymbol{e}_1}(\zeta)| \leq \frac{5.81}{\rho}\mathrm{e}^{\frac{5.81}{\rho}|\xi|}$. One shows in the same way that $\widehat{w}_{\boldsymbol{e}_1}$ has at most exponential growth of order 1 at infinity on $\overset{\bullet}{\mathscr{R}}^{(1)}$. $\square$

6.1.3 The k-th series

We now turn to the k-th series, that is the terms $\widetilde{w}_{k\boldsymbol{e}_i} = \widetilde{w}_{k\boldsymbol{e}_i}^{[0]}$ of the transseries, for $k \geq 2$.

Lemma 6.4. *For every integer $k \geq 2$, the k-th series $\widetilde{w}_{k\boldsymbol{e}_i} \in z^{-2(k-1)}\mathbb{C}[[z^{-1}]]$ is 1-Gevrey, its minor $\widehat{w}_{k\boldsymbol{e}_i}$ defines a holomorphic function on $\overset{\bullet}{\mathscr{R}}^{(0)}$ with*

[1] We recall that $\tilde{\mathcal{B}}\left(\dfrac{A}{z-\tau}\right) = A\mathrm{e}^{\tau\xi}$.

at most exponential growth of order 1 at infinity. Moreover, $\widehat{w}_{\boldsymbol{k}\boldsymbol{e}_i}$ *can be analytically continued to the Riemann surface* $\mathscr{R}^{(1)}$*, with at most exponential growth of order 1 at infinity on* $\mathscr{R}^{(1)}$*.*

Proof. Once again from the invariance of the equation (3.6) under the symmetry $z \mapsto -z$, there is no loss of generality in studying only the k-th series $\widehat{w}_{\boldsymbol{k}\boldsymbol{e}_1}$.
We know that $\widehat{w}_{\mathbf{0}}, \widehat{w}_{\boldsymbol{e}_1}$ are holomorphic on $\overset{\bullet}{\mathscr{R}}{}^{(0)}$ and can be analytically continued to $\mathscr{R}^{(1)}$. Moreover, for every $0 < \rho < 1$,

$$\text{for every } \zeta \in \overset{\bullet}{\mathscr{R}}{}^{(0)}_{\rho},\ |\widehat{w}_{\mathbf{0}}(\zeta)| \leq \widehat{\mathsf{w}}_{\mathbf{0}}(\xi), \qquad |\widehat{w}_{\boldsymbol{e}_1}(\zeta)| \leq \widehat{\mathsf{w}}_{\boldsymbol{e}_1}(\xi), \qquad \xi = |\zeta|$$

and, for every $0 < \rho \leq 1/5$,

$$\text{for every } \zeta \in \overline{\mathscr{R}}{}^{(1)}_{\rho},\ |\widehat{w}_{\mathbf{0}}(\zeta)| \leq \widehat{\mathsf{w}}_{\mathbf{0}}(\xi), \qquad |\widehat{w}_{\boldsymbol{e}_1}(\zeta)| \leq \widehat{\mathsf{w}}_{\boldsymbol{e}_1}(\xi), \qquad \xi = \operatorname{leng}(\zeta),$$

where $\widehat{\mathsf{w}}_{\mathbf{0}}$ and $\widehat{\mathsf{w}}_{\boldsymbol{e}_1}$ are entire functions, real positive and non-decreasing on $\mathbb{R}^+$, with at most exponential growth of order 1 at infinity.
We know from lemma 5.8 and (5.48) that, for every $k \geq 2$,

$$\widetilde{w}_{\boldsymbol{k}\boldsymbol{e}_1}(z) = \sum_{l \geq 0} a_{\boldsymbol{k}\boldsymbol{e}_1,l} z^{-l} \in z^{-2(k-1)}\mathbb{C}[[z^{-1}]]$$

solves the differential equation

$$\mathfrak{D}_{\boldsymbol{k}\boldsymbol{e}_1}\widetilde{w}_{\boldsymbol{k}\boldsymbol{e}_1} = \sum_{\substack{k_1+k_2=k \\ k_1 \geq 1,\, k_2 \geq 1}} \frac{\widetilde{w}_{k_1\boldsymbol{e}_1}\widetilde{w}_{k_2\boldsymbol{e}_1}}{2!} \frac{\partial^2 F(z,\widetilde{w}_{\mathbf{0}})}{\partial w^2}. \tag{6.6}$$

We deduce that the formal Borel transform $\tilde{\mathcal{B}}(\widetilde{w}_{\boldsymbol{k}\boldsymbol{e}_1}) = a_{\boldsymbol{k}\boldsymbol{e}_1,0}\delta + \widehat{w}_{\boldsymbol{k}\boldsymbol{e}_1}$ satisfies the identity[2]:

$$\mathfrak{D}_{\boldsymbol{k}\boldsymbol{e}_1}\widehat{w}_{\boldsymbol{k}\boldsymbol{e}_1} = \sum_{\substack{k_1+k_2=k \\ k_1 \geq 1,\, k_2 \geq 1}} \frac{(a_{k_1\boldsymbol{e}_1,0}\delta + \widehat{w}_{k_1\boldsymbol{e}_1}) * (a_{k_2\boldsymbol{e}_1,0}\delta + \widehat{w}_{k_2\boldsymbol{e}_1})}{2!} * \frac{\partial^2 \widehat{F}(\zeta,\widehat{w}_{\mathbf{0}})}{\partial w^2} \tag{6.7}$$

where $\dfrac{\partial^2 \widehat{F}(\zeta,\widehat{w}_{\mathbf{0}})}{\partial w^2} = 2\widehat{f}_2(\zeta) = \zeta$, whereas

$$\mathfrak{D}_{\boldsymbol{k}\boldsymbol{e}_1}\widehat{w}_{\boldsymbol{k}\boldsymbol{e}_1} = P_{\boldsymbol{k}\boldsymbol{e}_1}(\partial)\widehat{w}_{\boldsymbol{k}\boldsymbol{e}_1} + 1 * Q_{\boldsymbol{k}\boldsymbol{e}_1}(\partial)\widehat{w}_{\boldsymbol{k}\boldsymbol{e}_1} + \left(\zeta R_{\boldsymbol{k}\boldsymbol{e}_1} - \frac{\partial \widehat{F}(\zeta,\widehat{w}_{\mathbf{0}})}{\partial w}\right) * \widehat{w}_{\boldsymbol{k}\boldsymbol{e}_1} \tag{6.8}$$

with $\dfrac{\partial \widehat{F}(\zeta,\widehat{w}_{\mathbf{0}})}{\partial w}$ given by (6.3).
These equations (6.7) can be seen as linear differential equations with a *regular* point at 0. They are all of the type

$$p(\zeta)\widehat{w}(\zeta) + 1 * [q(\zeta)\widehat{w}](\zeta) = \zeta * [r(\zeta)\widehat{w}](\zeta) + \sum_{n=0}^{N} \widehat{f}_n * \widehat{w}^{*n}(\zeta) \tag{6.9}$$

[2] Remember that $a_{\boldsymbol{k}\boldsymbol{e}_1,0} = 0$ as a rule, apart from the case $k = 1$ where $a_{\boldsymbol{e}_1,0} = 1$.

that has been investigated in Sect. 3.4 and Sect. 4.4. We will use the methods introduced there and make a proof by induction on k, considering the operators $\mathcal{N}_k$ defined as follows:

$$\mathcal{N}_k \widehat{v} = \frac{1}{P_{(k,0)}(-\zeta)} \left[-1 * \left[Q_{(k,0)}(-\zeta)\widehat{v}\right] + \left(-\zeta R_{(k,0)} + \frac{\partial \widehat{F}(\zeta, \widehat{w}_{\mathbf{0}})}{\partial w} \right) * \widehat{v} \right.$$
$$\left. + \sum_{\substack{k_1+k_2=K \\ k_1 \geq 1,\, k_2 \geq 1}} \frac{(a_{k_1 \boldsymbol{e}_1,0}\delta + \widehat{w}_{k_1\boldsymbol{e}_1}) * (a_{k_2 \boldsymbol{e}_1,0}\delta + \widehat{w}_{k_2\boldsymbol{e}_1})}{2!} * \frac{\partial^2 \widehat{F}(\zeta, \widehat{w}_{\mathbf{0}})}{\partial w^2} \right].$$

Case 6.1. Case $k = 2$

- For $R > 0$ and $0 < \rho < 1$, we consider the star-shaped domain $U_R = D(0,R) \cap \overset{\bullet}{\mathscr{R}}{}_\rho^{(0)}$ and we defines $B_r = \{\widehat{v} \in \mathcal{O}(\overline{U_R}), \|\widehat{v}\|_\nu \leq r\}$ for $r > 0$ and $\nu > 0$. We look at the mapping $\mathcal{N}_2 : \widehat{v} \in B_r \mapsto \mathcal{N}_2\widehat{v}$. We know that $\widehat{w}_{(1,0)} \in \mathcal{O}(\overset{\bullet}{\mathscr{R}}{}^{(0)})$ while $\dfrac{\partial \widehat{F}(\zeta, \widehat{w}_{\mathbf{0}})}{\partial w}$ and $\dfrac{\partial^2 \widehat{F}(\zeta, \widehat{w}_{\mathbf{0}})}{\partial w^2}$ belong to $\mathcal{M}\mathcal{O}(\overset{\bullet}{\mathscr{R}}{}_\rho^{(0)})$. Using lemma 6.1 and arguments already used in Sect. 3.4.3, one easily shows that $\mathcal{N}_2$ is a contractive map. Thus the equation (6.7) for $k = 1$ has a unique solution in B_r. This shows, by unicity, that $\widehat{w}_{2\boldsymbol{e}_1}$ defines a holomorphic function on $\overset{\bullet}{\mathscr{R}}{}^{(0)}$.
  When replacing U_R by the open set of L-points $\mathscr{U} = \mathscr{U}_{\rho,L} \subset \mathscr{R}^{(1)}$ and arguing like in Sect. 4.4.2, one shows that $\widehat{w}_{2\boldsymbol{e}_1}$ can be analytically continued to the Riemann surface $\mathscr{R}^{(1)}$.
- To get upper bounds, we notice that, for every $\zeta \in \overset{\bullet}{\mathscr{R}}{}_\rho^{(0)}$,

$$\left| \frac{\partial \widehat{F}(\zeta, \widehat{w}_{\mathbf{0}})}{\partial w} \right| \leq \left| \frac{\partial \widehat{F}}{\partial w} \right| (\xi, \widehat{\mathsf{w}}_{\mathbf{0}}) \quad \text{and} \quad \left| \frac{\partial^2 \widehat{F}(\zeta, \widehat{w}_{\mathbf{0}})}{\partial w^2} \right| \leq \left| \frac{\partial^2 \widehat{F}}{\partial w^2} \right| (\xi, \widehat{\mathsf{w}}_{\mathbf{0}})$$

  with $\xi = |\zeta|$, $\left| \dfrac{\partial \widehat{F}}{\partial w} \right| (\xi, \widehat{\mathsf{w}}_{\mathbf{0}}) = |\widehat{f}_1|(\xi) + 2|\widehat{f}_2| * \widehat{\mathsf{w}}_{\mathbf{0}}(\xi) = 4\xi + \xi * \widehat{\mathsf{w}}_{\mathbf{0}}(\xi)$ and $\left| \dfrac{\partial^2 \widehat{F}}{\partial w^2} \right| (\xi, \widehat{\mathsf{w}}_{\mathbf{0}}) = 2|f_2|(\xi) = \xi$. Using lemma 6.2 and the Grönwall lemma 3.9, we sees that for every $\zeta \in \overset{\bullet}{\mathscr{R}}{}_\rho^{(0)}$, $|\widehat{w}_{2\boldsymbol{e}_1}(\zeta)| \leq \widehat{\mathsf{w}}_{2\boldsymbol{e}_1}(\xi)$ with $\xi = |\zeta|$, where $\widehat{\mathsf{w}}_{2\boldsymbol{e}_1}$ is the entire function, real positive on $\mathbb{R}^+$, with at most exponential growth of order 1 at infinity, satisfying the linear equation:

$$\frac{1}{M_{\rho,(0)}} \widehat{\mathsf{w}}_{2\boldsymbol{e}_1} = \left(3 + \frac{9}{4} M_{\rho,(0)} \xi + \left| \frac{\partial \widehat{F}}{\partial w} \right| (\xi, \widehat{\mathsf{w}}_{\mathbf{0}}) \right) * \widehat{\mathsf{w}}_{2\boldsymbol{e}_1} + \frac{(\delta + \widehat{\mathsf{w}}_{\boldsymbol{e}_1})^{*2}}{2!} * \left| \frac{\partial^2 \widehat{F}}{\partial w^2} \right| (\xi, \widehat{\mathsf{w}}_{\mathbf{0}}) \tag{6.10}$$

  When working on $\mathscr{R}^{(1)}$, one rather argues with the Grönwall lemma 4.8 and one obtains that for every $\zeta \in \overline{\mathscr{R}}{}_\rho^{(1)}$ $|\widehat{w}_{2\boldsymbol{e}_1}(\zeta)| \leq \widehat{\mathsf{w}}_{2\boldsymbol{e}_1}(\xi)$ now with $\xi = \operatorname{leng}(\zeta)$, where $\widehat{\mathsf{w}}_{2\boldsymbol{e}_1}$ is the entire function, real positive and non-decreasing on $\mathbb{R}^+$, with at most exponential growth of order 1 at infinity, satisfying the linear equation:

$$\frac{1}{M_{\rho,(1)}}\widehat{\mathsf{w}}_{2\boldsymbol{e}_1} = \left(3+\frac{9}{4}M_{\rho,(1)}\xi + \left|\frac{\partial \widehat{F}}{\partial w}\right|(\xi,\widehat{\mathsf{w}}_{\mathbf{0}})\right)*\widehat{\mathsf{w}}_{2\boldsymbol{e}_1} + \frac{(\delta+\widehat{\mathsf{w}}_{\boldsymbol{e}_1})^{*2}}{2!}*\left|\frac{\partial^2 \widehat{F}}{\partial w^2}\right|(\xi,\widehat{\mathsf{w}}_{\mathbf{0}}). \tag{6.11}$$

Case 6.2. Induction We assume that for every integer k such that $0 \leq k < K$ with $K \geq 3$, $\widehat{w}_{k\boldsymbol{e}_1}$ is holomorphic on $\overset{\bullet}{\mathscr{R}}{}^{(0)}_{\rho}$, can be analytically continued to $\mathscr{R}^{(1)}$ and

$$\text{for every } \zeta \in \overset{\bullet}{\mathscr{R}}{}^{(0)}_{\rho},\ |\widehat{w}_{k\boldsymbol{e}_1}(\zeta)| \leq \widehat{\mathsf{w}}_{k\boldsymbol{e}_1}(\xi), \qquad \xi = |\zeta|,$$

$$\text{for every } \zeta \in \overline{\mathscr{R}}{}^{(1)}_{\rho},\ |\widehat{w}_{k\boldsymbol{e}_1}(\zeta)| \leq \widehat{\mathsf{w}}_{k\boldsymbol{e}_1}(\xi), \qquad \xi = \mathrm{leng}(\zeta),$$

where, in each case, $\widehat{\mathsf{w}}_{k\boldsymbol{e}_1}$ is an entire function, real positive and nondecreasing on $\mathbb{R}^+$, with at most exponential growth of order 1 at infinity.

- One easily shows that the mapping $\mathcal{N}_K : \widehat{v} \in B_r \mapsto \mathcal{N}_K\widehat{v}$ is a contractive, either working in $\mathcal{O}(\overline{U_R}), \|\widehat{v}\|_\nu$ or in $\mathcal{O}(\overline{\mathscr{U}_{\rho,L}}), \|\widehat{v}\|_\nu$. Thus, by unicity, $\widehat{w}_{K\boldsymbol{e}_1}$ is holomorphic on $\overset{\bullet}{\mathscr{R}}{}^{(0)}$ and can by analytically continued to $\mathscr{R}^{(1)}$.
- We get upper bounds, either in $\overset{\bullet}{\mathscr{R}}{}^{(0)}_{\rho}$ with the Grönwall lemma 3.9, or in $\overline{\mathscr{R}}{}^{(1)}_{\rho}$ with the Grönwall lemma 4.8. We get that for every $\zeta \in \overset{\bullet}{\mathscr{R}}{}^{(0)}_{\rho}$ $|\widehat{w}_{K\boldsymbol{e}_1}(\zeta)| \leq \widehat{\mathsf{w}}_{K\boldsymbol{e}_1}(\xi)$ with $\xi = |\zeta|$, where $\widehat{\mathsf{w}}_{K\boldsymbol{e}_1}$ is the entire function, real positive on $\mathbb{R}^+$, with at most exponential growth of order 1 at infinity, satisfying the linear equation:

$$\begin{aligned}\frac{1}{M_{\rho,(0)}}\widehat{\mathsf{w}}_{K\boldsymbol{e}_1} &= \left(3+\frac{9}{4}M_{\rho,(0)}\xi + \left|\frac{\partial \widehat{F}}{\partial w}\right|(\xi,\widehat{\mathsf{w}}_{\mathbf{0}})\right)*\widehat{\mathsf{w}}_{K\boldsymbol{e}_1} \\ &\quad + \sum_{\substack{k_1+k_2=K \\ k_1\geq 1,\, k_2\geq 1}} \frac{(a_{k_1\boldsymbol{e}_1,0}\delta+\widehat{\mathsf{w}}_{k_1\boldsymbol{e}_1})*(a_{k_2\boldsymbol{e}_1,0}\delta+\widehat{\mathsf{w}}_{k_2\boldsymbol{e}_1})}{2!}*\left|\frac{\partial^2 \widehat{F}}{\partial w^2}\right|(\xi,\widehat{\mathsf{w}}_{\mathbf{0}}).\end{aligned} \tag{6.12}$$

  Also, for every $\zeta \in \overline{\mathscr{R}}{}^{(1)}_{\rho}$, $|\widehat{w}_{K\boldsymbol{e}_1}(\zeta)| \leq \widehat{\mathsf{w}}_{K\boldsymbol{e}_1}(\xi)$ where $\xi = \mathrm{leng}(\zeta)$, with $\widehat{\mathsf{w}}_{K\boldsymbol{e}_1}$ an entire function, real positive and nondecreasing on $\mathbb{R}^+$, with at most exponential growth of order 1 at infinity, satisfying the linear equation:

$$\begin{aligned}\frac{1}{M_{\rho,(1)}}\widehat{\mathsf{w}}_{K\boldsymbol{e}_1} &= \left(3+\frac{9}{4}M_{\rho,(1)}\xi + \left|\frac{\partial \widehat{F}}{\partial w}\right|(\xi,\widehat{\mathsf{w}}_{\mathbf{0}})\right)*\widehat{\mathsf{w}}_{K\boldsymbol{e}_1} \\ &\quad + \sum_{\substack{k_1+k_2=K \\ k_1\geq 1,\, k_2\geq 1}} \frac{(a_{k_1\boldsymbol{e}_1,0}\delta+\widehat{\mathsf{w}}_{k_1\boldsymbol{e}_1})*(a_{k_2\boldsymbol{e}_1,0}\delta+\widehat{\mathsf{w}}_{k_2\boldsymbol{e}_1})}{2!}*\left|\frac{\partial^2 \widehat{F}}{\partial w^2}\right|(\xi,\widehat{\mathsf{w}}_{\mathbf{0}}).\end{aligned}$$

This ends the proof of lemma 6.4. □

6.1.4 The other k-th series

Looking at (5.53), one easily see that the above methods can be applied to study the minor $\widehat{w}_{\mathbf{1}} = \widehat{w}^{[0]}_{\mathbf{1}}$ of the $(1,1)$-series $\widetilde{w}_{\mathbf{1}}$. Thus, theorem 6.1 is shown for $\boldsymbol{k} = \mathbf{0}$ any $\boldsymbol{k} \in \Xi_{n+1,0} \setminus \Xi_{n,0}$ and with $n = 1$. The rest of the proof is made by induction on n, using the hierarchy of equations given in corollary 5.1 and the reasoning made above. This part holds no surprise and is left to the reader. This ends the proof of theorem 6.1.

6.2 Summability of the transseries for Painlevé I

We now restrict ourself to the transseries solution of Painlevé I, having in view of analysing their 1-summability. From the invariance of the equation (3.6) under the symmetry $z \mapsto -z$, it is enough to focus only on the transseries (5.41) associated with the multiplier $\lambda_1 = 1$, namely:

$$\widetilde{w}(z, U\boldsymbol{e}_1) = \sum_{k=0}^{\infty} V^k \widetilde{w}_{k\boldsymbol{e}_1}(z), \qquad V^k = U^k \mathrm{e}^{-\lambda_1 k z} z^{-\tau_1 k}. \tag{6.13}$$

6.2.1 A useful complement

We can complete lemma 6.4 with the following result.

Lemma 6.5. *In lemma 6.4, for every* $0 < \rho < 1$, *there exist* $A = A(\rho) > 0$ *and* $\tau = \tau(\rho) > 0$ *such that the following properties are satisfied for every integer* $k \geq 2$:

- *for every* $\zeta \in \overset{\bullet}{\mathscr{R}}{}^{(0)}_{\rho}$, $|\widehat{w}_{k\boldsymbol{e}_1}(\zeta)| \leq \widehat{\mathsf{w}}_{k\boldsymbol{e}_1}(\xi)$, $\xi = |\zeta|$, *where* $\widehat{\mathsf{w}}_{k\boldsymbol{e}_1}$ *is an entire function, real positive on* $\mathbb{R}^+$, *and* $\widehat{\mathsf{w}}_{k\boldsymbol{e}_1}(\xi) = O(\xi^{2k-3})$;
- *for every* $\xi \in \mathbb{C}$, *for every* $\xi \in \mathbb{C}$, $|\widehat{\mathsf{w}}_{k\boldsymbol{e}_1}(\xi)| \leq \left(\dfrac{3\sqrt{\rho}}{2}\right)^k A\mathrm{e}^{\tau|\xi|}$, *and for every positive integer* $1 \leq m \leq 2k-3$,

$$|\widehat{\mathsf{w}}_{k\boldsymbol{e}_1}(\xi)| \leq \left(\frac{3\sqrt{\rho}}{2}\right)^k A^{m+1} \left(\frac{\zeta^{m-1}}{(m-1)!} * \mathrm{e}^{\tau\zeta}\right)(|\xi|).$$

Moreover one can take $A = \tau = \dfrac{27}{4\rho}$ *in the above upper bounds.*

Proof. We know by theorem 3.2, lemma 6.3 and lemma 6.4 that, for every integer $k \in \mathbb{N}$, $\widehat{w}_{k\boldsymbol{e}_1}$ is holomorphic on $\overset{\bullet}{\mathscr{R}}{}^{(0)}$. Also, for every $0 < \rho < 1$,

$$\text{for every } \zeta \in \overset{\bullet}{\mathscr{R}}{}^{(0)}_{\rho},\ |\widehat{w}_{k\boldsymbol{e}_1}(\zeta)| \leq \widehat{\mathsf{w}}_{k\boldsymbol{e}_1}(\xi), \qquad \xi = |\zeta|$$

where $\widehat{\mathsf{w}}_{\boldsymbol{0}}(\xi) = A_{\boldsymbol{0}}\mathrm{e}^{\tau_{\boldsymbol{0}}\xi}$ and $\widehat{\mathsf{w}}_{\boldsymbol{e}_1}(\xi) = A_{\boldsymbol{e}_1}\mathrm{e}^{\tau_{\boldsymbol{e}_1}\xi}$ are convenient majorant functions while, for any integer $k \geq 2$, $\widehat{\mathsf{w}}_{k\boldsymbol{e}_1}$ solves the convolution equation (6.12). One first shows that for any integer $k \geq 2$, $\widehat{\mathsf{w}}_{k\boldsymbol{e}_1}(\xi) = O(\xi^{2k-3})$ and we reason by induction. Indeed, for $k = 2$ and using the fact that $\left|\frac{\partial^2 \widehat{F}}{\partial w^2}\right|(\xi, \widehat{\mathsf{w}}_{\boldsymbol{0}}) = O(\xi)$, one sees that $(\delta + \widehat{\mathsf{w}}_{\boldsymbol{e}_1})^{*2} * \left|\dfrac{\partial^2 \widehat{F}}{\partial w^2}\right|(\xi, \widehat{\mathsf{w}}_{\boldsymbol{0}}) = O(\xi)$, thus $\widehat{\mathsf{w}}_{2\boldsymbol{e}_1}(\zeta) = O(\zeta)$. Now by an induction hypothesis, for any $k_1, k_2 \geq 1$ and $k_1 + k_2 = k \geq 3$, $(a_{k_1\boldsymbol{e}_1,0}\delta + \widehat{\mathsf{w}}_{k_1\boldsymbol{e}_1}) * (a_{k_2\boldsymbol{e}_1,0}\delta + \widehat{\mathsf{w}}_{k_2\boldsymbol{e}_1}) = O(\xi^{2k-5})$ (we recall that $a_{k\boldsymbol{e}_1,0} = 0$ apart from $a_{\boldsymbol{e}_1,0} = 1$), thus $\widehat{\mathsf{w}}_{k\boldsymbol{e}_1}(\zeta) = O(\xi^{2k-3})$ by (6.12).

We now introduce the generating function $\widehat{\mathsf{w}}(\xi,V)=\sum_{k=2}^{\infty}V^k\widehat{\mathsf{w}}_{k\boldsymbol{e}_1}(\xi)$. One deduces from (6.12) that $\widehat{\mathsf{w}}$ satisfies the condition:

$$\frac{1}{M_{\rho,(0)}}\widehat{\mathsf{w}}=\left(3+\frac{9}{4}M_{\rho,(0)}\xi+\left|\frac{\partial\widehat{F}}{\partial w}\right|(\xi,\widehat{\mathsf{w}}_{\mathbf{0}})\right)*\widehat{\mathsf{w}}$$
$$+\sum_{k=2}^{\infty}V^k\sum_{\substack{k_1+k_2=k\\ k_1\geq 1,\,k_2\geq 1}}\frac{(a_{k_1\boldsymbol{e}_1,0}\delta+\widehat{\mathsf{w}}_{k_1\boldsymbol{e}_1})*(a_{k_2\boldsymbol{e}_1,0}\delta+\widehat{\mathsf{w}}_{k_2\boldsymbol{e}_1})}{2!}*\left|\frac{\partial^2\widehat{F}}{\partial w^2}\right|(\xi,\widehat{\mathsf{w}}_{\mathbf{0}}).$$

This can be written also as follows (recall: $a_{k\boldsymbol{e}_1,0}=0$ apart from $a_{\boldsymbol{e}_1,0}=1$):

$$\frac{1}{M_{\rho,(0)}}\widehat{\mathsf{w}}=$$
$$\left(3+\frac{9}{4}M_{\rho,(0)}\xi+\left|\frac{\partial\widehat{F}}{\partial w}\right|(\xi,\widehat{\mathsf{w}}_{\mathbf{0}})\right)*\widehat{\mathsf{w}}+\frac{\big(V\big(\delta+\widehat{\mathsf{w}}_{\boldsymbol{e}_1}\big)+\widehat{\mathsf{w}}\big)^{*2}}{2!}*\left|\frac{\partial^2\widehat{F}}{\partial w^2}\right|(\xi,\widehat{\mathsf{w}}_{\mathbf{0}}).$$

Explicitly, one can take $M_{\rho,(0)}=\dfrac{1}{\rho}$ (by lemma 6.1), $\widehat{\mathsf{w}}_{\mathbf{0}}(\xi)=4.22\mathrm{e}^{\frac{4.22}{\rho}\xi}$ (by theorem 3.2), $\widehat{\mathsf{w}}_{\boldsymbol{e}_1}(\xi)=\dfrac{5.81}{\rho}\mathrm{e}^{\frac{5.81}{\rho}\xi}$ (by lemma 6.3), and we recall that $\left|\dfrac{\partial\widehat{F}}{\partial w}\right|(\xi,\widehat{\mathsf{w}}_{\mathbf{0}})=4\xi+\xi*\widehat{\mathsf{w}}_{\mathbf{0}}(\xi)$ and $\left|\dfrac{\partial^2\widehat{F}}{\partial w^2}\right|(\xi,\widehat{\mathsf{w}}_{\mathbf{0}})=\xi$. Therefore, $\widehat{\mathsf{w}}$ solves the convolution equation:

$$\rho\widehat{\mathsf{w}}=\left(3+\left(4+\frac{9}{4\rho}\right)\xi+4.22\xi*\mathrm{e}^{\frac{4.22}{\rho}\xi}\right)*\widehat{\mathsf{w}}+\frac{\xi}{2!}*\left(V\big(\delta+\frac{4.63}{\rho}\mathrm{e}^{\frac{4.63}{\rho}\xi}\big)+\widehat{\mathsf{w}}\right)^{*2}.$$

The generating function $\widehat{\mathsf{w}}(\xi,V)$ is the Borel transform of $\widetilde{\mathsf{w}}(\zeta,V)$, solution of the algebraic equation

$$\rho\widetilde{\mathsf{w}}=\left(\frac{3}{z}+\left(4+\frac{9}{4\rho}\right)\frac{1}{z^2}+\frac{4.22}{z^2}\frac{1}{z-\frac{4.22}{\rho}}\right)\widetilde{\mathsf{w}}$$
$$+\frac{1}{2z^2}\left[V\left(1+\frac{5.81}{\rho}\frac{1}{z-\frac{5.81}{\rho}}\right)+\widetilde{\mathsf{w}}\right]^2 \tag{6.14}$$

with

$$\widetilde{\mathsf{w}}(z,V)\simeq\frac{1}{2\rho}\left[\frac{V}{z}\left(1+\frac{5.81}{\rho}\frac{1}{z-\frac{5.81}{\rho}}\right)\right]^2\ \text{when } V\to 0 \text{ with } |z| \text{ large enough.}$$

We view (6.14) as a fixed point problem, $\mathsf{w}=\mathcal{N}(\mathsf{w})$. We set $U=D(\infty,\frac{4\rho}{27})\times D(0,\frac{2}{3\sqrt{\rho}})$. We equip the space $O(\overline{U})$ with the maximum norm and we consider the closed ball $B_1=\{\mathsf{w}\in O(\overline{U}),\ \|\mathsf{w}\|\leq 1\}$ of the Banach algebra $\big(O(\overline{U}),\|\,\|\big)$. One easily shows that $\mathcal{N}:B_1\to B_1$ is a contractive map (remember that $\rho<1$), hence the fixed-point problem $\mathsf{w}=\mathcal{N}(\mathsf{w})$ has a unique solution $\widetilde{\mathsf{w}}=\widetilde{\mathsf{w}}(z,V)$ in B_1. Its Taylor expansion with respect to V at 0 reads $\widetilde{\mathsf{w}}(z,V)=\sum_{k=2}^{\infty}V^k\widetilde{\mathsf{w}}_{k\boldsymbol{e}_1}(z)$, where $(\widetilde{\mathsf{w}}_{k\boldsymbol{e}_1})_{k\geq 2}$ is a sequence of holomorphic functions on the disc $D(\infty, \frac{4\rho}{27})$ and, by the Cauchy inequalities, for every integer $k \geq 2$, $\sup_{|z|>\frac{27}{4\rho}} |\widetilde{\mathsf{w}}_{k\boldsymbol{e}_1}(z)| \leq \left(\frac{3\sqrt{\rho}}{2}\right)^k$. Moreover, since $\widehat{\mathsf{w}}_{k\boldsymbol{e}_1}(\xi) = O(\xi^{2k-3})$, $\widetilde{\mathsf{w}}_{k\boldsymbol{e}_1}(z) = O(z^{-2(k-1)})$. We end the proof with lemma 3.5: $\widetilde{\mathsf{w}}_{k\boldsymbol{e}_1}$ is an entire function, for every $\xi \in \mathbb{C}$, $|\widetilde{\mathsf{w}}_{k\boldsymbol{e}_1}(\xi)| \leq \left(\frac{3\sqrt{\rho}}{2}\right)^k \frac{27}{4\rho} \mathrm{e}^{\frac{27}{4\rho}|\xi|}$ and for every positive integer $1 \leq m \leq 2k-3$,

$$|\widetilde{\mathsf{w}}_{k\boldsymbol{e}_1}(\xi)| \leq \left(\frac{3\sqrt{\rho}}{2}\right)^k \left(\frac{27}{4\rho}\right)^{m+1} \left(\frac{\zeta^{m-1}}{(m-1)!} * \mathrm{e}^{\frac{27}{4\rho}\zeta}\right)(|\xi|).$$

This ends the proof. $\square$

6.2.2 Summability of the transseries

We start with a definition.

Definition 6.2. One says that the transseries $\widetilde{w}(z,V) = \sum_{k=0}^{\infty} V^k \widetilde{w}_k(z)$ is Borel-Laplace summable in a direction $\theta \in \underset{\bullet}{\mathbb{S}}^1$ if each $\widetilde{w}_k$ is Borel-Laplace summable in that direction and if the series of function $\sum_{k=0}^{\infty} V^k \mathscr{S}^\theta \widetilde{w}_k(z)$ converges on a domain in the usual sense (uniform convergence on every compact subset of that domain). In that case, one denotes by $\mathscr{S}^\theta \widetilde{w}(z,V)$ the Borel-Laplace sum of the transseries.

We have of course in mind to consider the Borel-Laplace sums of the transseries
$\widetilde{w}(z, U\boldsymbol{e}_1) = \sum_{k=0}^{\infty} (U\mathrm{e}^{-z} z^{3/2})^k \widetilde{w}_{k\boldsymbol{e}_1}(z)$ and $\widetilde{w}(z, U\boldsymbol{e}_2) = \sum_{k=0}^{\infty} (U\mathrm{e}^{z} z^{3/2})^k \widetilde{w}_{k\boldsymbol{e}_2}(z)$ given by proposition 5.2. Notice that the mapping $z \mapsto \mathrm{e}^{\pm z} z^{3/2}$ is well-defined on $\underset{\bullet}{\mathbb{C}}$ and we remind the reader that the domain Π_τ^θ of $\underset{\bullet}{\mathbb{C}}$ has been defined in definition 3.11.

Definition 6.3. Let $g : \underset{\bullet}{\mathbb{C}} \to \mathbb{C}$ and $\kappa : \mathbb{R} \to \mathbb{R}^{+\star}$ be two continuous functions, and $\theta \in \underset{\bullet}{\mathbb{S}}^1$, $\tau \in \mathbb{R}$. We define

$$\mho^\theta(g, \tau, \kappa) = \bigcup_{c>\tau} \{z \in \Pi_c^\theta \, , \, |g(z)| < \kappa(c)\}.$$

Let $I \subset \underset{\bullet}{\mathbb{S}}^1$ be an arc, $\gamma : I \to \mathbb{R}$ locally bounded and $\mathcal{K} : I \to \mathcal{C}^0(\mathbb{R}, \mathbb{R}^{+\star})$ a continuous function. We note

$$\mathscr{V}(I, g, \gamma, \mathcal{K}) = \bigcup_{\theta \in I} \mho^\theta\big(g, \gamma(\theta), \mathcal{K}(\theta)\big) \subset \underset{\bullet}{\mathbb{C}}.$$

Theorem 6.2. *The transseries*

$$\widetilde{w}(z,U\boldsymbol{e}_i)=\sum_{k=0}^{\infty}\big(V_i(U)\big)^k(z)\widetilde{w}_{k\boldsymbol{e}_i}(z),\qquad \big(V_i(U)\big)(z)=U\mathrm{e}^{-\lambda_i z}z^{-\tau_i},\qquad i=1,2,\tag{6.15}$$

of the prepared equation (3.6) associated with the first Painlevé equation, are Borel-Laplace summable in any direction $\theta\in\mathbb{R}\setminus\pi\mathbb{Z}$ *and any* $U\in\mathbb{C}$*, and their Borel-Laplace sum are holomorphic solutions of (3.6). More precisely, for any* $R>0$*, for any open arc* $I_j=]j\pi,(j+1)\pi[$, $j\in\mathbb{Z}$, $w_{tru,j,i}(z,U)=\mathscr{S}^{I_j}\widetilde{w}(z,U\boldsymbol{e}_i)$ *defines a holomorphic function with respect to* (z,U)*, on a domain of the form* $\mathscr{V}(I_j,V_i(R),\tau,\mathcal{K})\times D(0,R)$*. Moreover one can choose* $\tau(\theta)=\dfrac{27}{4|\sin(\theta)|}$ *and* $\mathcal{K}(\theta):c\in\mathbb{R}\mapsto\dfrac{2c^2}{3\tau(\theta)^2\sqrt{\sin(\theta)}}$.

Proof. The theorem is a consequence of theorem 3.2, lemma 6.3, lemma 6.4 and lemma 6.5. Let us precise the reasoning for $i=1$ and the arc $I_0=]0,\pi[$.

We know from lemmas 6.4 and 6.5 (applied with $m=2k-3$) that for any $\delta\in]0,\dfrac{\pi}{2}[$ and any integer $k\geq 2$, for every $\zeta\in\mathfrak{s}_0^{\infty}(]\delta,\pi-\delta[)$ (cf. definition 7.1),

$$|\widehat{w}_{k\boldsymbol{e}_1}(\zeta)|\leq\left(\frac{3\sqrt{\sin(\delta)}}{2}\right)^k A_\delta^{2k-2}\left(\frac{\xi^{2k-4}}{(2k-4)!}*\mathrm{e}^{\tau_\delta\xi}\right)(\xi),\quad \xi=|\zeta|,\tag{6.16}$$

with $A_\delta=\tau_\delta=\dfrac{27}{4\sin(\delta)}$. We now fix a direction $\theta\in I_0$ and, for $k\geq 2$, we consider the Borel-Laplace sum

$$\mathscr{S}^\theta\,\widetilde{w}_{k\boldsymbol{e}_1}(z)=\int_0^{\infty\mathrm{e}^{\mathrm{i}\theta}}\mathrm{e}^{-z\zeta}\widehat{w}_{k\boldsymbol{e}_1}(\zeta)\,d\zeta=\int_0^{+\infty}\mathrm{e}^{-z\xi\mathrm{e}^{\mathrm{i}\theta}}\widehat{w}_{k\boldsymbol{e}_1}(\xi\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{e}^{\mathrm{i}\theta}d\xi.$$

For any $c>\tau_\theta$ and any $z\in\overline{\Pi}_c^\theta$, $|\mathrm{e}^{-z\xi\mathrm{e}^{\mathrm{i}\theta}}|\leq\mathrm{e}^{-c\xi}$, for $\xi\geq 0$. Therefore, for $z\in\overline{\Pi}_c^\theta$ and $\xi\geq 0$,

$$\left|\mathrm{e}^{-z\xi\mathrm{e}^{\mathrm{i}\theta}}\widehat{w}_{k\boldsymbol{e}_1}(\xi\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{e}^{\mathrm{i}\theta}\right|\leq\left(\frac{3\sqrt{\sin(\theta)}}{2}\right)^k A_\theta^{2k-2}\mathrm{e}^{-c\xi}\left(\frac{\xi^{2k-4}}{(2k-4)!}*\mathrm{e}^{\tau_\theta\xi}\right)(\xi).$$

We deduce that $\mathscr{S}^\theta\,\widetilde{w}_{k\boldsymbol{e}_1}$ is holomorphic on Π_c^θ and, for every $z\in\overline{\Pi}_c^\theta$,

$$|\mathscr{S}^\theta\,\widetilde{w}_{k\boldsymbol{e}_1}(z)|\leq\left(\frac{3\sqrt{\sin(\theta)}}{2}\right)^k\left(\frac{A_\theta}{c}\right)^{2k-2}\frac{c}{c-\tau_\theta}.$$

We turn to the series of function $\displaystyle\sum_{k\geq 2}\left(U\mathrm{e}^{-z}z^{3/2}\right)^k\mathscr{S}^\theta\,\widetilde{w}_{k\boldsymbol{e}_1}(z)$. From what precedes, for any $R>0$, for any $c'>c>\tau_\theta$, for every $(z,U)\in\Pi_{c'}^\theta\times D(0,R)$, the series is normally convergent when $|R\mathrm{e}^{-z}z^{3/2}|\leq\dfrac{2c^2}{3A_\theta^2\sqrt{\sin(\theta)}}$. We end with theorem 3.2 and lemma 6.3: for any direction $\theta\in I_0$, for any $c>\tau_\theta$, the series of function $\displaystyle\sum_{k\geq 0}\left(U\mathrm{e}^{-z}z^{3/2}\right)^k\mathscr{S}^\theta\,\widetilde{w}_{k\boldsymbol{e}_1}(z)$ defines a holomorphic function on the domain $\mho^\theta\times D(0,R)$ with

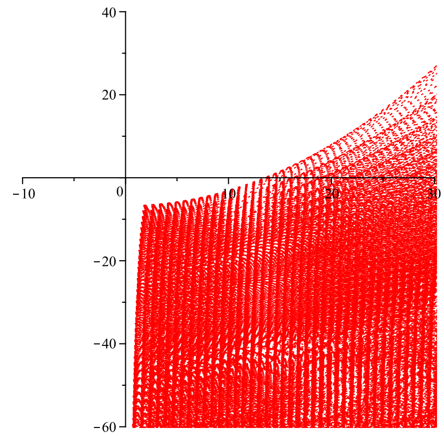


Fig. 6.1 The (shaded) domain $\mathscr{V}(I_0, V_1(0.5), \tau, \mathcal{K})$ on projection, for $\tau(\theta) = \dfrac{27}{4|\sin(\theta)|}$, $\big(\mathcal{K}(\theta)\big)(c) = \dfrac{2c^2}{3\tau(\theta)^2\sqrt{\sin(\theta)}}$ and $V_1(U)(z) = U\mathrm{e}^{-z}z^{3/2}$.

$$\mho^\theta = \bigcup_{c>\tau_\theta} \{z \in \Pi_c^\theta\,,\ |R\mathrm{e}^{-z}z^{3/2}| < \frac{2c^2}{3A_\theta^2\sqrt{\sin(\theta)}}\}.$$

Making θ varying on I_0, these functions glue together to provide a holomorphic function $\mathscr{S}^{I_0}\widetilde{w}(z, U\boldsymbol{e}_1)$ on the domain $\mathscr{V}(I_0, V_1(R), \tau, \mathcal{K}) \times D(0, R)$ with $\tau(\theta) = \dfrac{27}{4|\sin(\theta)|}$ and $\mathcal{K}(\theta) : c \in \mathbb{R} \mapsto \dfrac{2c^2}{3\tau(\theta)^2\sqrt{\sin(\theta)}}$ (since $A_\theta = \tau_\theta$), see Fig. 6.1. $\square$

Remark 6.1. The theorem 6.2 can be shown by other means, see the comments in Sect. 6.5.

6.2.3 Remarks

1. We know by proposition 5.2 that $\widetilde{w}_{k\boldsymbol{e}_2}(z) = \widetilde{w}_{k\boldsymbol{e}_1}(-z)$ for every $k \geq 0$. One deduces that for any $j \in \mathbb{Z}$, for any $\theta \in I_j$, for every $z \in \Pi^{\pi-\theta}_{\tau(\pi-\theta)}$, $z\mathrm{e}^{\mathrm{i}\pi} \in \Pi^{-\theta}_{\tau(-\theta)}$ and $\mathscr{S}^{\pi-\theta}\,\widetilde{w}_{k\boldsymbol{e}_2}(z) = \mathscr{S}^{-\theta}\,\widetilde{w}_{k\boldsymbol{e}_1}(z\mathrm{e}^{\mathrm{i}\pi})$. Therefore, for any $\theta \in I_j$, for every $z \in \Pi^{\pi-\theta}_{\tau(\pi-\theta)}$,

$$\mathscr{S}^{\pi-\theta}\widetilde{w}(z, U\boldsymbol{e}_2) = \mathscr{S}^{-\theta}\widetilde{w}(z\mathrm{e}^{\mathrm{i}\pi}, U\mathrm{e}^{\mathrm{i}\pi/2}\boldsymbol{e}_1)$$

   and, as a consequence, for any $j \in \mathbb{Z}$,

$$\begin{aligned}&\text{for every } z \in \mathscr{V}(I_j, V_2(U), \tau, \mathcal{K}),\ w_{tru,j,2}(z, U) = w_{tru,j-1,1}(z\mathrm{e}^{\mathrm{i}\pi}, U\mathrm{e}^{\mathrm{i}\pi/2})\\ &\text{for every } z \in \mathscr{V}(I_j, V_1(U), \tau, \mathcal{K}),\ w_{tru,j,1}(z, U) = w_{tru,j-1,2}(z\mathrm{e}^{\mathrm{i}\pi}, U\mathrm{e}^{\mathrm{i}\pi/2})\end{aligned} \tag{6.17}$$

2. Here we adopt the convention : for $z = re^{i\alpha} \in \dot{\mathbb{C}}$, we note $\overline{z} = r^{-i\alpha} \in \dot{\mathbb{C}}$. We know by proposition 5.2 that $\widetilde{w}_{k\boldsymbol{e}_i}(z) \in \mathbb{R}[[z^{-1}]]$ for any $k \in \mathbb{N}$, $i = 1, 2$. Thus, for any $j \in \mathbb{Z}$ and any $\theta \in I_j$, for $z \in \Pi^\theta_{\tau(\theta)}$,

$$\overline{\mathscr{S}^\theta\,\widetilde{w}_{k\boldsymbol{e}_i}}(z) = \mathscr{S}^{-\theta}\,\widetilde{w}_{k\boldsymbol{e}_i}(\overline{z}).$$

Therefore, for any $j \in \mathbb{Z}$, for every $z \in \mathscr{V}(I_j, V_i(U), \tau, \mathcal{K})$,

$$\overline{w_{tru,j,i}}(z,U) = w_{tru,(-j-1),i}(\overline{z},\overline{U})$$

and with (6.17) we deduce that, for every $z \in \mathscr{V}(I_j, V_1(U), \tau, \mathcal{K})$ and $z \in \mathscr{V}(I_j, V_2(U), \tau, \mathcal{K})$ respectively,

$$\begin{aligned} \overline{w_{tru,j,1}}(z,U) &= w_{tru,j,2}\big(\overline{z}\mathrm{e}^{-(2j+1)\mathrm{i}\pi}, \overline{U}\mathrm{e}^{-(j+1/2)\mathrm{i}\pi}\big) \\ \overline{w_{tru,j,2}}(z,U) &= w_{tru,j,1}\big(\overline{z}\mathrm{e}^{-(2j+1)\mathrm{i}\pi}, \overline{U}\mathrm{e}^{-(j+1/2)\mathrm{i}\pi}\big). \end{aligned} \tag{6.18}$$

6.2.4 Considerations on the domain

Viewing (6.17) and (6.18), it will be enough for our purpose to consider the domain $\mathscr{V}(I_0, V_1, \tau, \mathcal{K})$ with $I_0 =]0,\pi[$, $\big(V_1(U)\big)(z) = U\mathrm{e}^{-z}z^{3/2}$ with $|U| > 0$, $\tau(\theta) = \dfrac{27}{4|\sin(\theta)|}$, $\big(\mathcal{K}(\theta)\big)(c) = \dfrac{2c^2}{3\tau(\theta)^2\sqrt{\sin(\theta)}}$. We would like to describe the boundary of this domain. As a matter of fact, we will restrict ourself to describing its subdomain $\mho^\theta\big(V_1(U), \tau(\theta), \mathcal{K}(\theta)\big)$ with $\theta = \pi/2$. Considered by projection on $\mathbb{C}$, this domain reads: $z = x + \mathrm{i}y$, $(x,y) \in \mathbb{R}^2$, belongs to $\mho^{\frac{\pi}{2}}\big(V_1, \tau(\frac{\pi}{2}), \mathcal{K}(\frac{\pi}{2})\big)$ if and only if there exists $\lambda > 1$ so that

$$\begin{cases} y < -\dfrac{27}{4}\lambda \\ |U|\mathrm{e}^{-x}(x^2+y^2)^{3/4} < \dfrac{2}{3}\lambda^2. \end{cases}$$

(We take $c = \frac{27}{4}\lambda > \tau(\pi/2)$). We now fix $y = -\frac{27}{4}\lambda$ with $\lambda > 1$ and we remark that $z = x + \mathrm{i}y$ belongs to $\mho^{\frac{\pi}{2}}\big(V_1(U), \tau(\frac{\pi}{2}), \mathcal{K}(\frac{\pi}{2})\big)$ iff $x > X$ with X such that

$$|U|\mathrm{e}^{-X}(X^2+y^2)^{3/4} = \frac{2}{3}\left(\frac{4}{27}y\right)^2. \tag{6.19}$$

Indeed, just see that the real mapping $x \mapsto \mathrm{e}^{-x}(x^2+y^2)^p$ is decreasing when $|y| \geq p$, and use an argument of continuity. With the implicit function theorem, these arguments show the existence of a unique solution $X : y \in]-\infty, -\frac{3}{4}[\mapsto X(y)$ of (6.19), of class $\mathcal{C}^\infty$ and increasing with y, which can be described as follows. The above equality is equivalent to writing

$$\left(1 + \frac{X^2}{y^2}\right)^3 = \alpha y^2 \mathrm{e}^{4X}, \qquad \alpha = \left(\frac{32}{2187|U|}\right)^4. \tag{6.20}$$

and we can remark that $X(-\alpha^{-1/2}) = 0$ if $-\alpha^{-1/2} < -\frac{3}{4}$. When assuming $y^2 \gg X^2$, we get $X = -\dfrac{\ln(\alpha y^2)}{4} + \varepsilon$, $\varepsilon = o(1)$ as a first approximation. Plugging this in (6.20), one gets

$$X = -\frac{\ln(\alpha y^2)}{4} + 3\frac{\ln^2(\alpha y^2)}{4^2 y^2} + o(y^{-2})$$

and one can keep on this way to get an asymptotic expansion at any order of the solution[3]. To put it in a nutshell:

Corollary 6.1. *In theorem 6.2, the sum* $w_{tru,0,1}(z,U) = \mathscr{S}^{I_0}\widetilde{w}(z,U\boldsymbol{e}_1)$ *defines, for any* $U \in \mathbb{C}^\star$*, a holomorphic function with respect to* z *on a domain which contains, by projection on* $\mathbb{C}$*, a subdomain of the form* $\left\{z = x+iy,\ y < -\dfrac{27}{4},\ x > X(y)\right\}$ *where* X *is an increasing* $\mathcal{C}^\infty$ *function on* $]-\infty, -\frac{3}{4}[$*, whose asymptotics when* $y \to -\infty$ *is given by:*

$$X(y) = -\frac{\ln(\alpha y^2)}{4} + 3\frac{\ln^2(\alpha y^2)}{4^2 y^2} + o(y^{-2}), \qquad \alpha = \left(\frac{32}{2187|U|}\right)^4 \tag{6.21}$$

and so that $X(-\alpha^{-1/2}) = 0$ *if* $-\alpha^{-1/2} < -\frac{3}{4}$.

6.3 Summability of the formal integral

We saw with corollary 5.2 that the formal integral can be interpreted as a formal transformation $w = \widetilde{\Phi}(z,\mathbf{u})$,

$$\widetilde{\Phi}(z,\mathbf{u}) = \sum_{\boldsymbol{k}\in\mathbb{N}^2} \mathbf{u}^{\boldsymbol{k}} \widetilde{w}_{\boldsymbol{k}}^{[0]}(z), \tag{6.22}$$

that formally transforms the prepared ODE (3.6) into the normal form equation (5.66). It is then natural to wonder whether this formal transformation gives rise to an analytic transformations $\Phi_\theta(z,\mathbf{u})$ by Borel-Laplace summation,

$$\Phi_\theta(z,\mathbf{u}) = \mathscr{S}^\theta \widetilde{\Phi}(z,\mathbf{u}) = \sum_{\boldsymbol{k}\in\mathbb{N}^2} \mathbf{u}^{\boldsymbol{k}} \mathscr{S}^\theta \widetilde{w}_{\boldsymbol{k}}^{[0]}(z),$$

with a definition of the sum similar to that of definition 6.2. One could give a positive answer to this question, for the price of some further effort.

One has to extend lemma 6.5 to the whole $\boldsymbol{k}$-th series $\widetilde{w}_{\boldsymbol{k}}^{[0]}$. It is worth for this matter to complete the Banach spaces detailed by proposition 3.7 by other "focusing algebras" for which we refer to [6], in particular those based on L^1_ν-norms.

This does not mean that the formal integral is Borel-Laplace summable, which this is wrong, due to the effect of the exponentials. Only the restrictions of the formal integral to convenient submanifolds is 1-summable, which means here just considering one of the two the transseries. However, the sums of the two transseries share no common domain of convergence and thus, the formal integral cannot be summed by Borel-Laplace summation.

We do not pursue toward this direction and we conclude this chapter with the truncated solutions.

[3] One can also describe the solution in term of the Lambert function, the compositional inverse of the function $x\mathrm{e}^x$.

6.4 Truncated solutions for the first Painlevé equation

We have demonstrated with theorem 6.2 that, for any $j \in \mathbb{Z}$ and $i = 1, 2$, the sum $w_{tru,j,i}(z, U) = \mathscr{S}^{I_j} \widetilde{w}(z, U\boldsymbol{e}_i)$ is a holomorphic solution of (3.6), for z on a domain of the form $\mathscr{V}(I_j, V_i(U), \tau, \mathcal{K})$ with $I_j =]j\pi, (j+1)\pi[$. From its very definition and from corollary 6.1, the domain $\mathscr{V}(I_j, V_i(U), \tau, \mathcal{K})$ contains a sectorial neighbourhood of ∞ with aperture $\check{I}_j^i$ where (see Fig. 6.1):

- when $i = 1$, $\check{I}_j^1 =]-\frac{1}{2}\pi, +\frac{1}{2}\pi[-j\pi$ for j even, $\check{I}_j^1 =]-\frac{3}{2}\pi, -\frac{1}{2}\pi[-j\pi$ for j odd;
- when $i = 2$, $\check{I}_j^2 =]-\frac{1}{2}\pi, +\frac{1}{2}\pi[-j\pi$ for j odd, $\check{I}_j^2 =]-\frac{3}{2}\pi, -\frac{1}{2}\pi[-j\pi$ for j even.

To go back to the the first Painlevé equation (2.1), we use the transformation $\mathscr{T}$ of definition 3.12.

Definition 6.4. The conformal mapping $\mathscr{T}$ sends the domain $\mathscr{V}(I, g, \gamma, \mathcal{K})$ onto the domain $\mathscr{T}\big(\mathscr{V}(I, g, \gamma, \mathcal{K})\big)$ and we set

$$\mathcal{S}(I, g, \gamma, \mathcal{K}) = \mathscr{T}\big(\mathscr{V}(I, g, \gamma, \mathcal{K})\big), \qquad \overset{\bullet}{\mathcal{S}}(I, g, \gamma, \mathcal{K}) = \pi\Big(\mathcal{S}(I, g, \gamma, \mathcal{K})\Big). \tag{6.23}$$

The domain $\mathcal{S}(I_j, V_i(U), \tau, \mathcal{K})$ contains a sectorial neighbourhood of ∞ with aperture K_j^i (see Fig. 6.2):

- when $i = 1$, $K_j^1 =]-\frac{7}{5}\pi, -\frac{3}{5}\pi[-\frac{4}{5}j\pi$ for j even, $K_j^1 =]-\frac{11}{5}\pi, -\frac{7}{5}\pi[-\frac{4}{5}j\pi$ for j odd;
- when $i = 2$, $K_j^2 =]-\frac{7}{5}\pi, -\frac{3}{5}\pi[-\frac{4}{5}j\pi$ for j odd, $K_j^2 =]-\frac{11}{5}\pi, -\frac{7}{5}\pi[-\frac{4}{5}j\pi$ for j even.

In any case, the domains $\mathcal{S}(I_j, V_i(U), \tau, \mathcal{K})$ are in connection: for every $j \in \mathbb{Z}$,

$$\mathcal{S}(I_{j+1}, V_2(U), \tau, \mathcal{K}) = \mathrm{e}^{-4\mathrm{i}\pi/5} \mathcal{S}(I_j, V_1(U), \tau, \mathcal{K}).$$

From (3.4), (2.6), (2.7), the transformation $z \in \mathscr{V}(I_j, V_i(U), \tau, \mathcal{K}) \leftrightarrow x \in \mathcal{S}(I_j, V_i(U), \tau, \mathcal{K})$,

$$w_{tru,j,i}(z, U) \leftrightarrow u_{tru,j,i}(x, U) = \frac{\mathrm{e}^{i\frac{\pi}{2}} x^{\frac{1}{2}}}{\sqrt{6}} \left(1 - \frac{4}{25\big(\mathscr{T}^{-1}(x)\big)^2} + \frac{w_{tri,j,i}\big(\mathscr{T}^{-1}(x), U\big)}{\big(\mathscr{T}^{-1}(x)\big)^2}\right).$$

provides the solutions $u_{tru,j,i}(x, U)$ for the first Painlevé equation. These are the **truncated solutions**.

The property (6.17) translates into the following relationships between truncated solutions: for any $j \in \mathbb{Z}$, for every $x \in \mathcal{S}(I_j, V_1(U), \tau, \mathcal{K})$, respectively $x \in \mathcal{S}(I_j, V_2(U), \tau, \mathcal{K})$,

$$\begin{aligned} u_{tru,j,1}(x, U) &= \mathrm{e}^{2\mathrm{i}\pi/5} u_{tru,j+1,2}(x\mathrm{e}^{-4\mathrm{i}\pi/5}, U\mathrm{e}^{-\mathrm{i}\pi/2}) \\ u_{tru,j,2}(x, U) &= \mathrm{e}^{2\mathrm{i}\pi/5} u_{tru,j+1,1}(x\mathrm{e}^{-4\mathrm{i}\pi/5}, U\mathrm{e}^{-\mathrm{i}\pi/2}) \end{aligned} \tag{6.24}$$

These are the symmetries discussed in Sect. 2.5. In the same way from (6.18), for any $j \in \mathbb{Z}$, for every $x \in \mathcal{S}(I_j, V_1(U), \tau, \mathcal{K})$, respectively $x \in \mathcal{S}(I_j, V_2(U), \tau, \mathcal{K})$,

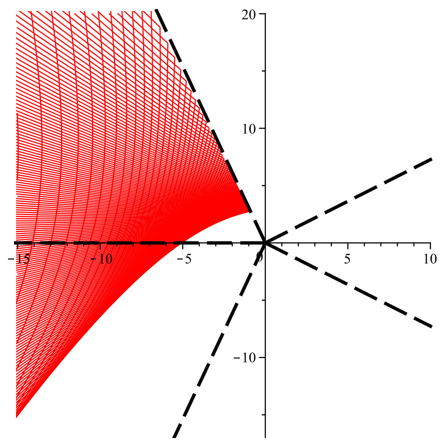


Fig. 6.2 The (shaded) domain $\overset{\bullet}{\mathcal{S}}(I_0, V_1(U), \tau, \mathcal{K})$ for $\tau(\theta) = \dfrac{27}{4|\sin(\theta)|}$, $\big(\mathcal{K}(\theta)\big)(c) = \dfrac{2c^2}{3\tau(\theta)^2\sqrt{\sin(\theta)}}$ and $V_1(U)(z) = U\mathrm{e}^{-z}z^{3/2}$.

$$\begin{aligned}\overline{u_{tru,j,1}}(x,U) &= \mathrm{e}^{\frac{2}{5}(2j+1)\mathrm{i}\pi}u_{tru,j,2}(\overline{x}\mathrm{e}^{-\frac{2}{5}(4j+7)\mathrm{i}\pi}, \overline{U}\mathrm{e}^{-(j+1/2)\mathrm{i}\pi}), \\ \overline{u_{tru,j,2}}(x,U) &= \mathrm{e}^{\frac{2}{5}(2j+1)\mathrm{i}\pi}u_{tru,j,1}(\overline{x}\mathrm{e}^{-\frac{2}{5}(4j+7)\mathrm{i}\pi}, \overline{U}\mathrm{e}^{-(j+1/2)\mathrm{i}\pi}).\end{aligned} \tag{6.25}$$

6.5 Comments

We mentionned in Sect. 5.5 the existence of formal transforms of the type $\mathbf{v} = \widetilde{T}(z,\mathbf{u})$, $\widetilde{T}(z,\mathbf{u}) = \sum_{\boldsymbol{k}\in\mathbb{N}^n}\mathbf{u}^{\boldsymbol{k}}\widetilde{\mathbf{v}}_{\boldsymbol{k}}(z)$, $\widetilde{\mathbf{v}}_{\boldsymbol{k}}(z) \in \mathbb{C}^n[[z^{-1}]]$ that brings differential and difference systems to their linear normal form, under some convenient hypotheses. For differential equations of type (5.67), the series $\widetilde{\mathbf{v}}_{\boldsymbol{k}}$ are in general not 1-summable but multisummable [10]. The first results in that direction, concerning the multisummability of the formal series solutions, were obtained by Braaksma [1] then by Ramis & Sibuya [11]. A resurgent approach for 1-level differential equations were undertaken by Costin [4], with the proof of the 1-summability of the formal integral on restriction to convenient submanifolds. These results were then generalized to differential and difference equations, see e.g. [2, 9, 7, 5] and references therein, at least for the cases where no resonance occurs. The question of the (multi)summability of the above formal transforms may be delicate, even for 1-level differential systems or ODEs, when **quasi-resonance** occur, giving rise to *small divisors*.

If $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_n)$ stands for the multipliers and in absence of resonance, it may happen that $\boldsymbol{\lambda}.\boldsymbol{k}$ comes close to one multiplier, for some $\boldsymbol{k} \in \mathbb{N}^n$. Thus, the construction of the formal integral gives rise to division by small factors. One has "quasi-resonance" when there exists an increasing sequence ($\boldsymbol{k}_j \in \mathbb{N}^n$) such that $\lim_{j\to\infty}\boldsymbol{\lambda}.\boldsymbol{k}_j = 0$ fast enough, a condition that translates into diophantine relations on the sequence.

More details on this subject can be found in [8].

We finally mention a general upshot, that of the formation of singularities near the anti-Stokes rays. Considering the Borel-Laplace sum of a transseries stemming from (resurgent) 1-level differential or difference equations, it is possible, as shown in [7] (see also [6]) to analyze its behavior on the boundary of its domain of convergence, by a suitable use of a multi-scale analysis. This is detailed in [5] for the first Painlevé equation.

Acknowledgements I warmly thank my student Julie Belpaume who helped me to work out this chapter.

References

1. B.L.J. Braaksma, *Multisummability of formal power series solutions of nonlinear meromorphic differential equations.* Ann. Inst. Fourier (Grenoble) **42** (1992), no. 3, 517-540.
2. B.L.J. Braaksma, *Transseries for a class of nonlinear difference equations. In memory of W. A. Harris, Jr.* J. Differ. Equations Appl. **7** (2001), no. 5, 717-750.
3. B. Candelpergher, C. Nosmas, F. Pham, *Approche de la résurgence*, Actualités mathématiques, Hermann, Paris (1993).
4. O. Costin, *On Borel summation and Stokes phenomena for rank-*1 *nonlinear systems of ordinary differential equations.* Duke Math. J. **93** (1998), no. 2, 289-344.
5. O. Costin, *Correlation between pole location and asymptotic behavior for Painlevé I solutions.* Comm. Pure Appl. Math. **52** (1999), no. 4, 461-478.
6. O. Costin, *Asymptotics and Borel summability*, Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, 141. CRC Press, Boca Raton, FL, 2009.
7. O. Costin, R. Costin, *On the formation of singularities of solutions of nonlinear differential systems in antistokes directions.* Invent. Math. **145** (2001), no. 3, 425-485.
8. J. Ecalle, *Les fonctions résurgentes. Tome III : l'équation du pont et la classification analytique des objets locaux.* Publ. Math. d'Orsay, Université Paris-Sud, 1985.05 (1985).
9. G.R. Kuik, *Transseries in Difference and Differential Equations.* PhD thesis, Rijksuniversiteit Groningen (2003).
10. M. Loday-Richaud, *Divergent series and differential equations.* To appear.
11. J.-P. Ramis, Y Sibuya, *A new proof of multisummability of formal solutions of nonlinear meromorphic differential equations.* Ann. Inst. Fourier (Grenoble) **44** (1994), no. 3, 811-848.

Chapter 7
Supplements to resurgence theory

Abstract This chapter is devoted to some general nonsense in resurgence theory that will be useful to study furthermore the first Painlevé equation from the viewpoint of the resurgent analysis. We define sectorial germs of analytic functions (Sect. 7.2) and we introduce the sheaf of microfunctions (Sect. 7.3). This provides an approach to the notion of singularities : this is the purpose of Sect. 7.4. We define the formal Laplace transform for microfunctions and for singularities and, conversely, the inverse formal Borel transform acting on asymptotic classes (Sect. 7.5). We make some links with the Borel-Ritt theorem. The main properties of the Laplace transform that we need in this course are developed in Sect. 7.6. We finally introduce some spaces of resurgent functions and define the alien operators (Sect. 7.7, 7.8 and 7.9).

7.1 Introduction

At its very root, one can rely the Borel-Laplace summation scheme to the simple formula

$$\frac{1}{z^n} = \mathcal{L}^\theta\left(\frac{\zeta^{n-1}}{\Gamma(n)}\right) = \int_0^{\infty e^{i\theta}} e^{-z\zeta}\frac{\zeta^{n-1}}{\Gamma(n)}d\zeta, \qquad n \in \mathbb{N}^\star, \;\; z \in \dot{\Pi}_0^\theta.$$

Consider a holomorphic function $\widehat{\varphi} \in \mathcal{O}(D(0,R))$ with Taylor expansion $\displaystyle\sum_{n\geq 1} a_n \frac{\zeta^{n-1}}{\Gamma(n)}$ at the origin. We take an open arc $I =]-\alpha+\theta, \theta+\alpha[$, $0 < \alpha \leq \pi/2$, bisected by the direction θ, and we note $I^\star =]-\alpha-\theta, -\theta+\alpha[\subseteq \check{\theta}$. For some $r \geq 0$, we set $\dot{\boldsymbol{\mathfrak{s}}}^\infty = \dot{\boldsymbol{\mathfrak{s}}}_r^\infty(I^\star)$. For any cut-off $\kappa \in]0,R[$, the truncated Laplace integral $\displaystyle\varphi_\kappa(z) = \int_0^{\kappa e^{i\theta}} e^{-z\zeta}\widehat{\varphi}(\zeta)d\zeta$ provides an element of $\overline{\mathcal{A}}_1(\dot{\boldsymbol{\mathfrak{s}}}^\infty)$ whose 1-Gevrey asymptotics $T_{1,\dot{\boldsymbol{\mathfrak{s}}}^\infty}\varphi_\kappa(z)$ (see [14]) is given by the 1-Gevrey series $\displaystyle\sum_{n\geq 1}\frac{a_n}{z^n} \in \mathbb{C}[[z^{-1}]]_1$: this is essentially the Borel-Ritt theorem for 1-Gevrey asymptotics. For two cut-off points $\kappa_1, \kappa_2 \in]0,R[$, the difference $\varphi_{\kappa_1} - \varphi_{\kappa_2}$ belongs to $\overline{\mathcal{A}}^{\leq -1}(\dot{\boldsymbol{\mathfrak{s}}}^\infty)$, the differential ideal of $\overline{\mathcal{A}}_1(\dot{\boldsymbol{\mathfrak{s}}}^\infty)$ made of 1-exponentially flat functions on $\dot{\boldsymbol{\mathfrak{s}}}^\infty$.

One gets this way a morphism $\tilde{\mathcal{L}}(I) : \widehat{\varphi} \in \mathcal{O}_0 \mapsto \mathrm{cl}(\varphi_\kappa) \in \mathcal{A}_1(I^\star)/\mathcal{A}^{\leq -1}(I^\star)$, where here $\mathcal{O}_0$ stands for the constant sheaf (of convolution algebras) over $\mathbb{S}^1$. By (obvious) compatibility with the restriction maps, one obtains[1] a morphism of sheaves of differential algebras, $\tilde{\mathcal{L}} : \mathcal{O}_0 \to \mathcal{A}_1/\mathcal{A}^{\leq -1}$, where the quotient sheaf $\mathcal{A}_1/\mathcal{A}^{\leq -1}$ over $\mathbb{S}^1$ is known to be isomorphic to the constant sheaf $\mathbb{C}[[z^{-1}]]_1$ (Borel-Ritt theorem 3.3, see [14, 17]). The formal Laplace transfom $\tilde{\mathcal{L}}$ is an isomorphism, the inverse morphism being the formal Borel transform $\tilde{\mathcal{B}} : \mathbb{C}[[z^{-1}]]_1 \to \mathcal{O}_0$ (seen as a morphism of sheaves).

One can extend the theory by considering the properties of Laplace integrals defined along Hankel contours. For instance, standard formulae provide

$$\Gamma(\sigma) = \frac{1}{1 - \mathrm{e}^{-2i\pi\sigma}} \int_{\gamma_{[-2\pi,0],\varepsilon}} \mathrm{e}^{-\zeta} \zeta^{\sigma-1} d\zeta, \qquad \sigma \in \mathbb{C} \setminus \mathbb{N}, \tag{7.1}$$

where the integration contour $\gamma_{[-2\pi,0],\varepsilon}$ is the (endless) Hankel contour drawn on Fig. 7.1, while $\zeta^{\sigma-1} = \mathrm{e}^{(\sigma-1)\log\zeta}$ and $\log\zeta$ is the branch of the logarithm so that $\arg(\log\zeta) \in]-2\pi, 0[$. Performing a change of variable, one gets the identity

$$\frac{1}{z^\sigma} = \mathcal{L}^0 \overset{\vee}{I}_\sigma (z) = \int_{\gamma_{[-2\pi,0],\varepsilon}} \mathrm{e}^{-z\zeta} \overset{\vee}{I}_\sigma (\zeta) d\zeta, \quad z \in \overset{\bullet}{\Pi}{}^0_0, \tag{7.2}$$

with $z^\sigma = \mathrm{e}^{\sigma \log z}$ where this time $\log z$ is the branch of the logarithm so that $\arg(\log z) \in]-\pi, \pi[$, while

$$\overset{\vee}{I}_\sigma (\zeta) = \begin{cases} \dfrac{\zeta^{\sigma-1}\log(\zeta)}{2i\pi\Gamma(\sigma)} & \text{for } \sigma - 1 \in \mathbb{N} \\ \dfrac{\zeta^{\sigma-1}}{(1-\mathrm{e}^{-2i\pi\sigma})\Gamma(\sigma)} & \text{for } \sigma - 1 \in \mathbb{C} \setminus \mathbb{N}. \end{cases}$$

The definition of $\overset{\vee}{I}_\sigma$ that we give for $\sigma - 1 \in \mathbb{C} \setminus \mathbb{N}$ is well-defined when $-\sigma \notin \mathbb{N}$. It can be analytically continued to the case $-\sigma \in \mathbb{N}$ by the reflection formula.

This example provides another one that will be used later on : for any $m \in \mathbb{N}$, any $\sigma \in \mathbb{C} \setminus \mathbb{N}^\star$, for $z \in \overset{\bullet}{\Pi}{}^0_0$, $(-1)^m z^{-\sigma} (\log z)^m = \mathcal{L}^0 \overset{\vee}{J}_{\sigma,m}$, $\overset{\vee}{J}_{\sigma,m} = \left(\dfrac{\partial}{\partial\sigma}\right)^m \overset{\vee}{I}_\sigma$ with the above convention for the $\log z$. Remark however that $\mathcal{L}^0 \overset{\vee}{I}_\sigma = \mathcal{L}^0 \big(\overset{\vee}{I}_\sigma + \mathrm{hol}\big)$ when hol is any holomorphic function on a half-strip containing the origin, with at most exponential growth of order 1 at infinity. This justifies the introduction of the spaces of microfunctions and singularities that we do in the next sections.

This chapter can be seen as a sequel of the resurgence theory developed in [24]. For most of the materials presented here, we mainly refer to [7, 9, 10, 1, 24] for this chapter, see also [4, 23, 20].

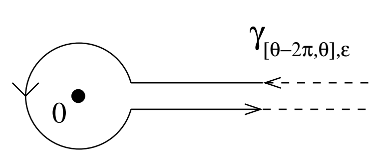


Fig. 7.1 The Hankel contour $\gamma_{[\theta-2\pi,\theta],\varepsilon}$ for $\theta = 0$.

[1] Modulo the quite innocent complex conjugation $I \to I^\star$.

7.2 Sectorial germs

7.2.1 Sectors

We remind that

$$\underset{\bullet}{\mathbb{C}} = \{\zeta = r\mathrm{e}^{\mathrm{i}\theta} \mid r > 0,\ \theta \in \mathbb{R}\}, \qquad \pi : \xi \in \underset{\bullet}{\mathbb{C}} \mapsto \overset{\bullet}{\zeta} = r\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C}^\star$$

is the Riemann surface of the logarithm (definition 3.10). We complete definition 3.7 by defining sectors on $\underset{\bullet}{\mathbb{C}}$.

Definition 7.1. For an open arc I of $\underset{\bullet}{\mathbb{S}}^1$ and $0 \le r < R \le \infty$, $\mathfrak{s}_r^R(I)$ denotes the simply connected domain of $\underset{\bullet}{\mathbb{C}}$ of the form $\mathfrak{s}_r^R(I) = \{\zeta = \xi \mathrm{e}^{\mathrm{i}\theta} \mid \theta \in I, \xi \in]r, R[\}$.
We simply write $\mathfrak{s}_0(I)$ –resp. $\mathfrak{s}^\infty(I)$– for a domain of type $\mathfrak{s}_0^R(I)$ –resp. $\mathfrak{s}_r^\infty(I)$– for some $0 \le r < R \le \infty$. We even write $\mathfrak{s}_0$ and $\mathfrak{s}^\infty$ for such domains, when there is no need to indicate the arc I.
For $0 < r < R < \infty$, we set $\bar{\mathfrak{s}}_0^R(I) = \{\zeta = \xi \mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C} \mid \theta \in \bar{I},\ 0 < \xi \le R\}$ and $\bar{\mathfrak{s}}_r^\infty(I) = \{\zeta = \xi \mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C} \mid \theta \in \bar{I},\ r \le \xi < \infty\}$.
For a continuous function $\mathrm{R} : \mathbb{R} \to]0, +\infty[$, we note $\mathfrak{s}_0^{\mathrm{R}}(\underset{\bullet}{\mathbb{S}}^1)$ the simply connected domain $\mathfrak{s}_0^{\mathrm{R}}(\underset{\bullet}{\mathbb{S}}^1) = \{\zeta = r\mathrm{e}^{\mathrm{i}\theta},\ 0 < r < \mathrm{R}(\theta)\} \subset \underset{\bullet}{\mathbb{C}}$. We simply write $\mathfrak{s}_0(\underset{\bullet}{\mathbb{S}}^1)$ for such a domain, when there is no need to specify the function R.

7.2.2 Sectorial germs

Definition 7.2 (Sectorial germs-1). For I an open arc of $\mathbb{S}^1$, one says that two functions $\varphi_1 \in \mathcal{O}(\overset{\bullet}{\mathfrak{s}}_0^{R_1}(I))$, $\varphi_2 \in \mathcal{O}(\overset{\bullet}{\mathfrak{s}}_0^{R_2}(I))$ define the same **sectorial germ** $\overset{\vee}{\varphi}$ **of direction I at** 0, when φ_1 and φ_2 coincide on a same domain of type $\overset{\bullet}{\mathfrak{s}}_0(I)$. We note $\overline{\mathcal{O}}^0(I) = \varinjlim_{R \to 0} \mathcal{O}(\overset{\bullet}{\mathfrak{s}}_0^R(I))$ the space of germs of direction I at 0, and $\mathcal{O}^0$ the sheaf over $\mathbb{S}^1$ associated with the presheaf $\overline{\mathcal{O}}^0$.

As a rule in this paper for the (pre)sheafs one encounters, the restriction maps are the usual restrictions of functions. We warn the reader that the presheaf $\overline{\mathcal{O}}^0$ is not a sheaf over $\mathbb{S}^1$ (see for instance a counter example given in [14]) : for an open arc I, a section $\overset{\vee}{\varphi} \in \mathcal{O}^0(I) = \Gamma(I, \mathcal{O}^0)$ is a collection of holomorphic functions $\varphi_i \in \mathcal{O}(\overset{\bullet}{\mathfrak{s}}_0^{R_i}(I_i))$ that glue together on their intersection domains, the set $\{I_i\}$ being an open covering of I

Example 7.1. We denote by $\mathbb{C}\{\zeta, \zeta^{-1}\}$ the space of Laurent series $\sum_{n \in \mathbb{Z}} a_n \zeta^n$ which converge on a punctured disc $D(0, R)^\star$. This space can also be seen as a constant sheaf over $\mathbb{S}^1$ and the space $\mathcal{O}^0(\mathbb{S}^1)$ of global sections of $\mathcal{O}^0$ on $\mathbb{S}^1$ coincides with $\mathbb{C}\{\zeta, \zeta^{-1}\}$.
For $n \in \mathbb{N}^\star$ and a given direction $\theta_0 \in \mathbb{S}^1$, we now consider the sectorial germ $\overset{\vee}{\varphi}_{\theta_0}(\zeta) = \overset{\vee}{I}_n(\zeta) = \dfrac{\zeta^{n-1} \log(\zeta)}{2i\pi \Gamma(n)} \in \mathcal{O}_{\theta_0}^0$, for any given determination of the log. Here $\mathcal{O}_{\theta_0}^0$ denotes the stalk at θ_0 of the sheaf $\mathcal{O}^0$. When making θ varying from θ_0 on $I =]-\pi + \theta_0, \theta_0 + \pi[$ on $\mathbb{S}^1$, the sectorial germs $\overset{\vee}{\varphi}_\theta \in \mathcal{O}_\theta^0$ glue

together and defined a section $\overset{\vee}{\varphi} \in \Gamma(I, \mathcal{O}^0)$ which cannot be prolonged to a global section.

This last example illustrates the need of defining sectorial germs for functions defined on sectors of $\underset{\bullet}{\mathbb{C}}$. The covering map $\dot{\pi} : \underset{\bullet}{\mathbb{S}}^1 \to \mathbb{S}^1$ (see definition 3.10) allows to consider the sheaf $\dot{\pi}^\star \mathcal{O}^0$ over $\underset{\bullet}{\mathbb{S}}^1$, that is the inverse image by $\dot{\pi}$ of the sheaf $\mathcal{O}^0$ (see [1, 12, 3]). For J an open arc of $\underset{\bullet}{\mathbb{S}}^1$, an element $\overset{\vee}{\varphi}$ of $\dot{\pi}^\star \mathcal{O}^0(J)$ appears as an element of the **space** $\Gamma(J, \mathcal{O}^0)$ **of multivalued sections** of $\mathcal{O}^0$ on J, that is $\overset{\vee}{\varphi} = s(J)$ where s is a continuous map such that $s \circ p = \dot{\pi}$:

$$\begin{array}{ccl} & & \widetilde{\mathcal{O}}^0 = \bigsqcup_{\theta \in \mathbb{S}^1} \mathcal{O}^0_\theta \\ & s \nearrow & \downarrow p \\ \underset{\bullet}{\mathbb{S}}^1 & \underset{\dot{\pi}}{\longrightarrow} & \mathbb{S}^1 \end{array}$$

We say that in another way in the following definition:

Definition 7.3 (Sectorial germs-2). For J an open arc of $\underset{\bullet}{\mathbb{S}}^1$, one says that two functions $\varphi_1 \in \mathcal{O}(\mathfrak{s}_0^{R_1}(J))$, $\varphi_2 \in \mathcal{O}(\mathfrak{s}_0^{R_2}(J))$ define the same sectorial germ $\overset{\vee}{\varphi}$ of direction J at 0 when φ_1 and φ_2 coincide on a same domain of type $\mathfrak{s}_0(J)$. We note $\Gamma(J, \mathcal{O}^0)$ the space of multivalued sections of germs of direction J.

Remark 7.1. For any $\omega \in \mathbb{C}$ and by translation, one can of course define $\mathcal{O}^\omega$, the sheaf over $\mathbb{S}^1$ of sectorial germs at ω, associated with the presheaf $\overline{\mathcal{O}}^\omega$.

7.3 Microfunctions

In this section, we introduce the sheaf of microfunctions $\mathscr{C}_\omega$ at $\omega \in \mathbb{C}$, in the spirit of [1] to whom we refer. Since $\mathscr{C}_\omega$ is deduced from $\mathscr{C} = \mathscr{C}_0$ by translation, we make the focus on the case $\omega = 0$.

7.3.1 Microfunctions, definitions

We complete definition 3.6.

Definition 7.4. For a direction θ, for an open arc $I =]\alpha, \beta[$ (of $\mathbb{S}^1$ or $\underset{\bullet}{\mathbb{S}}^1$), we note:

1. $\theta^* = -\theta$ and $I^\star =]-\beta, -\alpha[$ the complex conjugate arc;
2. $\breve{\theta} =]-\frac{\pi}{2} - \theta, -\theta + \frac{\pi}{2}[$ and $\breve{I} = \bigcup_{\theta \in I} \breve{\theta}$;
3. $\breve{\theta} =]\theta - 3\pi/2, \theta - \pi/2[$ the "copolar" of θ;
4. $\breve{I} =]\alpha - 3\pi/2, \beta - \pi/2[= \bigcup_{\theta \in I} \breve{\theta}$ the "copolar" of I;
5. for $|I| > \pi$, $\hat{I} =]\alpha + \pi/2, \beta - \pi/2[$; for $|I| < \pi$, $\hat{I} =]\beta - \pi/2, \alpha + \pi/2[$. When $|I| = \pi$, we set $\hat{I} = \{\beta - \pi/2\}$.

We would like to define "microfunctions of codirection I at 0". For an open arc I of $\mathbb{S}^1$ of aperture $\leq \pi$, we first notice that its copolar $\check{I}$ is of aperture $\leq \pi$, thus can be seens as an arc of $\mathbb{S}^1$. For such an arc, we note $\check{\overline{\mathcal{O}}}^0(I) = \overline{\mathcal{O}}^0(\check{I})$. We now remark that for two arcs $I_2 \subsetneq I_2$ of $\mathbb{S}^1$, of aperture $\leq \pi$, one has $\check{I}_2 \subseteq \check{I}_1$. The restriction map $\rho_{\check{I}_2,\check{I}_1} : \overline{\mathcal{O}}^0(\check{I}_1) \to \overline{\mathcal{O}}^0(\check{I}_2)$ gives rise to a restriction map $\check{\rho}_{I_2,I_1} = \rho_{\check{I}_2,\check{I}_1}$ from $\check{\overline{\mathcal{O}}}^0(I_1)$ into $\check{\overline{\mathcal{O}}}^0(I_2)$. This justifies the following definition.

Definition 7.5 (Microfunctions). For I an open arc of $\mathbb{S}^1$ of aperture $\leq \pi$, one calls $\check{\overline{\mathcal{O}}}^0(I) = \overline{\mathcal{O}}^0(\check{I})$ the space of germs of codirection I at 0, and $\check{\mathcal{O}}^0$ the corresponding sheaf over $\mathbb{S}^1$.
Viewing $\mathcal{O}_0$ as a constant sheaf over $\mathbb{S}^1$, we set $\mathscr{C} = \check{\mathcal{O}}^0/\mathcal{O}_0$. This quotient sheaf over $\mathbb{S}^1$ is the sheaf of **microfunctions** at 0 and $\mathscr{C}(I) = \Gamma(I, \mathscr{C})$ is the space of sections of **microfunctions of codirection** I at 0.

The sheaf of microfunctions $\mathscr{C}$ and makes allusion to Sato's microlocal analysis, see, e.g. [22, 13, 18]. We mention that microfunctions depending on parameters can be also defined, see for instance [4] for a resurgent context.

We mention that $\mathscr{C}(I) = \check{\mathcal{O}}^0(I)/\mathcal{O}_0$, that is the quotient sheaf coincide with the pre-quotient sheaf, because $\mathcal{O}_0$ is a constant sheaf.

In what follows, we transpose with some abuse the notations for singularities (Sect. 7.4) to that for microfunctions.

Definition 7.6. For an open arc I of $\mathbb{S}^1$ of aperture $\leq \pi$, we note $\overset{\triangledown}{\varphi} = \mathrm{sing}_0^I \overset{\vee}{\varphi} \in \mathscr{C}(I)$ the microfunction of codirection I at 0 defined by the sectorial germ $\overset{\vee}{\varphi} \in \check{\overline{\mathcal{O}}}^0(I)$ of codirection I.

When I is an arc of aperture $> \pi$, then $\check{I}$ is of aperture larger than 2π and should be seen as an arc of $\underset{\bullet}{\mathbb{S}^1}$. In that case, a microfunction $\overset{\triangledown}{\varphi}$ of $\mathscr{C}(I)$ is represented by an element $\overset{\vee}{\varphi}$ of $\Gamma(\check{I}, \mathcal{O}^0)$.

For I an arc of $\mathbb{S}^1$ of aperture $> \pi$, one can define the variation map, $\mathrm{var} : \mathscr{C}(I) \to \Gamma(\hat{I}, \mathcal{O}^0)$,

$$\mathrm{var} : \overset{\triangledown}{\varphi} \in \mathscr{C}(I) \mapsto \widehat{\varphi} \in \Gamma(\hat{I}, \mathcal{O}^0), \;\; \widehat{\varphi}(\zeta) = \overset{\vee}{\varphi}(\zeta) - \overset{\vee}{\varphi}(\zeta e^{-2i\pi}).$$

Example 7.2. 1. For $n \in \mathbb{N}$, the sectorial germ $\overset{\vee}{I}_{-n}(\zeta) = \dfrac{(-1)^n}{2i\pi}\dfrac{n!}{\zeta^{n+1}}$ can be seen as a global section of the sheaf $\mathcal{O}^0$. The associated microfunction is equally denoted by $\overset{\triangledown}{I}_{-n}$, $\delta^{(n)}$ or $\mathrm{sing}_0 \overset{\vee}{I}_{-n}$.
   Notice that for any holomorphic germ $\widehat{\varphi} \in \mathcal{O}_0$, the sectorial germ $\widehat{\varphi}\, \overset{\vee}{I}_0$ defines a microfunction $\mathrm{sing}_0(\widehat{\varphi}\, \overset{\vee}{I}_0)$ equal to $\widehat{\varphi}(0)\delta^{(0)} = \widehat{\varphi}(0)\delta$.
2. More generally, the constant sheaf $\mathbb{C}\{\zeta, \zeta^{-1}\}$ over $\mathbb{S}^1$ can be seen as a subsheaf of $\mathscr{C}$ (of vector spaces). Any microfunction $\overset{\triangledown}{\psi}$ of $\mathbb{C}\{\zeta, \zeta^{-1}\}$ can be written as a sum $\sum_{n\geq 0} a_n \overset{\triangledown}{I}_{-n} = \sum_{n\geq 0} a_n \delta^{(n)}$, where the Laurent series $\overset{\vee}{\psi}(\zeta) = \sum_{n\geq 0} a_n \frac{(-1)^n}{2i\pi}\frac{n!}{\zeta^{n+1}}$ converges for $|\zeta| > 0$.
3. We assume that $\widehat{\varphi} \in \mathcal{O}_0$ is a germ of holomorphic function. For a direction $\theta_0 \in \mathbb{S}^1$, we consider the microfunction $\overset{\triangledown}{\phi}_{\theta_0} = \mathrm{sing}_0^{\theta_0}\left(\widehat{\varphi}\frac{\log}{2i\pi}\right) \in \mathscr{C}_{\theta_0}$ (where $\mathscr{C}_{\theta_0}$ is the stalk at θ_0 of the sheaf $\mathscr{C}$), represented by the sectorial germ

$\overset{\vee}{\phi}_{\theta_0}= \widehat{\varphi}\frac{\log}{2i\pi}$ of $\check{\mathcal{O}}^0_{\theta_0} = \mathcal{O}^0(\check{\theta}_0)$, for any given determination of the log (remark that $\overset{\triangledown}{\phi}_{\theta_0}$ does not depend on the chosen determination). Making θ varying from θ_0 up to $\theta_0 + 2\pi$ on $\mathbb{S}^1$, the microfunctions $\overset{\triangledown}{\phi}_{\theta}= \operatorname{sing}^{\theta}_0 \left(\widehat{\varphi}\frac{\log}{2i\pi}\right) \in \mathscr{C}_\theta$ glue together and $\overset{\triangledown}{\phi}_{\theta_0}=\overset{\triangledown}{\phi}_{\theta_0+2\pi}$. This provides a global section denoted by $\overset{\triangledown}{\phi}= \operatorname{sing}_0 \left(\widehat{\varphi}\frac{\log}{2i\pi}\right) \in \Gamma(\mathbb{S}^1, \mathscr{C})$ which does not depend of the chosen determination of the log one started with.
It can be shown (through the variation map) that the space of global sections $\Gamma(\mathbb{S}^1, \mathscr{C})$ of the sheaf of microfunctions, is composed of microfunctions of the form $\overset{\triangledown}{\psi} + \operatorname{sing}_0 \left(\widehat{\varphi}\frac{\log}{2i\pi}\right)$, with $\overset{\triangledown}{\psi} \in \mathbb{C}\{\zeta, \zeta^{-1}\}$ and $\widehat{\varphi} \in \mathcal{O}_0$, see [1].

4. We suppose $\sigma - 1 \in \mathbb{C} \setminus \mathbb{N}$. For a direction $\theta \in \mathbb{S}^1$, the microfunction $\overset{\triangledown}{\phi}_{\theta}= \operatorname{sing}^{\theta}_0 \left(\overset{\vee}{I}_\sigma\right)$, represented by the sectorial germ $\overset{\vee}{I}_\sigma(\zeta) = \dfrac{\zeta^{\sigma-1}}{(1-\mathrm{e}^{-2i\pi\sigma})\Gamma(\sigma)}$, is well-defined once the determination of the log has been chosen. Let us now fix the arc $I =]0, 2\pi[$, consider the arc $\check{I} =]-3\pi/2, 3\pi/2[$ as an arc of $\overset{\bullet}{\mathbb{S}}{}^1$ and $\overset{\vee}{I}_\sigma \in \Gamma(\check{I}, \mathcal{O}^0)$ as a (uniquely well-defined) multivalued section of $\mathcal{O}^0$ on $\check{I}$. One can apply to its associated microfunction $\overset{\triangledown}{I}_\sigma \in \mathscr{C}(I)$ the variation map and $\operatorname{var}(\overset{\triangledown}{I}_\sigma) = \widehat{I}_\sigma \in \Gamma(\hat{I}, \mathcal{O}^0)$, $\hat{I} =]\pi/2, 3\pi/2[$, is given by $\widehat{I}_\sigma(\zeta) = \dfrac{\zeta^{\sigma-1}}{\Gamma(\sigma)}$.

7.3.2 Convolution product of microfunctions

This subsection is devoted to convolution products of microfunctions. We start with some geometrical preliminaries.

7.3.2.1 Geometrical Preliminaries

Definition 7.7. For $\varepsilon > 0$, for an open sector $I \subset \mathbb{S}^1$ of aperture $< \pi$, we set $S_\varepsilon(\hat{I}) = \displaystyle\bigcup_{\eta \in \overset{\bullet}{\mathfrak{s}}{}^{\infty}_0(\hat{I})} D(\eta, \varepsilon)$, the "$\varepsilon$-neighbourhood" in $\mathbb{C}$ of the sector $\overset{\bullet}{\mathfrak{s}}{}^{\infty}_0(\hat{I})$. When the open arc I is of aperture $= \pi$, then $\hat{I} = \{\theta\}$ and we set $S_\varepsilon(\hat{I}) = \displaystyle\bigcup_{s \in \mathbb{R}^+} D(s\mathrm{e}^{\mathrm{i}\theta}, \varepsilon)$.

We set $\overset{\bullet}{\mathfrak{S}}_\varepsilon(I) = \mathbb{C} \setminus \overline{S}_\varepsilon(\hat{I})$ and we denote $-\partial\overset{\bullet}{\mathfrak{S}}_\varepsilon(I) = \partial S_\varepsilon(\hat{I})$ the oriented boundary.

We denote $\Gamma_{I,\varepsilon,\eta_1,\eta_2}$ the path that follows the oriented boundary $-\partial\overset{\bullet}{\mathfrak{S}}_\varepsilon(I)$ from η_1 to η_2. We set $\Gamma_{I,\varepsilon}$ the endless path that follows the oriented boundary $-\partial\overset{\bullet}{\mathfrak{S}}_\varepsilon(I)$.

Lemma 7.1. *We note $\zeta - \overline{S}_\varepsilon(\hat{I})$ the convex domain deduced from $\overline{S}_\varepsilon(\hat{I})$ by the point reflection centered on $\zeta/2 \in \mathbb{C}$. If $\operatorname{dist}(\zeta, S_\varepsilon(\hat{I})) \geq 2\varepsilon$, then*

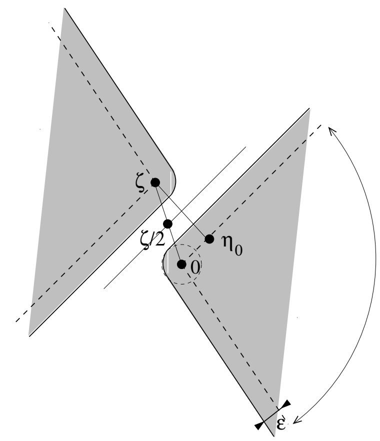


Fig. 7.2 The domain $S_\varepsilon(\hat{I})$ (left-hand side shaded domain), the domain $\zeta - \overline{S}_\varepsilon(\hat{I})$ (right-hand side shaded domain.

$\zeta - \overline{S}_\varepsilon(\hat{I}) \subset \dot{\mathfrak{S}}_\varepsilon(I)$. *In particular, for every* $\zeta \in \dot{\mathfrak{S}}_{2\varepsilon}(I)$, *for every* $\eta \in (-\partial \dot{\mathfrak{S}}_\varepsilon(I))$, *one has* $\zeta - \eta \in \dot{\mathfrak{S}}_\varepsilon(I)$.

Proof. We only consider the case where $I \subset \mathbb{S}^1$ is an open arc of aperture $< \pi$. We take an open sector $\dot{\mathfrak{s}}_0^\infty(\hat{I})$ and $\zeta \in \mathbb{C} \setminus \dot{\mathfrak{s}}_0^\infty(\hat{I})$. Then $\zeta/2 \in \mathbb{C} \setminus \dot{\mathfrak{s}}_0^\infty(\hat{I})$ as well. Denote by $\zeta - \dot{\mathfrak{s}}_0^\infty(\hat{I})$ the convex domain deduced from $\dot{\mathfrak{s}}_0^\infty(\hat{I})$ by the point reflection centered on $\zeta/2 \in \mathbb{C}$. One sees that for every $\xi \in \zeta - \dot{\mathfrak{s}}_0^\infty(\hat{I})$, for every $\eta \in \dot{\mathfrak{s}}_0^\infty(\hat{I})$, $\mathrm{dist}(\zeta, \dot{\mathfrak{s}}_0^\infty(\hat{I})) \leq \mathrm{dist}(\xi, \eta)$ (dist is the euclidean distance). Indeed, by the projection theorem for convex sets, there exist a unique point η_0 on the closure of $\dot{\mathfrak{s}}_0^\infty(\hat{I})$ so that $\mathrm{dist}(\zeta, \eta_0) = \mathrm{dist}\big(\zeta, \dot{\mathfrak{s}}_0^\infty(\hat{I})\big)$, see Fig. 7.2. One easily shows that the perpendicular bisector of the segment $[\zeta, \eta_0]$ separates the two convex sets $\dot{\mathfrak{s}}_0^\infty(\hat{I})$ and $\zeta - \dot{\mathfrak{s}}_0^\infty(\hat{I})$. Therefore, if $\mathrm{dist}(\zeta, S_\varepsilon(\hat{I})) \geq 2\varepsilon$, then $\zeta - \overline{S}_\varepsilon(\hat{I}) \subset \dot{\mathfrak{S}}_\varepsilon(I)$. $\square$

Lemma 7.2. *Let* $I =]\alpha, \beta[\subset \mathbb{S}^1$ *be an open sector of aperture* $\leq \pi$ *and* $\varepsilon > 0$. *We consider* $\eta_1 \in (-\partial \dot{\mathfrak{S}}_\varepsilon(I))$ *and we note* $r = |\eta_1|$. *We suppose that* $(\varepsilon/r) < 1$ *and we set* $\delta = \arcsin(\varepsilon/r) \in]0, \pi/2[$.

1. *if* $\hat{J} =]\beta - \pi/2, \alpha + \pi/2 + \delta[$ *is an open sector of aperture* $< \pi$, *we set* $h = r\sin(\hat{J})$. *Then, for any* $\zeta \in D(0,h)$, $\zeta - \eta_1 \in \dot{\mathfrak{s}}_0^\infty(\check{I})$.

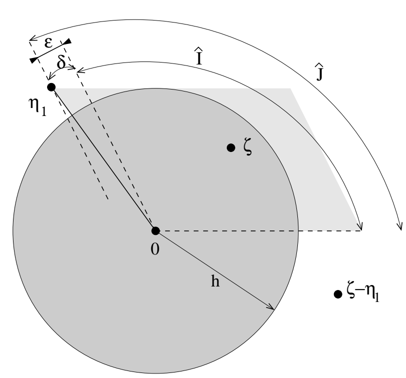


Fig. 7.3 Picture associated with the proof of lemma 7.2.

2. *if* $\hat{J} =]\beta - \pi/2, \alpha + \pi/2 + \delta[$ *is an open sector of aperture* $\leq \pi/2$*, then, for any* $\zeta \in D(0,r)$*,* $\zeta - \eta_1 \in \overset{\bullet}{\mathfrak{s}}{}_0^{\infty}(\check{I})$*.*

Proof. Left as an easy exercise. Just look at Fig. 7.3. □

7.3.2.2 Convolution product of microfunctions

We take two microfunctions $\overset{\triangledown}{\varphi}$ and $\overset{\triangledown}{\psi}$ of codirection I, where I is an open arc of aperture $< \pi$. For any strict subarc $I_1 \subset I$, these microfunctions can be represented by functions $\overset{\vee}{\varphi}$ and $\overset{\vee}{\psi}$ belonging to $\mathcal{O}\big(\overset{\bullet}{\mathfrak{s}}{}_0^{R+r}(\check{I}_1)\big)$ with $R > r > 0$ small enough. In what follows, we take $\varepsilon \in]0, \frac{r}{2}\sin(\pi - |\hat{I}|)[$.
We remark that both $\overset{\bullet}{\mathfrak{S}}_{2\varepsilon}(I) \cap D(0,r)$ and $\overset{\bullet}{\mathfrak{S}}_{\varepsilon}(I_1) \cap D(0,R)$ are non empty domains and that $\overset{\bullet}{\mathfrak{S}}_{\varepsilon}(I_1) \cap D(0,R) \subset \overset{\bullet}{\mathfrak{s}}{}_0^{R+r}(\check{I}_1)$.
We consider a path $\Gamma = \Gamma_{I_1,\varepsilon,\eta_1,\eta_2}$ that follows the oriented boundary $-\partial\overset{\bullet}{\mathfrak{S}}_{\varepsilon}(I_1)$ from η_1 to η_2 with $r < |\eta_1| < R$, $r < |\eta_2| < R$, drawn on Fig. 7.4.

For any $\eta \in \Gamma_{I_1,\varepsilon,\eta_1,\eta_2}$ and any $\zeta \in \overset{\bullet}{\mathfrak{S}}_{2\varepsilon}(I) \cap D(0,r)$, $|\zeta - \eta| < R + r$ and we know by lemma 7.1 that $\zeta - \eta \in \overset{\bullet}{\mathfrak{S}}_{\varepsilon}(I)$. Therefore, the function

$$\overset{\vee}{\varphi} *_{\Gamma} \overset{\vee}{\psi}(\zeta) = \int_{\Gamma_{I_1,\varepsilon,\eta_1,\eta_2}} \overset{\vee}{\varphi}(\eta)\, \overset{\vee}{\psi}(\zeta - \eta)\, d\eta \tag{7.3}$$

is well-defined for all $\zeta \in \overset{\bullet}{\mathfrak{S}}_{2\varepsilon}(I) \cap D(0,r)$ and is holomorphic on this domain (which is non empty since $2\varepsilon < r$).

Notice that $\overset{\vee}{\varphi} *_{\Gamma} \overset{\vee}{\psi}$ can be analytically continued to $\overset{\bullet}{\mathfrak{S}}_{2\varepsilon}(I) \cup D(0,r)$ when $\overset{\vee}{\psi}$ is holomorphic on $D(0, R+r)$, because $|\zeta - \eta| < R + r$ for η on the integration contour and $\zeta \in D(0,r)$. Thus, by linearity, adding to $\overset{\vee}{\psi}$ an element of $\mathcal{O}\big(D(0,R+r)\big)$ results in the addition of an element of $\mathcal{O}\big(D(0,r)\big)$ for $\overset{\vee}{\varphi} *_{\Gamma} \overset{\vee}{\psi}$. Similarly when $\overset{\vee}{\varphi}$ is holomorphic on $D(0,R+r)$, then $\overset{\vee}{\varphi} *_{\Gamma} \overset{\vee}{\psi}$ can be analytically continued to $\overset{\bullet}{\mathfrak{S}}_{2\varepsilon}(I) \cup D(0,r)$: by an homotopy in $D(0,R)$, just deform the contour $\Gamma_{I_1,\varepsilon,\eta_1,\eta_2}$ into an arc Γ' running

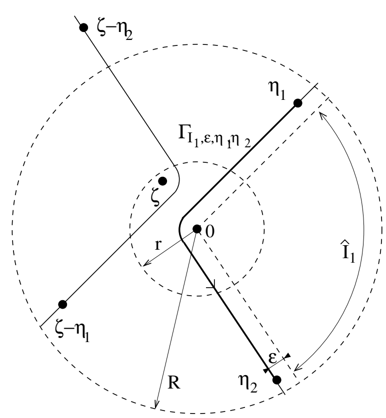


Fig. 7.4 The path of integration $\Gamma_{I_1,\varepsilon,\eta_1,\eta_2}$.

from η_1 to η_2 in $\{\eta = s e^{i\theta} \mid s \in]r,R[,\ \theta \in \hat{\check{I}}\} \subset \overline{S}_\varepsilon(\hat{I})$; by Cauchy, the two functions $\int_{\Gamma_{I_1,\varepsilon,\eta_1,\eta_2}} \overset{\vee}{\varphi}(\eta) \overset{\vee}{\psi}(\zeta-\eta)d\eta$ and $\int_{\Gamma'} \overset{\vee}{\varphi}(\eta) \overset{\vee}{\psi}(\zeta-\eta)d\eta$ coincide for $\zeta \in \overset{\bullet}{\mathfrak{S}}_{2\varepsilon}(I) \cap D(0,r)$, while the second integral is holomorphic on $D(0,r)$.

Replacing η_1, η_2 by η_1', η_2' on $-\partial\overset{\bullet}{\mathfrak{S}}_\varepsilon(I_1)$, with $r < |\eta_1'| < R$, $r < |\eta_2'| < R$, results in modifying $\overset{\vee}{\chi}_{I_1,\varepsilon,\eta_1,\eta_2}$ by an element of $\mathcal{O}\big(D(0,h)\big)$ for $h > 0$ small enough : writing $\Gamma' = \Gamma_{I_1,\varepsilon,\eta_1',\eta_2'}$, the difference

$$\overset{\vee}{\varphi} *_\Gamma \overset{\vee}{\psi}(\zeta) - \overset{\vee}{\varphi} *_{\Gamma'} \overset{\vee}{\psi}(\zeta) = \left(\int_{\eta_1}^{\eta_1'} + \int_{\eta_2'}^{\eta_2}\right) \overset{\vee}{\varphi}(\eta) \overset{\vee}{\psi}(\zeta-\eta)d\eta \tag{7.4}$$

can be analytically continued from $\overset{\bullet}{\mathfrak{S}}_{2\varepsilon}(I) \cap D(0,r)$ to $D(0,h)$. Indeed, using the condition on ε and by lemma 7.2, we see that for η on the two segment contours and for $\zeta \in D(0,h)$ with $0 < h \le r\sin(\hat{I})$, $\zeta - \eta$ remains in $\overset{\bullet}{\mathfrak{s}}{}_0^\infty(\check{I}_1) \cap D(0,R+r)$ where $\overset{\vee}{\psi}$ is holomorphic.

Finally replacing ε by a another $\varepsilon' \in]0, \frac{r}{2}\sin(\pi - |\hat{I}|)[$ yields the same conclusion : for ζ on the intersection domain $\overset{\bullet}{\mathfrak{S}}_{2\varepsilon}(I) \cap \overset{\bullet}{\mathfrak{S}}_{2\varepsilon'}(I) \cap D(0,r)$, one can compare the two functions $\overset{\vee}{\varphi} *_\Gamma \overset{\vee}{\psi}$ and $\overset{\vee}{\varphi} *_{\Gamma'} \overset{\vee}{\psi}$, $\Gamma' = \Gamma_{I_1,\varepsilon',\eta_1',\eta_2'}$. By Cauchy, the difference reads like (7.4) with the same conclusion.

In particular, we can let $\varepsilon \to 0$ in the above construction: the family of functions $\overset{\vee}{\varphi} *_\Gamma \overset{\vee}{\psi}(\zeta)$ glue together modulo the elements of $\mathcal{O}_0$, thus providing a microfunction of codirection I_1. Making the arcs $I_1 \subset I$ recovering I, one sees that these microfunctions glue together to give a microfunction of codirection I.

Definition 7.8. Let be I an open arc I of aperture $< \pi$. We consider two microfunctions of codirection I, $\overset{\triangledown}{\varphi}$ and $\overset{\triangledown}{\psi}$, represented by the sectorial germ of codirection I, $\overset{\vee}{\varphi}$ and $\overset{\vee}{\psi}$ respectively. For a covering of I by open arcs $I_1 \subset I$, the family of functions $\overset{\vee}{\varphi} *_\Gamma \overset{\vee}{\psi}(\zeta)$ defined by (7.3) with $\Gamma = \Gamma_{I_1,\varepsilon,\eta_1,\eta_2}$, glue together modulo $\mathcal{O}_0$ and provide a microfunction of codirection I denoted by $\overset{\triangledown}{\varphi} * \overset{\triangledown}{\psi}$. It is called the convolution product of $\overset{\triangledown}{\varphi}$ and $\overset{\triangledown}{\psi}$.

Proposition 7.1. *The sheaf of microfunctions $\mathscr{C}$ is a sheaf of $\mathbb{C}$-differential convolution algebras, for the derivation $\partial : \mathrm{sing}_0^I(\overset{\vee}{\psi}) \mapsto \mathrm{sing}_0^I(-\zeta \overset{\vee}{\psi})$. These algebras are commutative, associative and with unit $\delta = \mathrm{sing}_0\left(\frac{1}{2i\pi}\frac{1}{\zeta}\right)$.*

Proof. In what follows we use the previous notations : $\overset{\triangledown}{\varphi}$ and $\overset{\triangledown}{\psi}$ are the microfunctions of codirection I, an open arc of aperture $< \pi$. One takes a subarc $I_1 \subset I$ and the microfunctions can be represented by functions $\overset{\vee}{\varphi}$ and $\overset{\vee}{\psi}$ belonging to $\mathcal{O}\big(\overset{\bullet}{\mathfrak{s}}{}_0^{R+r}(\check{I}_1)\big)$ with $R > r > 0$ small enough.

We consider the microfunction $\overset{\triangledown}{\psi}_0 = \delta \in \mathscr{C}(\mathbb{S}^1)$ that we represent by $\overset{\vee}{\psi}_0(\zeta) = \widehat{\varphi}_0(\zeta) \overset{\vee}{I}_0(\zeta) = \frac{\widehat{\varphi}_0(\zeta)}{2i\pi\zeta}$ with $\widehat{\varphi}_0 \in \mathcal{O}\big(D(0,R+r)\big)$ subject to the condition $\widehat{\varphi}_0(0) = 1$. Thus $\overset{\vee}{\varphi} *_\Gamma \overset{\vee}{\psi}_0$ reads:

$$\overset{\vee}{\varphi} *_{\Gamma} \overset{\vee}{\psi}_0 (\zeta) = \frac{1}{2i\pi} \int_{\Gamma_{I_1,\varepsilon,\eta_1,\eta_2}} \overset{\vee}{\varphi} (\eta) \frac{\widehat{\varphi}_0(\zeta-\eta)}{\zeta-\eta} d\eta.$$

By Cauchy and the residue formula, one easily gets that for all $\zeta \in \overset{\bullet}{\mathfrak{s}}{}_0^{R+r}(\check{I}_1) \cap D(0,r)$, $\overset{\vee}{\varphi} *_{\Gamma} \overset{\vee}{\psi}_0 = \overset{\vee}{\varphi}$ +hol, where hol can be analytically continued to $D(0,r)$. This implies that $\overset{\triangledown}{\varphi} * \delta = \overset{\triangledown}{\varphi}$.

We now consider the integral:

$$\overset{\vee}{\varphi} *_{\Gamma\times\Gamma'} \overset{\vee}{\psi} (\zeta) = \frac{1}{2i\pi} \int_{\Gamma\times\Gamma'} \frac{\widehat{\varphi}_0(\zeta-(\xi_1+\xi_2))}{\zeta-(\xi_1+\xi_2)} \overset{\vee}{\varphi} (\xi_1) \overset{\vee}{\psi} (\xi_2) d\xi_1 d\xi_2, \quad (7.5)$$
$$\widehat{\varphi}_0 \in \mathcal{O}\big(D(0,R+r)\big), \quad \widehat{\varphi}_0(0) = 1,$$

where $\Gamma = \Gamma_{I_1,\varepsilon,\eta_1,\eta_2}$, $\Gamma' = \Gamma_{I_1,\varepsilon',\eta'_1,\eta'_2}$. We remark that for any $(\xi_1,\xi_2) \in \Gamma\times\Gamma'$ one has $(\xi_1+\xi_2) \in \overline{S}_{\varepsilon+\varepsilon'}(\hat{I}_1) \cap D(0,2R)$. Thus $\overset{\vee}{\varphi} *_{\Gamma\times\Gamma'} \overset{\vee}{\psi}$ defines a holomorphic function on the simply connected domain $\overset{\bullet}{\mathfrak{S}}_{\varepsilon+\varepsilon'}(I_1)$: just apply the Lebesgue dominated convergence theorem for ζ on any connected compact subset of $\overset{\bullet}{\mathfrak{S}}_{\varepsilon+\varepsilon'}(I_1)$. This also allows to use the Fubini theorem:

$$\begin{aligned}\overset{\vee}{\varphi} *_{\Gamma\times\Gamma'} \overset{\vee}{\psi} (\zeta) &= \int_{\Gamma} \left(\frac{1}{2i\pi} \int_{\Gamma'} \frac{\widehat{\varphi}_0(\zeta-(\xi_1+\xi_2))}{\zeta-(\xi_1+\xi_2)} \overset{\vee}{\psi} (\xi_2) d\xi_2 \right) \overset{\vee}{\varphi} (\xi_1) d\xi_1 \\ &= \int_{\Gamma'} \left(\frac{1}{2i\pi} \int_{\Gamma} \frac{\widehat{\varphi}_0(\zeta-(\xi_1+\xi_2))}{\zeta-(\xi_1+\xi_2)} \overset{\vee}{\varphi} (\xi_1) d\xi_1 \right) \overset{\vee}{\psi} (\xi_2) d\xi_2.\end{aligned}$$

From the previous considerations, we recognize $\overset{\vee}{\varphi} *_{\Gamma\times\Gamma'} \overset{\vee}{\psi} = \overset{\vee}{\varphi} *_{\Gamma} \overset{\vee}{\psi}$ +hol for the first equality, $\overset{\vee}{\varphi} *_{\Gamma\times\Gamma'} \overset{\vee}{\psi} = \overset{\vee}{\psi} *_{\Gamma'} \overset{\vee}{\varphi}$ +hol for the second equality, where hol is a holomorphic function that can be analytically continued to a neighbourhood of 0. As a consequence,

$$\overset{\triangledown}{\varphi} * \overset{\triangledown}{\psi} = \overset{\triangledown}{\psi} * \overset{\triangledown}{\varphi},$$

that is the convolution product of microfunctions is commutative. One easily shows in the same way that the convolution product of microfunctions is associative. The fact that ∂ is a derivation is obvious. □

We have previously seen two kind of integral representations, $\overset{\vee}{\varphi} *_{\Gamma} \overset{\vee}{\psi}$ (equation (7.3)) and $\overset{\vee}{\varphi} *_{\Gamma\times\Gamma'} \overset{\vee}{\psi}$ (equation (7.5)) for the convolution product $\overset{\triangledown}{\varphi} * \overset{\triangledown}{\psi}$ of two microfunctions. Other representations can be obtained under convenient hypotheses as exemplified by the next proposition.

Proposition 7.2. *One considers a $\overset{\triangledown}{\psi}$ a microfunction of codirection I, an open arc I of aperture $< \pi$, represented by the sectorial germ $\overset{\vee}{\psi}$ of codirection I. Let be $\overset{\triangledown}{\varphi} \in \Gamma(\mathbb{S}^1, \mathscr{C})$ a microfunction of the form $\mathrm{sing}_0\left(\widehat{\varphi}\frac{\log}{2i\pi}\right)$ with $\widehat{\varphi} \in \mathcal{O}_0$. Then, the microfunction $\overset{\triangledown}{\varphi} * \overset{\triangledown}{\psi}$ of codirection I can be represented modulo $\mathcal{O}_0$ by a family of functions of the form*

$$\int_0^{\eta_1} \widehat{\varphi}(\eta) \overset{\vee}{\psi} (\zeta-\eta) d\eta \quad \textit{and} \quad \int_0^{\eta_2} \widehat{\varphi}(\eta) \overset{\vee}{\psi} (\zeta-\eta) d\eta \qquad (7.6)$$

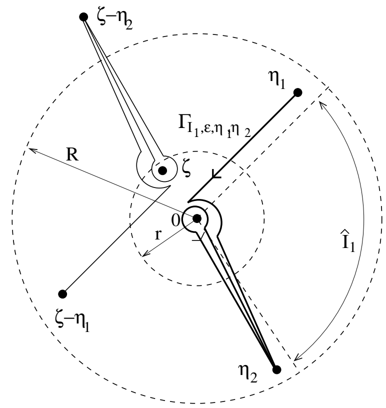


Fig. 7.5 Decomposition of the path $\Gamma_{I_1,\varepsilon,\eta_1,\eta_2}$.

with η_1, η_2 *as for definition 7.8.*

The proof is left as an exercise. (See [23]). Starting with the integral representation (7.3), the idea is to decompose the path $\Gamma_{I_1,\varepsilon,\eta_1,\eta_2}$ as on Fig. 7.5 and to use the integrability of the log at the origin.

7.4 Space of singularities

We now turn to classical notions and notations in resurgence theory [9, 10, 24, 23, 20].

7.4.1 Singularities

Definition 7.9. For $\theta \in \mathbb{R}$ and $\alpha > 0$, we denote by $\mathrm{ANA}_{\theta,\alpha}$ the space of sections $\Gamma(\check{J}, \mathcal{O}^0)$, where $\check{J} =]\theta - \alpha - 2\pi, \theta + \alpha[\subset \underset{\bullet}{\mathbb{S}}^1$, and $\mathrm{ANA} = \Gamma(\underset{\bullet}{\mathbb{S}}^1, \mathcal{O}^0)$.

Thus ANA is the space of sectorial germs at 0 that are represented by functions $\overset{\vee}{\varphi}$ holomorphic on a simply connected domain of the form $\mathfrak{s}_0(\underset{\bullet}{\mathbb{S}}^1)$.

Definition 7.10. One defines $\mathrm{SING}_{\theta,\alpha} = \mathrm{ANA}_{\theta,\alpha}/\mathcal{O}_0$ and $\mathrm{SING} = \mathrm{ANA}/\mathcal{O}_0$. The elements of these quotient spaces are called **singularities** at 0. One denotes by sing_0 the canonical projection,

$$\mathrm{sing}_0 : \begin{cases} \mathrm{ANA} \to \mathrm{SING} \\ \overset{\vee}{\varphi} \quad\;\; \mapsto \overset{\triangledown}{\varphi} \end{cases}, \qquad \mathrm{sing}_0 : \begin{cases} \mathrm{ANA}_{\theta,\alpha} \to \mathrm{SING}_{\theta,\alpha} \\ \overset{\vee}{\varphi} \qquad\quad \mapsto \overset{\triangledown}{\varphi} \end{cases}.$$

If $\mathrm{sing}_0(\overset{\vee}{\varphi}) = \overset{\triangledown}{\varphi}$, then $\overset{\vee}{\varphi}$ is called a **major** of the singularity $\overset{\triangledown}{\varphi}$.

In particular, with these notations:

Proposition 7.3. *The space of singularities* $\mathrm{SING}_{\theta,\alpha}$ *can be identified with the space* $\Gamma(J, \mathscr{C})$ *of multivalued sections of* $\mathscr{C}$ *by* $\dot{\pi}$, *with* $J =]-\frac{\pi}{2} - \alpha + \theta, \theta + \alpha + \frac{\pi}{2}[$.

Notice that $\mathrm{SING}_{\theta,\alpha}$ and SING are naturally $\mathcal{O}_0$-modules.

Definition 7.11. One defines the spaces SING_ω, resp. $\mathrm{SING}_{\omega,\theta,\alpha}$ of singularities at $\omega \in \mathbb{C}$, by translation from SING, resp. $\overline{\mathrm{SING}}_{\theta,\alpha}$.

It is of course enough to study the spaces of singularities at O and this is what we do in what follows.

Definition 7.12. For $f \in \mathcal{O}_0$ and $\overset{\triangledown}{\varphi} = \mathrm{sing}_0 \overset{\vee}{\varphi}$ in SING or $\mathrm{SING}_{\theta,\alpha}$, one defines the product $f \overset{\triangledown}{\varphi}$ by $f \overset{\triangledown}{\varphi} = \mathrm{sing}_0(f \overset{\vee}{\varphi})$.

Definition 7.13. One defines the **variation map** by

$$\mathrm{var} : \begin{cases} \mathrm{SING} & \to \mathrm{ANA} \\ \overset{\triangledown}{\varphi} = \mathrm{sing}_0(\overset{\vee}{\varphi}) & \mapsto \widehat{\varphi}, \quad \widehat{\varphi}(\zeta) = \overset{\vee}{\varphi}(\zeta) - \overset{\vee}{\varphi}(\zeta \mathrm{e}^{-2i\pi}) \end{cases}$$

and $\widehat{\varphi} = \mathrm{var}(\overset{\triangledown}{\varphi})$ is called the **minor** of the singularity $\overset{\triangledown}{\varphi}$.
The variation map var operates similarly on every element $\overset{\triangledown}{\varphi} \in \mathrm{SING}_{\theta,\alpha}$, with $\widehat{\varphi} = \mathrm{var}(\overset{\triangledown}{\varphi})$ in $\Gamma(\hat{J}, \mathcal{O}^0)$, where $\hat{J} =]\theta - \alpha, \theta + \alpha[\subset \underset{\bullet}{\mathbb{S}^1}$.
A minor is said to be **regular** when it belongs to $\mathcal{O}_0$.

We illustrate the notion of singularities by the following examples. (The reader will recognize sectorial germs used in the introduction of this chapter).

Definition 7.14. The singularities $\overset{\triangledown}{I}_\sigma, \overset{\triangledown}{J}_{\sigma,m} \in \mathrm{SING}$, $\sigma \in \mathbb{C}$, $m \in \mathbb{N}$ are defined as follows.

- For $\sigma \in \mathbb{C} \setminus \mathbb{N}^\star$, $\overset{\triangledown}{I}_\sigma = \mathrm{sing}_0(\overset{\vee}{I}_\sigma)$ where $\overset{\vee}{I}_\sigma(\zeta) = \frac{\zeta^{\sigma-1}}{(1-\mathrm{e}^{-2i\pi\sigma})\Gamma(\sigma)}$.
  In particular, $\overset{\triangledown}{I}_{-n} = \delta^{(n)} = \mathrm{sing}_0\left(\dfrac{(-1)^n}{2i\pi}\dfrac{n!}{\zeta^{n+1}}\right), \quad n \in \mathbb{N}$.
- For $n \in \mathbb{N}^\star$, $\overset{\triangledown}{I}_n = \mathrm{sing}_0(\overset{\vee}{I}_n)$ with $\overset{\vee}{I}_n(\zeta) = \frac{\zeta^{n-1}\log(\zeta)}{2i\pi\Gamma(n)}$.
- For $m \in \mathbb{N}$ and $\sigma \in \mathbb{C}$, $\overset{\triangledown}{J}_{\sigma,m} = \left(\frac{\partial}{\partial\sigma}\right)^m \overset{\triangledown}{I}_\sigma$.

It is useful to define the following subspaces of “integrable singularities”, $\mathrm{SING}^{\mathrm{int}} \subset \mathrm{SING}$ and $\mathrm{SING}^{\mathrm{int}}_{\theta,\alpha} \subset \mathrm{SING}_{\theta,\alpha}$.

Definition 7.15. An **integrable minor** is a germ $\widehat{\varphi} \in \mathrm{ANA}$ holomorphic in the domain $\mathfrak{s}_0(\underset{\bullet}{\mathbb{S}^1}) \subset \underset{\bullet}{\mathbb{C}}$ which has a primitive $\widehat{\phi}$ such that $\widehat{\phi} \to 0$ uniformaly in any proper subsector $\overline{\mathfrak{s}}_0 \Subset \mathfrak{s}_0(\underset{\bullet}{\mathbb{S}^1})$. The space of integrable minors is denoted by $\mathrm{ANA}^{\mathrm{int}}$.
An **integrable singularity** is a singularity $\overset{\triangledown}{\varphi} \in \mathrm{SING}$ which admits a major $\overset{\vee}{\varphi}$ holomorphic in the domain $\mathfrak{s}_0(\underset{\bullet}{\mathbb{S}^1}) \subset \mathfrak{s}_0(\underset{\bullet}{\mathbb{S}^1})$ such that $\lim\limits_{\zeta \to 0} \zeta \overset{\vee}{\varphi}(\zeta) = 0$ uniformaly in any proper subsector $\overline{\mathfrak{s}}_0 \Subset \mathfrak{s}_0(\underset{\bullet}{\mathbb{S}^1})$. One denotes by $\mathrm{SING}^{\mathrm{int}}$ the space of integrable singularities.

There is a natural injection $\mathcal{O}_0 \hookrightarrow \mathrm{ANA}^{\mathrm{int}}$ from the space of germs of holomorphic functions to the space $\mathrm{ANA}^{\mathrm{int}}$ of integrable minors. The space $\mathrm{ANA}^{\mathrm{int}}$ can be equipped with a convolution product, by extending the usual law convolution on $\mathcal{O}_0$.

It is not hard to show that integrable singularities satisfy the following property:

Proposition 7.4. *By restriction, the variation map* var *induces a linear isomorphism* $\mathrm{SING}^{\mathrm{int}} \to \mathrm{ANA}^{\mathrm{int}}$*. The inverse map is denoted by* $^\flat : \widehat{\varphi} \in \mathrm{ANA}^{\mathrm{int}} \mapsto {}^\flat\widehat{\varphi} \in \mathrm{SING}^{\mathrm{int}}$*.*

This allows to transports the convolution law from $\mathrm{ANA}^{\mathrm{int}}$ to $\mathrm{SING}^{\mathrm{int}}$ by the variation map.

Definition 7.16. The convolution product of $\widehat{\varphi}_1, \widehat{\varphi}_2 \in \mathrm{ANA}^{\mathrm{int}}$ is defined by $\widehat{\varphi}_1 * \widehat{\varphi}_2(\zeta) = \int_0^\zeta \widehat{\varphi}_1(\eta)\widehat{\varphi}_1(\zeta-\eta)d\eta$.
The convolution of two integrable singularities $\overset{\triangledown}{\varphi}_1 = {}^\flat\widehat{\varphi}_1, \overset{\triangledown}{\varphi}_2 = {}^\flat\widehat{\varphi}_2 \in \mathrm{SING}^{\mathrm{int}}$ is given by : $\overset{\triangledown}{\varphi}_1 * \overset{\triangledown}{\varphi}_2 = {}^\flat\big(\widehat{\varphi}_1 * \widehat{\varphi}_2\big)$.

Quite similarly:

Definition 7.17. A minor $\widehat{\varphi}$ holomorphic in the domain $\mathfrak{s}_0(\hat{I}) \subset \underset{\bullet}{\mathbb{C}}$ is said to be integrable if $\widehat{\varphi}$ has a primitive $\widehat{\phi}$ such that $\widehat{\phi} \to 0$ uniformaly in any proper subsector $\bar{\mathfrak{s}}_0 \Subset \mathfrak{s}_0(\hat{I})$. One denotes by $\mathrm{ANA}^{\mathrm{int}}_{\theta,\alpha}$ the space of these integrable minors.
An integrable singularity is a singularity $\overset{\triangledown}{\varphi} \in \mathrm{SING}_{\theta,\alpha}$ which has a major $\overset{\vee}{\varphi}$ holomorphic in the domain $\mathfrak{s}_0(\check{I}) \subset \underset{\bullet}{\mathbb{C}}$ and such that $\lim\limits_{\zeta\to 0} \zeta \overset{\vee}{\varphi}(\zeta) = 0$ uniformaly in any proper subsector $\bar{\mathfrak{s}}_0 \Subset \mathfrak{s}_0(\check{I})$. One denotes $\mathrm{SING}^{\mathrm{int}}_{\theta,\alpha}$ the space of these integrable singularities.

Proposition 7.5. *By restriction, the variation map* var *induces a linear isomorphism* $\mathrm{SING}^{\mathrm{int}}_{\theta,\alpha} \to \mathrm{ANA}^{\mathrm{int}}_{\theta,\alpha}$*.*
The inverse map is denoted by $^\flat : \widehat{\varphi} \in \mathrm{ANA}^{\mathrm{int}}_{\theta,\alpha} \mapsto {}^\flat\widehat{\varphi} \in \mathrm{SING}^{\mathrm{int}}_{\theta,\alpha}$*.*

We end with further definitions.

Definition 7.18. Any singularity $\overset{\triangledown}{\varphi}$ of the form $\overset{\triangledown}{\varphi} = a\delta + {}^\flat\widehat{\varphi}$ with $\widehat{\varphi} \in \mathcal{O}_0$ is said to be **simple**. The space of simple singularities is denoted by $\mathrm{SING}^{\mathrm{simp}}$. The space $\mathrm{SING}^{\mathrm{s.ram}}$ of **simply ramified** singularities is the vector space spanned by $\mathrm{SING}^{\mathrm{simp}}$ and the set of singularities $\{\overset{\triangledown}{I}_{-n},\, n \in \mathbb{N}\}$.

7.4.2 Convolution product of singularities at 0

The resurgence theory asserts that the space of singularities SING can be equipped with a convolution product [7, 8, 24], see also [1, 21]. Since $\mathrm{SING}_{\theta,\alpha}$ can be identified with the space $\Gamma(J,\mathscr{C})$ of multivalued sections of $\mathscr{C}$ by $\dot{\pi}$, with $J =]-\frac{\pi}{2}-\alpha+\theta, \theta+\alpha+\frac{\pi}{2}[$, the convolution product for microfunctions (proposition 7.1) allows to transport this product to $\mathrm{SING}_{\theta,\alpha}$: for any two singularities $\overset{\triangledown}{\varphi}, \overset{\triangledown}{\psi} \in \mathrm{SING}_{\theta,\alpha}$ and any strict subarc $I \subset J$ of aperture $= \pi$, one can find two majors $\overset{\vee}{\varphi}, \overset{\vee}{\psi} \in \mathrm{ANA}_{\theta,\alpha}$ that can be represented by holomorphic functions on a sector $\mathfrak{s}_0(\check{I})$. By projection on $\mathbb{C}$, one can think of $\overset{\vee}{\varphi}, \overset{\vee}{\psi}$ as belonging to $\mathcal{O}(\overset{\bullet}{\mathfrak{s}}_0(\check{I}))$, that is sectorial germs of codirection I. By restriction, $\overset{\triangledown}{\varphi}, \overset{\triangledown}{\psi}$ are seen as microfunctions of codirection I, whose convolution product $\overset{\triangledown}{\varphi} * \overset{\triangledown}{\psi} \in \Gamma(I,\mathscr{C})$ can be represented either by

$$\overset{\vee}{\varphi} *_{\Gamma} \overset{\vee}{\psi}(\zeta) = \int_{\Gamma} \overset{\vee}{\varphi}(\eta) \overset{\vee}{\psi}(\zeta - \eta) d\eta \tag{7.7}$$

or by

$$\overset{\vee}{\varphi} *_{\Gamma\times\Gamma} \overset{\vee}{\psi}(\zeta) = \frac{1}{2i\pi} \int_{\Gamma\times\Gamma} \frac{f(\zeta - (\xi_1+\xi_2))}{\zeta - (\xi_1+\xi_2)} \overset{\vee}{\varphi}(\xi_1) \overset{\vee}{\psi}(\xi_2) d\xi_1 d\xi_2, \tag{7.8}$$

with $f \in \mathcal{O}_0$ and $f(0) = 1$ (cf. (7.3) and (7.5)), where $\Gamma = \Gamma_{I,\varepsilon,\eta_1,\eta_2}$ is as in definition 7.7. Considering a covering of J by such intervals I, these sections glue together to give the convolution product $\overset{\triangledown}{\varphi} * \overset{\triangledown}{\psi}$ as a multivalued section of $\mathscr{C}$ over J.

Proposition 7.6. *The space* SING *can be equipped with a convolution product* $*$ *that makes it a commutative convolution algebra, with unit* $\delta = \mathrm{sing}_0\left(\dfrac{1}{2i\pi\zeta}\right) = \overset{\triangledown}{I}_0$. *Moreover:*

1. *the linear operator,* $\partial : \overset{\triangledown}{\varphi} = \mathrm{sing}_0(\overset{\vee}{\varphi}) \in \mathrm{SING} \mapsto \partial\, \overset{\triangledown}{\varphi} = \mathrm{sing}_0(-\zeta\, \overset{\vee}{\varphi}) \in \mathrm{SING}$, *is a derivation.*
2. *if* $\overset{\triangledown}{\varphi}$ *and* $\overset{\triangledown}{\psi}$ *belong to* $\mathrm{SING}^{\mathrm{int}}$, *then* $\overset{\triangledown}{\varphi} * \overset{\triangledown}{\psi}$ *belongs to* $\mathrm{SING}^{\mathrm{int}}$ *and* ${}^\flat\widehat{\varphi} * {}^\flat\widehat{\varphi} = {}^\flat(\widehat{\varphi} * \widehat{\varphi})$. *In particular, the space of simple singularities* $\mathrm{SING}^{\mathrm{simp}}$ *is a convolution subalgebra.*

Theses properties remain true when one considers $\mathrm{SING}_{\theta,\alpha}$ *instead of* SING.

Proof. We have already demonstrated that $\mathrm{SING}_{\theta,\alpha}$ (thus SING) is a commutative convolution algebra for the convolution product with unit δ. The equality ${}^\flat\widehat{\varphi} * {}^\flat\widehat{\varphi} = {}^\flat(\widehat{\varphi} * \widehat{\varphi})$ for integrable singularities, emerges from considerations on integrals and is left as an exercise. (Start with proposition 7.2. See [23]). □

7.5 Formal Laplace transform, formal Borel transform

7.5.1 Formal Laplace transform for microfunctions at 0

We start with the following definition.

Definition 7.19. For an open arc $I \subset \mathbb{S}^1$ and $r \geq 0$, we note:

1. $\overline{\mathcal{A}}^{\leq 0}(\overset{\bullet}{\mathfrak{s}}{}_r^{\infty}(I))$ the $\mathbb{C}$-differential algebra of holomorphic functions φ on $\overset{\bullet}{\mathfrak{s}}{}_r^{\infty}(I)$ that satisfy the property : for any proper subdomain $\overset{\bullet}{\mathfrak{s}}{}^{\infty} \Subset \overset{\bullet}{\mathfrak{s}}{}_r^{\infty}(I)$, for any $\varepsilon > 0$, there exists $C > 0$ so that, for all $z \in \overset{\bullet}{\mathfrak{s}}{}^{\infty}$, $|\varphi(z)| \leq C\mathrm{e}^{\varepsilon|z|}$;
2. we set $\overline{\mathcal{A}}^{\leq 0}(I) = \varinjlim_{r\to\infty} \overline{\mathcal{A}}^{\leq 0}(\overset{\bullet}{\mathfrak{s}}{}_r^{\infty}(I))$. This defines a presheaf $\overline{\mathcal{A}}^{\leq 0}$;
3. we denote by $\mathcal{A}^{\leq 0}$ the sheaf over $\mathbb{S}^1$ associated with the presheaf $\overline{\mathcal{A}}^{\leq 0}$.

Remark 7.2. The fact that $\mathcal{A}^{\leq 0}$ is indeed a sheaf of differential algebras is an exercise left to the reader. (We stress that the derivation considered is the usual one for holomorphic functions).

The sheaf $\mathcal{A}^{\leq 0}$ should not be confused with the sheaf $\mathcal{A}^{<0}$ of flat germs at infinity (definition 3.9). As a matter of fact, $\overline{\mathcal{A}}^{<0}(I) \subset \overline{\mathcal{A}}(I) \subset \overline{\mathcal{A}}^{\leq 0}(I)$ where $\overline{\mathcal{A}}$ stands for the presheaf of asymptotic functions (see definition 3.9 and [14, 16, 17]).
We mention that our definition of $\mathcal{A}^{\leq 0}$ differs from that of Malgrange in [16] where $\mathcal{A}^{\leq 0}$ is defined as the sheaf of sectorial germs that admit an asymptotics belonging to the formal Nilsson class, that is of the form $\sum \widetilde{w}(z)\frac{\log^m(z)}{z^\sigma}$, $\sigma \in \mathbb{C}$, $m \in \mathbb{N}$, $\widetilde{w} \in \mathbb{C}[[z^{-1}]]$. Our sheaf $\mathcal{A}^{\leq 0}$ contains this sheaf as a subsheaf. However, the constructions in the sequel resemble in much aspect to that of Malgrange [16].

The following Lemma is left to the reader as an exercise. This will allow us in a moment to properly define the quotient sheaf $\mathcal{A}^{\leq 0}/\mathcal{A}^{\leq -1}$ over $\mathbb{S}^1$.

Lemma 7.3. *The space $\overline{\mathcal{A}}^{\leq -1}(\overset{\bullet}{\mathfrak{s}}{}^\infty)$ –resp. $\overline{\mathcal{A}}^{\leq -1}(I)$– of 1-exponentially flat functions on $\overset{\bullet}{\mathfrak{s}}{}^\infty$ –resp. of 1-exponentially flat germs at infinity over I–, is a differential ideal of $\overline{\mathcal{A}}^{\leq 0}(\overset{\bullet}{\mathfrak{s}}{}^\infty(I))$ –resp. of $\overline{\mathcal{A}}^{\leq 0}$.*

Definition 7.20. For a direction θ (of $\mathbb{S}^1$ or $\underset{\bullet}{\mathbb{S}}^1$), we note R_θ the ray $]0, \mathrm{e}^{\mathrm{i}\theta}\infty[$.
For $\kappa > \varepsilon \geq 0$, we note $R_{\theta,\varepsilon} =]\varepsilon \mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{\mathrm{i}\theta}\infty[$ and $R_{\theta,\varepsilon;\kappa} =]\varepsilon\mathrm{e}^{\mathrm{i}\theta}, \kappa\mathrm{e}^{\mathrm{i}\theta}[$.
For a closed arc $\bar{J} = [\theta_1, \theta_2]$, we denote by $\gamma_{\bar{J},\varepsilon}$ (resp. $\gamma_{\bar{J},\varepsilon;\kappa}$), the Hankel contour (resp. truncated Hankel contour) which consists in following:

1. $R_{\theta_1,\varepsilon}$, resp. $R_{\theta_1,\varepsilon;\kappa}$, backward,
2. then the circular arc $\delta_{\bar{J},\varepsilon} = \{\varepsilon\mathrm{e}^{\mathrm{i}\theta} \mid \theta \in \bar{J}\}$ oriented in the anti-clockwise way,
3. finally $R_{\theta_2,\varepsilon}$, resp. $R_{\theta_2,\varepsilon;\kappa}$, forward.

We take an open arc I of $\mathbb{S}^1$ of aperture $\leq \pi$, and a microfunction $\overset{\triangledown}{\varphi} \in \mathscr{C}(I)$ of codirection I, represented by the germ $\overset{\vee}{\varphi} \in \check{\mathcal{O}}^0(I)$. For any open arc $I_1 =]\alpha_1, \beta_1[$ with $\bar{I}_1 \subset I$, one can find $R > 0$ so that the restriction of $\overset{\vee}{\varphi}$ to $\check{I}_1 =]\alpha_1 - 3\pi/2, \beta_1 - \pi/2[\subset \mathbb{S}^1$ is represented by a function (still denoted by $\overset{\vee}{\varphi}$) holomorphic in the sector $\mathfrak{s}_0^R(\check{I}_1)$. We take another open arc $I_2 =]\alpha_2, \beta_2[$, $\bar{I}_2 \subset I_1$, so that $\check{I}_1 \setminus \check{\bar{I}}_2$ has two connected components. We take one arbitrary direction in each component, $\theta_1 \in]\alpha_1 - 3\pi/2, \alpha_2 - 3\pi/2[$, $\theta_2 \in]\beta_2 - \pi/2, \beta_1 - \pi/2[$. For $R > \kappa > \varepsilon > 0$, we consider the truncated Laplace integral $\varphi_{\theta_1,\theta_2,\kappa}(z) = \displaystyle\int_{\gamma_{[\theta_1,\theta_2],\varepsilon;\kappa}} \mathrm{e}^{-z\zeta} \overset{\vee}{\varphi}(\zeta)d\zeta$, see Fig. 7.6.

The function $\varphi_{\theta_1,\theta_2,\kappa}$ satisfies the following properties:

- $\varphi_{\theta_1,\theta_2,\kappa}$ is an entire function, since one integrates on a (relatively) compact path of the domain of holomorphy of $\overset{\vee}{\varphi}$.
- for $\varepsilon > 0$ chosen as small as we want, set $M = \sup_{\bar{\mathfrak{s}}_\varepsilon^\kappa(]\theta_1,\theta_2[)} | \overset{\vee}{\varphi} |$ with $\bar{\mathfrak{s}}_\varepsilon^\kappa(]\theta_1,\theta_2[) = \{\zeta = \xi\mathrm{e}^{\mathrm{i}\theta} \mid \theta \in [\theta_1,\theta_2], \xi \in [\varepsilon,\kappa]\}$. then:
  - for all $z \in \mathbb{C}$, $\left| \displaystyle\int_{\delta_{[\theta_1,\theta_2],\varepsilon}} \mathrm{e}^{-z\zeta} \overset{\vee}{\varphi}(\zeta)d\zeta \right| \leq \varepsilon|\check{I}_1|M\mathrm{e}^{\varepsilon|z|}$ where $|\check{I}_1| = \beta_1 - \alpha_1 + \pi$;
  - we observe that for any $r > 0$, for every $z \in \overset{\bullet}{\Pi}{}_r^{\theta_1}$, $\left| \displaystyle\int_{R_{\theta_1,\varepsilon;\kappa}} \mathrm{e}^{-z\zeta} \overset{\vee}{\varphi}(\zeta)d\zeta \right| \leq \kappa M\mathrm{e}^{-\varepsilon r}$.

  Similarly, for every $z \in \overset{\bullet}{\Pi}{}_r^{\theta_2}$, $\left| \displaystyle\int_{R_{\theta_2,\varepsilon;\kappa}} \mathrm{e}^{-z\zeta} \overset{\vee}{\varphi}(\zeta)d\zeta \right| \leq \kappa M\mathrm{e}^{-\varepsilon r}$.

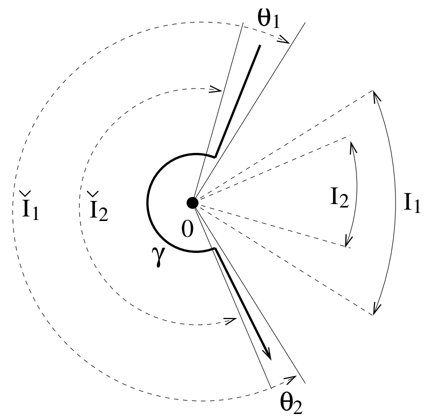


Fig. 7.6 Formal Laplace transform. The open arcs I_1, I_2, $\check{I}_1$, $\check{\check{I}}_2$, and the path $\gamma = \gamma_{[\theta_1,\theta_2],\varepsilon;\kappa}$.

– the domain $\overset{\bullet}{\Pi}{}^{\theta_1}_{r}$ contains any closed sector of the form $\overset{\bullet}{\mathfrak{s}}{}^{\infty}_{r'}(J_1)$ with J_1 an open arc so that $\bar{J}_1 \subset]-\frac{\pi}{2}-\theta_1, -\theta_1+\frac{\pi}{2}[$ and $r' > 0$ large enough. Since $\beta_2 - \frac{\pi}{2} < \theta_1 < \alpha_2 + \frac{\pi}{2}$, one deduces that $\overset{\bullet}{\Pi}{}^{\theta_1}_{r}$ contains any closed sector of the form $\overset{\bullet}{\mathfrak{s}}{}^{\infty}_{r'}(I_2^\star)$ with $r' > 0$ large enough. Similarly, $\overset{\bullet}{\Pi}{}^{\theta_2}_{r}$ contains any closed sector of the form $\overset{\bullet}{\mathfrak{s}}{}^{\infty}_{r'}(I_2^\star)$ with $r' > 0$ large enough.

From this analysis, since $\varepsilon > 0$ can be chosen arbitrarily small, we retain that $\varphi_{\theta_1,\theta_2,\kappa}$ belongs to the space $\overline{\mathcal{A}}^{\leq 0}(\overset{\bullet}{\mathfrak{s}}{}^{\infty}_{r}(I_2^\star))$, $r > 0$ large enough.

- Furthermore, looking at the above analysis and by Cauchy, we can observe that for two cut-off points $\kappa, \kappa' \in]\varepsilon, R[$, for two directions $\theta_1' \in]\alpha_1 - 3\pi/2, \alpha_2 - 3\pi/2[$, $\theta_2' \in]\beta_2 - \pi/2, \beta_1 - \pi/2[$ the difference $\varphi_{\theta_1,\theta_2,\kappa} - \varphi_{\theta_1',\theta_2',\kappa'}$ belongs to $\overline{\mathcal{A}}^{\leq -1}(\overset{\bullet}{\mathfrak{s}}{}^{\infty}_{r}(I_2^\star))$ with $r > 0$ large enough. We finally remark that adding to $\overset{\vee}{\varphi}$ a function holomorphic on $D(0,R)$ only affects $\varphi_{\theta_1,\theta_2,\kappa}(z)$ by the addition of an element of $\overline{\mathcal{A}}^{\leq -1}(\overset{\bullet}{\mathfrak{s}}{}^{\infty}_{r}(I_2^\star))$, $r > 0$ large enough.

One thus obtains a morphism , $\tilde{\mathcal{L}}(I, I_2) : \overset{\triangledown}{\varphi} \in \mathscr{C}(I) \mapsto \overset{\vartriangle}{\varphi} \in \overline{\mathcal{A}}^{\leq 0}(I_2^\star)/\overline{\mathcal{A}}^{\leq -1}(I_2^\star)$, $\widetilde{\varphi} = \mathrm{cl}(\varphi_{\theta_1,\theta_2,\kappa})$, which is obviosuly compatible with the restriction maps. This allows to move up to stalks, $\tilde{\mathcal{L}}_\alpha : \mathscr{C}_\alpha \to \left(\mathcal{A}^{\leq 0}/\mathcal{A}^{\leq -1}\right)_{\alpha^*}$ and finally[2] to a morphism of sheaves $\tilde{\mathcal{L}} : \mathscr{C} \to \mathcal{A}^{\leq 0}/\mathcal{A}^{\leq -1}$.

Definition 7.21. One calls **formal Laplace transform** for microfunctions at 0, the morphism of sheaves $\tilde{\mathcal{L}} : \mathscr{C} \to \mathcal{A}^{\leq 0}/\mathcal{A}^{\leq -1}$. The quotient sheaf $\mathcal{A}^{\leq 0}/\mathcal{A}^{\leq -1}$ over $\mathbb{S}^1$ is called the **sheaf of asymptotic classes**. An asymptotic class is usually denoted by $\overset{\vartriangle}{\varphi}$.

The term "sheaf of asymptotic classes" is borrowed from [1] where the sheaf $\mathcal{A}^{\leq 0}$ is denoted by $\mathcal{E}^0$, and the sheaf $\mathcal{A}^{\leq -1}$ is denoted by $\mathcal{E}^-$. The notation $\overset{\vartriangle}{\varphi}$ is own.

Example 7.3. For $(\sigma, m) \in \mathbb{C} \times \mathbb{N}$ and $I =]-\pi/2, \pi/2[\in \mathbb{S}^1$, we consider the microfunction $\overset{\triangledown}{J}_{\sigma,m} = \mathrm{sing}_0^I \left(\overset{\vee}{J}_{\sigma,m}\right) \in \mathscr{C}(I)$ represented by the sectorial germ $\overset{\vee}{J}_{\sigma,m} = \left(\dfrac{\partial}{\partial\sigma}\right) \overset{\vee}{I}_\sigma \in \check{\mathcal{O}}^0(I) = \mathcal{O}^0(\check{I})$, $\check{I} =]-2\pi, 0[$ and the branch of the log

[2] Modulo complex conjugation

that $\arg(\log\zeta) \in \check{I}$. By standard formulae recalled in Sect. 7.1, one readily gets that its formal Laplace transform $\overset{\vartriangle}{J}_{\sigma,m} = \tilde{\mathcal{L}}(I) \overset{\triangledown}{J}_{\sigma,m}$ is an asymptotic class that can be represented by the (sectorial germ at infinity of) holomorphic function(s) $(-1)^m \dfrac{\log^m(z)}{z^\sigma} \in \overline{\mathcal{A}}^{\leq 0}(I^\star)$, $I^\star =]-\pi/2, \pi/2[$ with the determination of the log so that $\arg(\log z) \in I^\star$.

The following proposition is a straight consequence of the very construction of the formal Laplace transform.

Proposition 7.7. *The formal Laplace transform satisfies the identity :* $\tilde{\mathcal{L}}\partial = \partial\tilde{\mathcal{L}}$.

7.5.2 Formal Borel transform for asymptotic classes

We take an open arc $I^\star$ of $\mathbb{S}^1$ of aperture $\leq \pi$, and a sectorial germ at infinity $\varphi \in \mathcal{A}^{\leq 0}(I^\star)$. For any open arc $I_1^\star$ with $\bar{I}_1^\star \subset I^\star$, one can find $r > 0$ so that the restriction of φ to $I_1^\star$ is (represented by) a holomorphic function (still denoted by φ) on the domain $\overset{\bullet}{\mathfrak{s}}{}_r^\infty(I_1^\star)$. For $z_1 \in \overset{\bullet}{\mathfrak{s}}{}_r^\infty(I_1^\star)$ and a direction $\alpha \in I_1^\star$, we note $\overset{\vee}{\varphi}_{z_1,\alpha}(\zeta) = -\dfrac{1}{2i\pi}\displaystyle\int_{R_{\alpha,z_1}} \mathrm{e}^{z\zeta}\varphi(z)dz$, see Fig. 7.7. We can make the following observations about this Laplace integral $\overset{\vee}{\varphi}_{z_1,\alpha}$:

- since $\varphi \in \overline{\mathcal{A}}^{\leq 0}(\overset{\bullet}{\mathfrak{s}}{}_r^\infty(I_1^\star))$, we know that for any proper subsector $\overline{\overset{\bullet}{\mathfrak{s}}}{}_{r_1}^\infty(J^\star) \Subset \overset{\bullet}{\mathfrak{s}}{}_r^\infty(I_1^\star)$, for any $\varepsilon > 0$, there exists $C > 0$ so that, for all $z \in \overline{\overset{\bullet}{\mathfrak{s}}}{}^\infty$, $|\varphi(z)| \leq C\mathrm{e}^{\varepsilon|z|}$. Assume that $z_1 \in \overline{\overset{\bullet}{\mathfrak{s}}}{}_{r_1}^\infty(J^\star)$ and take $\alpha \in \bar{J}^\star$. This implies that $\overset{\vee}{\varphi}_{z_1,\alpha}$ belongs to $\mathcal{O}(\overset{\bullet}{\Pi}{}_\varepsilon^{\alpha+\pi})$. Making α varying in $J^\star$ and since $\varepsilon > 0$ can be chosen arbitrarily small, these functions glue together by Cauchy, and provide a holomorphic function $\overset{\vee}{\varphi}_{z_1,J^\star}$ on $\overset{\bullet}{\mathscr{D}}(J^\star, 0) = \overset{\bullet}{\mathfrak{s}}{}_0^\infty(\check{J})$. Notice that for two points $z_1, z_2 \in \overline{\overset{\bullet}{\mathfrak{s}}}{}_{r_1}^\infty(J^\star)$, the difference $\overset{\vee}{\varphi}_{z_2,J^\star} - \overset{\vee}{\varphi}_{z_1,J^\star}$ defines an entire function (with at most exponential growth of order 1 at infinity). Therefore, localising near the origin, we get a sectorial germ $\overset{\vee}{\varphi}_{z_1,I^\star} \in \mathcal{O}(\check{I}) = \check{\mathcal{O}}(I)$, defined modulo the elements of $\mathcal{O}_0$, that is a microfunction of codirection I;
- when φ belongs to $\overline{\mathcal{A}}^{\leq -1}(I^\star)$, one easily sees from the above analysis that $\overset{\vee}{\varphi}_{z_1,I^\star}$ is holomorphic on a domain containing a full neighbourhood of the origin, thus by localisation, an element of $\mathcal{O}_0$.

To conclude, we have defined a morphism (of $\mathbb{C}$-differential algebras), $\tilde{\mathcal{B}}(I^\star) : \overset{\vartriangle}{\varphi} \in \mathcal{A}^{\leq 0}(I^\star)/\mathcal{A}^{\leq -1}(I^\star) \mapsto \overset{\triangledown}{\varphi} = \mathrm{cl}\,(\overset{\vee}{\varphi}_{z_1,I^\star}) \in \mathscr{C}(I)$ whose compatibility with the restriction maps is easy to check.

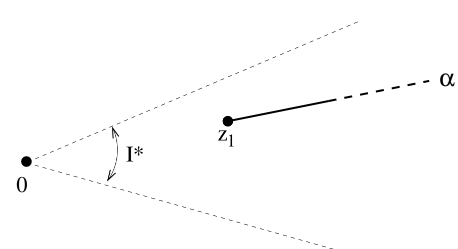


Fig. 7.7 Formal Borel transform. The open arcs $I^\star$, and the path R_{α,z_1}.

Definition 7.22. One calls **formal Borel transform** the morphism of sheaves $\tilde{\mathcal{B}} : \mathcal{A}^{\leq 0}/\mathcal{A}^{\leq -1} \to \mathscr{C}$.

The formal Laplace transform for microfunctions and the formal Borel transform for asymptotic classes are isomorphisms of sheaves, as shown in [1]:

Proposition 7.8. *The morphisms* $\tilde{\mathcal{L}} : \mathscr{C} \to \mathcal{A}^{\leq 0}/\mathcal{A}^{\leq -1}$ *and* $\tilde{\mathcal{B}} : \mathcal{A}^{\leq 0}/\mathcal{A}^{\leq -1} \to \mathscr{C}$ *are isomorphisms of sheaves and* $\tilde{\mathcal{L}} \circ \tilde{\mathcal{B}} = \mathrm{Id}$, $\tilde{\mathcal{B}} \circ \tilde{\mathcal{L}} = \mathrm{Id}$.

Remark 7.3. We have seen that we have an injective morphism of sheaves, $\widehat{\varphi} \in \mathcal{O}_0 \mapsto \overset{\nabla}{\varphi} = \mathrm{sing}_0^I \left(\widehat{\varphi} \frac{\log}{2i\pi}\right) \in \mathscr{C}(I)$, and the following commutative diagram makes a link between the formal Laplace transform for regular minor –resp. formal Borel transform for 1-Gevrey formal series– and the formal Laplace transform for microfunctions –resp. formal Borel transform for asymptotic classes:

$$\begin{array}{ccc} \mathcal{O}_0 & \hookrightarrow & \mathscr{C} \\ \tilde{\mathcal{L}} \downarrow\uparrow \tilde{\mathcal{B}} & & \tilde{\mathcal{B}} \uparrow\downarrow \tilde{\mathcal{L}} \\ \mathcal{A}_1/\mathcal{A}^{\leq -1} & \hookrightarrow & \mathcal{A}^{\leq 0}/\mathcal{A}^{\leq -1}. \end{array}$$

7.5.3 *Formal Laplace transform for singularities and back to convolution product*

In the sequel, we translate to singularities what have obtained so far for microfunctions.

7.5.3.1 Formal Laplace transform for singularities at 0

We start with two definitions.

Definition 7.23. Let be $\theta \in \mathbb{S}^1_\bullet$ and $\alpha > 0$. We denote by $\mathrm{ASYMP}_{\theta,\alpha}$ the space of asymptotic classes defined as multivalued sections of $\mathcal{A}^{\leq 0}/\mathcal{A}^{\leq -1}$ on $J^\star =]-\pi/2 - \alpha - \theta, -\theta + \alpha + \pi/2[$. We denote by ASYMP the space of asymptotic classes given by global sections of $\mathcal{A}^{\leq 0}/\mathcal{A}^{\leq -1}$ on $\mathbb{S}^1_\bullet$.

Definition 7.24. Let be $\sigma \in \mathbb{C}$ and $m \in \mathbb{N}$. We denote by $\overset{\triangle}{I}_\sigma \in$ ASYMP the asymptotic class represented by $1/z^\sigma$. We denote $\overset{\triangle}{J}_{\sigma,m} \in$ ASYMP the asymptotic class represented by $(-1)^m \dfrac{\log^m(z)}{z^\sigma}$. We often simply write $1/z^\sigma$ instead of $\overset{\triangle}{I}_\sigma$ and similarly for $\overset{\triangle}{J}_{\sigma,m}$.

We have already said that the space of singularities $\mathrm{SING}_{\theta,\alpha}$ can be identified with the space $\Gamma(J, \mathscr{C})$ of multivalued sections of $\mathscr{C}$ by $\dot{\pi}$, with $J =]-\frac{\pi}{2} - \alpha + \theta, \theta + \alpha + \frac{\pi}{2}[$. The formal Laplace transform for microfunctions thus extends to singularities, by inverse image:

$$\begin{array}{ccccc} & & \bigsqcup_{\dot{\beta} \in \mathbb{S}^1} \mathscr{C}_{\dot{\beta}} & \overset{\tilde{\mathcal{L}}}{\to} & \bigsqcup_{\dot{\beta}^\star \in \mathbb{S}^1} \left(\mathcal{A}^{\leq 0}/\mathcal{A}^{\leq -1}\right)_{\dot{\beta}^\star} \\ & s \nearrow & \downarrow p & & \downarrow p \\ \mathbb{S}^1_\bullet \supset J \ni \beta & \underset{\dot{\pi}}{\longrightarrow} & \mathbb{S}^1 \ni \dot{\beta} & \underset{\star}{\to} & \mathbb{S}^1 \ni \dot{\beta}^\star \end{array}$$

When returning to the very construction of the formal Laplace transform (Sect. 7.5.1), one sees that for a singularity $\overset{\triangledown}{\varphi}\in \mathrm{SING}_{\theta,\alpha}$, for any direction $\beta \in \hat{J} =]-\alpha+\theta, \theta+\alpha[$ and for $\breve{\beta}^\star =]-\frac{\pi}{2}+\beta, \beta+\frac{\pi}{2}[$, the formal Laplace transform $\tilde{\mathcal{L}}(\breve{\beta}^\star)\overset{\triangledown}{\varphi}$ is given as the class $\overset{\vartriangle}{\varphi}= \mathrm{cl}(\varphi_{\beta-2\pi,\beta,\kappa}) \in \mathcal{A}^{\leq 0}(\breve{\beta})/\mathcal{A}^{\leq -1}(\breve{\beta})$, $\breve{\beta} =]-\frac{\pi}{2}-\beta, -\beta+\frac{\pi}{2}[$, where $\varphi_{\beta-2\pi,\beta,\kappa}(z) = \int_{\gamma_{[\beta-2\pi,\beta],\varepsilon;\kappa}} \mathrm{e}^{-z\zeta}\overset{\vee}{\varphi}(\zeta)d\zeta$, with $\overset{\vee}{\varphi}$ any major of $\overset{\triangledown}{\varphi}$. This introduces the following definition. (Notice that $\breve{\hat{J}} = J^\star$).

Definition 7.25. The morphism $\tilde{\mathcal{L}}^\beta = \tilde{\mathcal{L}}(\breve{\beta}^\star) : \mathrm{SING}_{\theta,\alpha} \to \mathcal{A}^{\leq 0}(\breve{\beta})/\mathcal{A}^{\leq -1}(\breve{\beta})$ is called the formal Laplace transform in the direction $\beta \in \hat{J} =]-\alpha+\theta, \theta+\alpha[$. For any singularity $\overset{\triangledown}{\varphi}\in \mathrm{SING}_{\theta,\alpha}$, one denotes by $\tilde{\mathcal{L}}^{\hat{J}}\overset{\triangledown}{\varphi}\in \mathrm{ASYMP}_{\theta,\alpha}$ the asymptotic class given by the collection $\left(\tilde{\mathcal{L}}^\beta\overset{\triangledown}{\varphi}\right)_{\beta\in\hat{J}}$.

Example 7.4. We continue the example 7.3 but for the fact that we now consider $\overset{\triangledown}{J}_{\sigma,m}$ as a singularity in $\mathrm{SING}_{0,\pi}$. The formal Laplace transform $\tilde{\mathcal{L}}^{]-\pi,\pi[}\overset{\triangledown}{J}_{\sigma,m}$ is the asymptotic class $\overset{\vartriangle}{J}_{\sigma,m}\in \mathrm{ASYMP}_{0,\pi}$ seen by restriction as an element of $\Gamma(]-3\pi/2, 3\pi/2[, \mathcal{A}^{\leq 0}/\mathcal{A}^{\leq -1})$.

Let us linger for a moment to the cases of singularities of the form $\overset{\triangledown}{\varphi}= {}^\flat\widehat{\varphi} \in \mathrm{SING}^{\mathrm{int}}_{\theta,\alpha}$. For any direction $\beta \in]-\alpha+\theta, \theta+\alpha[$, the formal Laplace transform $\overset{\vartriangle}{\varphi}= \tilde{\mathcal{L}}^\beta\overset{\triangledown}{\varphi}\in \mathcal{A}^{\leq 0}(\breve{\beta})/\mathcal{A}^{\leq -1}(\breve{\beta})$ can be represented by the function

$$\varphi_{\beta-2\pi,\beta,\kappa}(z) = \int_{\gamma_{[\beta-2\pi,\beta],\varepsilon;\kappa}} \mathrm{e}^{-z\zeta}\overset{\vee}{\varphi}(\zeta)d\zeta = \int_{R_{\beta,0;\kappa}} \mathrm{e}^{-z\zeta}\widehat{\varphi}(\zeta)d\zeta, \tag{7.9}$$

and we thus recover the "usual" formal Laplace transform (see Sect. 7.1). In particular, we recall that we have extended the convolution law to $\mathrm{SING}^{\mathrm{int}}_{\theta,\alpha}$ by the variation map: for $\overset{\triangledown}{\varphi}_1= {}^\flat\widehat{\varphi}_1, \overset{\triangledown}{\varphi}_2= {}^\flat\widehat{\varphi}_2 \in \mathrm{SING}^{\mathrm{int}}_{\theta,\alpha}$, $\overset{\triangledown}{\varphi}_1 * \overset{\triangledown}{\varphi}_2= {}^\flat\big(\widehat{\varphi}_1 * \widehat{\varphi}_2\big)$. The above remark (7.9) shows that

$$\tilde{\mathcal{L}}^\beta(\overset{\triangledown}{\varphi}_1 * \overset{\triangledown}{\varphi}_2) = (\tilde{\mathcal{L}}^\beta\overset{\triangledown}{\varphi}_1)(\tilde{\mathcal{L}}^\beta\overset{\triangledown}{\varphi}_2),$$

by the properties of the "usual" formal Laplace transform.

We now assume that $\overset{\triangledown}{\varphi}$ is a simple singularity, $\overset{\triangledown}{\varphi}= a\delta + {}^\flat\widehat{\varphi} \in \mathrm{SING}^{\mathrm{simp}}$ with $\widehat{\varphi} \in \mathcal{O}_0$. For any arc $\hat{J} =]-\alpha+\theta, \theta+\alpha[$, the formal Laplace transform $\overset{\vartriangle}{\varphi}= \tilde{\mathcal{L}}^{\hat{J}}(a\delta+\overset{\triangledown}{\varphi})$ is an asymptotic class that belongs, more precisely, to $\Gamma(J^\star, \mathcal{A}_1/\mathcal{A}^{\leq -1})$. This again comes from (an analogue of) the identity (7.9) and classical arguments recalled in the introduction of this chapter.

Definition 7.26. One denotes by $\mathrm{ASYMP}^{\mathrm{simp}}$ the subspace of asymptotic classes obtained by injection of the global sections $\Gamma(\underset{\bullet}{\mathbb{S}}^1, \mathcal{A}_1/\mathcal{A}^{\leq -1})$ into ASYMP.

Proposition 7.9. *The restriction of the formal Laplace transform $\tilde{\mathcal{L}}$ to* $\mathrm{SING}^{\mathrm{simp}}$ *has* $\mathrm{ASYMP}^{\mathrm{simp}}$ *for its range.*

Remark 7.4. Consider a formal series expansion $\widetilde{\varphi} \in \mathbb{C}[[z^{-1}]]$ and an open arc of the form $J^\star =]-\pi/2-\alpha-\theta, -\theta+\alpha+\pi/2[\subset \underset{\bullet}{\mathbb{S}}^1$. By the Borel-Ritt

theorem, there are infinitely many $\varphi \in \overline{\mathcal{A}}(J^\star)$ whose Poincaré asymptotics $T(J^\star)\varphi$ is given by $\widetilde{\varphi}$ on $J^\star$. These various φ differ by flat germs, that is elements of $\overline{\mathcal{A}}^{<0}(J^\star)$. Therefore as a rule, these germs φ represent different asymptotic classes $\overset{\vartriangle}{\varphi} \in \mathrm{ASYMP}_{\theta,\alpha}$.
Now suppose that $\widetilde{\varphi}$ is 1-Gevrey and choose a (good) covering (I_i) of $J^\star$ where each I_i is an open arc of aperture less than π. By the Borel-Ritt theorem for 1-Gevrey asymptotics and for each subscript i, there exists $\varphi_i \in \overline{\mathcal{A}}_1(I_i)$ whose 1-Gevrey asymptotics $T_1(I_i)\varphi_i$ is φ. Moreover, each φ_i is uniquely defined this way up to 1-exponentially flat germs, that is up to elements of $\overline{\mathcal{A}}^{\leq -1}(I_i)$. One thus gets a uniquely defined section $\overset{\vartriangle}{\varphi} \in \Gamma(J^\star, \mathcal{A}_1/\mathcal{A}^{\leq -1})$ that can be thought of as an asymptotic class. One can characterize another way this asymptotic class $\overset{\vartriangle}{\varphi} \in \mathrm{ASYMP}^{\mathrm{simp}}$ by settling $\overset{\vartriangle}{\varphi} = \tilde{\mathcal{L}}(a\delta + \overset{\triangledown}{\varphi})$ where $\overset{\triangledown}{\varphi} = {}^\flat\widehat{\varphi}$ with $\widehat{\varphi}$ the minor of $\widetilde{\varphi}$ while a is its constant term.

Definition 7.27. The mapping $\natural : \widetilde{\varphi} \in \mathbb{C}[[z^{-1}]]_1 \mapsto \overset{\vartriangle}{\varphi} = {}^\natural\widetilde{\varphi} \in \mathrm{ASYMP}^{\mathrm{simp}}$ is defined by $\overset{\vartriangle}{\varphi} = \tilde{\mathcal{L}}(a\delta + \overset{\triangledown}{\varphi})$ where $\overset{\triangledown}{\varphi} = {}^\flat\widehat{\varphi}$, whereas $\widehat{\varphi}$ stands for the minor of $\widetilde{\varphi}$ and a its constant term.

Obviously, the mapping $\natural$ is an isomorphism, the inverse map being the (1-Gevrey) Taylor map. This allows to merge ${}^\natural\widetilde{\varphi}$ with $\widetilde{\varphi}$ in practice.

7.5.3.2 Back to convolution product

We have said without proof that $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{B}}$ are morphisms of sheaves of algebras. Thus it is certainly worthy to prove the following proposition.

Proposition 7.10. *For any two singularities* $\overset{\triangledown}{\varphi}_1, \overset{\triangledown}{\varphi}_2 \in \mathrm{SING}_{\theta,\alpha}$ *and any direction* $\beta \in]-\alpha+\theta, \theta+\alpha[$, $(\tilde{\mathcal{L}}^\beta\ \overset{\triangledown}{\varphi}_1)(\tilde{\mathcal{L}}^\beta\ \overset{\triangledown}{\varphi}_1) = \tilde{\mathcal{L}}^\beta(\overset{\triangledown}{\varphi}_1 * \overset{\triangledown}{\varphi}_2)$. *Moreover,* $\tilde{\mathcal{L}}^\beta(\partial\ \overset{\triangledown}{\varphi}) = \partial\tilde{\mathcal{L}}^\beta\ \overset{\triangledown}{\varphi}$.

Proof. (Adapted from [1]). We take two singularities $\overset{\triangledown}{\varphi}_1, \overset{\triangledown}{\varphi}_2 \in \mathrm{SING}_{\theta,\alpha}$ with major $\overset{\vee}{\varphi}_1, \overset{\vee}{\varphi}_2$. Choosing a direction $\beta \in \hat{J} =]-\alpha+\theta, \theta+\alpha[$, we can consider the formal Laplace transforms $\overset{\vartriangle}{\varphi}_1 = \tilde{\mathcal{L}}^\beta\ \overset{\triangledown}{\varphi}_1$ and $\overset{\vartriangle}{\varphi}_2 = \tilde{\mathcal{L}}^\beta\ \overset{\triangledown}{\varphi}_2$. These are elements of $\mathcal{A}^{\leq 0}(\breve{\beta})/\mathcal{A}^{\leq -1}(\breve{\beta})$ which can be represented respectively by

$$\varphi_1(z) = \int_{\gamma_1} \mathrm{e}^{-z\zeta}\ \overset{\vee}{\varphi}_1\ (\zeta)d\zeta \in \overline{\mathcal{A}}^{\leq 0}(\overset{\bullet}{\mathfrak{s}}{}_r^\infty(\breve{\beta})),\ \varphi_2(z) = \int_{\gamma_2} \mathrm{e}^{-z\zeta}\ \overset{\vee}{\varphi}_2\ (\zeta)d\zeta \in \overline{\mathcal{A}}^{\leq 0}(\overset{\bullet}{\mathfrak{s}}{}_r^\infty(\breve{\beta})),$$

with $\gamma_1 = \gamma_{[\beta-2\pi,\beta],\varepsilon_1;\kappa_1}$, $\gamma_2 = \gamma_{[\beta-2\pi,\beta],\varepsilon_2;\kappa_2}$ and some $r > 0$ large enough. The product $\overset{\vartriangle}{\varphi}_1\overset{\vartriangle}{\varphi}_2 \in \mathcal{A}^{\leq 0}(\breve{\beta})/\mathcal{A}^{\leq -1}(\breve{\beta})$ is thus represented by

$$\varphi_1\varphi_2(z) = \int_{\gamma_1\times\gamma_2} \mathrm{e}^{-z(\zeta_1+\zeta_2)}\ \overset{\vee}{\varphi}_1\ (\zeta_1)\ \overset{\vee}{\varphi}_2\ (\zeta_2)d\zeta_1 d\zeta_2 \in \overline{\mathcal{A}}^{\leq 0}(\overset{\bullet}{\mathfrak{s}}{}_r^\infty(\breve{\beta})).$$

Let us look at the formal Borel transform $\tilde{\mathcal{B}}(\breve{\beta})(\overset{\vartriangle}{\varphi}_1\overset{\vartriangle}{\varphi}_2) \in \mathscr{C}(\breve{\beta}^\star)$. This Borel transform can be represented by the integral $\overset{\vee}{(\varphi_1\varphi_2)}_{z_1,\alpha_1}\ (\zeta) = -\dfrac{1}{2i\pi}\displaystyle\int_{R_{\alpha_1,z_1}} \mathrm{e}^{z\zeta}\varphi_1\varphi_2(z)dz$, with $z_1 \in \overset{\bullet}{\mathfrak{s}}{}_{r_1}^\infty(\breve{\beta}^\star)$, for $r_1 > r$, and for any direction $\alpha_1 \in \breve{\beta}^\star$. The function $(\varphi_1 \overset{\vee}{} \varphi_2)_{z_1,\alpha_1}\,(\zeta)$ is holomorphic on $\overset{\bullet}{\Pi}{}_0^{\alpha_1+\pi}$ (go back to the construction of the formal Borel transform, Sect. 7.5.2). Taking $\zeta \in \overset{\bullet}{\Pi}{}_{2\varepsilon}^{\alpha_1+\pi}$ with $\varepsilon > \varepsilon_1 + \varepsilon_2$, we can apply Fubini.

Remark that $\zeta_1 + \zeta_2$ (or rather $\overset{\bullet}{\zeta}_1 + \overset{\bullet}{\zeta}_2$) remains in the bounded strip $\{\zeta \in \mathbb{C} \mid \mathrm{dist}(\zeta, \mathrm{e}^{i\beta}[0,\kappa]) \leq \varepsilon_1 + \varepsilon_2\}$, for $(\zeta_1, \zeta_2) \in \gamma_1 \times \gamma_2$. Thus $\zeta - (\zeta_1 + \zeta_2)$ remains in the domain $\overset{\bullet}{\Pi}{}_{\varepsilon}^{\alpha_1+\pi}$ for $\zeta \in \overset{\bullet}{\Pi}{}_{2\varepsilon}^{\alpha_1+\pi}$ and this ensures the integrability conditions.

This way, we get:

$$\begin{aligned}
(\varphi_1 \overset{\vee}{} \varphi_2)_{z_1,\alpha_1}\,(\zeta) &= -\frac{1}{2i\pi}\int_{R_{\alpha_1,z_1}} \mathrm{e}^{z\zeta}\left(\int_{\gamma_1\times\gamma_2} \mathrm{e}^{-z(\zeta_1+\zeta_2)}\,\overset{\vee}{\varphi}_1\,(\zeta_1)\,\overset{\vee}{\varphi}_2\,(\zeta_2)d\zeta_1 d\zeta_2\right)dz \\
&= \int_{\gamma_1\times\gamma_2} \frac{\mathrm{e}^{z_1(\zeta-\zeta_1-\zeta_2)}}{2i\pi(\zeta-\zeta_1-\zeta_2)}\,\overset{\vee}{\varphi}_1\,(\zeta_1)\,\overset{\vee}{\varphi}_2\,(\zeta_2)d\zeta_1 d\zeta_2 \\
&= \int_{\gamma_1}\left(\int_{\gamma_2} \frac{\mathrm{e}^{z_1(\zeta-\zeta_1-\zeta_2)}}{2i\pi(\zeta-\zeta_1-\zeta_2)}\,\overset{\vee}{\varphi}_2\,(\zeta_2)d\zeta_2\right)\overset{\vee}{\varphi}_1\,(\zeta_1)d\zeta_1
\end{aligned}$$

Returning to the very construction of the convolution product for singularities, we see that $(\varphi_1 \overset{\vee}{} \varphi_2)_{z_1,\alpha_1}$ is nothing but a major of the singularity $\mathrm{sing}_0\left(\dfrac{\mathrm{e}^{z_1\zeta}}{2i\pi\zeta}\right) * \overset{\triangledown}{\varphi}_1 * \overset{\triangledown}{\varphi}_2$. But $\mathrm{sing}_0\left(\dfrac{\mathrm{e}^{z_1\zeta}}{2i\pi\zeta}\right) = \delta$ and therefore $\mathrm{sing}_0\left((\varphi_1 \overset{\vee}{} \varphi_2)_{z_1,\alpha_1}\right) = \overset{\triangledown}{\varphi}_1 * \overset{\triangledown}{\varphi}_2$. From Proposition 7.8, we know that $\tilde{\mathcal{B}} \circ \tilde{\mathcal{L}} = \mathrm{Id}$ (when considering $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{L}}$ as morphisms of sheaves), thus the conclusion. The last statement as been already seen. $\square$

Example 7.5. We know by theorem 3.2 that the formal series $\widetilde{w}_{(0,0)}$ solution of the prepared ODE (3.6) associated with the first Painlevé equation, is 1-Gevrey. Its formal Borel transform $\widehat{w}_{(0,0)} = \tilde{\mathcal{B}}\widetilde{w}_{(0,0)}$ is thus a germ of holomorphic functions at the origin and we set $\overset{\triangledown}{w}_{(0,0)} = {}^\flat\widehat{w}_{(0,0)} \in \mathrm{SING}^{\mathrm{simp}}$. We now consider the singularity $\overset{\triangledown}{I}_\sigma * \overset{\triangledown}{w}_{(0,0)} \in \mathrm{SING}$, for any $\sigma \in \mathbb{C}$. By proposition 7.10, for an arbitrary direction $\beta \in \underset{\bullet}{\mathbb{S}}^1$, the formal Laplace transform $\tilde{\mathcal{L}}^\beta(\overset{\triangledown}{I}_\sigma * \overset{\triangledown}{w}_{(0,0)}) \in \mathcal{A}^{\leq 0}(\breve{\beta})/\mathcal{A}^{\leq -1}(\breve{\beta})$ is the asymptotic class of direction $\breve{\beta}$ which also reads:

$$\tilde{\mathcal{L}}^\beta(\overset{\triangledown}{I}_\sigma * \overset{\triangledown}{w}_{(0,0)}) = \tilde{\mathcal{L}}^\beta(\overset{\triangledown}{I}_\sigma)\tilde{\mathcal{L}}^\beta(\overset{\triangledown}{w}_{(0,0)}).$$

On the one hand, $\tilde{\mathcal{L}}^\beta\,\overset{\triangledown}{I}_\sigma$ is the asymptotic class $\overset{\vartriangle}{I}_\sigma \in \Gamma(\breve{\beta}, \mathcal{A}^{\leq 0}/\mathcal{A}^{\leq -1})$. On the other hand, $\tilde{\mathcal{L}}^\beta\,\overset{\triangledown}{w}_{(0,0)} = {}^\natural\widetilde{w}_{(0,0)}$. Therefore, $\tilde{\mathcal{L}}^\beta(\overset{\triangledown}{I}_\sigma * \overset{\triangledown}{w}_{(0,0)}) = \overset{\vartriangle}{I}_\sigma\,{}^\natural\widetilde{w}_{(0,0)}$ that can be identified with $\dfrac{1}{z^\sigma}\widetilde{w}_{(0,0)}$ with the branch of z^σ determined by the condition $\arg z \in \breve{\beta}$.

Example 7.6. We now use the notations of Sect. 3.5.2.3 but for the fact that we consider arcs on $\underset{\bullet}{\mathbb{S}}^1$. We write $\hat{I}_0 =]0,\pi[$ and $I_0^\star =]-3\pi/2, \pi/2[\subset \underset{\bullet}{\mathbb{S}}^1$ and in what follows with think of the Laplace-Borel sum $w_{tri,0} = \mathscr{S}^{\hat{I}_0}\widetilde{w}_{(0,0)}$ as (representing) a multivalued section of $\mathcal{A}_1$ on $I_0^\star$. Similarly, we set $\hat{I}_1 =]\pi, 2\pi[$

and $I_1^\star =]-5\pi/2, -\pi/2[\subset \underset{\bullet}{\mathbb{S}}^1$ and think of $w_{tri,1} = \mathscr{S}^{\hat{I}_1}\widetilde{w}_{(0,0)}$ as an element of $\Gamma(I_1^\star, \mathcal{A}_1)$. Notice that $I_0^\star \cap I_1^\star =]-3\pi/2, -\pi/2[$ on $\underset{\bullet}{\mathbb{S}}^1$. Since both $w_{tri,0}$ and $w_{tri,1}$ are asymptotic to the 1-Gevrey series $\widetilde{w}_{(0,0)}$, we know that the difference $w_{tri,0} - w_{tri,1}$ is a multivalued section of $\mathcal{A}^{\leq -1}$ on $I_0^\star \cap I_1^\star$. Therefore, for any $\sigma \in \mathbb{C}$, $\dfrac{1}{z^\sigma} w_{tri,0}$ and $\dfrac{1}{z^\sigma} w_{tri,1}$ glue together to give a multivalued section of $\mathcal{A}_1/\mathcal{A}^{\leq -1}$ on $I_0^\star \cup I_1^\star$, that can be identified with the asymptotic class $\overset{\triangle}{I}_\sigma \natural \widetilde{w}_{(0,0)} \in \mathrm{ASYMP}_{\pi,\pi}$. The formal Borel transform $\tilde{\mathcal{B}}(I_0^\star)\big(\overset{\triangle}{I}_\sigma \natural \widetilde{w}_{(0,0)}\big)$ is the multivalued section of $\mathscr{C}$ on $I_0 =]-\pi/2, 3\pi/2[$ which can be thought of as a singularity in $\mathrm{SING}_{\pi/2,\pi/2}$, and is given by $\tilde{\mathcal{B}}(I_0^\star)\big(\overset{\triangle}{I}_\sigma \natural \widetilde{w}_{(0,0)}\big) = \overset{\triangledown}{I}_\sigma * \overset{\triangledown}{w}_{(0,0)}$. Similarly, the formal Borel transform $\tilde{\mathcal{B}}(I_1^\star)\big(\overset{\triangle}{I}_\sigma \natural \widetilde{w}_{(0,0)}\big)$ is the multivalued section of $\mathscr{C}$ on $I_1 =]\pi/2, 5\pi/2[$ which provides a singularity in $\mathrm{SING}_{3\pi/2,\pi/2}$, of the form $\tilde{\mathcal{B}}(I_1^\star)\big(\overset{\triangle}{I}_\sigma \natural \widetilde{w}_{(0,0)}\big) = \overset{\triangledown}{I}_\sigma * \overset{\triangledown}{w}_{(0,0)}$. These two singularities glue together as the element $\overset{\triangledown}{I}_\sigma * \overset{\triangledown}{w}_{(0,0)}$ of $\mathrm{SING}_{\pi,\pi}$.

7.5.3.3 Formal Laplace transform for singularities at ω

The spaces SING_ω, resp. $\mathrm{SING}_{\omega,\theta,\alpha}$ of singularities at $\omega \in \mathbb{C}$ are the translated of SING, resp. $\mathrm{SING}_{\theta,\alpha}$. (See definition 7.11). By its very construction, the formal Laplace transform transforms the translation into the multilplication by an exponential.

Definition 7.28. The formal Laplace transform $\tilde{\mathcal{L}}$ sends SING_ω, resp. $\mathrm{SING}_{\omega,\theta,\alpha}$, onto the space denoted by $\mathrm{e}^{-\omega z}\mathrm{ASYMP}$, resp. $\mathrm{e}^{-\omega z}\mathrm{ASYMP}_{\theta,\alpha}$, made of **asymptotic classes with support based at** ω.

We mention the following result that can be thought of as an analogue of the Watson's lemma [14].

Lemma 7.4. *For any* $\omega \in \mathbb{C}^\star$*, the sum of the* $\mathbb{C}$*-vector spaces* $\mathrm{ASYMP}_{\theta,\alpha}$ *and* $\mathrm{e}^{-\omega z}\mathrm{ASYMP}_{\theta,\alpha}$ *is direct.*

Proof. We consider an asymptotic class $\overset{\triangle}{\varphi} \in \mathrm{ASYMP}_{\theta,\alpha}$. By definition, one can find a (good) open covering (J_j) of $J^\star =]-\pi/2 - \alpha - \theta, -\theta + \alpha + \pi/2[$ and a "0-cochain" $\big(\varphi_j \in \mathcal{A}^{\leq 0}(J_j)\big)_j$ with associated "1-coboundary" $\big(\varphi_{j+1} - \varphi_j \in \mathcal{A}^{\leq -1}(J_{j+1} \cap J_j)\big)_j$ that represents $\overset{\triangle}{\varphi}$. Now assume that $\overset{\triangle}{\varphi}$ also belongs to $\mathrm{e}^{-\omega z}\mathrm{ASYMP}_{\theta,\alpha}$. Considering a refinement of (J_j) if necessary, one deduces that $\varphi_j \in \mathcal{A}^{\leq -1}(J_j)$ for at least one j, since $J^\star$ is an arc of aperture $> \pi$. This implies that the formal Borel transform $\overset{\triangledown}{\varphi} \in \mathrm{SING}_{\theta,\alpha}$ has a major $\overset{\vee}{\varphi}$ that can be analytically continued to 0, thus $\overset{\triangledown}{\varphi} = 0$ and as a consequence $\overset{\triangle}{\varphi} = 0$. □

7.6 Laplace transforms

We develop here only matters convenient for this course. For more general nonsense on Laplace transforms in the framework of resurgent analysis, see [1, 2, 7, 8, 16].

7.6.1 Laplace transforms

Definition 7.29. For an open arc $I \subset \mathbb{S}^1$ and $r \geq 0$, we note:

1. $\mathcal{E}^{\leq 1}(I)$ the $\mathbb{C}$-differential algebra of holomorphic functions φ on $\dot{\mathfrak{s}}_0^\infty(I)$ with 1-exponential growth at infinity on the direction I : for any proper subsector $\dot{\mathfrak{s}}^\infty \Subset \dot{\mathfrak{s}}_0^\infty(I)$, there exist $C > 0$ and $\tau > 0$ so that, for all $z \in \dot{\mathfrak{s}}^\infty$, $|\varphi(z)| \leq C\mathrm{e}^{\tau|z|}$;
2. for any open arc $I \subset \mathbb{S}^1$ of aperture $\leq \pi$, we set $\check{\mathcal{E}}^{\leq 1}(I) = \mathcal{E}^{\leq 1}(\check{I})$, the space of holomorphic functions φ on $\dot{\mathfrak{s}}_0^\infty(\check{I})$, with 1-exponential growth at infinity on the codirection I.
3. we note $\mathcal{E}^{\leq 1}$ –resp. $\check{\mathcal{E}}^{\leq 1}$ – the sheaf over $\mathbb{S}^1$ corresponding to the family $\big(\mathcal{E}^{\leq 1}(I)\big)$ –resp. $\check{\mathcal{E}}^{\leq 1}(I)$–;
4. we note $\mathcal{O}(\mathbb{C})^{\leq 1}$ the space of entire functions with 1-exponential growth at infinity on every direction.

We take an open arc I of $\mathbb{S}^1$ of aperture $\leq \pi$, and a function $\overset{\vee}{\varphi} \in \check{\mathcal{E}}^{\leq 1}(I)$. Thus $\overset{\vee}{\varphi}$ is holomorphic on $\dot{\mathfrak{s}}_0^\infty(\check{I})$ and for any open arc I_1 so that $\bar{I}_1 \subset I$, for any $\varepsilon > 0$, there exist $C > 0$ and $\tau > 0$ so that, for all $\zeta \in \dot{\mathfrak{s}}_\varepsilon^\infty(\check{I}_1)$, $|\overset{\vee}{\varphi}(\zeta)| \leq C\mathrm{e}^{\tau|\zeta|}$. We consider the Laplace integral,

$$\varphi_{I_1}(z) = \int_{\gamma_{[\theta_1,\theta_2],\varepsilon}} \mathrm{e}^{-z\zeta}\, \overset{\vee}{\varphi}(\zeta)d\zeta = \left(-\int_{R_{\theta_1,\varepsilon}} + \int_{\delta_{[\theta_1,\theta_2],\varepsilon}} + \int_{R_{\theta_2,\varepsilon}}\right) \overset{\vee}{\varphi}(\zeta)d\zeta$$

where $\check{I}_1 =]\theta_1, \theta_2[$. This Laplace integral can be decomposed as follows:

- by classical arguments, the integral $\displaystyle\int_{R_{\theta_1,\varepsilon}} \mathrm{e}^{-z\zeta}\, \overset{\vee}{\varphi}(\zeta)d\zeta$ defines a holomorphic function on $\dot{\Pi}_\tau^{\theta_1}$ and we observe that for any $r > \tau$, for every $z \in \overline{\dot{\Pi}_r^{\theta_1}}$,
$$\left|\int_{R_{\theta_1,\varepsilon}} \mathrm{e}^{-z\zeta}\, \overset{\vee}{\varphi}(\zeta)d\zeta\right| \leq \int_\varepsilon^\infty \mathrm{e}^{-sr}C\mathrm{e}^{\tau s}ds \leq \frac{C}{r-\tau}\mathrm{e}^{-\varepsilon(r-\tau)}.$$
In the same way, the integral $\displaystyle\int_{R_{\theta_2,\varepsilon}} \mathrm{e}^{-z\zeta}\, \overset{\vee}{\varphi}(\zeta)d\zeta$ defines a holomorphic function on $\dot{\Pi}_\tau^{\theta_2}$ and for any $r > \tau$, for every $z \in \overline{\dot{\Pi}_r^{\theta_2}}$, $\left|\displaystyle\int_{R_{\theta_2,\varepsilon}} \mathrm{e}^{-z\zeta}\, \overset{\vee}{\varphi}(\zeta)d\zeta\right| \leq \dfrac{C}{r-\tau}\mathrm{e}^{-\varepsilon(r-\tau)}$;
- the integral $\displaystyle\int_{\delta_{[\theta_1,\theta_2],\varepsilon}} \mathrm{e}^{-z\zeta}\, \overset{\vee}{\varphi}(\zeta)d\zeta$ defines an entire function and $\left|\displaystyle\int_{\delta_{[\theta_1,\theta_2],\varepsilon}} \mathrm{e}^{-z\zeta}\, \overset{\vee}{\varphi}(\zeta)d\zeta\right| \leq C|\check{I}_1|\varepsilon\mathrm{e}^{\tau\varepsilon}\mathrm{e}^{\varepsilon|z|}$.
- by arguments already encounter (see Sect. 7.5.1), both $\dot{\Pi}_\tau^{\theta_1}$ and $\dot{\Pi}_\tau^{\theta_2}$ contains any proper subsector $\dot{\mathfrak{s}}^\infty$ of $\dot{\mathfrak{s}}_r^\infty(I_1^\star)$, once $r > 0$ is chosen large enough.

Therefore, φ_{I_1} belongs to the space $\overline{\mathcal{A}}^{\leq 0}(\overset{\bullet}{\mathfrak{s}}{}^{\infty}_{r_1}(I_1^\star))$ for $r_1 > 0$ large enough, because $\varepsilon > 0$ can be chosen arbitrarily small.

It is easy to see that adding to $\overset{\vee}{\varphi}$ any element of $\mathcal{O}(\mathbb{C})^{\leq 1}$, does not affect the function φ_{I_1} (just deform the contour of integration, by Cauchy).
The family of functions $(\varphi_{I_1})_{I_1 \subset I}$ obtained this way glue together analytically, by Cauchy.
The above construction gives a morphism, $\mathcal{L}(I) : \check{\mathcal{E}}^{\leq 1}(I)/\mathcal{O}(\mathbb{C})^{\leq 1} \to \mathcal{A}^{\leq 0}(I^\star)$, compatible with the restriction maps, which provides a morphism of sheaves[3].

Definition 7.30. The morphism of sheaves $\mathcal{L} : \check{\mathcal{E}}^{\leq 1}/\mathcal{O}(\mathbb{C})^{\leq 1} \to \mathcal{A}^{\leq 0}$ is called the strict Laplace transform[4].

We now return to the construction we did to get the formal Borel transform, Sect. 7.5.2. We take an open arc $I^\star$ of $\mathbb{S}^1$ of aperture $\leq \pi$ and $\varphi \in \mathcal{A}^{\leq 0}(I^\star)$. For $z_1 \in \overset{\bullet}{\mathfrak{s}}{}^{\infty}_{r}(I^\star)$, $r > 0$ large enough, for a direction $\alpha \in I^\star$, we consider $\overset{\vee}{\varphi}_{z_1,\alpha}(\zeta) = -\dfrac{1}{2i\pi}\displaystyle\int_{R_{\alpha,z_1}} \mathrm{e}^{z\zeta}\varphi(z)dz$. We have seen that, making α varying, one gets an element of $\check{\mathcal{E}}^{\leq 1}(I)$, while $\overset{\vee}{\varphi}_{z_1,\alpha}$ depends on z_1 only modulo an element of $\mathcal{O}(\mathbb{C})^{\leq 1}$. We thus get a morphism of sheaves $\mathcal{B} : \mathcal{A}^{\leq 0} \to \check{\mathcal{E}}^{\leq 1}/\mathcal{O}(\mathbb{C})^{\leq 1}$ which has the following property (we refer to [1] for the proof):

Proposition 7.11. *The morphisms* $\mathcal{L} : \check{\mathcal{E}}^{\leq 1}/\mathcal{O}(\mathbb{C})^{\leq 1} \to \mathcal{A}^{\leq 0}$ *and* $\mathcal{B} : \mathcal{A}^{\leq 0} \to \check{\mathcal{E}}^{\leq 1}/\mathcal{O}(\mathbb{C})^{\leq 1}$ *are isomorphisms of sheaves of* $\mathbb{C}$*-differential algebras, and* $\mathcal{L} \circ \mathcal{B} = \mathrm{Id}$, $\mathcal{B} \circ \mathcal{L} = \mathrm{Id}$. *Moreover,* $\mathcal{L}\partial = \partial\mathcal{L}$.

7.6.2 Singularities and Laplace transform

7.6.2.1 Summable singularities

We remind that $\mathrm{SING}_{\theta,\alpha}$ can be identified with the space $\Gamma(J,\mathscr{C})$ of multivalued sections of $\mathscr{C}$ over $J =]-\pi/2-\alpha+\theta, \theta+\alpha+\pi/2[\subset \underset{\bullet}{\mathbb{S}}^1$. In particular, any singularity $\overset{\triangledown}{\varphi} \in \mathrm{SING}_{\theta,\alpha}$ can be represented by a major $\overset{\vee}{\varphi} \in \mathrm{ANA}_{\theta,\alpha} = \Gamma(\check{J},\mathcal{O}^0)$, with $\check{J} =]\theta-\alpha-2\pi, \theta+\alpha[\subset \underset{\bullet}{\mathbb{S}}^1$.

Definition 7.31. An element $\overset{\vee}{\varphi} \in \mathrm{ANA}_{\theta,\alpha} = \Gamma(\check{J},\mathcal{O}^0)$ is said **summable** in the direction $\beta \in \hat{J} =]-\alpha+\theta, \theta+\alpha[$ if there exists a neighbourhood $\hat{J}_1 \subset \hat{J}$ of β so that the two restrictions $\overset{\vee}{\varphi}_1 \in \Gamma(\hat{J}_1,\mathcal{O}^0)$ and $\overset{\vee}{\varphi}_2 \in \Gamma(\hat{J}_2,\mathcal{O}^0)$ of $\overset{\vee}{\varphi}$ over $\hat{J}_1$ and $\hat{J}_2 = -2\pi + \hat{J}_1$ respectively, can be represented by elements of $\Gamma(\hat{J}_1,\mathcal{E}^{\leq 1})$ and $\Gamma(\hat{J}_2,\mathcal{E}^{\leq 1})$ respectively.

A singularity $\overset{\triangledown}{\varphi} \in \mathrm{SING}_{\theta,\alpha}$ is **summable** in the direction $\hat{J}$ if for any $\beta \in \hat{J}$, $\overset{\triangledown}{\varphi}$ has a major $\overset{\vee}{\varphi} \in \mathrm{ANA}_{\theta,\alpha}$ which summable in the direction β.

We note $\mathrm{SING}^{\mathrm{sum}}_{\theta,\alpha}$ the space of singularities $\overset{\triangledown}{\varphi} \in \mathrm{SING}_{\theta,\alpha}$ that are summable in the direction $\hat{J}$.

---

[3] As usual, modulo complex conjugation

[4] We abide a notation of [1], although the construction therein slightly differs from ours.

7.6.2.2 Laplace transforms of summable singularities

We take a singularity $\overset{\triangledown}{\varphi} \in \mathrm{SING}^{\mathrm{sum}}_{\theta,\alpha}$ and we consider a direction $\beta \in \hat{J}$. We note $\overset{\vee}{\varphi}$ a major of $\overset{\triangledown}{\varphi}$ which is summable in the direction β. Using the notations of the definition 7.31, we consider the following Laplace integral

$$\begin{aligned}\varphi_\beta(z) &= \int_{\gamma_{[\beta-2\pi,\beta],\varepsilon}} \mathrm{e}^{-z\zeta}\, \overset{\vee}{\varphi}(\zeta)d\zeta \qquad (7.10)\\ &= \int_{\delta_{[\beta-2\pi,\beta],\varepsilon}} \mathrm{e}^{-z\zeta}\, \overset{\vee}{\varphi}(\zeta)d\zeta - \int_{R_{\beta-2\pi,\varepsilon}} \mathrm{e}^{-z\zeta}\, \overset{\vee}{\varphi}_2(\zeta)d\zeta + \int_{R_{\beta,\varepsilon}} \mathrm{e}^{-z\zeta}\, \overset{\vee}{\varphi}_1(\zeta)d\zeta\\ &= \int_{\delta_{[\beta-2\pi,\beta],\varepsilon}} \mathrm{e}^{-z\zeta}\, \overset{\vee}{\varphi}(\zeta)d\zeta + \int_{R_{\beta,\varepsilon}} \mathrm{e}^{-z\zeta}\widehat{\varphi}(\zeta)d\zeta\end{aligned}$$

($\varepsilon > 0$ small enough. In the last equality, $\widehat{\varphi} = \mathrm{var}\, \overset{\triangledown}{\varphi}$). From the arguments used in Sect. 7.6.1, we see φ_β defines an element of $\overline{\mathcal{A}}^{\leq 0}(\breve{\beta})$. Moreover, if $\overset{\vee}{\psi}$ is another major of $\overset{\triangledown}{\varphi}$ which is summable in the direction β (for instance $\overset{\vee}{\varphi} - \overset{\vee}{\psi} \in \mathcal{O}(\mathbb{C})^{\leq 1}$), then its Laplace integral ψ_β coincide with φ_β as elements of $\overline{\mathcal{A}}^{\leq 0}(\breve{\beta})$. Thus φ_β is independent of the chosen summable major and only depends on $\overset{\triangledown}{\varphi} \in \mathrm{SING}^{\mathrm{sum}}_{\theta,\alpha}$. This allows us to write $\varphi_\beta = \mathcal{L}^\beta\, \overset{\triangledown}{\varphi}$.

Making β varying in $\hat{I}$, the functions $\mathcal{L}^\beta\, \overset{\triangledown}{\varphi}$ obviously glue together analytically (by Cauchy and using the independence of $\mathcal{L}^\beta\, \overset{\triangledown}{\varphi}$ with respect to the chosen summable major), to give and element $\mathcal{L}^{\hat{J}}\, \overset{\triangledown}{\varphi}$ of $\Gamma(J^\star, \mathcal{A}^{\leq 0})$.

Definition 7.32. The morphism $\mathcal{L}^\beta : \mathrm{SING}^{\mathrm{sum}}_{\theta,\alpha} \to \overline{\mathcal{A}}^{\leq 0}(\breve{\beta})$ is called the Laplace transform in the direction $\beta \in \hat{J} =]-\alpha+\theta, \theta+\alpha[$. The morphism $\mathcal{L}^{\hat{J}} : \mathrm{SING}^{\mathrm{sum}}_{\theta,\alpha} \to \Gamma(J^\star, \mathcal{A}^{\leq 0})$ is called the Laplace transform in the direction $\hat{J} =]-\alpha+\theta, \theta+\alpha[$.

We recover with the following proposition the examples given in the introduction of the chapter, see also [24].

Proposition 7.12. *The singularities* $\overset{\triangledown}{I}_\sigma$ *and* $\overset{\triangledown}{J}_{\sigma,m}$ *belong to* $\mathrm{SING}^{\mathrm{sum}}_{\theta,\alpha}$ *for any direction* θ *and any* $\alpha > 0$. *Moreover, for any direction* $\beta \in \underset{\bullet}{\mathbb{S}^1}$,

$$\mathcal{L}^\beta\, \overset{\triangledown}{I}_\sigma(z) = \frac{1}{z^\sigma}, \qquad \mathcal{L}^\beta\, \overset{\triangledown}{J}_{\sigma,m}(z) = (-1)^m \frac{\log^m(z)}{z^\sigma}, \qquad z \in \Pi_0^\beta \subset \underset{\bullet}{\mathbb{C}}.$$

This has the following consequences:

Proposition 7.13. *For all* $\sigma_1, \sigma_2 \in \mathbb{C}$, *for all* $m_1, m_2 \in \mathbb{N}$ $\overset{\triangledown}{I}_{\sigma_1} * \overset{\triangledown}{I}_{\sigma_2} = \overset{\triangledown}{I}_{\sigma_1+\sigma_2}$ *and* $\overset{\triangledown}{J}_{\sigma_1,m_1} * \overset{\triangledown}{J}_{\sigma_2,m_2} = \overset{\triangledown}{J}_{\sigma_1+\sigma_2,m_1+m_2}$.

Proof. From proposition 7.12, we deduce that $\tilde{\mathcal{L}}\, \overset{\triangledown}{I}_{\sigma_1} = \dfrac{1}{z^{\sigma_1}}$ and $\tilde{\mathcal{L}}\, \overset{\triangledown}{I}_{\sigma_2} = \dfrac{1}{z^{\sigma_2}}$. Thus by proposition 7.10, $\tilde{\mathcal{L}}\, \overset{\triangledown}{I}_{\sigma_1} * \overset{\triangledown}{I}_{\sigma_2} = \dfrac{1}{z^{\sigma_1+\sigma_2}}$ and one concludes by formal Borel transform. Same proof for the other equality. □

In definition 7.32, we meant morphisms of vector spaces. As a matter of fact, these are morphisms of $\mathbb{C}$-differential algebras. This is the matter of the following proposition.

Proposition 7.14. *The space* $\mathrm{SING}^{\mathrm{sum}}_{\theta,\alpha}$ *is a commutative and associative algebra with unit* δ*. The Laplace transform* $\mathcal{L}^\beta : \mathrm{SING}^{\mathrm{sum}}_{\theta,\alpha} \to \overline{\mathcal{A}}^{\leq 0}(\breve{\beta})$ *is compatible with the convolution of singularities:* $\mathcal{L}^\beta \overset{\triangledown}{\varphi} * \overset{\triangledown}{\psi} = \big(\mathcal{L}^\beta \overset{\triangledown}{\varphi}\big)\big(\mathcal{L}^\beta \overset{\triangledown}{\psi}\big)$*. Moreover,* $\mathcal{L}^\beta(\partial \overset{\triangledown}{\varphi}) = \partial \mathcal{L}^\beta \overset{\triangledown}{\varphi}$*.*

Proof. We go back to the very definition of the convolution product of microfunctions and singularities. For $\overset{\triangledown}{\varphi}, \overset{\triangledown}{\psi} \in \mathrm{SING}_{\theta,\alpha}$, for any $\beta \in \hat{J} =]-\alpha+\theta, \theta+\alpha[$, the convolution product $\overset{\triangledown}{\varphi} * \overset{\triangledown}{\psi}$ can be represented, for $\zeta \in \dot{\mathfrak{S}}_{2\varepsilon}(]\beta - 2\pi, \beta[)$ with $\varepsilon > 0$ as small as we want, by

$$\overset{\vee}{\varphi} *_{\Gamma\times\Gamma} \overset{\vee}{\psi}(\zeta) = \frac{1}{2i\pi}\int_{\Gamma\times\Gamma} \frac{\mathrm{e}^{\nu(\zeta-(\xi_1+\xi_2))}}{\zeta-(\xi_1+\xi_2)} \overset{\vee}{\varphi}(\xi_1)\overset{\vee}{\psi}(\xi_2) d\xi_1 d\xi_2, \qquad (7.11)$$

(see 7.8), where $\Gamma = \Gamma_{\beta,\varepsilon,\eta_1,\eta_2}$ is as in definition 7.7 and where $\overset{\vee}{\varphi}, \overset{\vee}{\psi}$ are thought of as belonging to $\mathcal{O}(\dot{\mathfrak{s}}_0(]\beta-2\pi,\beta[))$. In (7.11), $\nu \in \mathbb{C}$ is a free parameter which can be chosen at our convenience.
We now assume that $\overset{\triangledown}{\varphi}, \overset{\triangledown}{\psi} \in \mathrm{SING}^{\mathrm{sum}}_{\theta,\alpha}$ and that $\overset{\vee}{\varphi}, \overset{\vee}{\psi}$ are summable majors in the direction β. In that case, choosing $\nu = |\nu|\mathrm{e}^{-i\beta}$ with $|\nu|$ large enough to ensure the integrability, one can rather consider the convolution product $\overset{\triangledown}{\varphi} * \overset{\triangledown}{\psi}$ as represented by (7.11), but this time with an endless path $\Gamma = \Gamma_{\beta,\varepsilon}$ (see definition 7.7). This construction gives a major of $\overset{\triangledown}{\varphi}, \overset{\triangledown}{\psi}$ which is summable in the direction β. Moreover, the arguments used in the proof of the proposition 7.10 show that $\mathcal{L}^\beta \overset{\triangledown}{\varphi} * \overset{\triangledown}{\psi} = \big(\mathcal{L}^\beta \overset{\triangledown}{\varphi}\big)\big(\mathcal{L}^\beta \overset{\triangledown}{\psi}\big)$. □

Example 7.7. We consider the formal Borel transform $\widehat{w}_{(0,0)} = \tilde{\mathcal{B}}\widetilde{w}_{(0,0)}$ where $\widetilde{w}_{(0,0)}$ is the formal series solution of the prepared ODE (3.6) associated with the first Painlevé equation. We know by theorem 3.2 that $\widehat{w}_{(0,0)}$ can be analytically continued to the star-shaped domain $\dot{\mathscr{R}}^{(0)}$ with at most exponential growth of order 1 at infinity along non-horizontal directions. We set $\overset{\triangledown}{w}_{(0,0)} = {}^\flat\widehat{w}_{(0,0)} \in \mathrm{SING}^{\mathrm{int}}$. Then $\overset{\triangledown}{w}_{(0,0)} \in \mathrm{SING}^{\mathrm{sum}}_{\pi/2,\pi/2}$ (or $\overset{\triangledown}{w}_{(0,0)} \in \mathrm{SING}^{\mathrm{sum}}_{-\pi/2,\pi/2}$) : just consider the major $\overset{\vee}{w}_{(0,0)}(\zeta) = \widehat{w}_{(0,0)}(\zeta)\frac{\log(\zeta)}{2i\pi}$. The Laplace transform $\mathcal{L}^{]0,\pi[}\,\overset{\triangledown}{w}_{(0,0)}$ is well-defined and gives a section of $\mathcal{A}^{\leq 0}$ on $]-3\pi/2,\pi/2[$. As a matter of fact,

$$\mathcal{L}^{]0,\pi[}\,\overset{\triangledown}{w}_{(0,0)} = \mathcal{L}^{]0,\pi[}\widehat{w}_{(0,0)} = \mathscr{S}^{]0,\pi[}\widetilde{w}_{(0,0)}$$

and $\mathcal{L}^{]0,\pi[}\,\overset{\triangledown}{w}_{(0,0)}$ can be thought of as belonging to the space of sections $\Gamma(]-3\pi/2,\pi/2[,\mathcal{A}_1)$.
We now consider the singularity $\overset{\triangledown}{I}_\sigma * \overset{\triangledown}{w}_{(0,0)}$, for any $\sigma \in \mathbb{C}$. Using propositions 7.12 and 7.14, this singularity belongs (for instance) to $\mathrm{SING}^{\mathrm{sum}}_{\pi/2,\pi/2}$ and

$$\mathcal{L}^{]0,\pi[}\,\overset{\triangledown}{I}_\sigma * \overset{\triangledown}{w}_{(0,0)} = \big(\mathcal{L}^{]0,\pi[}\,\overset{\triangledown}{I}_\sigma\big)\big(\mathcal{L}^{]0,\pi[}\,\overset{\triangledown}{w}_{(0,0)}\big) = \frac{1}{z^\sigma}\mathscr{S}^{]0,\pi[}\widetilde{w}_{(0,0)},$$

this time viewed as a multivalued section $\mathcal{A}^{\leq 0}$ on $]-3\pi/2, \pi/2[\subset \underset{\bullet}{\mathbb{S}}^1$.

7.7 Spaces of resurgent functions

7.7.1 Preliminaries

We refer the reader to [1] (Pré I.3, lemme 3.0) for the proof of the following key-lemma, the idea of which being due to Ecalle.

Lemma 7.5 (Key-lemma). *Let $\Gamma \subset \mathbb{C}$ be an embedded curve, transverse to the circles $|\zeta| = R$ for all $R \geq R_0 > 0$. Let Φ be a holomorphic function on a neighbourhood of Γ. Then, for any continuous function $m : \mathbb{R}^+ \to \mathbb{R}^+$ so that $\inf\{m([0,\xi])\} > 0$ for all $\xi > 0$, there exists $\Psi \in \mathcal{O}(\mathbb{C})$ such that, for all $\zeta \in \Gamma$, $|\Phi(\zeta) + \Psi(\zeta)| \leq m(|\zeta|)$.*

In what follows, we use the notations introduced in definition 7.7. We also remind that $\widetilde{\mathbb{C}\setminus\mathbb{Z}}$ stands for the universal covering of $\mathbb{C}\setminus\mathbb{Z}$. One may also think of $\widetilde{\mathbb{C}\setminus\mathbb{Z}}$ as the universal covering of $\underset{\bullet}{\mathbb{C}} \setminus \bigcup\limits_{\theta=\pi k,\, k\in\mathbb{Z}} \{m\mathrm{e}^{\mathrm{i}\theta} \mid m \in \mathbb{N}^\star\}$.

Lemma 7.6. *Let $\overset{\triangledown}{\varphi} \in \mathrm{SING}$ be a singularity which can be determined by a major that can be analytically continued to $\widetilde{\mathbb{C}\setminus\mathbb{Z}}$. Then, for any direction θ and any $\varepsilon > 0$ small enough, the singularity $\overset{\triangledown}{\varphi}$ has a major $\overset{\vee}{\varphi}$ with the following properties:*

1. *the restriction of $\overset{\vee}{\varphi}$ as a sectorial germ of codirection $I =]-\pi/2+\theta, \theta+\pi/2[$, can be represented by a function Φ holomorphic on the cut plane $\mathbb{C}\setminus[0, \mathrm{e}^{\mathrm{i}\theta}\infty[= \overset{\bullet}{\mathfrak{s}}{}_0^{\infty}(\check{I})$, $\check{I} =]-2\pi+\theta, \theta[$;*
2. *Φ is bounded on $\overset{\bullet}{\mathfrak{S}}_{\varepsilon'}(I)$, for every $\varepsilon' > \varepsilon$.*
3. *Φ can be analytically continued to $\widetilde{\mathbb{C}\setminus\mathbb{Z}}$.*

Proof. Let $\overset{\vee}{\varphi}_1$ be a major of $\overset{\triangledown}{\varphi}$ that can be analytically continued to $\widetilde{\mathbb{C}\setminus\mathbb{Z}}$. This major can be represented by a function Φ_1 holomorphic on $\overset{\bullet}{\mathfrak{s}}{}_0^{R}(\check{I}) \cup S_{2\varepsilon}(\hat{I}) \setminus [0, \mathrm{e}^{\mathrm{i}\theta}\infty[$, for $R > 0$ and $\varepsilon > 0$ small enough, and Φ_1 can be analytically continued to $\widetilde{\mathbb{C}\setminus\mathbb{Z}}$. The boundary $\Gamma_{I,\varepsilon} = -\partial\overset{\bullet}{\mathfrak{S}}_{\varepsilon}(I)$ can be seen as an embedded curve $H_0 : \mathbb{R} \to \mathbb{C}$ that fulfills the condition of lemma 7.5 : one can find a function $\Psi_1 \in \mathcal{O}(\mathbb{C})$ so that $\Phi_2 = \Phi_1 + \Psi_1$ satisfies $|\Phi_2(\eta)| \leq \exp(-|\eta|)$ for all $\eta \in \Gamma_{I,\varepsilon}$. One can also assume that $|H_0'(s)|$ is bounded and these conditions ensure the integrability for the integral $\Phi(\zeta) = \dfrac{1}{2i\pi}\displaystyle\int_{H_0} \frac{\Phi_2(\eta)}{\zeta-\eta} d\eta$ which thus, defines a holomorphic function on $\overset{\bullet}{\mathfrak{S}}_{\varepsilon}(I)$. Moreover, one easily sees by Cauchy that $\Phi = \Phi_2 + \Psi_2$ where $\Psi_2 \in \mathcal{O}_0$. One observes that $|\zeta - \eta| \geq \varepsilon' - \varepsilon$ for $(\zeta, \eta) \in \overset{\bullet}{\mathfrak{S}}_{\varepsilon'}(I) \times \Gamma_{I,\varepsilon}$, with $\varepsilon' > \varepsilon$. Thus Φ is bounded on $\overset{\bullet}{\mathfrak{S}}_{\varepsilon'}(I)$. Notice that Φ_2 inherit from Φ_1 the property of being analytically continuable on $\widetilde{\mathbb{C}\setminus\mathbb{Z}}$. Thus one can analytically con-

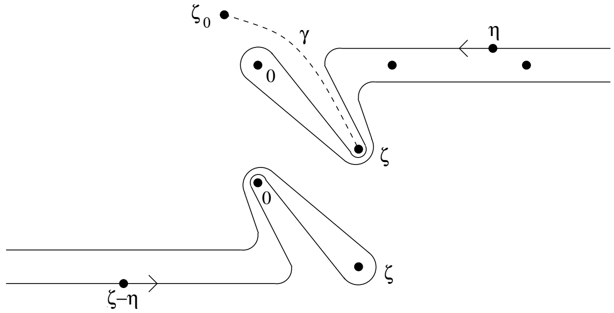


Fig. 7.8 Deformation of the contour $\Gamma_{I,\varepsilon}$ by an isotopy equal to the identity in a neighbourhood of infinity, for $\theta = 0$.

tinue Φ on $\widetilde{\mathbb{C}\setminus\mathbb{Z}}$ by Cauchy, by deformation of the contour by isotopies[5] $H : (s,t) \in \mathbb{R}\times[0,1] \mapsto H(s,t) = H_t(s) \in \mathbb{C}\setminus\mathbb{Z}$ that are equal to the identity in a neighbourhood of infinity, Fig. 7.8.

Finally, from the fact that $\Phi = \Phi_1 + \Psi$ with $\Psi_1 + \Psi_2 \in \mathcal{O}_0$, we see that Φ defines a sectorial germ $\overset{\vee}{\varphi}$ of codirection $I =]-\pi/2+\theta, \theta+\pi/2[$ whose associated microfunction coincides with the restriction of $\overset{\triangledown}{\varphi}$ to the codirection I. □

Lemma 7.7. *Let $\overset{\triangledown}{\varphi} \in \mathrm{SING}$ be a singularity which can be determined by a major that can be analytically continued to $\widetilde{\mathbb{C}\setminus\mathbb{Z}}$. Then, for any direction θ and for any $\varepsilon > 0$ small enough, the singularity $\overset{\triangledown}{\varphi}$ has a major $\overset{\vee}{\varphi}$ with the following properties:*

1. *the restriction of $\overset{\vee}{\varphi}$ as a sectorial germ of codirection $I =]-\pi/2+\theta, \theta+\pi/2[$, can be represented by a function Φ holomorphic on the cut plane $\mathbb{C}\setminus[0, \mathrm{e}^{\mathrm{i}\theta}\infty[= \dot{\mathfrak{s}}_0^\infty(\check{I})$, $\check{I} =]-2\pi+\theta, \theta[$;*
2. *$|\Phi(\eta)| \le \exp(-|\eta|)$ for all $\eta \in \Gamma_{I,\varepsilon}$, where $\Gamma_{I,\varepsilon} = -\partial\dot{\mathfrak{S}}_\varepsilon(I) \subset \dot{\mathfrak{s}}_0^\infty(\check{I})$;*
3. *Φ can be analytically continued to $\widetilde{\mathbb{C}\setminus\mathbb{Z}}$.*

Proof. Just consider first the function Φ_1 given by lemma 7.6, then use lemma 7.5 to define Φ from Φ_1. □

The above lemmas 7.6 and 7.7 motivate the introduction of new Riemann surfaces that will be used in a moment.

Definition 7.33. Let $\theta \in \mathbb{S}^1$ be a direction and $\zeta_0 \in \mathbb{C}\setminus[0, \mathrm{e}^{\mathrm{i}\theta}\infty[$. We note $\mathfrak{R}^\theta_{\zeta_0}$ the set of paths of the form $\lambda = \lambda_1\lambda_2$ where $\underline{\lambda}_1 : [0,1] \to \mathbb{C}\setminus[0, \mathrm{e}^{\mathrm{i}\theta}\infty[$ with $\lambda(0) = \zeta_0$, and $\underline{\lambda}_2 : [0,1] \to \mathbb{C}\setminus\mathbb{Z}$.
For $\lambda \in \mathfrak{R}^\theta_{\zeta_0}$, we note $\mathrm{cl}(\lambda)$ its equivalence class for the relation of homotopy $\sim_{\mathfrak{R}^\theta_{\zeta_0}}$ of paths in $\mathfrak{R}^\theta_{\zeta_0}$ with fixed extremities. We set

$$\mathcal{R}^\theta_{\zeta_0} = \{\zeta = \mathrm{cl}(\lambda) \mid \lambda \in \mathfrak{R}^\theta_{\zeta_0}\} \text{ and } \mathfrak{p} : \zeta = \mathrm{cl}(\lambda) \mapsto \dot{\zeta} = \underline{\lambda}(1) \in \mathbb{C}^\star.$$

---

[5] That is H is a homotopy and for each $t \in [0,1]$, H_t is an embedding. We remind that we see $\Gamma_{I,\varepsilon}$ as an embedded curve $H_0 : \mathbb{R} \to \mathbb{C}$.

Proposition 7.15. *The space $\mathcal{R}^{\theta}_{\zeta_0}$ can be equipped with a separated topology that makes $(\mathcal{R}^{\theta}_{\zeta_0}, \mathfrak{p})$ an étalé space. The space $\mathcal{R}^{\theta}_{\zeta_0}$ is arcconnected and simply connected, thus defines a Riemann surface by pulling back by $\mathfrak{p}$ the complex structure of $\mathbb{C}$. Moreover, for two points $\zeta_0, \zeta_1 \in \mathbb{C} \backslash [0, \mathrm{e}^{\mathrm{i}\theta}\infty[$, the two Riemann surfaces $\mathcal{R}^{\theta}_{\zeta_0}$ and $\mathcal{R}^{\theta}_{\zeta_1}$ are isomorphic.*

The proof of proposition 7.15 is left as an exercise. We complete the above proposition with a definition.

Definition 7.34. We note $\mathcal{R}^{\theta}$ for the equivalent class of Riemann surfaces $\mathcal{R}^{\theta}_{\zeta_0}$ given by proposition 7.15. We note $\overset{\bullet}{\mathcal{R}}{}^{\theta,(0)} = \mathbb{C} \setminus [0, \mathrm{e}^{\mathrm{i}\theta}\infty[$. We call "principal sheet" the unique domain $\mathcal{R}^{\theta,(0)} \subset \mathcal{R}^{\theta}$ so that the resctriction $\mathfrak{p}|_{\mathcal{R}^{\theta,(0)}}$ realizes a homeomorphism between $\mathcal{R}^{\theta,(0)}$ and the simply connected domain $\overset{\bullet}{\mathcal{R}}{}^{\theta,(0)}$.

7.7.2 Resurgent functions

Various spaces of so-called resurgent functions can be defined and used according to the context. We start with the notion of resurgent singularities.

7.7.2.1 Resurgent singularities, resurgent asymptotic classes

Definition 7.35. A singularity $\overset{\triangledown}{\varphi} \in \mathrm{SING}$ is said to be **$\mathbb{Z}$-resurgent** when it can be determined by a major $\overset{\vee}{\varphi} \in \mathrm{ANA}$ that can be analytically continued to $\widetilde{\mathbb{C} \setminus \mathbb{Z}}$. We denote by $\mathrm{RES}_{\mathbb{Z}}$ or simply RES the space of $\mathbb{Z}$-resurgent singularities.

A $\mathbb{Z}$-resurgent singularity is often simply called a $\mathbb{Z}$-resurgent function. Throughout this course we will usually write "resurgent singularity" in place of $\mathbb{Z}$-resurgent singularity.

Remark 7.5. It is important to keep in mind that the minor $\widehat{\varphi}$ of any resurgent singularity $\overset{\triangledown}{\varphi} \in \mathrm{RES}$, can be analytically continued to $\widetilde{\mathbb{C} \setminus \mathbb{Z}}$, since the minor $\widehat{\varphi}$ does not depend on the chosen major.

Definition 7.36. One says that $\overset{\triangledown}{\varphi} \in \mathrm{RES}$ is a **resurgent constant** when $\overset{\triangledown}{\varphi}$ has a major which can be analytically continued to $\underset{\bullet}{\mathbb{C}}$. The space of resurgent constants is denoted by CONS.

Definition 7.37. An asymptotic class $\overset{\vartriangle}{\varphi} \in \mathrm{ASYMP}$ is called a **$\mathbb{Z}$-resurgent asymptotic class**, <u>*resp.*</u> a **resurgent constant**, when its formal Borel transform $\overset{\triangledown}{\varphi}$ is a $\mathbb{Z}$-resurgent singularity, <u>*resp.*</u> a resurgent constant. We denote by $\widetilde{\mathrm{RES}}_{\mathbb{Z}}$ or simply $\widetilde{\mathrm{RES}}$ the space made of $\mathbb{Z}$-resurgent asymptotic classes. We denote by $\widetilde{\mathrm{CONS}}$ the sub-space of resurgent constants.

A $\mathbb{Z}$-resurgent asymptotic class is often simply called a $\mathbb{Z}$-resurgent function or even a resurgent function.

Example 7.8. The singularities $\overset{\triangledown}{I}_{\sigma}$ and $\overset{\triangledown}{J}_{\sigma,m}$ are resurgent constants, as well as their associated asymptotic classes $\overset{\vartriangle}{I}_{\sigma}$ and $\overset{\vartriangle}{J}_{\sigma,m}$.

7.7.2.2 Resurgent functions, resurgent series

We remind the following simple definition, for objects much discussed in [24].

Definition 7.38. The $\mathbb{C}$-differential commutative and associative convolution algebra $\mathbb{C}\delta \oplus \hat{\mathscr{R}}_{\mathbb{Z}}$ with unit δ, is called a space of $\mathbb{Z}$-resurgent functions. We denote by $\overset{\triangledown}{\mathscr{R}}_{\mathbb{Z}} \subset \mathrm{RES}$ the $\mathbb{C}$-differential commutative and associative convolution algebra made of resurgent singularities of the form $\overset{\triangledown}{\varphi} = a\delta + {}^\flat\widehat{\varphi}$ with $\widehat{\varphi} \in \hat{\mathscr{R}}_{\mathbb{Z}}$.

Since $\mathbb{C}\delta \oplus \hat{\mathscr{R}}_{\mathbb{Z}}$ is a convolution algebra, the identity ${}^\flat\widehat{\varphi} * {}^\flat\widehat{\varphi} = {}^\flat (\widehat{\varphi} * \widehat{\varphi})$ (proposition 7.6) implies that $\overset{\triangledown}{\mathscr{R}}_{\mathbb{Z}}$ is indeed a convolution algebra. One usually uses abridged notation $\overset{\triangledown}{\mathscr{R}}$ in this course.

Definition 7.39. A series expansion $\widetilde{\varphi} \in \mathbb{C}[[z^{-1}]]$ is a $\mathbb{Z}$-resurgent series when its formal Borel transform $\tilde{\mathcal{B}}\widetilde{\varphi}$ is a $\mathbb{Z}$-resurgent function or, equivalently, when the asymptotic class ${}^\natural\widetilde{\varphi}$ belongs to $\widetilde{\mathrm{RES}}_{\mathbb{Z}}$. We denote by $\widetilde{\mathscr{R}}_{\mathbb{Z}}$ the $\mathbb{C}$-differential commutative and associative algebra made of $\mathbb{Z}$-resurgent series.

Throughout this course we usually simply write "resurgent functions" or "resurgent series" instead of $\mathbb{Z}$-resurgent functions or $\mathbb{Z}$-resurgent series, since there is no risk of misunderstanding.

7.7.2.3 Resurgent singularities and convolution

Theorem 7.1. *The space* RES *is a $\mathbb{C}$-differential commutative and associative convolution algebra with unit δ, and* CONS $\subset$ RES *is a subalgebra. Therefore, the space* $\widetilde{\mathrm{RES}}$ *is a $\mathbb{C}$-differential commutative and associative algebra and* $\widetilde{\mathrm{CONS}} \subset \widetilde{\mathrm{RES}}$ *is a subalgebra.*

Proof. (Adapted from [8, 1]. The reader should look before at the reasoning made for the proof of proposition 4.6).
It is enough to only show that RES is a convolution space. We take two singularities $\overset{\triangledown}{\varphi}, \overset{\triangledown}{\psi} \in \mathrm{RES}$, we choose a direction θ and we suppose $0 < \varepsilon \ll 1$. By lemma 7.7 –<u>resp</u>. lemma 7.6– $\overset{\triangledown}{\varphi}$ –<u>resp</u>. $\overset{\triangledown}{\psi}$– has a major such that its restriction as a sectorial germ of codirection $I =]-\pi/2+\theta, \theta+\pi/2[$, can be represented by a function $\overset{\vee}{\varphi}$ –<u>resp</u>. $\overset{\vee}{\psi}$–, holomorphic on $\overset{\bullet}{\mathcal{R}}{}^{\theta,(0)}$, that can be analytically continued to the Riemann surface $(\mathcal{R}^\theta, \mathfrak{p})$ and moreover, satisfies the condition:

1. $|\overset{\vee}{\varphi}(\eta)| \leq \exp(-|\eta|)$ for all $\eta \in \mathit{\Gamma}_{I,\varepsilon}$, where $\mathit{\Gamma}_{I,\varepsilon} = -\partial \overset{\bullet}{\mathfrak{S}}_\varepsilon(I) \subset \overset{\bullet}{\mathcal{R}}{}^{\theta,(0)}$;
2. $\overset{\vee}{\psi}$ is bounded on $\overset{\bullet}{\mathfrak{S}}_\varepsilon(I)$.

We know by lemma 7.1 that $\zeta - \mathit{\Gamma}_{I,\varepsilon} \subset \overset{\bullet}{\mathfrak{S}}_\varepsilon(I)$ for every $\zeta \in \overset{\bullet}{\mathfrak{S}}_{2\varepsilon}(I)$. We also think of $\mathit{\Gamma}_{I,\varepsilon}$ as an embedded curve $H_0 : \mathbb{R} \to \mathbb{C}$ with $|H_0'(s)|$ bounded. Therefore, the above properties and the dominated Lebesgue theorem, ensure that the integral

$$\chi(\zeta) = \overset{\vee}{\varphi} *_{H_0} \overset{\vee}{\psi}(\zeta) = \int_{H_0} \overset{\vee}{\varphi}(\eta)\overset{\vee}{\psi}(\zeta - \eta)d\eta \tag{7.12}$$

defines a holomorphic function on $\overset{\bullet}{\mathfrak{S}}_{2\varepsilon}(I) \subset \overset{\bullet}{\mathcal{R}}^{\theta,(0)}$ which by (7.7), represents the convolution product $\overset{\triangledown}{\varphi} * \overset{\triangledown}{\psi}$. We want to show that χ can be analytically continued onto the Riemann surface $(\mathcal{R}^\theta, \mathfrak{p})$ (thus to $\widetilde{\mathbb{C}\setminus\mathbb{Z}}$ as well).

We choose a point $\zeta_0 \in \overset{\bullet}{\mathfrak{S}}_{2\varepsilon}(I)$ so that $\{\zeta_0 - H_0\} \cap \mathbb{Z} = \emptyset$, and we view χ as a germ of holomorphic functions at ζ_0: for $\xi \in \mathbb{C}$ close to 0, $\chi(\zeta_0 + \xi) = \displaystyle\int_{H_0} \overset{\vee}{\varphi}(\eta)\overset{\vee}{\psi}(\xi + \zeta_0 - \eta)d\eta$. We take a smooth path $\gamma : [0,1] \to \mathbb{C}\setminus\mathbb{Z}$ starting from $\zeta_0 = \gamma(0)$. We fix $R \gg \varepsilon$ so that $\gamma([0,1]) \subset D(0,R)$ and $\mathrm{length}(\gamma) < R$. We will get the analytic continuation of χ along γ by continuously deforming H_0 through an isotopy $H : (s,t) \in \mathbb{R}\times[0,1] \mapsto H_t(s) \in \mathbb{C}\setminus\mathbb{Z}$ that is equal to the identity for $|s|$ large enough. One introduces a $\mathcal{C}^1$ function $\eta : \mathbb{C} \to [0,1]$ satisfying $\{\zeta \in \mathbb{C} \mid \eta(\zeta) = 0\} = \mathbb{Z}$. We also set a $\mathcal{C}^1$ function $\rho : \mathbb{C} \to [0,1]$ with compact support so that the conditions $\rho|_{D(0,5R)} = 1$ and $\rho|_{\mathbb{C}\setminus D(0,6R)} = 0$ are fulfilled. In what follows, we see H_0 as an embedded curve $\mathbb{R} \to \mathbb{C}$ and there is no loss of generality in supposing the existence of $s_0 > 0$ so that $H_0(s) \in D(0,3R)$ for $|s| < s_0$, else $H_0(s) \in \mathbb{C}\setminus D(0,3R)$.

One considers the non-autonomous vector field $X(\zeta,t) = \dfrac{\eta(\zeta)\rho(\zeta)}{\eta(\zeta) + \eta\big(\gamma(t) - \zeta\big)}\gamma'(t)$.

We note $g : (t_0, t, \zeta_0) \in [0,1]^2 \times \mathbb{C} \mapsto g(t_0,t,\zeta_0) = g^{t_0,t}(\zeta_0) \in \mathbb{C}$ the (well-defined global) flow of the vector field, that is $t \in [0,1] \mapsto \zeta(t) = g^{t_0,t}(\zeta_0)$ is the unique integral curve satisfying both $\dfrac{\mathrm{d}\zeta}{\mathrm{d}t} = X(\zeta,t)$ and the datum $\zeta(t_0) = \zeta_0$. One finally notes $\phi_t(\zeta) = g^{0,t}(\zeta)$.

Notice that any integral curve $\zeta(t)$ of X has length less than $\mathrm{length}(\gamma) < R$, since $|X(\zeta,t)| \le |\gamma'(t)|$. With this remark and arguments detailed in [24], we can observe the following properties, for every $t \in [0,1]$:

1. $\phi_t(\gamma(0)) = \gamma(t)$, that is γ is an integral curve. (Notice that $\rho\big(\gamma(t)\big) = 1$ because $\gamma([0,1]) \subset D(0,R)$).
2. $\phi_t(\mathbb{C}\setminus\mathbb{Z}) \subset \mathbb{C}\setminus\mathbb{Z}$. (One has $\phi_t(\omega) = \omega$ for any $\omega \in \mathbb{Z}$ since $\eta(\omega) = 0$).
3. $\phi_t(\zeta) = \zeta$ for any $\zeta \in \mathbb{C}\setminus D(0,6R)$ (since $\rho|_{\mathbb{C}\setminus D(0,6R)} = 0$).
4. for every $\zeta \in D(0,3R)$, $\phi_t\big(\gamma(0) - \zeta\big) = \gamma(t) - \phi_t\big(\zeta\big)$. Indeed, if $t \mapsto \zeta(t)$ is an integral curve starting from $\zeta(0) \in D(0,3R)$, then $\zeta(t) \in D(0,4R)$ for every $t \in [0,1]$ (the integral curve have length $< R$), thus $\dfrac{\mathrm{d}\zeta}{\mathrm{d}t} = \dfrac{\eta(\zeta)}{\eta(\zeta) + \eta\big(\gamma(t) - \zeta\big)}\gamma'(t)$. Consider $\xi(t) = \gamma(t) - \zeta(t)$; one has $\dfrac{\mathrm{d}\xi}{\mathrm{d}t} = \dfrac{\eta(\xi)}{\eta(\xi) + \eta\big(\gamma(t) - \xi\big)}\gamma'(t) = \dfrac{\eta(\xi)\rho(\xi)}{\eta(\xi) + \eta\big(\gamma(t) - \xi\big)}\gamma'(t)$ because $|\xi(t)| < 5R$ for every $t \in [0,1]$, thus ξ is an integral curve of X.
5. for every $\zeta \in \mathbb{C}\setminus D(0,3R)$, $|\gamma(t) - \phi_t\big(\zeta\big)| > R$. As a matter of fact, observe that if $t \mapsto \zeta(t)$ is an integral curve starting from $\zeta(0) \in \mathbb{C}\setminus D(0,3R)$, then $|\zeta(t)| > 2R$ for every $t \in [0,1]$ and therefore $|\gamma(t) - \phi_t\big(\zeta\big)| > R$.

We define the isotopy $H : (s,t) \in \mathbb{R}\times[0,1] \mapsto H(s,t) = H_t(s)$ by setting $H_t(s) = \phi_t\big(H_0(s)\big)$. Since H_0 avoids $\mathbb{Z}$, one has $H_t(s) \in \mathbb{C}\setminus\mathbb{Z}$ by property 2. By property 3, we remark that for $|s|$ large enough, H is a constant map. Notice also that $H_0 \subset \overset{\bullet}{\mathcal{R}}^{\theta,(0)}$ can be lifted uniquely with respect to $\mathfrak{p}$ on the principal sheet $\mathcal{R}^{\theta,(0)}$ of $\mathcal{R}^\theta$. We note $\mathcal{H}_0$ this lifting. We can use the lifting theorem for homotopies [11, 5] to get the continuous mapping $\mathcal{H} : (s,t) \in \mathbb{R}\times[0,1] \mapsto \mathcal{H}(s,t) = \mathcal{H}_t(s) \in \mathcal{R}^\theta$ which makes commuting the following diagram:

$$
\begin{array}{ccc}
 & & \mathcal{R}^\theta \\
 & \mathcal{H} \nearrow & \downarrow \mathfrak{p} \\
\mathbb{R}\times[0,1] & \underset{H}{\longrightarrow} & \mathbb{C}.
\end{array}
\tag{7.13}
$$

We now set $K:(s,t)\in\mathbb{R}\times[0,1]\mapsto K(s,t)=K_t(s)=\gamma(t)-H_t(s)$. We know that $K_0(s)=\gamma(0)-H_0(s)\in\overset{\bullet}{\mathfrak{S}}_\varepsilon(I)\subset\overset{\bullet}{\mathcal{R}}^{\theta,(0)}$ for every $s\in\mathbb{R}$. In particular, one can lift K_0 uniquely with respect to $\mathfrak{p}$ into an embedded curve $\mathcal{K}_0$ on the principal sheet $\mathcal{R}^{\theta,(0)}$ of $\mathcal{R}^\theta$. Moreover $K_0(s)\in\mathbb{C}\setminus\mathbb{Z}$, for every $s\in\mathbb{R}$. Property 5 ensures that $K_t(s)$ stays in $\overset{\bullet}{\mathfrak{S}}_\varepsilon(I)$ for $|s|\geq s_0$, otherwise by property 4, $K_t(s)$ belongs to $\mathbb{C}\setminus\mathbb{Z}$. This implies that K_t can be lifted uniquely with respect to $\mathfrak{p}$ into an embedded curve $\mathcal{K}_t$ which lies on the principal sheet $\mathcal{R}^{\theta,(0)}$ of $\mathcal{R}^\theta$ for $|s|\geq s_0$. Applying again the lifting theorem for homotopies, one obtains a continuous mapping $\mathcal{K}:(s,t)\in\mathbb{R}\times[0,1]\mapsto\mathcal{H}(s,t)=\mathcal{H}_t(s)\in\mathcal{R}^\theta$ that makes commuting the following diagram:

$$
\begin{array}{ccc}
 & & \mathcal{R}^\theta \\
 & \mathcal{K} \nearrow & \downarrow \mathfrak{p} \\
\mathbb{R}\times[0,1] & \underset{K}{\longrightarrow} & \mathbb{C}.
\end{array}
\tag{7.14}
$$

We finally introduce the two holomorphic functions $\Phi,\Psi\in\mathcal{O}(\mathcal{R}^\theta)$ such that $\Phi(\zeta)=\overset{\vee}{\varphi}\big(\mathfrak{p}(\zeta)\big)$, $\Psi(\zeta)=\overset{\vee}{\psi}\big(\mathfrak{p}(\zeta)\big)$ for $\zeta\in\mathcal{R}^{\theta,(0)}$. With these notations, the germ of holomorphic functions χ at $\zeta_0=\gamma(0)$ reads

$$\chi(\gamma(0)+\xi)=\int_{\mathbb{R}}\Phi\big(\mathcal{H}_0(s)\big)\Psi(\xi+\mathcal{K}_0(s))H_0'(s)ds$$

and its analytic continuation along γ is obtained by

$$\chi(\gamma(t)+\xi)=\int_{\mathbb{R}}\Phi\big(\mathcal{H}_t(s)\big)\Psi(\xi+\mathcal{K}_t(s))H_t'(s)ds. \tag{7.15}$$

Indeed, remark that for $|s|$ large enough, $\Phi\big(\mathcal{H}_t(s)\big)=\overset{\vee}{\varphi}\big(H_t(s)\big)$ and $|\overset{\vee}{\varphi}\big(H_t(s)\big)|\leq\exp(-|H_t(s)|)$. Also, for $|s|\geq s_0$, $\Psi\big(\xi+\mathcal{K}_t(s)\big)=\overset{\vee}{\psi}\big(K_t(s)\big)$ which is bounded since $K_t(s)\in\overset{\bullet}{\mathfrak{S}}_\varepsilon(I)$. Thus the integral (7.15) is well-defined. The fact that (7.15) provides the analytic continuations comes from the Cauchy formula, see analogous arguments in, e.g. [24]. □

7.7.2.4 Supplements

One often uses other spaces in practice as we now exemplify.

The space $\mathbf{RES}^{(\theta,\alpha)}(L)$ The space $\hat{\mathscr{R}}^{(\theta,\alpha)}(L)$ was introduced by definition 4.18 and we know by proposition 4.6 that $\mathbb{C}\delta\oplus\hat{\mathscr{R}}^{(\theta,\alpha)}(L)$ is a convolution algebra. The following definition thus makes sense.

Definition 7.40. We denote by $\overset{\nabla}{\mathscr{R}}^{(\theta,\alpha)}(L)\supset\overset{\nabla}{\mathscr{R}}$ the $\mathbb{C}$-differential commutative and associative convolution algebra made of singularities of the form

$\overset{\triangledown}{\varphi} = a\delta + {}^\flat\widehat{\varphi} \in \mathrm{SING}$ with $\widehat{\varphi} \in \hat{\mathscr{R}}^{(\theta,\alpha)}(L)$. The associated space of formal series is denoted by $\widetilde{\mathscr{R}}^{(\theta,\alpha)}(L)$.

By its very definition, any element $\widehat{\varphi} \in \hat{\mathscr{R}}^{(\theta,\alpha)}(L)$ is a germ of holomorphic functions at 0 that can be analytically continued to the Riemann surface $\mathscr{R}^{(\theta,\alpha)}(L)$. This means that any $\overset{\triangledown}{\varphi} \in \overset{\triangledown}{\mathscr{R}}^{(\theta,\alpha)}(L)$ is a simple singularity that has a major $\overset{\vee}{\varphi}$ which can be analytically continued to a Riemann surface $\underset{\bullet}{\mathscr{R}}^{(\theta,\alpha)}(L)$ constructed from $\mathscr{R}^{(\theta,\alpha)}(L) \setminus \{0\}$ as an étalé space above $\underset{\bullet}{\mathbb{C}}$ that makes commuting the diagram:

$$\begin{array}{ccc} \underset{\bullet}{\mathscr{R}}^{(\theta,\alpha)}(L) & & \\ \downarrow & \searrow & \\ \mathscr{R}^{(\theta,\alpha)}(L) \setminus \{0\} & & \underset{\bullet}{\mathbb{C}} \\ \downarrow & \swarrow & \\ \mathbb{C}^\star & & \end{array}$$

. (The construction of $\underset{\bullet}{\mathscr{R}}^{(\theta,\alpha)}(L)$ is obvious and is left to the reader).

Since $\overset{\triangledown}{\mathscr{R}}^{(\theta,\alpha)}(L)$ is a convolution algebra, we know that for any two singularities $\overset{\triangledown}{\varphi}, \overset{\triangledown}{\psi} \in \overset{\triangledown}{\mathscr{R}}^{(\theta,\alpha)}(L)$, their convolution product $\overset{\triangledown}{\varphi} * \overset{\triangledown}{\psi}$ belongs to $\overset{\triangledown}{\mathscr{R}}^{(\theta,\alpha)}(L)$ as well, thus has a major that can be analytically continued to $\underset{\bullet}{\mathscr{R}}^{(\theta,\alpha)}(L)$. In substance, this comes from the property that ${}^\flat\widehat{\varphi} * {}^\flat\widehat{\varphi} = {}^\flat (\widehat{\varphi} * \widehat{\varphi})$ for two integrable singularities (proposition 7.6). Now, what about the convolution product $\overset{\triangledown}{\varphi} * \overset{\triangledown}{\psi}$ of two singularities $\overset{\triangledown}{\psi} \in \overset{\triangledown}{\mathscr{R}}^{(\theta,\alpha)}(L)$ and $\overset{\triangledown}{\varphi} \in \mathrm{RES}$? To give the answer, we prefer to shift to a more general case and we introduce a new definition.

Definition 7.41. Let be $\theta \in \{0, \pi\} \subset \mathbb{S}^1$, $\alpha \in]0, \pi/2]$ and $L > 0$. We denote by $\mathrm{RES}^{(\theta,\alpha)}(L)$ the space made of singularities that have majors that can be analytically continued to the Riemann surface $\underset{\bullet}{\mathscr{R}}^{(\theta,\alpha)}(L)$. The associated space of asymptotic classes is denoted by $\widetilde{\mathrm{RES}}^{(\theta,\alpha)}(L) \subset \mathrm{ASYMP}$.

Proposition 7.16. *The space* $\mathrm{RES}^{(\theta,\alpha)}(L)$ *is a* $\mathbb{C}$*-differential commutative and associative convolution algebra with unit* δ*, contained* RES *and* $\overset{\triangledown}{\mathscr{R}}^{(\theta,\alpha)}(L)$ *as subalgebras.*

Proof. The proof follows that of theorem 7.16 but for the fact that one adds the arguments used at the end of the proof of proposition 4.6.

The spaces $\mathbf{RES}^{(k)}$ The spaces $\hat{\mathscr{R}}^{(k)}$ were introduced by definition 4.15. They provide new spaces of singularities that are worthy of attention.

Definition 7.42. For $k \in \mathbb{N}^\star$, we denote by $\overset{\triangledown}{\mathscr{R}}^{(k)}$ the space of singularities of the form $\overset{\triangledown}{\varphi} = a\delta + {}^\flat\widehat{\varphi} \in \mathrm{SING}$ with $\widehat{\varphi} \in \hat{\mathscr{R}}^{(k)}$. The associated space of formal series is denoted by $\widetilde{\mathscr{R}}^{(k)}$.

Remark 7.6. Notice that the set of spaces $(\overset{\triangledown}{\mathscr{R}}^{(k)})_{k\in\mathbb{N}}$ provides an inverse system of spaces whose inverse limit $\varprojlim \overset{\triangledown}{\mathscr{R}}^{(k)} = \bigcap_k \overset{\triangledown}{\mathscr{R}}^{(k)}$ is $\overset{\triangledown}{\mathscr{R}}$. This is why we sometimes write $\overset{\triangledown}{\mathscr{R}}^{(\infty)} = \overset{\triangledown}{\mathscr{R}}$.
The space $\overset{\triangledown}{\mathscr{R}}^{(1)}$ is of particular interest since, from propositions 4.3 and 7.6, $\overset{\triangledown}{\mathscr{R}}^{(1)}$ makes a convolution algebra.

The space $\overset{\triangledown}{\mathscr{R}}^{(k)}$ is made of simple singularities that have majors that can be analytically continued to a Riemann surface $\underset{\bullet}{\mathscr{R}}^{(k)}$ above $\underset{\bullet}{\mathbb{C}}$ obviously deduced from $\mathscr{R}^{(k)} \setminus \{0\}$ and that makes commuting the diagram:

$$\begin{tikzcd} \underset{\bullet}{\mathscr{R}}^{(k)} \arrow[d] \arrow[dr] & \\ \mathscr{R}^{(k)} \setminus \{0\} \arrow[d] & \underset{\bullet}{\mathbb{C}} \arrow[dl] \\ \mathbb{C}^{\star} & \end{tikzcd}$$

. (The details are left to the reader).

We now consider larger spaces of singularities.

Definition 7.43. We denote by $\mathrm{RES}^{(k)}$ the space of singularities that have majors that can be analytically continued to the Riemann surface $\underset{\bullet}{\mathscr{R}}^{(k)}$, for $k \in \mathbb{N}^{\star}$. We denote by $\widetilde{\mathrm{RES}}^{(k)} \subset \mathrm{ASYMP}$ the space of asymptotic classes whose formal Borel transform belongs to $\mathrm{RES}^{(k)}$.

Remark 7.7. Notice again that $\underset{\leftarrow}{\lim} \mathrm{RES}^{(k)} = \bigcap_{k} \mathrm{RES}^{(k)} = \mathrm{RES}$, and we sometimes write $\mathrm{RES}^{(\infty)} = \mathrm{RES}$.

We will have a special interest in $\mathrm{RES}^{(1)}$ because of the following analogous to proposition 7.16.

Proposition 7.17. *The space* $\mathrm{RES}^{(1)}$ *is a* $\mathbb{C}$*-differential commutative and associative convolution algebra with unit* δ*. It contains* RES *and* $\overset{\triangledown}{\mathscr{R}}^{(1)}$ *as subalgebras.*

We omit the (rather lengthy) proof of this proposition. The main idea is to consider the integral representation (7.12) used in the proof of theorem 7.1 and to adapt the construction made in Sect. 4.2.

Conjecture 7.1. We conjecture that any space $\mathrm{RES}^{(k)}$ makes a convolution algebra as well.

7.8 Alien operators

Alien operators are powerful tools for analysing the singularities of resurgent functions. These operators are carefully defined and discussed in [24], especially when they operate on the algebra $\mathbb{C}\delta \oplus \hat{\mathscr{R}}^{\mathrm{simp}}$ of simple resurgent functions. Most of the arguments there can be easily adapted for alien operators acting on $\mathrm{RES}_{\mathbb{Z}}$, once the study of singularities had been made. This is the reason why we introduce the alien operators in a rather sketchy manner in what follows.

7.8.1 Alien operators associated with a triple

7.8.1.1 Mains definitions

We consider two directions $\theta_1, \theta_2 \in \mathbb{S}^1$, a point $\omega \in \mathbb{Z}$ and a sectorial germ $\overset{\vee}{\varphi} \in \mathcal{O}^0_{\theta_1}$ of direction θ_1. We can think of $\overset{\vee}{\varphi}$ as a sectorial germ on a sector

$\dot{\boldsymbol{\mathfrak{s}}}_0^{R_1}(I_1)$ for $0 < R_1 < 1$ and $I_1 \subset \mathbb{S}^1$ an open arc bisected by θ_1, and this is what we do in what follows.
We now assume that $\overset{\vee}{\varphi}$ can be analytically continued to $\widetilde{\mathbb{C} \setminus \mathbb{Z}}$. We consider a path $\gamma : J \to \mathbb{C} \setminus \mathbb{Z}$ starting from $\zeta_1 \in \dot{\boldsymbol{\mathfrak{s}}}_0^{R_1}(I_1)$ and ending at ζ_2 close to ω so that $\zeta_2 - \omega \in \dot{\boldsymbol{\mathfrak{s}}}_0^{R_2}(I_2)$ with $0 < R_2 < 1$ and $I_2 \subset \mathbb{S}^1$ an open arc bisected by θ_2. See Fig. 7.9.
By hypotheses, the analytic continuation $(\mathrm{cont}_\gamma \overset{\vee}{\varphi})$ of $\overset{\vee}{\varphi}$ along γ is a well-defined germ of holomorphic functions at ζ_2 that only depends on the homotopy class of γ (for the relation of homotopy of paths in $\mathbb{C} \setminus \mathbb{Z}$ with fixed extremities). Moreover, if $\overset{\vee}{\psi} \in \mathcal{O}_{\zeta_2 - \omega}$ stands for the germ of holomorphic functions at $\zeta_2 - \omega$ defined by $\overset{\vee}{\psi}(\xi) = \big(\mathrm{cont}_\gamma \overset{\vee}{\varphi}\big)(\omega + \xi)$ then, still by analytic continuations, $\overset{\vee}{\psi}$ determines a unique sectorial germ on $\dot{\boldsymbol{\mathfrak{s}}}_0^{R_2}(I_2)$ and thus, by restriction, a unique sectorial germ $\overset{\vee}{\psi} \in \mathcal{O}^0_{\theta_2}$.

This justifies the following definition adapted from [24].

Definition 7.44. Let be $\theta_1, \theta_2 \in \mathbb{S}^1$, $\omega \in \mathbb{Z}$ and $\overset{\vee}{\varphi} \in \mathcal{O}^0_{\theta_1}$ a sectorial germ of direction θ_1 that can be analytically continued to $\widetilde{\mathbb{C} \setminus \mathbb{Z}}$. Let $\gamma : J \to \mathbb{C} \setminus \mathbb{Z}$ be a path starting from a sufficiently small sector $\dot{\boldsymbol{\mathfrak{s}}}_0(I_1)$ bisected by θ_1 and ending close to ω in a sufficiently small sector of the form $\omega + \dot{\boldsymbol{\mathfrak{s}}}_0(I_2)$ where I_2 bisects θ_2. Then, one denotes by $\dot{\mathcal{A}}^{\gamma}_{\omega}(\theta_2, \theta_1)\, \overset{\vee}{\varphi} \in \mathcal{O}^0_{\theta_2}$ the sectorial germ of direction θ_2 represented by $\overset{\vee}{\psi}(\xi) = \big(\mathrm{cont}_\gamma \overset{\vee}{\varphi}\big)(\omega + \xi)$ for $\xi \in \dot{\boldsymbol{\mathfrak{s}}}_0(I_2)$.

We now consider two directions $\theta_1, \theta_2 \in \underset{\bullet}{\mathbb{S}}^1$ and a singularity $\overset{\triangledown}{\varphi} \in \mathrm{RES}_{\mathbb{Z}}$. Thinking of $\overset{\triangledown}{\varphi}$ as a singularity of $\mathrm{SING}_{\theta_1, \alpha_1}$ (for some $\alpha_1 > 0$), its minor $\widehat{\varphi}$ can be seen as representing a sectorial germ $\widehat{\varphi} \in \mathcal{O}^0_{\dot{\theta}_1}$ of direction $\dot{\theta}_1 = \dot{\pi}(\theta_1) \in \mathbb{S}^1$ that can be analytically continued to $\widetilde{\mathbb{C} \setminus \mathbb{Z}}$. Therefore, under the conditions of definition 7.44, the sectorial germ $\overset{\vee}{\psi}_{\dot{\theta}_2} = \dot{\mathcal{A}}^{\gamma}_{\omega}(\dot{\theta}_2, \dot{\theta}_1)\widehat{\varphi}$ of direction $\dot{\theta}_2 = \dot{\pi}(\theta_2) \in \mathbb{S}^1$ is well-defined. Even, by analytic continuations, one can deduce from $\overset{\vee}{\psi}_{\dot{\theta}_2}$ a sectorial germ of direction $I_{\dot{\theta}_2} =]-\pi + \dot{\theta}_2, \dot{\theta}_2 + \pi[\subset \mathbb{S}^1$ that we denote by $\overset{\vee}{\psi}_{I_{\dot{\theta}_2}} \in \Gamma(I_{\dot{\theta}_2}, \mathcal{O}^0)$. By inverse image by $\dot{\pi}$ of the sheaf $\mathcal{O}^0$, this sectorial germ $\overset{\vee}{\psi}_{I_{\dot{\theta}_2}}$ determined a uniquely defined sectorial germ of direction $I_{\theta_2} =]-\pi + \theta_2, \theta_2 + \pi[\subset \underset{\bullet}{\mathbb{S}}^1$ that we denote by $\overset{\vee}{\psi}_{I_{\theta_2}}$. Still by analytic continuations, this sectorial germ gives rise to a (multivalued) section on any

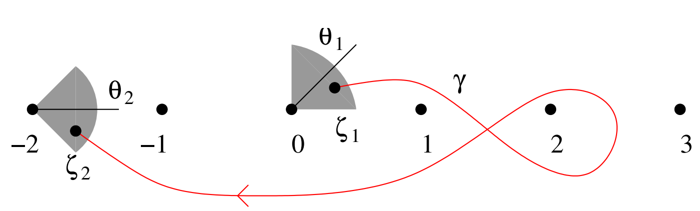


Fig. 7.9 A triple $(\gamma, \theta_1, \theta_2)$ defining the operator $\mathcal{A}^{\gamma}_{\omega}(\theta_1, \theta_2)$ at $\omega = -2$.

arc of the form $]-\alpha-2\pi+(\theta_2+\pi),(\theta_2+\pi)+\alpha[\in \underset{\bullet}{\mathbb{S}}^1$, $\alpha>0$, that is to an element $\overset{\vee}{\psi}$ of $\mathrm{ANA}=\bigcap_{\alpha>0}\mathrm{ANA}_{(\theta_2+\pi),\alpha}$, whose singularity $\overset{\triangledown}{\psi}$ belongs to $\mathrm{RES}_{\mathbb{Z}}$.

Definition 7.45. Let be $\theta_1,\theta_2\in\underset{\bullet}{\mathbb{S}}^1$ and $\omega\in\mathbb{Z}$. Let $\gamma: J\to\mathbb{C}\setminus\mathbb{Z}$ be a path starting from a sufficiently small sector $\overset{\bullet}{\mathfrak{s}}_0(I_1)$ bisected by $\dot\theta_1=\dot\pi(\theta_1)$ and ending close to ω in a sufficiently small sector of the form $\omega+\overset{\bullet}{\mathfrak{s}}_0(I_2)$ where I_2 bisects $\dot\theta_2=\dot\pi(\theta_2)$. For any singularity $\overset{\triangledown}{\varphi}\in\mathrm{RES}_{\mathbb{Z}}$, one denotes by $\mathcal{A}^{\gamma}_{\omega}(\theta_2,\theta_1)\,\overset{\triangledown}{\varphi}$ the singularity $\overset{\triangledown}{\psi}$ that can be represented by a major $\overset{\vee}{\psi}\in\mathrm{ANA}=\Gamma(\underset{\bullet}{\mathbb{S}}^1,\mathcal{O}^0)$ whose restriction $\overset{\vee}{\psi}_{\theta_2}\in\mathcal{O}^0_{\theta_2}$ is the sectorial germ of direction θ_2 determined by $\overset{\vee}{\psi}_{\dot\theta_2}=\overset{\bullet}{\mathcal{A}}{}^{\gamma}_{\omega}(\dot\theta_2,\dot\theta_1)\widehat{\varphi}$, where $\widehat{\varphi}$ is the minor of $\overset{\triangledown}{\varphi}$.
The linear operator $\mathcal{A}^{\gamma}_{\omega}(\theta_2,\theta_1):\mathrm{RES}_{\mathbb{Z}}\to\mathrm{RES}_{\mathbb{Z}}$ is called the **alien operator at ω associated with the triple** $(\gamma,\theta_1,\theta_2)$.

The alien operators have their counterparts on asymptotic classes through formal Borel and Laplace transforms.

Definition 7.46. The alien operator $\mathcal{A}^{\gamma}_{\omega}(\theta_2,\theta_1)$ at ω associated with the triple $(\gamma,\theta_1,\theta_2)$ is defined on asymptotic classes by making the following diagram commuting:
$$\begin{array}{ccc} \mathrm{RES} & \overset{\mathcal{A}^{\gamma}_{\omega}(\theta_2,\theta_1)}{\longrightarrow} & \mathrm{RES} \\ \tilde{\mathcal{L}}\downarrow\uparrow\tilde{\mathcal{B}} & & \tilde{\mathcal{L}}\downarrow\uparrow\tilde{\mathcal{B}} \\ \widetilde{\mathrm{RES}} & \overset{\mathcal{A}^{\gamma}_{\omega}(\theta_2,\theta_1)}{\longrightarrow} & \widetilde{\mathrm{RES}} \end{array}.$$

7.8.1.2 The spaces $\mathrm{RES}^{(\dot\theta,\alpha)}(L)$ and $\mathrm{RES}^{(k)}$

Alien operators acting on $\mathrm{RES}^{(\dot\theta,\alpha)}(L)$ We would like to define alien operators acting on the space $\mathrm{RES}^{(\dot\theta,\alpha)}(L)$. We take $\theta\in\{\pi k, k\in\mathbb{Z}\}\subset\underset{\bullet}{\mathbb{S}}^1$, $\alpha\in]0,\pi/2]$, $L>0$ and $m\in\{1,\cdots,\lceil L\rceil\}$. We set $\dot\theta=\dot\pi(\theta)\in\{0,\pi\}$, we consider a singularity $\overset{\triangledown}{\varphi}\in\mathrm{RES}^{(\dot\theta,\alpha)}(L)$ whose minor is $\widehat{\varphi}$. By the very definition 7.41 of the space $\mathrm{RES}^{(\dot\theta,\alpha)}(L)$, the sectorial germ $\overset{\vee}{\psi}_{\dot\theta_2}=\overset{\bullet}{\mathcal{A}}{}^{\gamma}_{\omega}(\dot\theta_2,\dot\theta)\widehat{\varphi}\in\mathcal{O}^0_{\dot\theta_2}$ is well defined under the following conditions:

1. $\omega=m\mathrm{e}^{\mathrm{i}\dot\theta}$ and the path γ is of type $\gamma^{\dot\theta}_{\boldsymbol{\varepsilon}}$ with $\boldsymbol{\varepsilon}=(\pm)_{m-1}\in\{+,-\}^{m-1}$. In that case, $\dot\theta_2$ should be $\dot\theta-\pi$;
2. however, starting from $\overset{\vee}{\psi}_{\dot\theta-\pi}$ and be analytic continuations, one can consider as well sectorial germs $\overset{\vee}{\psi}_{\dot\theta_2}$ with $\dot\theta_2\in I_{\dot\theta}=]-2\pi+\dot\theta,\dot\theta[\subset\mathbb{S}^1$.

By a construction already done, the various sectorial germs $\overset{\vee}{\psi}_{\dot\theta_2}$ glue together and provide a sectorial germ $\overset{\vee}{\psi}_{I_{\dot\theta}}\in\Gamma(I_{\dot\theta},\mathcal{O}^0)$ of direction $I_{\dot\theta}$. Still by analytic continuations and moving to multivalued sectorial germs by inverse image by $\dot\pi$ of the sheaf $\mathcal{O}^0$, one eventually gets an element $\overset{\vee}{\psi}$ of $\mathrm{ANA}_{\theta,\alpha}$ with $\dot\pi(\theta)=\dot\theta$.
This gives sense to the following definition.

Definition 7.47. Let be $\theta \in \{\pi k, k \in \mathbb{Z}\} \subset \underset{\bullet}{\mathbb{S}^1}$, $\alpha \in]0, \pi/2]$ and $L > 0$. We write $\dot{\theta} = \dot{\pi}(\theta) \in \{0, \pi\} \subset \mathbb{S}^1$. We pick $m \in \{1, \cdots, \lceil L \rceil\}$, we set $\omega = m\mathrm{e}^{\mathrm{i}\dot{\theta}}$ and we assume that the path γ is of type $\gamma^{\dot{\theta}}_{(\pm)_{m-1}}$.
For any singularity $\overset{\triangledown}{\varphi} \in \mathrm{RES}^{(\dot{\theta},\alpha)}(L)$, one denotes by $\mathcal{A}^{\gamma}_{\omega}(\theta, \theta)\, \overset{\triangledown}{\varphi}$ the singularity $\overset{\triangledown}{\psi} \in \mathrm{SING}_{\theta,\alpha}$ that can be represented by a major $\overset{\vee}{\psi} \in \mathrm{ANA}_{\theta,\alpha}$ whose restriction $\overset{\vee}{\psi}_{\theta-\pi} \in \mathcal{O}^0_{\theta-\pi}$ is the sectorial germ of direction $\theta - \pi$ determined by $\overset{\vee}{\psi}_{\theta-\pi} = \overset{\bullet}{\mathcal{A}}{}^{\gamma}_{\omega}(\dot{\theta} - \pi, \dot{\theta})\widehat{\varphi}$ where $\widehat{\varphi}$ stands for the minor of $\overset{\triangledown}{\varphi}$.
This gives rise to a linear operator $\mathcal{A}^{\gamma}_{\omega}(\theta, \theta) : \mathrm{RES}^{(\dot{\theta},\alpha)}(L) \to \mathrm{SING}_{\theta,\alpha}$, still called the alien operator at ω associated with the triple (γ, θ, θ).

Alien operators acting on $\mathbf{RES}^{(k)}$ We now work on the spaces $\mathrm{RES}^{(k)}$ given by definition 7.43. We want to demonstrate that alien operators can be defined on $\mathrm{RES}^{(k)}$, associated with triples of the form (γ, θ, θ) with γ of type $\gamma^{\dot{\theta}}_{(+)_m}$ or $\gamma^{\dot{\theta}}_{(-)_m}$.

We start with $\mathrm{RES}^{(1)}$. Let be $\theta_1 \in \{\pi k, k \in \mathbb{Z}\} \subset \underset{\bullet}{\mathbb{S}^1}$ and set $\omega_1 = \mathrm{e}^{\mathrm{i}\dot{\theta}_1}$ with $\dot{\theta}_1 = \dot{\pi}(\theta_1)$. The very definition of $\mathrm{RES}^{(1)}$ and the above reasoning lead straight to the following linear operators, for any integer $m_1 \geq 2$ and any $\varepsilon \in \{-, +\}$:

$$\mathcal{A}^{\gamma^{\dot{\theta}_1}_{()}}_{\omega_1}(\theta_1, \theta_1) : \mathrm{RES}^{(1)} \to \mathrm{SING}_{\theta_1,\pi}, \quad \mathcal{A}^{\gamma^{\dot{\theta}_1}_{(\varepsilon)_{m_1-1}}}_{m_1\omega_1}(\theta_1, \theta_1) : \mathrm{RES}^{(1)} \to \mathrm{SING}_{\theta_1+\pi/2,\pi/2} \tag{7.16}$$

Let us now move to the next case $k = 2$, that is we consider the space $\mathrm{RES}^{(2)} \subset \mathrm{RES}^{(1)}$. Of course the above operators (7.16) still act on $\mathrm{RES}^{(2)}$ but, however, their ranges can be made more precise. By the very definition of $\mathrm{RES}^{(2)}$, the minor $\widehat{\varphi}$ of any singularity $\overset{\triangledown}{\varphi} \in \mathrm{RES}^{(2)}$, when considered as a sectorial germ, can be analytically continued along any path γ of type $\gamma^{\dot{\theta}_1}_{\boldsymbol{\varepsilon}^{\boldsymbol{n}_1}}$ with

$$\boldsymbol{\varepsilon}^{\boldsymbol{n}_1} \in \{\big((\pm)^{n_1}, (+)_{m_1-1}\big), \big((\pm)^{n_1}, (-)_{m_1-1}\big) \mid (n_1, m_1) \in (\mathbb{N}^\star)^2\}.$$

Moreover, introducing $\dot{\theta}_2 = \dot{\theta}_1 + (n-1)\pi$, $\omega_1 = \mathrm{e}^{\mathrm{i}\dot{\theta}_1}$, and $\omega_2 - \omega_1 = \mathrm{e}^{\mathrm{i}\dot{\theta}_2}$, the analytic continuation $\mathrm{cont}_\gamma\, \widehat{\varphi}$ of $\widehat{\varphi}$ along γ is a germ of holomorphic functions that can be analytically continued onto the simply connected domain $\mathfrak{p}(\mathscr{R}^{\boldsymbol{\varepsilon}^{\boldsymbol{n}_1},\dot{\theta}}) = \mathbb{C} \setminus \{] - \infty, p] \cup [p + 1, +\infty[\}$ where $]p, (p+1)[=]\omega_1, \omega_2[$ when $m_1 = 1$, $]p, (p+1)[=](m_1 - 1)\omega_2, m_1\omega_2[$ when $m_1 \geq 2$. Considering only odd values for n_1 (thus $\dot{\theta}_2 = \dot{\theta}_1$ on $\mathbb{S}^1$), one immediately sees that (7.16) becomes:

$$\begin{aligned} \mathcal{A}^{\gamma^{\dot{\theta}_1}_{()}}_{\omega_1}(\theta_1, \theta_1) &: \mathrm{RES}^{(2)} \to \mathrm{RES}^{(1)}, \\ \mathcal{A}^{\gamma^{\dot{\theta}_1}_{(\varepsilon)_1}}_{2\omega_1}(\theta_1, \theta_1) &: \mathrm{RES}^{(2)} \to \mathrm{SING}_{\theta_1,\pi} \\ \mathcal{A}^{\gamma^{\dot{\theta}_1}_{(\varepsilon)_{m_1-1}}}_{m_1\omega_1}(\theta_1, \theta_1) &: \mathrm{RES}^{(2)} \to \mathrm{SING}_{\theta_1+\pi/2,\pi/2}, \quad m_1 \geq 3. \end{aligned} \tag{7.17}$$

Notice in particular that the operator $\mathcal{A}^{\gamma^{\dot{\theta}_1}_{()}}_{\omega_1}(\theta_2, \theta_1)$ now acts on $\mathrm{RES}^{(2)}$ as well, for any direction $\theta_2 \in \underset{\bullet}{\mathbb{S}^1}$.

The reasoning generalizes and we give the result.

Lemma 7.8. *Let be $\theta_1 \in \{\pi k, k \in \mathbb{Z}\} \subset \underset{\bullet}{\mathbb{S}}^1$. For any integer $k \geq 1$, any $\varepsilon \in \{-,+\}$ and any $m_1 \in \mathbb{N}^\star$, setting $\omega_1 = \mathrm{e}^{\mathrm{i}\dot{\theta}_1}$, the alien operator $\mathcal{A}_{m_1\omega_1}^{\gamma_{(\varepsilon)m_1-1}^{\dot{\theta}_1}}(\theta_1,\theta_1)$ is well defined on $\mathrm{RES}^{(k)}$ with the range:*

$$\begin{aligned}
&\mathcal{A}_{m_1\omega_1}^{\gamma_{(\varepsilon)m_1-1}^{\dot{\theta}_1}}(\theta_1,\theta_1) : \mathrm{RES}^{(k)} \to \mathrm{RES}^{(k-m_1)}, \ \ 1 \leq m_1 \leq k-1 \qquad (7.18)\\
&\mathcal{A}_{m_1\omega_1}^{\gamma_{(\varepsilon)m_1-1}^{\dot{\theta}_1}}(\theta_1,\theta_1) : \mathrm{RES}^{(k)} \to \mathrm{SING}_{\theta_1,\pi}, \ \ m_1 = k\\
&\mathcal{A}_{m_1\omega_1}^{\gamma_{(\varepsilon)m_1-1}^{\dot{\theta}_1}}(\theta_1,\theta_1) : \mathrm{RES}^{(k)} \to \mathrm{SING}_{\theta_1+\pi/2,\pi/2}, \ \ m_1 \geq k+1.
\end{aligned}$$

7.8.1.3 Miscellaneous properties

We start with a simple result which is a consequence of the very definitions.

Proposition 7.18. *For any alien operator of the form $\mathcal{A}_\omega^\gamma(\theta_2,\theta_1) : \mathrm{RES}_\mathbb{Z} \to \mathrm{RES}_\mathbb{Z}$, acting on $\mathrm{RES}_\mathbb{Z}$, $\mathrm{RES}^{(\dot{\theta},\alpha)}(L)$ or $\mathrm{RES}^{(k)}$, for any singularity $\overset{\triangledown}{\varphi}$:*

$$\mathcal{A}_\omega^\gamma(\theta_2,\theta_1)\big(\partial \overset{\triangledown}{\varphi}\big) = (\partial-\omega)\mathcal{A}_\omega^\gamma(\theta_2,\theta_1)\overset{\triangledown}{\varphi}. \tag{7.19}$$

In other words, $[\mathcal{A}_\omega^\gamma(\theta_2,\theta_1),\partial] = -\omega\mathcal{A}_\omega^\gamma(\theta_2,\theta_1)$.

We introduce new definitions before keeping on.

Definition 7.48. For any $k \in \mathbb{Z}$, one denotes by $\varrho_k \in \mathrm{Aut}(\pi)$ the deck transformation of the cover $(\underset{\bullet}{\mathbb{C}},\pi)$, defined by:

$$\varrho_k : \zeta = r\mathrm{e}^{\mathrm{i}\theta} \in \underset{\bullet}{\mathbb{C}} \mapsto \varrho_k(\zeta) = r\mathrm{e}^{\mathrm{i}\theta+2\mathrm{i}\pi k} \in \underset{\bullet}{\mathbb{C}}.$$

For any singularity of the form $\overset{\triangledown}{\varphi} = \mathrm{sing}_0\overset{\vee}{\varphi} \in \mathrm{SING}$, $\overset{\vee}{\varphi} \in \mathrm{ANA}$, we write $\varrho_k.\overset{\triangledown}{\varphi} = \mathrm{sing}_0\,(\overset{\vee}{\varphi}\circ\varrho_k) \in \mathrm{SING}$.
More generally, for any $r \in \mathbb{R}$, one defines

$$\varrho_r : \zeta = r\mathrm{e}^{\mathrm{i}\theta} \in \underset{\bullet}{\mathbb{C}} \mapsto \varrho_r(\zeta) = r\mathrm{e}^{\mathrm{i}\theta+2\mathrm{i}\pi r} \in \underset{\bullet}{\mathbb{C}}$$

and $\varrho_r.\overset{\triangledown}{\varphi} = \mathrm{sing}_0\,(\overset{\vee}{\varphi}\circ\varrho_r) \in \mathrm{SING}$.

Remark 7.8. With this notation, the variation map $\mathrm{var} : \mathrm{SING} \to \mathrm{ANA}$ reads $\mathrm{var} = id - \varrho_{-1}.$.

The alien operators associated with a triple satisfy some identities as can be easily observed:

Proposition 7.19. *For any given alien operator $\mathcal{A}_\omega^\gamma(\theta_2,\theta_1) : \mathrm{RES}_\mathbb{Z} \to \mathrm{RES}_\mathbb{Z}$ and for any $k \in \mathbb{Z}$,*

$$\mathcal{A}_\omega^\gamma(\theta_2,\theta_1+2\pi k) = \mathcal{A}_\omega^\gamma(\theta_2,\theta_1)\varrho_k.\,, \qquad \mathcal{A}_\omega^\gamma(\theta_2+2\pi k,\theta_1) = \varrho_{-k}.\mathcal{A}_\omega^\gamma(\theta_2,\theta_1). \tag{7.20}$$

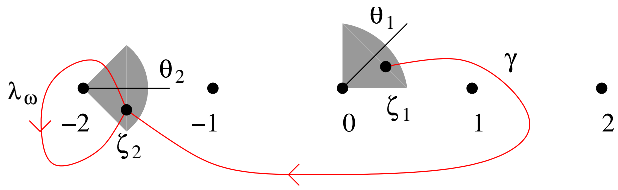


Fig. 7.10 Two triples $(\gamma, \theta_1, \theta_2)$ and $(\gamma\lambda_\omega, \theta_1, \theta_2)$ for the point $\omega = -2$, with λ_ω a closed path of winding number $\mathrm{wind}_\omega(\lambda_\omega) = 1$ at ω.

Let us now consider a point $\omega \in \mathbb{Z}$ and a given triple $(\gamma, \theta_1, \theta_2)$. One can prolongs the path γ into the path $\gamma\lambda_\omega^k$ where λ_ω^k is a closed path near ω that surrounds that point like on Fig. 7.10, with winding number $\mathrm{wind}_\omega(\lambda_\omega^k) = k \in \mathbb{Z}$ at that point. One can as well consider the path $\lambda_0^k\gamma$ where λ_0^k is a closed path surrounding the origin with winding number $\mathrm{wind}_\omega(\lambda_0^k) = k \in \mathbb{Z}$. A little thought provides the following result.

Proposition 7.20. *We consider a triple $(\gamma, \theta_1, \theta_2)$ defining alien operator $\mathcal{A}_\omega^\gamma(\theta_2, \theta_1) : \mathrm{RES}_\mathbb{Z} \to \mathrm{RES}_\mathbb{Z}$ at ω. We assume that $\gamma\lambda_\omega^k$, resp. $\lambda_0^k\gamma$ is a product of paths so that λ_ω^k, resp. λ_0^k, is a closed path surrounding ω, resp. 0, and close to that point, with winding number $\mathrm{wind}_\omega(\lambda_\omega^k) = k$, resp. $\mathrm{wind}_0(\lambda_0^k) = k$, $k \in \mathbb{Z}$. Then,*

$$\mathcal{A}_\omega^{\lambda_0^k\gamma}(\theta_2, \theta_1) = \mathcal{A}_\omega^\gamma(\theta_2, \theta_1)\varrho_k., \qquad \mathcal{A}_\omega^{\gamma\lambda_\omega^k}(\theta_2, \theta_1) = \varrho_k.\mathcal{A}_\omega^\gamma(\theta_2, \theta_1). \tag{7.21}$$

In particular,

$$\mathcal{A}_\omega^\gamma(\theta_2, \theta_1 + 2\pi k) = \mathcal{A}_\omega^{\lambda_0^k\gamma}(\theta_2, \theta_1), \qquad \mathcal{A}_\omega^\gamma(\theta_2 + 2\pi k, \theta_1) = \mathcal{A}_\omega^{\gamma\lambda_\omega^{-k}}(\theta_2, \theta_1). \tag{7.22}$$

We end with the following property.

Proposition 7.21. *For any alien operator of the form $\mathcal{A}_\omega^\gamma(\theta, \theta)$ acting on $\mathrm{RES}_\mathbb{Z}$ or $\mathrm{RES}^{(\dot\theta,\alpha)}(L)$, for any singularity $\overset{\triangledown}{\varphi}$ and any resurgent constant $\overset{\triangledown}{\mathrm{const}} \in \mathrm{CONS}$,*

$$\mathcal{A}_\omega^\gamma(\theta, \theta)\big(\overset{\triangledown}{\mathrm{const}} * \overset{\triangledown}{\varphi}\big) = \overset{\triangledown}{\mathrm{const}} * \big(\mathcal{A}_\omega^\gamma(\theta, \theta)\overset{\triangledown}{\varphi}\big). \tag{7.23}$$

We stress that in proposition 7.21, only alien operators of the form $\mathcal{A}_\omega^\gamma(\theta_2, \theta_1)$ with $\theta_1 = \theta_2$ are considered. We omit the proof of this proposition which relies on a careful reading of what have been done for showing theorem 7.1.

7.8.2 Composition of alien operators

7.8.2.1 Alien operators on $\mathrm{RES}_\mathbb{Z}$

The following definition is adapted from [24].

Definition 7.49. One calls **alien operator at $\omega \in \mathbb{Z}$ associated with the couple** (θ_1^1, θ_2^m) any linear combination of composite operators of the form

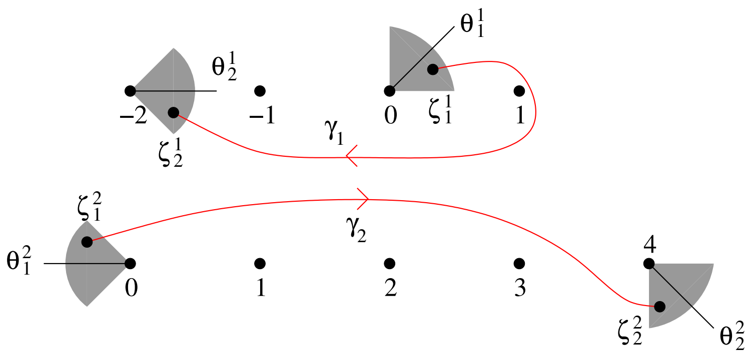


Fig. 7.11 The triple $(\gamma_1, \theta_1^1, \theta_2^1)$ for the point $\omega_1 = -2$, the triple $(\gamma_2, \theta_2^1, \theta_2^2)$ for the point $\omega_2 - \omega_1 = 4$, with $\theta_1^2 = \theta_2^1 + \pi$.

$$\mathcal{A}^{\gamma_m}_{\omega_m - \omega_{m-1}}(\theta_2^m, \theta_1^m) \circ \cdots \circ \mathcal{A}^{\gamma_2}_{\omega_2 - \omega_1}(\theta_2^2, \theta_1^2) \circ \mathcal{A}^{\gamma_1}_{\omega_1}(\theta_2^1, \theta_1^1) : \mathrm{RES}_{\mathbb{Z}} \to \mathrm{RES}_{\mathbb{Z}}$$

where $(\omega_1, \cdots, \omega_m) \in \mathbb{Z}^m$, $m \in \mathbb{N}^\star$ with $\omega = \omega_m = \sum_{j=1}^m \omega_j - \omega_{j-1}$ and the convention $\omega_0 = 0$.

Example 7.9. We exemplify the above definition. We take $\omega_1 = -2$ and $\omega_2 = 2$. The alien operator $\mathcal{A}^{\gamma_1}_{\omega_1}(\theta_2^1, \theta_1^1)$ at the point $\omega_1 = -2$ is associated with the triple $(\gamma_1, \theta_1^1, \theta_2^1)$ drawn on Fig. 7.11. The alien operator $\mathcal{A}^{\gamma_1}_{\omega_2 - \omega_1}(\theta_2^2, \theta_1^2)$ at the point $\omega_2 - \omega_1 = 4$ is associated with the triple $(\gamma_1, \theta_1^1, \theta_2^1)$ drawn on Fig. 7.11. We furthemore assume that $\theta_1^2 - \theta_2^1 \in [0, 2\pi[$ to fix our mind.

From the very definitions of the alien operators and of a minor, one easily checks that the composite alien operator $\mathcal{A}^{\gamma_2}_{\omega_2 - \omega_1}(\theta_2^2, \theta_1^2) \circ \mathcal{A}^{\gamma_1}_{\omega_1}(\theta_2^1, \theta_1^1)$ at ω_2, can be written as the difference of two simple alien operators, namely

$$\mathcal{A}^{\gamma_2}_{\omega_2 - \omega_1}(\theta_2^2, \theta_1^2) \circ \mathcal{A}^{\gamma_1}_{\omega_1}(\theta_2^1, \theta_1^1) = \mathcal{A}^{\Gamma^+}_{\omega_2}(\theta_2^2, \theta_1^1) - \mathcal{A}^{\Gamma^-}_{\omega_2}(\theta_2^2, \theta_1^1).$$

In this equality, Γ^+ and Γ^{-1} stands for the (homotopy class of the) product of paths $\Gamma^+ = \gamma_1 \lambda^+_{\omega_1}(\omega_1 + \gamma_2)$ and $\Gamma^- = \gamma_1 \lambda^-_{\omega_1}(\omega_1 + \gamma_2)$ respectively, where the paths $\lambda^+_{\omega_1}$ and $\lambda^-_{\omega_1}$ drawn on Fig. 7.12, are homotopic to small arcs so that $(\lambda^-_{\omega_1})^{-1}\lambda^+_{\omega_1}$ makes a loop around ω_1 counterclockwise.

Typically, the end point of γ_1 is $\zeta_2^1 = \omega_1 + r e^{i\dot{\theta}_1^1}$ while the starting point of γ_2 is $\zeta_1^2 = r e^{i\dot{\theta}_1^2}$ with $0 < r \ll 1$. Then, $\lambda^+_{\omega_1} : \dot{\theta} \in [\dot{\theta}_2^1, \dot{\theta}_1^2] \mapsto \omega_1 + r e^{i\dot{\theta}}$ while $(\lambda^-_{\omega_1})^{-1} : \dot{\theta} \in [-2\pi + \dot{\theta}_1^2, \dot{\theta}_2^1] \mapsto \omega_1 + r e^{i\dot{\theta}}$.

From this result, one deduces from proposition 7.20 that for any $k \in \mathbb{Z}$,

$$\mathcal{A}^{\gamma_2}_{\omega_2 - \omega_1}(\theta_2^2, \theta_1^2 + 2\pi k) \circ \mathcal{A}^{\gamma_1}_{\omega_1}(\theta_2^1, \theta_1^1) = \mathcal{A}^{\Gamma_k^+}_{\omega_2}(\theta_2^2, \theta_1^1) - \mathcal{A}^{\Gamma_k^-}_{\omega_2}(\theta_2^2, \theta_1^1).$$

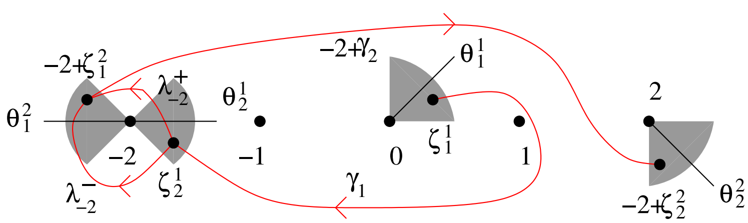


Fig. 7.12 The paths $\Gamma^+ = \gamma_1 \lambda^+_{\omega_1}(\omega_1 + \gamma_2)$ and $\Gamma^+ = \gamma_1 \lambda^-(\omega_1 + \gamma_2)$, $\omega_1 = -2$.

with $\Gamma_k^+ = \gamma_1 \lambda_{\omega_1}^k \lambda_{\omega_1}^+ (\omega_1 + \gamma_2)$ and $\Gamma_k^- = \gamma_1 \lambda_{\omega_1}^k \lambda_{\omega_1}^- (\omega_1 + \gamma_2)$ respectively, where $\lambda_{\omega_1}^k$ stands for a closed path around $\omega_1 = -2$ with winding number $\mathrm{wind}_{\omega_1}(\lambda_{\omega_1}^k) = k$ at that point.

What have been done in the above example can be generalized. This is the matter of the next proposition.

Proposition 7.22. *We consider the two alien operators* $\mathcal{A}_{\omega_1}^{\gamma_1}(\theta_2^1, \theta_1^1)$, $\mathcal{A}_{\omega_2-\omega_1}^{\gamma_2}(\theta_2^2, \theta_1^2)$ *and we assume that* $\theta_1^2 - \theta_2^1 \in [0, 2\pi[$. *Then, for any* $k \in \mathbb{Z}$,

$$\mathcal{A}_{\omega_2-\omega_1}^{\gamma_2}(\theta_2^2, \theta_1^2 + 2\pi k) \circ \mathcal{A}_{\omega_1}^{\gamma_1}(\theta_2^1, \theta_1^1) = \mathcal{A}_{\omega_2}^{\Gamma_k^+}(\theta_2^2, \theta_1^1) - \mathcal{A}_{\omega_2}^{\Gamma_k^-}(\theta_2^2, \theta_1^1).$$

with $\Gamma_k^+ = \gamma_1 \lambda_{\omega_1}^k \lambda_{\omega_1}^+ (\omega_1 + \gamma_2)$ *and* $\Gamma_k^- = \gamma_1 \lambda_{\omega_1}^k \lambda_{\omega_1}^- (\omega_1 + \gamma_2)$ *respectively, where* $\lambda_{\omega_1}^k$ *stands for a closed path around* ω_1 *with winding number* $\mathrm{wind}_{\omega_1}(\lambda_{\omega_1}^k) = k$ *at that point, whereas* $\lambda_{\omega_1}^+$ *and* $\lambda_{\omega_1}^-$ *follows small arcs so that* $(\lambda_{\omega_1}^-)^{-1} \lambda_{\omega_1}^+$ *makes a loop around* ω_1 *counterclockwise.*
As a consequence, any alien operator at a point $\omega \in \mathbb{Z}$ *associated with the couple* (θ_1, θ_2) *can be written as a linear combination of alien operators at* ω *associated with triples of the form* $(\gamma, \theta_1, \theta_2)$.

We now focus on paths of type $\gamma_{\boldsymbol{\varepsilon}\boldsymbol{n}}^{\dot\theta}$. For $m \in \mathbb{N}^\star$, we take a $(m-1)$-tuple of signs $\boldsymbol{\varepsilon} = (\varepsilon_1, \cdots, \varepsilon_{m-1}) \in \{+, -\}^{m-1}$ and $\boldsymbol{n} = (n_1, \cdots, n_{m-1}) \in (\mathbb{N}^\star)^{m-1}$. We choose a direction $\theta_1 \in \{\pi k, k \in \mathbb{Z}\}$. Following definition 4.4, to a path of type $\gamma_{\boldsymbol{\varepsilon}\boldsymbol{n}}^{\dot\theta_1}$ one associates a sequence of points and directions defined as follows :

$$\begin{aligned} \dot\theta_{j+1} &= \dot\theta_j + \varepsilon_j (n_j - 1)\pi \qquad 1 \le j \le m-1 \\ \omega_{j+1} - \omega_j &= \mathrm{e}^{\mathrm{i}\dot\theta_{j+1}} \qquad 0 \le j \le m-1 \\ \omega_0 &= 0. \end{aligned} \tag{7.24}$$

These data thus provide a uniquely defined alien operator $\mathcal{A}_{\omega_m}^{\gamma_{\boldsymbol{\varepsilon}\boldsymbol{n}}^{\dot\theta_1}}(\theta_m, \theta_1)$, once the direction $\theta_m \in \underset{\bullet}{\mathbb{S}}^1$, $\dot\theta_m = \dot\pi(\theta_m)$ is chosen.

Theorem 7.2. *Let* $m \in \mathbb{N}^\star$ *be a positive integer,* $\boldsymbol{\varepsilon} \in \{+, -\}^{m-1}$, $\boldsymbol{n} \in (\mathbb{N}^\star)^{m-1}$ *and* $\theta_1 \in \{\pi k, k \in \mathbb{Z}\}$. *Let* γ *be a path of type* $\gamma_{\boldsymbol{\varepsilon}\boldsymbol{n}}^{\dot\theta_1}$, ω_m *and* $\dot\theta_m$ *given by (7.24), and* $\theta_m \in \underset{\bullet}{\mathbb{S}}^1$ *so that* $\dot\theta_m = \dot\pi(\theta_m)$. *Then the alien operator* $\mathcal{A}_{\omega_m}^{\gamma}(\theta_m, \theta_1)$ *at* ω_m *associated with the triple* $(\gamma, \theta_1, \theta_m)$ *can be written as a* $\mathbb{Z}$*-linear combination of composite operators of the form*

$$\mathcal{A}_{\omega'_k - \omega'_{k-1}}^{\gamma_k}(\theta_m, \theta'_k) \circ \cdots \circ \mathcal{A}_{\omega'_2 - \omega'_1}^{\gamma_2}(\theta'_2, \theta'_2) \circ \mathcal{A}_{\omega'_1}^{\gamma_1}(\theta'_1, \theta'_1)$$

that satisfy the properties:

- $(\omega'_1, \cdots, \omega'_k) \in \mathbb{Z}^k$, $k \in \mathbb{N}^\star$ *and* $\omega'_k = \omega_m$;
- $\dot\theta_m = \dot\theta'_k$;
- *for every* $j = 1, \cdots, k$, *the path* γ_j *is of type* $\gamma_{(+)_{m_j - 1}}^{\dot\theta'_j}$, $m_j \in \mathbb{N}^\star$;
- $\sum_{j=1}^k m_j \le m$.

This theorem is of a purely geometric nature. We omit its proof (see [1] Sect. Rés II-2, see also [24, 21]) and we rather produce two examples that explain the algorithm.

Example 7.10. We consider a path γ of type $\gamma_{\boldsymbol{\varepsilon}}^{\dot{\theta}_1}$ for $\boldsymbol{\varepsilon} = (+,-,+)$ and we take $\dot{\theta}_1 = 0$, see Fig. 4.2. To the path γ one associates by (7.24) the sequence of points and directions: $\begin{cases} \dot{\theta}_j = 0, & 1 \le j \le 4 \\ \omega_0 = 0, & \omega_{j+1} - \omega_j = 1 \quad 0 \le j \le 3 \end{cases}$. One takes $\theta_j = \theta' = 0$ for any $j \in [1,4]$. We want to decompose the alien operator $\mathcal{A}_{\omega_4}^{\gamma}(\theta_4, \theta_1)$. From the very definition of the alien operators, one observes that

$$\mathcal{A}_{\omega_4-\omega_2}^{\gamma_{(+)}^{\dot{\theta}_3}}(\theta_4,\theta_3) \circ \mathcal{A}_{\omega_2}^{\gamma_{(+)}^{\dot{\theta}_1}}(\theta_2,\theta_1) = \mathcal{A}_{\omega_4}^{\gamma_{(+)^3}^{\dot{\theta}_1}}(\theta_4,\theta_1) - \mathcal{A}_{\omega_4}^{\gamma_{(+,-,+)}^{\dot{\theta}_1}}(\theta_4,\theta_1),$$

and therefore

$$\mathcal{A}_{\omega_4}^{\gamma_{(+,-,+)}^{\dot{\theta}_1}}(\theta_4,\theta_1) = \mathcal{A}_{\omega_4}^{\gamma_{(+)^3}^{\dot{\theta}'}}(\theta',\theta') - \mathcal{A}_{\omega_4-\omega_2}^{\gamma_{(+)}^{\dot{\theta}'}}(\theta',\theta') \circ \mathcal{A}_{\omega_2}^{\gamma_{(+)}^{\dot{\theta}'}}(\theta',\theta')$$

Example 7.11. A bit more difficult, we consider a path γ of type $\gamma_{\boldsymbol{\varepsilon}^{\boldsymbol{n}}}^{\dot{\theta}_1}$ for $\boldsymbol{\varepsilon} = (+,-,+)$, $\boldsymbol{n} = (1,3,1)$ and $\dot{\theta}_1 = 0$, see Fig. 7.13. The algorithm (7.24) still provides $\begin{cases} \dot{\theta}_j = 0, & 1 \le j \le 4 \\ \omega_0 = 0, & \omega_{j+1} - \omega_j = 1 \quad 0 \le j \le 3 \end{cases}$. One takes again $\theta_j = \theta' = 0$ for any $j \in [1,4]$. Since

$$\mathcal{A}_{\omega_4-\omega_2}^{\gamma_{(+)}^{\dot{\theta}_3}}(\theta_4,\theta_3-2\pi) \circ \mathcal{A}_{\omega_2}^{\gamma_{(+)}^{\dot{\theta}_1}}(\theta_2,\theta_1) = \mathcal{A}_{\omega_4}^{\gamma_{(+,-,+)}^{\dot{\theta}_1}}(\theta_4,\theta_1) - \mathcal{A}_{\omega_4}^{\gamma_{(+,-^2,+)}^{\dot{\theta}_1}}(\theta_4,\theta_1),$$

one deduces with the first example that

$$\begin{aligned}
\mathcal{A}_{\omega_4}^{\gamma_{(+,-^2,+)}^{\dot{\theta}_1}}(\theta_4,\theta_1) &= \mathcal{A}_{\omega_4}^{\gamma_{(+,-,+)}^{\dot{\theta}'}}(\theta',\theta') - \mathcal{A}_{\omega_4-\omega_2}^{\gamma_{(+)}^{\dot{\theta}'}}(\theta',\theta'-2\pi) \circ \mathcal{A}_{\omega_2}^{\gamma_{(+)}^{\dot{\theta}'}}(\theta',\theta') \\
&= \mathcal{A}_{\omega_4}^{\gamma_{(+)^3}^{\dot{\theta}'}}(\theta',\theta') - \mathcal{A}_{\omega_4-\omega_2}^{\gamma_{(+)}^{\dot{\theta}'}}(\theta',\theta') \circ \mathcal{A}_{\omega_2}^{\gamma_{(+)}^{\dot{\theta}'}}(\theta',\theta') \\
&\quad - \mathcal{A}_{\omega_4-\omega_2}^{\gamma_{(+)}^{\dot{\theta}'}}(\theta',\theta'-2\pi) \circ \mathcal{A}_{\omega_2}^{\gamma_{(+)}^{\dot{\theta}'}}(\theta',\theta').
\end{aligned}$$

Example 7.12. A step further, we consider a path γ of type $\gamma_{\boldsymbol{\varepsilon}^{\boldsymbol{n}}}^{\dot{\theta}_1}$ for $\boldsymbol{\varepsilon} = (-,+,+,+,-)$, $\boldsymbol{n} = (1,2,1,1,1)$ and take $\theta_1 = 0$, see Fig. 4.3. Using (7.24), we define:

$$\begin{cases} \dot{\theta}_1 = \dot{\theta}_2 = 0 \\ \dot{\theta}_3 = \cdots = \dot{\theta}_6 = \pi \\ \omega_0 = 0,\ \omega_1 - \omega_0 = \omega_2 - \omega_1 = 1 \\ \omega_3 - \omega_2 = \cdots = \omega_6 - \omega_5 = -1. \end{cases}$$

We set $\theta_1 = \theta_2 = \theta'_1 = 0$, $\theta_3 = \cdots = \theta_6 = \theta'_2 = \pi$. We start with the identity:

$$\mathcal{A}_{\omega_6-\omega_5}^{\gamma_{()}^{\dot{\theta}_6}}(\theta_6,\theta_6) \circ \mathcal{A}_{\omega_5}^{\gamma_{(-,+^2,+,+)}^{\dot{\theta}_1}}(\theta_5,\theta_1) = \mathcal{A}_{\omega_6}^{\gamma_{(-,+^2,+,+,+)}^{\dot{\theta}_1}}(\theta_6,\theta_1) - \mathcal{A}_{\omega_6}^{\gamma}(\theta_6,\theta_1).$$

Next, a little thought yields:

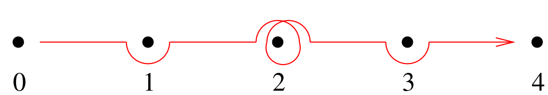


Fig. 7.13 A path of type $\gamma_{\boldsymbol{\varepsilon}^{\boldsymbol{n}}}^{\dot{\theta}_1}$ for $\boldsymbol{\varepsilon} = (+,-,+)$, $\boldsymbol{n} = (1,3,1)$ and $\dot{\theta}_1 = 0$.

$$\mathcal{A}_{\omega_6-\omega_2}^{\gamma_{(+,+,+)}^{\dot\theta_3}}(\theta_6,\theta_3)\circ\mathcal{A}_{\omega_2}^{\gamma_{(-)}^{\dot\theta_1}}(\theta_2,\theta_1)=\mathcal{A}_{\omega_6}^{\gamma_{(-,+^2,+,+,+)}^{\dot\theta_1}}(\theta_6,\theta_1)-\mathcal{A}_{\omega_6}^{\gamma_{(+)}^{\dot\theta_5}}(\theta_6,\theta_5),$$

$$\mathcal{A}_{\omega_5-\omega_2}^{\gamma_{(+,+)}^{\dot\theta_3}}(\theta_5,\theta_3)\circ\mathcal{A}_{\omega_2}^{\gamma_{(-)}^{\dot\theta_1}}(\theta_2,\theta_1)=\mathcal{A}_{\omega_5}^{\gamma_{(-,+^2,+,+)}^{\dot\theta_1}}(\theta_5,\theta_1)-\mathcal{A}_{\omega_5}^{\gamma_{()}^{\dot\theta_5}}(\theta_5,\theta_5).$$

Finally, $\mathcal{A}_{\omega_2-\omega_1}^{\gamma_{()}^{\dot\theta_2}}(\theta_2,\theta_2)\circ\mathcal{A}_{\omega_1}^{\gamma_{()}^{\dot\theta_1}}(\theta_1,\theta_1)=\mathcal{A}_{\omega_2}^{\gamma_{(+)}^{\dot\theta_1}}(\theta_2,\theta_1)-\mathcal{A}_{\omega_2}^{\gamma_{(-)}^{\dot\theta_1}}(\theta_2,\theta_1)$. Putting things together, one concludes:

$$\begin{aligned}\mathcal{A}_{\omega_6}^{\gamma}(\theta_6,\theta_1)=&\ \mathcal{A}_{\omega_6}^{\gamma_{(+)}^{\dot\theta_2'}}(\theta_2',\theta_2')\\
&+\mathcal{A}_{\omega_6-\omega_2}^{\gamma_{(+,+,+)}^{\dot\theta_2'}}(\theta_2',\theta_2')\circ\mathcal{A}_{\omega_2}^{\gamma_{(+)}^{\dot\theta_1'}}(\theta_1',\theta_1')-\mathcal{A}_{\omega_6-\omega_5}^{\gamma_{()}^{\dot\theta_2'}}(\theta_2',\theta_2')\circ\mathcal{A}_{\omega_5}^{\gamma_{()}^{\dot\theta_2'}}(\theta_2',\theta_2')\\
&-\mathcal{A}_{\omega_6-\omega_2}^{\gamma_{(+,+,+)}^{\dot\theta_2'}}(\theta_2',\theta_2')\circ\mathcal{A}_{\omega_2-\omega_1}^{\gamma_{()}^{\dot\theta_1'}}(\theta_1',\theta_1')\circ\mathcal{A}_{\omega_1}^{\gamma_{()}^{\dot\theta_1'}}(\theta_1',\theta_1')\\
&-\mathcal{A}_{\omega_6-\omega_5}^{\gamma_{()}^{\dot\theta_2'}}(\theta_2',\theta_2')\circ\mathcal{A}_{\omega_5-\omega_2}^{\gamma_{(+,+)}^{\dot\theta_2'}}(\theta_2',\theta_2')\circ\mathcal{A}_{\omega_2}^{\gamma_{(+)}^{\dot\theta_1'}}(\theta_1',\theta_1')\\
&+\mathcal{A}_{\omega_6-\omega_5}^{\gamma_{()}^{\dot\theta_2'}}(\theta_2',\theta_2')\circ\mathcal{A}_{\omega_5-\omega_2}^{\gamma_{(+,+)}^{\dot\theta_2'}}(\theta_2',\theta_2')\circ\mathcal{A}_{\omega_2-\omega_1}^{\gamma_{()}^{\dot\theta_1'}}(\theta_1',\theta_1')\circ\mathcal{A}_{\omega_1}^{\gamma_{()}^{\dot\theta_1'}}(\theta_1',\theta_1').\end{aligned}$$

7.8.2.2 Alien operators on $\mathrm{RES}^{(k)}$

We have seen with lemma 7.8 that the alien operators associated with triples of the form $(\gamma,\theta_1,\theta_1)$ act on $\mathrm{RES}^{(k)}$ for γ of type $\gamma_{(+)_m}^{\dot\theta_1}$ and $\gamma_{(-)_m}^{\dot\theta_1}$. We keep on this study according to the guiding line of this section.

We assume $\dot\theta_1\in\{0,\pi\}$ and take two integers l,k subject to the condition $2\leq l\leq k$. By the very definition of $\mathrm{RES}^{(k)}$, the minor $\widehat{\varphi}$ of any singularity $\overset{\triangledown}{\varphi}\in\mathrm{RES}^{(k)}$, once considered as a sectorial germ, can be analytically continued along any path γ of type $\gamma_{\boldsymbol{\varepsilon}^{\boldsymbol{n}_l}}^{\dot\theta_1}$ with

$$\boldsymbol{\varepsilon}^{\boldsymbol{n}_l}\in\{\big((\pm)_{l-1}^{\boldsymbol{n}_l},(\varepsilon)_{m_l-1}\big)\mid\varepsilon\in\{+,-\},\boldsymbol{n}_l=(n_1,\cdots,n_{l-1})\in(\mathbb{N}^\star)^{l-1},m_l\in\mathbb{N}^\star\}.$$

With the notations of (7.24), the analytic continuation $\mathrm{cont}_\gamma\,\widehat{\varphi}$ of $\widehat{\varphi}$ along γ is a germ of holomorphic functions that can be analytically continued onto the simply connected domain $\mathfrak{p}(\mathscr{R}^{\boldsymbol{\varepsilon}^{\boldsymbol{n}_l},\dot\theta})=\mathbb{C}\setminus\{]-\infty,p]\cup[p+1,+\infty[\}$ where $]p,(p+1)[=]\omega_{l-1},\omega_l[$ when $m_l=1$, $]p,(p+1)[=](m_l-1)\omega_l,m_l\omega_l[$ otherwise. These properties translate into the next statement (the details are left to the reader).

Proposition 7.23. *Let be $\theta_1\in\{\pi k,k\in\mathbb{Z}\}\subset\underset{\bullet}{\mathbb{S}}^1$ and $(l,k)\in\mathbb{N}$ with the condition $1\leq l\leq k$. The following alien operators are well-defined, for any $\varepsilon\in\{-,+\}$, any $\boldsymbol{n}_l\in\mathbb{N}^{l-1}$ and any $m_l\in\mathbb{N}^\star$. Setting $\dot\theta_l,\omega_l$ by (7.24) and $\theta_l\in\underset{\bullet}{\mathbb{S}}^1$ with $\dot\theta_l=\dot\pi(\theta_l)$,*

$$\begin{aligned}&\mathcal{A}_{m_l\omega_l}^{\gamma_{((\pm)_{l-1}^{\boldsymbol{n}_l}(\varepsilon)_{m_l-1})}^{\dot\theta_1}}(\theta_l,\theta_1):\mathrm{RES}^{(k)}\to\mathrm{RES}^{(k-l-m_l+1)},\quad 1\leq m_l\leq k-l\\
&\mathcal{A}_{m_l\omega_l}^{\gamma_{((\pm)_{l-1}^{\boldsymbol{n}_l}(\varepsilon)_{m_l-1})}^{\dot\theta_1}}(\theta_l,\theta_1):\mathrm{RES}^{(k)}\to\mathrm{SING}_{\theta_1,\pi},\quad m_l=k-l+1\\
&\mathcal{A}_{m_l\omega_l}^{\gamma_{((\pm)_{l-1}^{\boldsymbol{n}_l}(\varepsilon)_{m_l-1})}^{\dot\theta_1}}(\theta_l,\theta_1):\mathrm{RES}^{(k)}\to\mathrm{SING}_{\theta_1+\pi/2,\pi/2},\quad m_1\geq k-l+2.\end{aligned}\tag{7.25}$$

Equivalently, $\mathcal{A}^{\gamma^{\dot\theta_l}_{(\varepsilon)_{m_l-1}}}_{m_l\omega_l-\omega_{l-1}}(\theta_l,\theta_l)\circ\cdots\mathcal{A}^{\gamma^{\dot\theta_2}_{()}}_{\omega_2-\omega_1}(\theta_2,\theta_2)\circ\mathcal{A}^{\gamma^{\dot\theta_1}_{()}}_{\omega_1}(\theta_1,\theta_1)$ *are well-defined alien operators, with* $\dot\theta_j,\omega_j$ *given by (7.24) and* $\theta_j\in\underset{\bullet}{\mathbb{S}}^1$ *with* $\dot\theta_j=\dot\pi(\theta_j)$*, with the following ranges:*

$$
\begin{aligned}
&\mathcal{A}^{\gamma^{\dot\theta_l}_{(\varepsilon)_{m_l-1}}}_{m_l\omega_l-\omega_{l-1}}(\theta_l,\theta_l)\circ\cdots\circ\mathcal{A}^{\gamma^{\dot\theta_1}_{()}}_{\omega_1}(\theta_1,\theta_1):\mathrm{RES}^{(k)}\to\mathrm{RES}^{(k-l-m_l+1)},\ \ 1\le m_l\le k-l\\
&\mathcal{A}^{\gamma^{\dot\theta_l}_{(\varepsilon)_{m_l-1}}}_{m_l\omega_l-\omega_{l-1}}(\theta_l,\theta_l)\circ\cdots\circ\mathcal{A}^{\gamma^{\dot\theta_1}_{()}}_{\omega_1}(\theta_1,\theta_1):\mathrm{RES}^{(k)}\to\mathrm{SING}_{\theta_1,\pi},\ \ m_l=k-l+1\\
&\mathcal{A}^{\gamma^{\dot\theta_l}_{(\varepsilon)_{m_l-1}}}_{m_l\omega_l-\omega_{l-1}}(\theta_l,\theta_l)\circ\cdots\circ\mathcal{A}^{\gamma^{\dot\theta_1}_{()}}_{\omega_1}(\theta_1,\theta_1):\mathrm{RES}^{(k)}\to\mathrm{SING}_{\theta_1+\frac{\pi}{2},\frac{\pi}{2}},\ \ m_1\ge k-l+2.
\end{aligned}
\tag{7.26}
$$

We would like now to discuss a kind of converse of proposition 7.23 with the next two propositions.

Proposition 7.24. *Let* $k\in\mathbb{N}^\star$ *be a positive integer and* $\overset{\triangledown}{\varphi}\in\mathrm{RES}^{(k)}$*. We suppose that for any* $\theta\in\{\pi k,k\in\mathbb{Z}\}\subset\underset{\bullet}{\mathbb{S}}^1$ *one has* $\mathcal{A}^{\gamma^{\dot\theta}_{()}}_{\omega}(\theta,\theta)\ \overset{\triangledown}{\varphi}\in\mathrm{RES}^{(k)}$*, with* $\omega=\mathrm{e}^{\mathrm{i}\dot\theta}$*,* $\dot\theta=\dot\pi(\theta)$*. Then* $\overset{\triangledown}{\varphi}$ *belongs to* $\mathrm{RES}^{(k+1)}$*.*

Proof. There will be no loss of generality in assuming that $\overset{\triangledown}{\varphi}$ is a simple singularity and this assumption is easier to handle : $\overset{\triangledown}{\varphi}=a\delta+{}^\flat\widehat\varphi\in\overset{\triangledown}{\mathscr{R}}^{(k)}$ with $\widehat\varphi\in\hat{\mathscr{R}}^{(k)}$.
We consider a singularity $\overset{\triangledown}{\mathscr{R}}^{(1)}$. Thus, $\widehat\varphi$ can be analytically continued to $\mathscr{R}^{(1)}$. Equivalently, for any $\theta_1\in\{\pi k,k\in\mathbb{Z}\}$, $\widehat\varphi$ can be analytically continued along any path γ_1 of type $\gamma^{\dot\theta_1}_{(\varepsilon)_{m-1}}$, $m\in\mathbb{N}^\star$, $\varepsilon\in\{-,+\}$ and $\mathrm{cont}_{\gamma_1}\widehat\varphi$ is a germ that can be analytically continued to the star-shaped domain $\mathfrak{p}\big(\mathscr{R}^{(\varepsilon)_{m-1},\dot\theta_1}\big)$.

Let us assume that for any $\theta_1\in\{\pi k,k\in\mathbb{Z}\}$, $\mathcal{A}^{\gamma^{\dot\theta_1}_{()}}_{\omega}(\theta_1,\theta_1)\ \overset{\triangledown}{\varphi}$ belongs to $\mathrm{RES}^{(1)}$, where $\omega_1=\mathrm{e}^{\mathrm{i}\dot\theta_1}$. We claim that $\overset{\triangledown}{\varphi}$ belongs to $\mathrm{RES}^{(2)}$.

Our assumption results in the following property : for any $n_1\in\mathbb{N}^\star$ and any path γ of type $\gamma^{\dot\theta_1}_{(\pm)_1^{n_1}}$, denoting by $\lambda^-_{\omega_1}$ a clockwise loop around ω_1, the difference $\big(\mathrm{cont}_\gamma-\mathrm{cont}_{\gamma\lambda^-_{\omega_1}}\big)\widehat\varphi$ is a sectorial germ that can be analytically continued along any path γ_2 of type $\gamma^{\dot\theta_2}_{(\varepsilon)_{m-1}}$, $m\in\mathbb{N}^\star$, $\varepsilon\in\{-,+\}$, $\dot\theta_2=\dot\theta_1+(n_1-1)\pi$. Moreover $\mathrm{cont}_{\gamma_2}\big(\mathrm{cont}_\gamma-\mathrm{cont}_{\gamma\lambda^-_{\omega_1}}\big)\widehat\varphi$ is a germ of holomorphic functions that can be analytically continued to the star-shaped domain $\mathfrak{p}\big(\mathscr{R}^{((\pm)_1^{n_1},(\varepsilon)_{m-1}),\dot\theta_1}\big)$.
Start with $n_1=1$ and a path γ of type $\gamma^{\dot\theta_1}_{(+)_1}$, <u>resp</u>. $\gamma^{\dot\theta_1}_{(-)_1}$. Take a path γ_2 of type $\gamma^{\dot\theta_2}_{(+)_{m-1}}$, $\dot\theta_2=\dot\theta_1$, <u>resp</u>. $\gamma^{\dot\theta_2}_{(-)_{m-1}}$. Notice that $\gamma_1=\gamma\gamma_2$ is a path of type $\gamma^{\dot\theta_1}_{(\varepsilon)_m}$. Therefore from the above property, $\mathrm{cont}_{\gamma_2}\big(\mathrm{cont}_\gamma\widehat\varphi\big)=\mathrm{cont}_{\gamma_1}\widehat\varphi$ is well-defined and gives a germ that can be analytically continued to the domain $\mathfrak{p}\big(\mathscr{R}^{(+)_m,\dot\theta_1}\big)=\mathfrak{p}\big(\mathscr{R}^{((+)_1,(+)_{m-1}),\dot\theta_1}\big)$, <u>resp</u> $\mathfrak{p}\big(\mathscr{R}^{(-)_m,\dot\theta_1}\big)=\mathfrak{p}\big(\mathscr{R}^{((-)_1,(-)_{m-1}),\dot\theta_1}\big)$.
But this implies that $\mathrm{cont}_{\gamma_2}\big(\mathrm{cont}_{\gamma\lambda^-_{\omega_1}}\widehat\varphi\big)=\mathrm{cont}_{\gamma\lambda^-_{\omega_1}\gamma_2}\widehat\varphi$ is also well-defined and provides a germ that can be analytically continued to the domain $\mathfrak{p}\big(\mathscr{R}^{(+)_m,\dot\theta_1}\big)=\mathfrak{p}\big(\mathscr{R}^{((-)_1,(+)_{m-1}),\dot\theta_1}\big)$, <u>resp</u> $\mathfrak{p}\big(\mathscr{R}^{(-)_m,\dot\theta_1}\big)=\mathfrak{p}\big(\mathscr{R}^{((-)_1^3,(-)_{m-1}),\dot\theta_1}\big)$.
(Notice that the path $\gamma\lambda^-_{\omega_1}\gamma_2$ is a path of type $\gamma^{\dot\theta_1}_{((-)_1,(+)_{m-1})}$, <u>resp</u> $\gamma^{\dot\theta_1}_{((-)_1^3,(-)_{m-1})}$).

Of course, one could have chosen a path γ of type $\gamma^{\dot\theta_1}_{(+)_1}$ and a path γ_2 of type $\gamma^{\dot\theta_1}_{(-)_{m-1}}$. The path $\gamma_1=\gamma\lambda^-_{\omega_1}\gamma_2$ is a path of type $\gamma^{\dot\theta_1}_{(-)_m}$ and we conclude for the analytic continuation of $\widehat{\varphi}$ along the path $\gamma\gamma_2$ of type $\gamma^{\dot\theta_1}_{((+)_1,(-)_{m-1})}$.

One can pursue this way by induction on n_1 to show our claim. Here, we just add the case $n_1=2$ so as to deal with a subtlety. We thus consider a path γ of type $\gamma^{\dot\theta_1}_{(+)^2_1}$ and a path γ_2 of type $\gamma^{\dot\theta_2}_{(\varepsilon)_{m-1}}$, $\dot\theta_2=\dot\theta_1+\pi$. Notice that the path $\gamma\lambda^-_{\omega_1}\gamma_2$ is homotopic to a path of type $\gamma^{\dot\theta_1}_{()}$ when $m=1$, of type $\gamma^{\dot\theta_2}_{(\varepsilon)_{m-2}}$ when $m\geq 2$. Therefore, $\mathrm{cont}_{\gamma_2}\big(\mathrm{cont}_{\gamma\lambda^-_{\omega_1}}\widehat{\varphi}\big)$ is well-defined and one concludes that $\widehat{\varphi}$ can be analytically continued along the path $\gamma_1=\gamma\gamma_2$ of type $\gamma^{\dot\theta_1}_{((+)^2_1,(\varepsilon)_{m-1})}$ and moreover the germ $\mathrm{cont}_{\gamma_1}\widehat{\varphi}$ can be analytically continued to the star-shaped domain $\mathfrak{p}\big(\mathscr{R}^{((+)^2_1,(\varepsilon)_{m-1}),\dot\theta_1}\big)$.

The same reasoning can be generalized and give the proposition. □

A quite similar (and even simpler) reasoning gives the next result.

Proposition 7.25. *Let be $k\in\mathbb{N}^\star$ and $\overset{\triangledown}{\varphi}\in\mathrm{RES}^{(k)}$. We suppose that for any $\theta_1\in\{\pi k,k\in\mathbb{Z}\}\subset\underset{\bullet}{\mathbb{S}}^1$ and any $\boldsymbol{n}\in\mathbb{N}^{k-1}$, the singularity $\mathcal{A}^{\gamma^{\dot\theta_1}_{((\pm)^{\boldsymbol{n}}_{k-1})}}_{\omega_k}(\theta_k,\theta_1)\,\overset{\triangledown}{\varphi}$ belongs to $\mathrm{RES}^{(1)}$, where ω_k is given by (7.24). Then $\overset{\triangledown}{\varphi}$ belongs to $\mathrm{RES}^{(k+1)}$.*

We eventually use theorem 7.2 to reformulate proposition 7.25.

Corollary 7.1. *Let $k\in\mathbb{N}^\star$ be a positive integer and $\overset{\triangledown}{\varphi}\in\mathrm{RES}^{(k)}$. We suppose that $\mathcal{A}^{\gamma_k}_{\omega_k-\omega_{k-1}}(\theta_k,\theta_k)\circ\cdots\circ\mathcal{A}^{\gamma_2}_{\omega_2-\omega_1}(\theta_2,\theta_2)\circ\mathcal{A}^{\gamma_1}_{\omega_1}(\theta_1,\theta_1)\,\overset{\triangledown}{\varphi}$ belongs to $\mathrm{RES}^{(1)}$ for any composite operator that satisfies the properties:*

- *for every $j=1,\cdots,k$, the path γ_j is of type $\gamma^{\dot\theta_j}_{(+)_{m_j-1}}$, $m_j\in\mathbb{N}^\star$;*
- *$\sum_{j=1}^k m_j=k$.*

Then $\overset{\triangledown}{\varphi}$ belongs to $\mathrm{RES}^{(k+1)}$.

7.8.3 Alien derivations

We now specialize to some alien operators.

7.8.3.1 Definitions

Definition 7.50. Let be $\theta\in\{\pi k,k\in\mathbb{Z}\}\subset\underset{\bullet}{\mathbb{S}}^1$, $\alpha\in]0,\pi/2]$ and $L>0$. We denote $\dot\theta=\dot\pi(\theta)\in\{0,\pi\}\subset\mathbb{S}^1$. We set $\omega=m e^{\mathrm{i}\theta}\in\underset{\bullet}{\mathbb{C}}$ for $m\in\{1,\cdots,\lceil L\rceil\}$, <u>*resp.*</u> $m\in\mathbb{N}^\star$. The so-called alien operators at ω,

$$\varDelta^+_\omega,\varDelta_\omega:\mathrm{RES}^{(\dot\theta,\alpha)}(L)\to\mathrm{SING}_{\theta,\alpha},\quad \underline{\textit{resp.}}\ \ \varDelta^+_\omega,\varDelta_\omega:\mathrm{RES}\to\mathrm{RES},$$

are defined by (7.27),

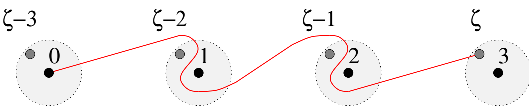


Fig. 7.14 Symmatrically contractile path H_b and contributions to $\Delta^+_\omega(\widehat{\varphi} * \widehat{\psi})$ for $\omega = 3$. Pinchings occur between 1 and $\zeta - 2$, and between 2 and $\zeta - 1$.

$$\begin{aligned}
\Delta^+_\omega \overset{\triangledown}{\varphi} &= \mathcal{A}^{\gamma^{\dot\theta}_{(+)_{m-1}}}_{\overset{\bullet}{\omega}}(\theta,\theta)\overset{\triangledown}{\varphi} && (7.27)\\
\Delta_\omega \overset{\triangledown}{\varphi} &= \sum_{\boldsymbol{\varepsilon}=(\varepsilon_1,\cdots,\varepsilon_{m-1})\in\{+,-\}^{m-1}} \frac{p(\boldsymbol{\varepsilon})!q(\boldsymbol{\varepsilon})!}{m!}\mathcal{A}^{\gamma^{\dot\theta}_{\boldsymbol{\varepsilon}}}_{\overset{\bullet}{\omega}}(\theta,\theta)\overset{\triangledown}{\varphi},
\end{aligned}$$

where $p(\boldsymbol{\varepsilon})$, resp. $q(\boldsymbol{\varepsilon}) = m-1-p(\boldsymbol{\varepsilon})$, denotes the number of "+" signs, resp. "−" signs in the sequence $\boldsymbol{\varepsilon}$.

Definition 7.51. The alien operators $\Delta^+_\omega, \Delta_\omega : \widetilde{\mathrm{RES}} \to \widetilde{\mathrm{RES}}$ for asymptotic classes are defined by making the following diagrams commuting:

$$\begin{array}{ccc}
\mathrm{RES} & \overset{\Delta^+_\omega,\Delta_\omega}{\longrightarrow} & \mathrm{RES}\\
\tilde{\mathcal{L}}\downarrow\uparrow\tilde{\mathcal{B}} & & \tilde{\mathcal{L}}\downarrow\uparrow\tilde{\mathcal{B}}\,.\\
\widetilde{\mathrm{RES}} & \overset{\Delta^+_\omega,\Delta_\omega}{\longrightarrow} & \widetilde{\mathrm{RES}}
\end{array}$$

7.8.3.2 Properties

Theorem 7.3. *Under the hypotheses of definition 7.50, the alien operators* $\Delta^+_\omega : \mathrm{RES}^{(\dot\theta,\alpha)}(L) \to \mathrm{SING}_{\theta,\alpha}$, resp. $\Delta^+_\omega : \mathrm{RES} \to \mathrm{RES}$, *satisfy the identity:*

$$\Delta^+_\omega(\overset{\triangledown}{\varphi} * \overset{\triangledown}{\psi}) = (\Delta^+_\omega \overset{\triangledown}{\varphi}) * \overset{\triangledown}{\psi} + \sum_{\omega_1+\omega_2=\omega}\left(\Delta^+_{\omega_1}\overset{\triangledown}{\varphi}\right) * \left(\Delta^+_{\omega_2}\overset{\triangledown}{\psi}\right) + \overset{\triangledown}{\varphi} * (\Delta^+_\omega \overset{\triangledown}{\psi}) \tag{7.28}$$

where the sum runs over all $\omega_1 = m_1 \mathrm{e}^{\mathrm{i}\theta}$, $\omega_2 = m_2\mathrm{e}^{\mathrm{i}\theta}$, *with* $m_1, m_2 \in \mathbb{N}^\star$ *such that* $m_1 + m_2 = m$.
The alien operators $\Delta_\omega : \mathrm{RES}^{(\dot\theta,\alpha)}(L) \to \mathrm{SING}_{\theta,\alpha}$, resp. $\Delta_\omega : \mathrm{RES} \to \mathrm{RES}$, *satisfy the Leibniz rule,* $\Delta_\omega(\overset{\triangledown}{\varphi} * \overset{\triangledown}{\psi}) = \left(\Delta_\omega\overset{\triangledown}{\varphi}\right) * \overset{\triangledown}{\psi} + \overset{\triangledown}{\varphi} * \left(\Delta_\omega\overset{\triangledown}{\psi}\right)$. *Moreover,* $\Delta_\omega(\partial\,\overset{\triangledown}{\varphi}) = (\partial - \overset{\bullet}{\omega})(\Delta_\omega\,\overset{\triangledown}{\varphi})$. *Eventually,* $\Delta^+_\omega\,\overset{\triangledown}{\mathrm{cons}} = \Delta_\omega\,\overset{\triangledown}{\mathrm{cons}} = 0$ *for any resurgent constant* $\overset{\triangledown}{\mathrm{cons}}$.

Proof. We give the proof for the identity (7.28) only, so as to exemplify the use of singularities. Moreover we work on the space $\overset{\triangledown}{\mathscr{R}}^{(\dot\theta,\alpha)}(L)$.

The reader is invited to compare with the proof made in [24] for simple resurgent functions.

There is no loss of generality in assuming that $\overset{\triangledown}{\varphi} = {}^\flat\widehat{\varphi}$, $\overset{\triangledown}{\psi} = {}^\flat\widehat{\psi}$ with $\widehat{\varphi}, \widehat{\psi} \in \widehat{\mathscr{R}}^{(\theta,\alpha)}(L)$. By proposition 7.6 one has ${}^\flat\widehat{\varphi} * {}^\flat\widehat{\varphi} = {}^\flat(\widehat{\varphi} * \widehat{\varphi})$, therefore we can use arguments developed in chapter 4 (see in particular the proof of proposition 4.3), which allow us some abuse of notations.

The analytic continuation of the convolution product $\widehat{\varphi} * \widehat{\psi}$ along a path γ of type $\gamma^{\dot\theta}_{(+)_{m-1}} = \pi(\gamma^{\theta}_{(+)_{m-1}})$, ending at $\zeta = \omega + \overset{\bullet}{\xi}_0$ near $](m-1)\mathrm{e}^{\mathrm{i}\dot\theta}, m\mathrm{e}^{\mathrm{i}\dot\theta}[$, is the germ of holomorphic function defined as follows:

$$(\mathrm{cont}_\gamma\, \widehat{\varphi}*\widehat{\psi})(\omega+\overset{\bullet}{\xi}) = \int_{H_b} \widehat{\varphi}(\eta_1)\widehat{\psi}(\eta_2+\overset{\bullet}{\xi}-\overset{\bullet}{\xi}_0)+\int_0^{\overset{\bullet}{\xi}-\overset{\bullet}{\xi}_0} \widehat{\varphi}(\zeta+\eta)\widehat{\psi}(\overset{\bullet}{\xi}-\overset{\bullet}{\xi}_0-\eta)d\eta.$$

Here H_b is a symmatrically contractile path deduced from γ, $\widehat{\varphi}(\eta_1) = \mathrm{cont}_{\underline{H}_b|[0,s]}\widehat{\varphi}\big(\underline{H}_b(s)\big)$, $\widehat{\psi}(\eta_2+\overset{\bullet}{\xi}-\overset{\bullet}{\xi}_0) = \mathrm{cont}_{\underline{H}_b^{-1}|[0,s]}\widehat{\psi}\big(\underline{H}_b^{-1}(s)+\overset{\bullet}{\xi}-\overset{\bullet}{\xi}_0\big)$ and $\widehat{\varphi}(\zeta+\eta) = \mathrm{cont}_{\underline{H}_b}\widehat{\varphi}\big(\underline{H}_b(1)+\eta\big)$. To get the associated singularity, that is $\Delta^+_\omega(\overset{\triangledown}{\varphi} * \overset{\triangledown}{\psi})$, one only needs to consider the restrictions:

1. of the first integral near the "pinching points" (see Fig. 7.14), where one easily recognizes convolution products for majors and these provide the contribution $\sum_{\omega_1+\omega_2=\omega}\big(\Delta^+_{\omega_1}\overset{\triangledown}{\varphi}\big) * \big(\Delta^+_{\omega_2}\overset{\triangledown}{\psi}\big)$ to the singularity $\Delta^+_\omega(\overset{\triangledown}{\varphi} * \overset{\triangledown}{\psi})$;
2. of the two integrals near the end points, which provide the missing contributions (use proposition 7.2).

This ends the proof. □

Definition 7.52. The linear operators Δ_ω are called alien derivations and RES is called a resurgent algebra (since stable under alien derivations).

We refer to [24, 1] for the proof of the next statements.

Theorem 7.4. *For any $\theta \in \{\ k\pi, k\in\mathbb{Z}\}$, $\omega \in \underset{\bullet}{\mathbb{C}}$ with $\arg(\omega)=\theta$,*

$$\Delta_\omega = \sum_{s\in\mathbb{N}^\star} \frac{(-1)^{s-1}}{s} \sum_{\substack{\arg(\omega_1)=\cdots=\arg(\omega_{s-1})=\theta \\ 0\prec\omega_1\prec\cdots\prec\omega_s\prec\omega}} \Delta^+_{\omega-\omega_{s-1}}\circ\cdots\circ\Delta^+_{\omega_2-\omega_1}\circ\Delta^+_{\omega_1}, \tag{7.29}$$

$$\Delta^+_\omega = \sum_{s\in\mathbb{N}^\star} \frac{1}{s!} \sum_{\substack{\arg(\omega_1)=\cdots=\arg(\omega_{s-1})=\theta \\ 0\prec\omega_1\prec\cdots\prec\omega_s\prec\omega}} \Delta_{\omega-\omega_{s-1}}\circ\cdots\circ\Delta_{\omega_2-\omega_1}\circ\Delta_{\omega_1}, \tag{7.30}$$

In the above theorem, $\prec$ stands for the total order on $[0,\omega]$ induced by $t\in[0,1]\mapsto t\omega\in[0,\omega]$.

The alien derivations own the property of generating the whole set of alien operators. We precise this claim with the following upshot from theorem 7.2 and theorem 7.4.

Theorem 7.5. *Let $m\in\mathbb{N}^\star$ be a positive integer, $\boldsymbol{\varepsilon}\in\{+,-\}^{m-1}$, $\boldsymbol{n}\in(\mathbb{N}^\star)^{m-1}$ and $\theta_1\in\{\pi k, k\in\mathbb{Z}\}$. Let γ be a path of type $\gamma^{\dot\theta_1}_{\boldsymbol{\varepsilon}^{\boldsymbol{n}}}$, $\overset{\bullet}{\omega}_m$ and $\dot\theta_m$ given by (7.24), and $\theta_m\in\underset{\bullet}{\mathbb{S}}^1$ so that $\dot\theta_m=\dot\pi(\theta_m)$. Then the alien operator $\mathcal{A}^\gamma_{\overset{\bullet}{\omega}_m}(\theta_m,\theta_1)$ at $\overset{\bullet}{\omega}_m$ associated with the triple $(\gamma,\theta_1,\theta_m)$ can be written as a $\mathbb{Z}$-linear, <u>resp.</u> $\mathbb{Q}$-linear combination of composite operators of the form*

$$\varrho_{k_n}.\big(\Delta^+_{\omega_n}\circ\cdots\circ\Delta^+_{\omega_2}\circ\Delta^+_{\omega_1}\big),\ \ \underline{resp.}\ \ \varrho_{k_n}.\big(\Delta_{\omega_n}\circ\cdots\circ\Delta_{\omega_2}\circ\Delta_{\omega_1}\big),$$

that satisfy the properties:

- $(\overset{\bullet}{\omega}_1,\cdots,\overset{\bullet}{\omega}_n)\in(\mathbb{Z}^\star)^n$, $n\in\mathbb{N}^\star$ *and* $\pi\Big(\sum_{j=1}^n\omega_j\Big)=\overset{\bullet}{\omega}_m$;
- $\dot\theta_m=\arg(\omega_n)+2\pi k_n$, $k_n\in\mathbb{Z}$;
- $\sum_{j=1}^n|\omega_j|\le m$.

Example 7.13. We continue the example 7.10. The path γ is of type $\gamma^0_{\boldsymbol{\varepsilon}}$ for $\boldsymbol{\varepsilon} = (+,-,+)$ and we know that

$$\mathcal{A}_4^{\gamma^0_{(+,-,+)}}(0,0) = \Delta_4^+ - \Delta_2^+ \circ \Delta_2^+.$$

(On the right-hand side of the equality, $(4,2)$ stands for $(4\mathrm{e}^{\mathrm{i}0}, 2\mathrm{e}^{\mathrm{i}0})$). Using theorem 7.4, one gets:

$$\begin{gathered}\mathcal{A}_4^{\gamma^0_{(+,-,+)}}(0,0) = \Delta_4 + \tfrac{1}{2!}\big(\Delta_3 \circ \Delta_1 + \Delta_2 \circ \Delta_2 + \Delta_1 \circ \Delta_3\big)\\ +\tfrac{1}{3!}\big(\Delta_2 \circ \Delta_1 \circ \Delta_1 + \Delta_1 \circ \Delta_2 \circ \Delta_1 + \Delta_1 \circ \Delta_1 \circ \Delta_1\big) + \tfrac{1}{4!}\Delta_1 \circ \Delta_1 \circ \Delta_1 \circ \Delta_1\\ -\big(\Delta_2 + \tfrac{1}{2!}\Delta_1 \circ \Delta_1\big) \circ \big(\Delta_2 + \tfrac{1}{2!}\Delta_1 \circ \Delta_1\big).\end{gathered}$$

(We do not simplify).

Example 7.14. We continue the example 7.11. The path γ is of type $\gamma^{\dot{\theta}_1}_{\boldsymbol{\varepsilon}\boldsymbol{n}}$ for $\boldsymbol{\varepsilon} = (+,-,+)$, $\boldsymbol{n} = (1,3,1)$ and we have shown the identity:

$$\mathcal{A}_4^{\gamma^0_{(+,-2,+)}}(0,0) = \Delta_4^+ - \Delta_2^+ \circ \Delta_2^+ - \varrho_{-1}.\Delta^+_{2\mathrm{e}^{-2\mathrm{i}\pi}} \circ \Delta_2^+.$$

This can be expressed in term of alien derivatives as well.

We end with an observation. By its very definition, any singularity $\overset{\triangledown}{\varphi} \in \overset{\triangledown}{\mathscr{R}}{}^{(\theta,\alpha)}(L)$ has a *regular* minor. This property involves the following relationships for the action of the alien operators. (These are essentially consequences of propositions 7.19 and 7.20).

Proposition 7.26. *We suppose* $\alpha \in]0,\pi/2]$, $L > 0$ *and* $m \in \{1,\cdots,\lceil L \rceil\}$. *The following equalities hold for any* $k \in \mathbb{Z}$*:*

- *for any* $\overset{\triangledown}{\varphi} \in \overset{\triangledown}{\mathscr{R}}{}^{(0,\alpha)}(L)$,

$$\Delta^+_{m\mathrm{e}^{\mathrm{i}\pi 2k}} \overset{\triangledown}{\varphi} = \varrho_{-k}.\big(\Delta^+_{m\mathrm{e}^{\mathrm{i}\pi 0}} \overset{\triangledown}{\varphi}\big), \qquad \Delta_{m\mathrm{e}^{\mathrm{i}\pi 2k}} \overset{\triangledown}{\varphi} = \varrho_{-k}.\big(\Delta_{m\mathrm{e}^{\mathrm{i}\pi 0}} \overset{\triangledown}{\varphi}\big);$$

- *for any* $\overset{\triangledown}{\varphi} \in \overset{\triangledown}{\mathscr{R}}{}^{(\pi,\alpha)}(L)$,

$$\Delta^+_{m\mathrm{e}^{\mathrm{i}\pi(2k+1)}} \overset{\triangledown}{\varphi} = \varrho_{-k}.\big(\Delta^+_{m\mathrm{e}^{\mathrm{i}\pi}} \overset{\triangledown}{\varphi}\big), \qquad \Delta_{m\mathrm{e}^{\mathrm{i}\pi(2k+1)}} \overset{\triangledown}{\varphi} = \varrho_{-k}.\big(\Delta_{m\mathrm{e}^{\mathrm{i}\pi}} \overset{\triangledown}{\varphi}\big);$$

- *moreover, if* $\overset{\triangledown}{\varphi} \in \overset{\triangledown}{\mathscr{R}}{}^{(0,\alpha)}(L) \cap \overset{\triangledown}{\mathscr{R}}{}^{(\pi,\alpha)}(L)$ *and if its minor* $\widehat{\varphi}$ *is even, then*

$$\Delta^+_{\mathrm{e}^{\mathrm{i}\pi}} \overset{\triangledown}{\varphi} = \varrho_{-1/2}.\big(\Delta^+_1 \overset{\triangledown}{\varphi}\big), \qquad \Delta_{\mathrm{e}^{\mathrm{i}\pi}} \overset{\triangledown}{\varphi} = \varrho_{-1/2}.\big(\Delta_1 \overset{\triangledown}{\varphi}\big)$$

  with $1 = \mathrm{e}^{\mathrm{i}0}$, *while if* $\widehat{\varphi}$ *is odd, then*

$$\Delta^+_{\mathrm{e}^{\mathrm{i}\pi}} \overset{\triangledown}{\varphi} = -\varrho_{-1/2}.\big(\Delta^+_1 \overset{\triangledown}{\varphi}\big), \qquad \Delta_{\mathrm{e}^{\mathrm{i}\pi}} \overset{\triangledown}{\varphi} = -\varrho_{-1/2}.\big(\Delta_1 \overset{\triangledown}{\varphi}\big)$$

Example 7.15. We take $\widehat{\varphi}(\zeta) = \dfrac{\zeta}{e^{2i\pi\zeta} - 1} \in \widehat{\mathscr{R}}$. This is a meromorphic function with simple poles at $\mathbb{Z}^\star$ whose residue at $m \in \mathbb{Z}^\star$ is $\mathrm{res}_m \widehat{\varphi} = m$. Introducing the singularity $\overset{\triangledown}{\varphi} = {}^\flat\widehat{\varphi}$, one easily deduces that for every $k \in \mathbb{Z}$ and every $m \in \mathbb{N}^\star$,

$$\Delta_{m\mathrm{e}^{\mathrm{i}\pi k}} \overset{\triangledown}{\varphi} = \Delta^+_{m\mathrm{e}^{\mathrm{i}\pi k}} \overset{\triangledown}{\varphi} = (-1)^k m\delta. \tag{7.31}$$

The formal Laplace transform $\tilde{\mathcal{L}} \overset{\triangledown}{\varphi}$ is an asymptotic class $\overset{\vartriangle}{\varphi}= \natural\widetilde{\varphi}$ that can be representend by a $\mathbb{Z}$-resurgent series $\widetilde{\varphi} \in \widetilde{\mathscr{R}}_{\mathbb{Z}}$ and (7.31) translates into

$$\Delta_{me^{i\pi k}} \overset{\vartriangle}{\varphi} = \Delta^{+}_{me^{i\pi k}} \overset{\vartriangle}{\varphi} = (-1)^k m. \tag{7.32}$$

We now look at the singularity $\overset{\triangledown}{\psi}_{\sigma,n} = \overset{\triangledown}{J}_{\sigma,n} * \overset{\triangledown}{\varphi}$ for $(\sigma, n) \in \mathbb{C} \times \mathbb{N}$. By the Leibniz rule and since $\overset{\triangledown}{J}_{\sigma,n}$ is a resurgent constant,

$$\Delta_{me^{i\pi k}} \overset{\triangledown}{\psi}_{\sigma,n} = \overset{\triangledown}{J}_{\sigma,n} * \Delta_{me^{i\pi k}} \overset{\triangledown}{\varphi} = (-1)^k m \overset{\triangledown}{J}_{\sigma,n} \in \bigcap_{\alpha>0} \mathrm{SING}_{\pi k,\alpha}. \tag{7.33}$$

The asymptotic class associated to $\overset{\triangledown}{\psi}_{\sigma,n}$ by formal Laplace transform is $\overset{\vartriangle}{\psi}_{\sigma,n} = \overset{\vartriangle}{J}_{\sigma,n} \overset{\vartriangle}{\varphi} \in \widetilde{\mathrm{RES}}$. The identity (7.33) provides:

$$\Delta_{me^{i\pi k}} \overset{\vartriangle}{\psi}_{\sigma,n} = (-1)^k m \overset{\vartriangle}{J}_{\sigma,n} \in \bigcap_{\alpha>0} \mathrm{ASYMP}_{\pi k,\alpha}. \tag{7.34}$$

7.8.3.3 The spaces $\mathrm{RES}^{(k)}$

We have already describe the action of the alien operators on the spaces $\mathrm{RES}^{(k)}$. We can draw some consequences from theorem 7.3.

Corollary 7.2. *Let be $k \in \mathbb{N}^\star$ and $\omega \in \underset{\bullet}{\mathbb{C}}$ so that $\overset{\bullet}{\omega}$ is an integer and $|\omega| \le k$. The alien operator Δ_ω acts on $\mathrm{RES}^{(k)}$ and*

$$\begin{aligned} &\Delta_\omega : \mathrm{RES}^{(k)} \to \mathrm{RES}^{(k-|\omega|)}, \quad \text{when } 1 \le |\omega| \le k-1 \\ &\Delta_\omega : \mathrm{RES}^{(k)} \to \mathrm{SING}_{\arg(\omega),\pi}, \quad \text{when } |\omega| = k. \end{aligned} \tag{7.35}$$

Moreover for any $\overset{\triangledown}{\varphi}, \overset{\triangledown}{\psi} \in \mathrm{RES}^{(k)}$:

- $\Delta_\omega(\partial \overset{\triangledown}{\varphi}) = (\partial - \overset{\bullet}{\omega})(\Delta_\omega \overset{\triangledown}{\varphi})$;
- $\Delta_\omega\big(\overset{\triangledown}{\varphi} * \overset{\triangledown}{\psi}\big)$ *belongs to* $\mathrm{RES}^{(1)}$ *when* $1 \le |\omega| \le k-1$ *and to* $\mathrm{SING}_{\arg(\omega),\pi}$ *when* $|\overset{\bullet}{\omega}| = k$ *and furthermore* $\Delta_\omega\big(\overset{\triangledown}{\varphi} * \overset{\triangledown}{\psi}\big) = \big(\Delta_\omega \overset{\triangledown}{\varphi}\big) * \overset{\triangledown}{\psi} + \overset{\triangledown}{\varphi} * \big(\Delta_\omega \overset{\triangledown}{\psi}\big)$ *(Leibniz rule).*

Proof. The identity (7.35) is a consequence of proposition 7.23. The commutation formula $[\Delta_\omega, \partial] = -\overset{\bullet}{\omega} \Delta_\omega$ ensues from proposition 7.18. Notice now that for any $k \in \mathbb{N}^\star$, any $L \in]k-1, k]$ and any $\alpha \in]0, \pi/2]$, one has $\mathrm{RES}^{(\dot{\theta},\alpha)}(L) \supset \mathrm{RES}^{(k)}$. Take two singularities $\overset{\triangledown}{\varphi}, \overset{\triangledown}{\psi} \in \mathrm{RES}^{(k)}$ and consider them as belonging to $\mathrm{RES}^{(\dot{\theta},\alpha)}(L)$. One can apply theorem 7.3 to get: $\Delta_\omega\big(\overset{\triangledown}{\varphi} * \overset{\triangledown}{\psi}\big) = \big(\Delta_\omega \overset{\triangledown}{\varphi}\big) * \overset{\triangledown}{\psi} + \overset{\triangledown}{\varphi} * \big(\Delta_\omega \overset{\triangledown}{\psi}\big) \in \mathrm{SING}_{\theta,\alpha}$. Also, we know that $\Delta_\omega \overset{\triangledown}{\varphi}$ and $\Delta_\omega \overset{\triangledown}{\psi}$ belong to $\mathrm{RES}^{(k-m)}$ or $\mathrm{SING}_{\theta,\pi}$ depending on $|\omega|$. Finally when $1 \le |\omega| \le k-1$, one can work in $\mathrm{RES}^{(1)} \supset \mathrm{RES}^{(k-m)}$ which is a convolution algebra by proposition 7.17 and this provides the conclusion. □

Definition 7.53. The alien operators $\Delta^+_\omega, \Delta_\omega : \widetilde{\mathrm{RES}}^{(k)} \to \widetilde{\mathrm{RES}}^{(k-|\omega|)}$ for $1 \leq |\omega| \leq k-1$, resp. $\Delta^+_\omega, \Delta_\omega : \widetilde{\mathrm{RES}}^{(k)} \to \mathrm{ASYMP}_{\arg(\omega),\pi}$, for $|\omega| = k$, for asymptotic classes are defined by making the following diagrams commuting:

$$\begin{array}{ccc} \mathrm{RES}^{(k)} & \overset{\Delta^+_\omega,\Delta_\omega}{\longrightarrow} & \mathrm{RES}^{(k-|\omega|)} \\ \tilde{\mathcal{L}} \downarrow\uparrow \tilde{\mathcal{B}} & & \tilde{\mathcal{L}} \downarrow\uparrow \tilde{\mathcal{B}} \\ \widetilde{\mathrm{RES}}^{(k)} & \overset{\Delta^+_\omega,\Delta_\omega}{\longrightarrow} & \widetilde{\mathrm{RES}}^{(k-|\omega|)}. \end{array} \quad , \text{resp.} \quad \begin{array}{ccc} \mathrm{RES}^{(k)} & \overset{\Delta^+_\omega,\Delta_\omega}{\longrightarrow} & \mathrm{SING}_{\arg(\omega),\pi} \\ \tilde{\mathcal{L}} \downarrow\uparrow \tilde{\mathcal{B}} & & \tilde{\mathcal{L}} \downarrow\uparrow \tilde{\mathcal{B}} \\ \widetilde{\mathrm{RES}}^{(k)} & \overset{\Delta^+_\omega,\Delta_\omega}{\longrightarrow} & \mathrm{ASYMP}_{\arg(\omega),\pi}. \end{array}$$

We add a property that will be useful in the sequel.

Corollary 7.3. *Let $k \in \mathbb{N}^\star$ be a positive integer and $\overset{\triangledown}{\varphi} \in \mathrm{RES}^{(k)}$. We suppose that for any $n \in \mathbb{N}^\star$, $\Delta_{\omega_n} \circ \cdots \circ \Delta_{\omega_2} \circ \Delta_{\omega_1} \overset{\triangledown}{\varphi}$ belongs to $\mathrm{RES}^{(1)}$ for any composite operator that satisfies the properties: $(\overset{\bullet}{\omega}_1, \cdots, \overset{\bullet}{\omega}_n) \in (\mathbb{Z}^\star)^n$ and $\sum_{j=1}^n |\omega_j| = k$. Then $\overset{\triangledown}{\varphi}$ belongs to $\mathrm{RES}^{(k+1)}$.*

Proof. This is a direct consequence of both corollary 7.1 and theorem 7.2. □

7.9 Ramified resurgent functions

As already said, one uses various spaces of resurgent functions, accordingly to the problem under consideration. We introduce some of them.

7.9.1 Simple and simply ramified resurgent functions

We start with the resurgent algebra of simple resurgent singularities [24, 1, 7] and we make use of proposition 7.6.

Definition 7.54. A $\mathbb{Z}$-resurgent singularity $\overset{\triangledown}{\varphi} \in \mathrm{RES}$ is said to be a **simple resurgent** singularity when $\overset{\triangledown}{\varphi} = a\delta + {}^\flat\widehat{\varphi} \in \mathrm{SING}^{\mathrm{simp}}$ and, for any alien operator $\mathcal{A}^\gamma_\omega(\theta_2,\theta_1)$, $\mathcal{A}^\gamma_\omega(\theta_2,\theta_1) \overset{\triangledown}{\varphi}$ belongs to $\mathrm{SING}^{\mathrm{simp}}$. The minor $\widehat{\varphi}$, resp. the 1-Gevrey series $\widetilde{\varphi} = a + \tilde{\mathcal{L}}\widehat{\varphi}$, associated with a simple $\mathbb{Z}$-resurgent singularity is a **simple resurgent function**, resp. a **simple resurgent series**, and one denotes by $\hat{\mathscr{R}}_\mathbb{Z}{}^{\mathrm{simp}}$, resp. $\widetilde{\mathscr{R}}_\mathbb{Z}{}^{\mathrm{simp}}$ the space of simple $\mathbb{Z}$-resurgent functions., resp. series. The resurgent subalgebra made of simple resurgent singularities is denoted by $\mathrm{RES}^{\mathrm{simp}}_\mathbb{Z}$ and the corresponding space of asymptotic classes is denoted by $\widetilde{\mathrm{RES}}^{\mathrm{simp}}_\mathbb{Z}$.

As usual in this course, we use abridged notations. One can make acting the alien operators on the space $\widetilde{\mathscr{R}}^{\mathrm{simp}}$.

Definition 7.55. The alien operators $\Delta^+_\omega, \Delta_\omega : \widetilde{\mathscr{R}}^{\mathrm{simp}} \to \widetilde{\mathscr{R}}^{\mathrm{simp}}$ are defined by making the following diagrams commuting: $\begin{array}{ccc} \widetilde{\mathrm{RES}}^{\mathrm{simp}} & \overset{\Delta^+_\omega,\Delta_\omega}{\longrightarrow} & \widetilde{\mathrm{RES}}^{\mathrm{simp}} \\ T_1 \downarrow\uparrow^\natural & & T_1 \downarrow\uparrow^\natural \\ \widetilde{\mathscr{R}}^{\mathrm{simp}} & \overset{\Delta^+_\omega,\Delta_\omega}{\longrightarrow} & \widetilde{\mathscr{R}}^{\mathrm{simp}} \end{array}$.

Obviously (from proposition 7.26), for any $\widetilde{\varphi} \in \widetilde{\mathscr{R}}^{\mathrm{simp}}$, the alien derivation $\Delta_\omega \widetilde{\varphi}$ only depends on $\overset{\bullet}{\omega}$, thus one could define $\Delta_\omega^+, \Delta_\omega : \widetilde{\mathscr{R}}^{\mathrm{simp}} \to \widetilde{\mathscr{R}}^{\mathrm{simp}}$ for $\omega \in \mathbb{Z}^\star$.

Before introducing the simply ramified resurgent functions, we need to state the following straightforward consequence of proposition 7.13.

Lemma 7.9. *The space* $\mathrm{SING}^{\mathrm{s.ram}}$ *of simply ramified singularities* $\overset{\triangledown}{\varphi} = \sum_{n=0}^{N} a_n \overset{\triangledown}{I}_{-n} + ^{\flat}\widehat{\varphi}$, $\widehat{\varphi} \in \mathcal{O}_0$, *is a convolution subalgebra.*

Definition 7.56. One denotes by $\mathrm{ASYMP}^{\mathrm{s.ram}}$ the space of asymptotic classes associated with $\mathrm{SING}^{\mathrm{s.ram}}$. The restriction of the Taylor map to $\mathrm{ASYMP}^{\mathrm{s.ram}}$ is denoted by $T_1^{\mathrm{s.ram}}$. One denotes by $^{\natural\,\mathrm{s.ram}}$ its composition inverse, that is the natural extension of the mapping $^\natural$ to $\mathbb{C}[z] \oplus \mathbb{C}[[z^{-1}]]_1$.

Definition 7.57. A $\mathbb{Z}$-resurgent singularity $\overset{\triangledown}{\varphi} \in \mathrm{RES}$ is a **simply ramified resurgent** singularity if $\overset{\triangledown}{\varphi} = \sum_{n=0}^{N} a_{-n} \overset{\triangledown}{I}_{-n} + ^{\flat}\widehat{\varphi} \in \mathrm{SING}^{\mathrm{s.ram}}$ and if, for any alien operator $\mathcal{A}_\omega^\gamma(\theta_2, \theta_1)$, $\mathcal{A}_\omega^\gamma(\theta_2, \theta_1) \overset{\triangledown}{\varphi}$ belongs to $\mathrm{SING}^{\mathrm{s.ram}}$. The resurgent subalgebra made of simply ramified resurgent singularities is denoted by $\mathrm{RES}_{\mathbb{Z}}^{\mathrm{s.ram}}$ to which corresponds the space of asymptotic classes $\widetilde{\mathrm{RES}}^{\mathrm{s.ram}}$. The space of the associated formal series $\widetilde{\varphi}(z) = \sum_{n=-N}^{\infty} a_n z^{-n}$ is denoted by $\widetilde{\mathscr{R}}_{\mathbb{Z}}{}^{\mathrm{s.ram}}$

One can define the the alien operators $\Delta_\omega^+, \Delta_\omega : \widetilde{\mathscr{R}}^{\mathrm{s.ram}} \to \widetilde{\mathscr{R}}^{\mathrm{s.ram}}$ in the same manner than in definition 7.55 and, again, for any $\widetilde{\varphi} \in \widetilde{\mathscr{R}}^{\mathrm{s.ram}}$, the alien derivation $\Delta_\omega \widetilde{\varphi}$ only depends on $\overset{\bullet}{\omega}$.

7.9.2 Ramified resurgent functions

The following definition makes sense by propositions 7.6 and 7.13.

Definition 7.58. We denote by $\mathrm{SING}^{\mathrm{ram}} \subset \mathrm{SING}$ the convolution subalgebra generated by the simple singularities and the set of singularities $\{\overset{\triangledown}{J}_{\sigma,m}, (\sigma, m) \in \mathbb{C} \times \mathbb{N}\}$. An element of $\overset{\triangledown}{\varphi} \in \mathrm{SING}^{\mathrm{ram}}$ is called a **ramified singularity** and reads as a finite sum $\overset{\triangledown}{\varphi} = \sum\limits_{(\sigma,m)} \overset{\triangledown}{J}_{\sigma,m} * \overset{\triangledown}{\varphi}_{(\sigma,m)}$ with $\overset{\triangledown}{\varphi}_{(\sigma,m)} \in \mathrm{SING}^{\mathrm{simp}}$. The associated space of asymptotic classes is denoted by $\mathrm{ASYMP}^{\mathrm{ram}} \subset \mathrm{ASYMP}$.

To a ramified singularity $\overset{\triangledown}{\varphi} = \sum\limits_{(\sigma,m)} \overset{\triangledown}{J}_{\sigma,m} * \overset{\triangledown}{\varphi}_{(\sigma,m)}$ is associated, by formal Laplace transform, an asymptotic class $\overset{\vartriangle}{\varphi} \in \mathrm{ASYMP}^{\mathrm{ram}}$ of the form $\overset{\vartriangle}{\varphi} = \sum\limits_{(\sigma,m)} \overset{\vartriangle}{J}_{\sigma,m} \overset{\vartriangle}{\varphi}_{(\sigma,m)}$ with $\overset{\vartriangle}{\varphi}_{(\sigma,m)} = {}^\natural \widetilde{\varphi}_{(\sigma,m)} \in \mathrm{ASYMP}^{\mathrm{simp}}$. This asymptotic class provides a formal expansion of the type

$$\widetilde{\varphi} = \sum_{(\sigma,m)} (-1)^m \frac{\log^m(z)}{z^\sigma} \widetilde{\varphi}_{(\sigma,m)} \in \bigoplus_{(\sigma,m)} \frac{\log^m(z)}{z^\sigma} \mathbb{C}[[z^{-1}]]_1$$

through the Taylor map, for any given arc of $\underset{\bullet}{\mathbb{S}}^1$.

We have encountered such formal expansions when we considered the formal integral for Painlevé I (theorem 5.1).

In the same way that $\mathbb{C}[[z^{-1}]]_1$ can be thought of as a constant sheaf on $\mathbb{S}^1$, the space $\bigoplus_{(\sigma,m)} \frac{\log^m(z)}{z^\sigma}\mathbb{C}[[z^{-1}]]_1$ can be seen as a constant sheaf on $\underset{\bullet}{\mathbb{S}}^1$. This justifies the following definition.

Definition 7.59. Let be $\theta \in \underset{\bullet}{\mathbb{S}}^1$ and $\alpha > 0$. We denote by $\widetilde{\mathrm{Nils}}_1$, resp. $\widetilde{\mathrm{Nils}}_{1,(\theta,\alpha)}$. the space of global sections of the sheaf $\bigoplus_{(\sigma,m)} \frac{\log^m(z)}{z^\sigma}\mathbb{C}[[z^{-1}]]_1$, resp. section on $J^\star =]-\pi/2-\alpha-\theta, -\theta+\alpha+\pi/2[$. We call $\widetilde{\mathrm{Nils}}_1$ the differential algebra of **1-Gevrey Nilsson series**.

The restriction of the Taylor map to $\mathrm{ASYMP}^{\mathrm{ram}}$ is denoted by $T_1^{\mathrm{s.ram}}$. One denotes by

$$\natural^{\mathrm{ram}}: \begin{array}{ll} \widetilde{\mathrm{Nils}}_1 & \to \mathrm{ASYMP}^{\mathrm{ram}} \\ \widetilde{\varphi} = \sum_{(\sigma,m)} \widetilde{J}_{\sigma,m}\widetilde{\varphi}_{(\sigma,m)} & \to \natural^{\mathrm{ram}}\widetilde{\varphi} = \sum_{(\sigma,m)} \overset{\vartriangle}{J}_{\sigma,m}\, {}^\natural\widetilde{\varphi}_{(\sigma,m)} \end{array}$$

its composition inverse, where $\widetilde{J}_{\sigma,m}(z) = (-1)^m \frac{\log^m(z)}{z^\sigma}$.

One can define the space $\widetilde{\mathrm{Nils}}$ as well, made of formal expansions of the form $\widetilde{\varphi} = \sum_{(\sigma,m)} \widetilde{J}_{\sigma,m}\widetilde{\varphi}_{(\sigma,m)}$ with $\widetilde{\varphi}_{(\sigma,m)} \in \mathbb{C}[[z^{-1}]]$. Let us consider an element $\widetilde{\varphi} \in \widetilde{\mathrm{Nils}}$ under the form $\widetilde{\varphi} = \sum_{i=1}^n \frac{\widetilde{\varphi}_i}{z^{\sigma_i}}$, $\widetilde{\varphi}_i \in \mathbb{C}[[z^{-1}]]$. We can of course assume that for any $i \neq j$, $\sigma_i - \sigma_j \notin \mathbb{Z}$. We denote $\omega_i = \mathrm{e}^{-2\mathrm{i}\pi\sigma_i}$ and we remark that $\omega_i - \omega_j \neq 0$ for any $i \neq j$. We set $\varrho.\widetilde{\varphi}(z) = \widetilde{\varphi}(z\mathrm{e}^{2\mathrm{i}\pi})$ and more generally $\varrho_k.\widetilde{\varphi}(z) = \widetilde{\varphi}(z\mathrm{e}^{2\mathrm{i}\pi k})$ for any $k \in \mathbb{Z}$. We notice that $\varrho_k.\widetilde{\varphi} = \displaystyle\sum_{i=1}^n \omega_i^k \frac{\widetilde{\varphi}_i}{z^{\sigma_i}}$. Therefore, ${}^t(\widetilde{\varphi}, \varrho_1.\widetilde{\varphi}, \cdots, \varrho_n.\widetilde{\varphi}) = A^t \left(\frac{\widetilde{\varphi}_1}{z^{\sigma_1}}, \frac{\widetilde{\varphi}_2}{z^{\sigma_2}}, \cdots, \frac{\widetilde{\varphi}_n}{z^{\sigma_n}}\right)$ where A stands for the $n \times n$ invertible Vandermonde matrix $A = \begin{pmatrix} 1 & \cdots & 1 \\ \omega_1 & \cdots & \omega_n \\ \vdots & & \vdots \\ \omega_1^n & \cdots & \omega_n^n \end{pmatrix}$. This implies that for each integer $i \in [1,n]$, $\dfrac{\widetilde{\varphi}_i}{z^{\sigma_i}}$ is a linear combination of $\widetilde{\varphi}, \varrho.\widetilde{\varphi}, \cdots, \varrho_n.\widetilde{\varphi}$. This observation can be generalized:

Lemma 7.10. *Let $\widetilde{\varphi} = \sum_i \sum_{m=0}^{r_i-1} \widetilde{J}_{\sigma_i,m}\widetilde{\varphi}_{(\sigma_i,m)}$ be an element of $\widetilde{\mathrm{Nils}}$. Then the series $\widetilde{\varphi}_{(\sigma_i,m)} \in \mathbb{C}[[z^{-1}]]$ are uniquely determined by $\widetilde{\varphi}$ and its monodromy (that is $\varrho.\widetilde{\varphi}$, $\varrho_2.\widetilde{\varphi}$, etc.) once one imposes that $\sigma_i - \sigma_j \notin \mathbb{Z}$ whenever $\widetilde{\varphi}_{(\sigma_i,m)}.\widetilde{\varphi}_{(\sigma_j,m)} \neq 0$.*

Proof. This is a well-known fact and we follow a reasoning from [19]. We only show how $\widetilde{\varphi}$ determines the series $\widetilde{\varphi}_{(\sigma_i,m)}$ since we will use this result in a moment.

If $\omega = \mathrm{e}^{-2\mathrm{i}\pi\sigma}$, observe that $(\varrho-\omega)\left(\dfrac{\log^m(z)}{z^\sigma}\right) = \omega \displaystyle\sum_{l=0}^{m-1} \binom{m}{l} (2\mathrm{i}\pi)^{m-l} \frac{\log^l(z)}{z^\sigma}$ and therefore $(\varrho-\omega)^m \left(\dfrac{\log^m(z)}{z^\sigma}\right) = m!\dfrac{\omega^m}{z^\sigma}$ while $(\varrho-\omega)^{m+1}\left(\dfrac{\log^m(z)}{z^\sigma}\right) = 0$.

As a consequence, for any $\widetilde{\varphi} \in \widetilde{\mathrm{Nils}}$ one has $P(\varrho)\widetilde{\varphi} \in \widetilde{\mathrm{Nils}}$ for any polynomial $P \in \mathbb{C}[X]$, and there exists a polynomial $P \in \mathbb{C}[X]$ such that $P(\varrho)\widetilde{\varphi} = 0$. We denote by $d(\widetilde{\varphi})$ the degree of the minimal polynomial of the action of ϱ on $\widetilde{\varphi}$. We then make a reasoning by induction on $d(\widetilde{\varphi})$.
Suppose that $d(\widetilde{\varphi}) = 1$. This means that there exists $\omega = \mathrm{e}^{-2\mathrm{i}\pi\sigma} \in \mathbb{C}$ such that $(\varrho - \omega)\widetilde{\varphi} = 0$, thus $\varrho\,(z^{\sigma}\widetilde{\varphi}) = z^{\sigma}\widetilde{\varphi}$. Therefore $\widetilde{\varphi}$ is of the form $\widetilde{\varphi} = \frac{\widetilde{\varphi}_{(\sigma_1,0)}}{z^{\sigma_1}}$ with $\widetilde{\varphi}_{(\sigma_1,0)} \in \mathbb{C}[[z^{-1}]]$ and a convenient choice of $\sigma_1 \in \mathbb{C}$ so that $\sigma_1 - \sigma \in \mathbb{Z}$. (Thus $\widetilde{\varphi}_{(\sigma_1,0)} = \varrho\,(z^{\sigma_1}\widetilde{\varphi})$).
Suppose now that for any $\widetilde{\varphi} \in \widetilde{\mathrm{Nils}}$ such that $d(\widetilde{\varphi}) \leq d$, its decomposition is (uniquely) determined by $\widetilde{\varphi}, \varrho.\widetilde{\varphi}, \cdots, \varrho_d.\widetilde{\varphi}$.
Take $\widetilde{\varphi} \in \widetilde{\mathrm{Nils}}$ with $d(\widetilde{\varphi}) = d+1$. The minimal polynomial of the action of ϱ on $\widetilde{\varphi}$ is $P(X) = \prod_i (X-\omega_i)^{r_i}$ with $\sum_i r_i = d+1$. Write $\widetilde{P}(X) = (X-\omega_1)^{r_1-1}\prod_{i\neq 1}(X-\omega_i)^{r_i} = (X-\omega_i)^{r_1-1}Q(X)$. From the fact that $(\rho - \omega_1)\widetilde{P}(\rho)\widetilde{\varphi} = 0$, we deduce the identity $\widetilde{P}(\rho)\widetilde{\varphi} = \dfrac{\widetilde{\phi}}{z^{\sigma_1}}$ with $\widetilde{\phi} \in \mathbb{C}[[z^{-1}]]_1$ and a convenient $\sigma_1 \in \mathbb{C}$ such that $\omega_1 = \mathrm{e}^{-2\mathrm{i}\pi\sigma_1}$. Since

$$\widetilde{P}(\varrho)\left(\frac{\log^{r_1-1}(z)}{z^{\sigma_1}}\right) = Q(\varrho)\left((r_1-1)!\frac{\omega_1^{r_1-1}}{z^{\sigma_1}}\right) = Q(\omega_1)(r_1-1)!\frac{\omega_1^{r_1-1}}{z^{\sigma_1}},$$

we see that necessarily $\widetilde{P}(\varrho)\left(\widetilde{J}_{\sigma_1,r_1-1}\widetilde{\varphi}_{\sigma_1,r_1-1}\right) = (-1)^{r_1-1}\dfrac{\widetilde{\phi}}{z^{\sigma_1}}$ and $\widetilde{\varphi}_{\sigma_1,r_1-1} = (-1)^{r_1-1}\dfrac{\widetilde{\phi}}{(r_1-1)!\omega_1^{r_1-1}Q(\omega_1)}$.
We finally observe that $\widetilde{P}(\varrho)\left(\widetilde{\phi} - \widetilde{J}_{\sigma_1,r_1-1}\widetilde{\varphi}_{\sigma_1,r_1-1}\right) = 0$ and we can apply the induction hypothesis on $\widetilde{\phi} - \widetilde{J}_{\sigma_1,r_1-1}\widetilde{\varphi}_{\sigma_1,r_1-1}$. This ends the proof. □

We are in good position to define the ramified resurgent functions [23, 7, 8], see also [15].

Definition 7.60. A $\mathbb{Z}$-resurgent singularity $\overset{\triangledown}{\varphi} \in \mathrm{RES}_{\mathbb{Z}}$ is a **ramified resurgent** singularity when $\overset{\triangledown}{\varphi} \in \mathrm{SING}^{\mathrm{ram}}$ whereas, for any alien operator $\mathcal{A}^{\gamma}_{\omega}(\theta_2,\theta_1)$, $\mathcal{A}^{\gamma}_{\omega}(\theta_2,\theta_1)\ \overset{\triangledown}{\varphi}$ belongs to $\mathrm{SING}^{\mathrm{ram}}$. The space of ramified resurgent singularities makes a resurgent subalgebra denoted by $\mathrm{RES}^{\mathrm{ram}}_{\mathbb{Z}}$. The corresponding space of asymptotic classes, resp. formal expansions, is denoted by $\widetilde{\mathrm{RES}}^{\mathrm{ram}}_{\mathbb{Z}}$, resp. $\widetilde{\mathscr{R}}_{\mathbb{Z}}^{\mathrm{ram}}$.

We state a result that derives directly from lemma 7.10

Proposition 7.27. *The formal expansion* $\widetilde{\varphi} = \sum_{(\sigma,m)} \widetilde{J}_{\sigma,m}\widetilde{\varphi}_{(\sigma,m)} \in \widetilde{\mathrm{Nils}}$ *belongs to* $\widetilde{\mathscr{R}}^{\mathrm{ram}}$ *if and only if each of its component* $\widetilde{\varphi}_{(\sigma,m)}$ *belongs to* $\widetilde{\mathscr{R}}^{\mathrm{ram}}$.

Definition 7.61. The alien operators $\Delta^+_{\omega}, \Delta_{\omega} : \widetilde{\mathscr{R}}^{\mathrm{ram}} \to \widetilde{\mathscr{R}}^{\mathrm{ram}}$ are defined by making the following diagrams commuting:

$$\begin{array}{ccc} \widetilde{\mathrm{RES}}^{\mathrm{ram}} & \overset{\Delta^+_{\omega},\Delta_{\omega}}{\longrightarrow} & \widetilde{\mathrm{RES}}^{\mathrm{ram}} \\ T_1^{\mathrm{ram}} \downarrow\uparrow^{\natural\,\mathrm{ram}} & & T_1^{\mathrm{ram}} \downarrow\uparrow^{\natural\,\mathrm{ram}} \\ \widetilde{\mathscr{R}}^{\mathrm{ram}} & \overset{\Delta^+_{\omega},\Delta_{\omega}}{\longrightarrow} & \widetilde{\mathscr{R}}^{\mathrm{ram}} \end{array}.$$

We eventually lay down a direct consequence of proposition 7.19. (We warn to the change of sign).

Proposition 7.28. *Let $\widetilde{\varphi}$ be an element of $\widetilde{\mathscr{R}}^{\mathrm{ram}}$. Then, for any $\omega \in \underset{\bullet}{\mathbb{C}}$ with $\overset{\bullet}{\omega} \in \mathbb{Z}^\star$, for any $k \in \mathbb{Z}$,*

$$\Delta_{\omega e^{2i\pi k}}\widetilde{\varphi} = \varrho_k.\Big(\Delta_\omega \varrho_{-k}.\widetilde{\varphi}\Big), \qquad \Delta_{\omega e^{i\pi}}\widetilde{\varphi} = \varrho_{1/2}.\Big(\Delta_\omega \varrho_{-1/2}.\widetilde{\varphi}\Big).$$

Example 7.16. Suppose that $\widetilde{\varphi} \in \mathbb{C}[[z^{-1}]]_1$ belongs to $\widetilde{\mathscr{R}}^{\mathrm{ram}}$ with $\Delta_\omega\widetilde{\varphi} = \frac{\log(z)}{z^\sigma}\widetilde{\psi}$, $\widetilde{\psi} \in \mathbb{C}[[z^{-1}]]$. For $k \in \mathbb{Z}$, $\varrho_{-k}.\widetilde{\varphi}(z) = \widetilde{\varphi}(z)$, then $\Delta_{\omega e^{2i\pi k}}\widetilde{\varphi}(z) = \frac{\log(z+2\pi k)}{z^\sigma e^{2i\pi k\sigma}}\widetilde{\psi}(z)$. Suppose furthermore that $\widetilde{\varphi}$ is even, so that $\varrho_{-1/2}.\widetilde{\varphi}(z) = \widetilde{\varphi}(z)$. On deduces that $\Delta_{\omega e^{i\pi}}\widetilde{\varphi}(z) = \frac{\log(z+\pi)}{z^\sigma e^{i\pi\sigma}}\widetilde{\psi}(-z)$.

7.10 Comments

We mentionned in Sect. 4.6 the generalisation of the resurgence theory with the notion of "endlessly continuable functions". The whole constructions made in this chapter can be extend as well to this context.

We of course owe the main ideas presented here from the work of Ecalle, who started his theory in the 1970's [6]. We have borrowed most of the materials to Pham *et al.* [1], in particular the microfunctions and the sheaf approach. To compare with other written papers devoted to resurgence theory, we have paid more attention to the sheaf and associated spaces of asymptotic classes. Finally, the possible mistakes are own.

References

1. B. Candelpergher, C. Nosmas, F. Pham, *Approche de la résurgence*, Actualités mathématiques, Hermann, Paris (1993).
2. O. Costin, *Asymptotics and Borel summability*, Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, 141. CRC Press, Boca Raton, FL, 2009.
3. P. Deligne, *Equations différentielles à points singuliers réguliers.* Lecture Notes in Mathematics, Vol. 163. Springer-Verlag, Berlin-New York, 1970.
4. E. Delabaere, F. Pham, *Resurgent methods in semi-classical asymptotics*, Ann. Inst. Henri Poincaré, Sect. A **71** (1999), no 1, 1-94.
5. W. Ebeling, *Functions of several complex variables and their singularities.* Translated from the 2001 German original by Philip G. Spain. Graduate Studies in Mathematics, 83. American Mathematical Society, Providence, RI, 2007.
6. J. Écalle, *Un analogue des fonctions automorphes: les fonctions résurgentes.* Séminaire Choquet, 17e année (1977/78), Initiation l'analyse, Fasc. 1, Exp. No. 11, 9 pp., Secrétariat Math., Paris, 1978.
7. J. Écalle, *Les algèbres de fonctions résurgentes*, Publ. Math. d'Orsay, Université Paris-Sud, 1981.05 (1981).
8. J. Ecalle, *Les fonctions résurgentes. Tome III : l'équation du pont et la classification analytique des objets locaux.* Publ. Math. d'Orsay, Université Paris-Sud, 1985.05 (1985).
9. J. Écalle, *Fonctions analysables et preuve constructive de la conjecture de Dulac.* Actualités mathématiques, Hermann, Paris (1992).
10. J. Écalle, *Six lectures on transseries, Analysable functions and the Constructive proof of Dulac's conjecture*, Bifurcations and periodic orbits of vector fields (Montreal, PQ, 1992), 75-184, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 408, Kluwer Acad. Publ., Dordrecht, 1993.
11. O. Forster, *Lectures on Riemann Surfaces*, Graduate texts in mathematics; 81, Springer, New York (1981).

12. R. Godement, *Topologie algébrique et théorie des faisceaux.* Publications de l'Institut de Mathématique de l'Université de Strasbourg, XIII. Actualités Scientifiques et Industrielles, No. 1252. Hermann, Paris, 1973.
13. M. Kashiwara, T. Kawai, T. Kimura, *Foundations of algebraic analysis.* Princeton Mathematical Series, **37**. Princeton University Press, Princeton, NJ, 1986.
14. M. Loday-Richaud, *Divergent series and differential equations.* To appear.
15. M. Loday-Richaud, P. Remy, *Resurgence, Stokes phenomenon and alien derivatives for level-one linear differential systems.* J. Differential Equations **250** (2011), no. 3, 1591-1630.
16. B. Malgrange, *Equations différentielles à coefficients polynomiaux.* Progress in Mathematics, 96. Birkhäuser Boston, Inc., Boston, MA, 1991.
17. B. Malgrange, *Sommation des séries divergentes.* Expositiones Mathematicae **13**, 163-222 (1995).
18. M. Morimoto, *An introduction to Sato's hyperfunctions.* Translated and revised from the 1976 Japanese original by the author. Translations of Mathematical Monographs, 129. American Mathematical Society, Providence, RI, 1993.
19. L. Narváez Macarro, *D-modules in dimension 1.* Algebraic approach to differential equations, 1-51, World Sci. Publ., Hackensack, NJ, 2010.
20. C. Olivé, D. Sauzin, T. Seara, *Resurgence in a Hamilton-Jacobi equation.* Proceedings of the International Conference in Honor of Frédéric Pham (Nice, 2002). Ann. Inst. Fourier (Grenoble) **53** (2003), no. 4, 1185-1235.
21. Y. Ou, *Sur la stabilité par produit de convolution d'algèbres de résurgence.* PhD thesis, Université d'Angers (2012).
22. F. Pham, *Intégrales singulières.* Savoirs Actuels (Les Ulis). EDP Sciences, Les Ulis; CNRS Editions, Paris, 2005
23. D. Sauzin, *Resurgent functions and splitting problems*, RIMS Kokyuroku 1493 (31/05/2006) 48-117
24. D. Sauzin, *Introduction to 1-summability and the resurgence theory.* To appear.
25. D. Sauzin, *On the stability under convolution of resurgent functions.* Funkcial. Ekvac. **56** (2013), no. 3, 397-413.
26. D. Sauzin, *Nonlinear analysis with resurgent functions.* Preprint 2013, 35 pages. http://hal.archives-ouvertes.fr/hal-00766749

Chapter 8
Resurgent structure for Painlevé I

Abstract We show the resurgence property for the formal series solution of the prepared form associated with the first Painlevé equation. The detailled resurgent structure is given in Sect. 8.1. Its proof is given using the so-called bridge equation (Sect. 8.4), after some preliminaries (Sect. 8.3). The Stokes phenomena is briefly analyzed in Sect. 8.2.

8.1 The main theorem

8.1.1 Some recalls

The formal integral of the prepared form (3.6) associated with the first Painlevé equation was described with theorem 5.1 and its corollary 5.1. It can be written under the following equivalent form:

$$\widetilde{w}(z,\boldsymbol{U}) = \widetilde{W}_{\boldsymbol{0}}(z) + \sum_{n=0}^{\infty} \sum_{\boldsymbol{k}\in \Xi_{n+1,0}\setminus \Xi_{n,0}} \boldsymbol{U}^{\boldsymbol{k}} \mathrm{e}^{-\boldsymbol{\lambda}.\boldsymbol{k}z} \widetilde{W}_{\boldsymbol{k}}(z), \tag{8.1}$$

where $\widetilde{W}_{\boldsymbol{0}} = \widetilde{w}_{\boldsymbol{0}} = \widetilde{w}_{\boldsymbol{0}}^{[0]}$ and for any $n \in \mathbb{N}$ and any $\boldsymbol{k} \in \Xi_{n+1,0} \setminus \Xi_{n,0}$,

$$\widetilde{W}_{\boldsymbol{k}} = \sum_{l=0}^{n} \frac{1}{l!} (\boldsymbol{\varkappa}.\boldsymbol{k})^l \log^l(z) \widetilde{W}_{\boldsymbol{k}-\boldsymbol{1}}^{[0]}, \quad \widetilde{W}_{\boldsymbol{k}}^{[0]} = z^{-\boldsymbol{\tau}.\boldsymbol{k}} \widetilde{w}_{\boldsymbol{k}}^{[0]}. \tag{8.2}$$

The formal series $\widetilde{w}_{\boldsymbol{0}} \in \mathbb{C}[[z^{-1}]]$ solves (3.6), namely

$$P(\partial)\widetilde{w}_{\boldsymbol{0}} + \frac{1}{z}Q(\partial)\widetilde{w}_{\boldsymbol{0}} = F(z,\widetilde{w}_{\boldsymbol{0}}) = f_0 + f_1\widetilde{w}_{\boldsymbol{0}} + f_2\widetilde{w}_{\boldsymbol{0}}^2, \tag{8.3}$$

$P(\partial) = \partial^2 - 1$, $Q(\partial) = -3\partial$, $f_0(z) = \dfrac{392}{625}z^{-2}$, $f_1(z) = -4z^{-2}$, $f_2(z) = \dfrac{1}{2}z^{-2}$, while the $\widetilde{W}_{\boldsymbol{k}}$ satisfy a hierarchy of equations given by lemma 5.3 that we recall:

$$\mathfrak{P}_{\boldsymbol{k}}\widetilde{W}_{\boldsymbol{k}} = \sum_{\substack{\boldsymbol{k}_1+\boldsymbol{k}_2=\boldsymbol{k} \\ |\boldsymbol{k}_i|\geq 1}} \frac{\widetilde{W}_{\boldsymbol{k}_1}\widetilde{W}_{\boldsymbol{k}_2}}{2!}\frac{\partial^2 F(z,\widetilde{w}_{\mathbf{0}})}{\partial w^2}, \tag{8.4}$$
$$\mathfrak{P}_{\boldsymbol{k}} = \mathfrak{P}_{\boldsymbol{k}}(\widetilde{w}_{\mathbf{0}}) = P(-\boldsymbol{\lambda}.\boldsymbol{k}+\partial) + \frac{1}{z}Q(-\boldsymbol{\lambda}.\boldsymbol{k}+\partial) - \frac{\partial F(z,\widetilde{w}_{\mathbf{0}})}{\partial w}.$$

To what concerns the $\boldsymbol{k}$-th series $\widetilde{w}^{[0]}_{\mathbf{k}} \in \mathbb{C}[[z^{-1}]]$, we have a result that ensues directly from theorem 6.1:

Proposition 8.1. *For any $\boldsymbol{k} \in \mathbb{N}^2$, the $\boldsymbol{k}$-th series $\widetilde{w}^{[0]}_{\mathbf{k}}$ belongs to $\widetilde{\mathscr{R}}^{(1)}$, the asymptotic class $\overset{\triangle}{W}_{\mathbf{k}} = {}^{\natural\,\mathrm{ram}}\widetilde{W}_{\mathbf{k}}$ belongs to $\widetilde{\mathrm{RES}}^{(1)}$ and the singularity $\overset{\triangledown}{W}_{\mathbf{k}} = \tilde{\mathcal{B}}\,\overset{\triangle}{W}_{\mathbf{k}}$ belongs to $\mathrm{RES}^{(1)}$.*

Notice that $\widetilde{I}_{-\boldsymbol{\tau}.\boldsymbol{k}}\widetilde{W}_{\boldsymbol{k}} = \sum_{l=0}^{n}\frac{1}{l!}(-\boldsymbol{\varkappa}.\boldsymbol{k})^l\,\widetilde{J}_{-\boldsymbol{\tau}.1,l}\widetilde{w}^{[0]}_{\boldsymbol{k}-\mathbf{1}}$. for any $n \in \mathbb{N}$ and any $\boldsymbol{k} \in \Xi_{n+1,0}\setminus\Xi_{n,0}$. Therefore, $\overset{\triangledown}{W}_{\mathbf{k}} = \sum_{l=0}^{n}\frac{1}{l!}(-\boldsymbol{\varkappa}.\boldsymbol{k})^l\,\overset{\triangledown}{J}_{-\boldsymbol{\tau}.1,l} * \overset{\triangledown}{w}_{\boldsymbol{k}-\mathbf{1}}{}^{[0]}$ where $\overset{\triangledown}{w}^{[0]}_{\mathbf{e}_i} = \delta + {}^{\flat}\widehat{w}_{\boldsymbol{e}_i}$ for $i =, 1, 2$, otherwise $\overset{\triangledown}{w}^{[0]}_{\boldsymbol{k}} = {}^{\flat}\widehat{w}^{[0]}_{\boldsymbol{k}}$.

8.1.2 The main theorem

We now formulate the main result of this chapter.

Theorem 8.1. *The formal integral $\widetilde{w}(z,\boldsymbol{U})$ of the prepared form (3.6) associated with the first Painlevé equation, is resurgent. More precisely, for any $\boldsymbol{k} \in \mathbb{N}^2$, $\widetilde{W}_{\mathbf{k}}$ belongs to the space $\widetilde{\mathscr{R}}_{\mathbb{Z}}{}^{\mathrm{ram}}$ of ramified resurgent formal expansions.*
We set $\omega_1^j = \mathrm{e}^{2\mathrm{i}\pi j}$ ($\overset{\bullet}{\omega}{}_1^j = \lambda_1$) and $\omega_2^j = \mathrm{e}^{2\mathrm{i}\pi(j+1/2)}$ ($\overset{\bullet}{\omega}{}_2^j = \lambda_2$) for any $j \in \mathbb{Z}$. Then, for every $\omega \in \underset{\bullet}{\mathbb{C}}$ of the form $\omega = k_0\omega_1^j$, <u>resp.</u> $\omega = k_0\omega_2^j$, with $k_0 \in \mathbb{N}^\star$, there exist two sequences of complex numbers $\big(A_n(\omega)\big)_{n\in\mathbb{N}}$ and $\big(B_n(\omega)\big)_{n\in\mathbb{N}}$, uniquely defined and only depending on ω such that, for any $\boldsymbol{k} = (k_1,k_2) \in \mathbb{N}^2$ and any $n \in \mathbb{N}$,

$$\begin{aligned}
\Delta_\omega\widetilde{W}_{\boldsymbol{k}+\mathbf{n}} &= \sum_{m=-1}^{n}\Big((k_1+m+k_0)A_{n-m}(\omega) + (k_2+m)B_{n-m}(\omega)\Big)\widetilde{W}_{\boldsymbol{k}+\mathbf{m}+k_0\boldsymbol{e}_1},\\
&\underline{\textit{resp.}}\\
\Delta_\omega\widetilde{W}_{\boldsymbol{k}+\mathbf{n}} &= \sum_{m=-1}^{n}\Big((k_2+m+k_0)A_{n-m}(\omega) + (k_1+m)B_{n-m}(\omega)\Big)\widetilde{W}_{\boldsymbol{k}+\mathbf{m}+k_0\boldsymbol{e}_2},
\end{aligned} \tag{8.5}$$

where by convention $\widetilde{W}_{(k_1,k_2)} = 0$ if $k_1 < 0$ or $k_2 < 0$.
The sequences $\big(A_n(\omega)\big)_{n\in\mathbb{N}}$ and $\big(B_n(\omega)\big)_{n\in\mathbb{N}}$ are subject to the conditions: $A_n(\omega) = 0$ when $|\omega| \geq n+2$ and $B_n(\omega) = 0$ when $|\omega| \geq n+1$. Also, $\big(A_n(\omega)\big)_{n\in\mathbb{N}}$ and $\big(B_n(\omega)\big)_{n\in\mathbb{N}}$ are known for every $\omega \in \underset{\bullet}{\mathbb{C}}$ once they are known for $\arg\omega = 0$ only. In particular, $A_0(\omega_i^j) = (-1)^j A_0(\omega_i^0)$ while $A_0(\omega_2^j) = -\mathrm{i}A_0(\omega_1^j)$.

The proof of this theorem will be given in Sect. 8.4.

Remark 8.1. We detail (8.5) for $n = 0$. For For any $j \in \mathbb{Z}$ and any $k_0 \in \mathbb{N}^\star$,

$$\begin{aligned}\Delta_{k_0\omega_1^j}\widetilde{w}_{\mathbf{0}} &= A_0(k_0\omega_1^j)\widetilde{W}_{\boldsymbol{e}_1} = A_0(k_0\omega_1^j)z^{3/2}\widetilde{w}_{\boldsymbol{e}_1} \\ \Delta_{k_0\omega_2^j}\widetilde{w}_{\mathbf{0}} &= A_0(k_0\omega_2^j)\widetilde{W}_{\boldsymbol{e}_2} = A_0(k_0\omega_2^j)z^{3/2}\widetilde{w}_{\boldsymbol{e}_2}\end{aligned} \tag{8.6}$$

and $A_0(k_0\omega_i^j) = 0$ when $k_0 \geq 2$.

When $\boldsymbol{k} \in \Xi_{1,0}$, we use abridged notations $\widetilde{w}_{\boldsymbol{k}} = \widetilde{w}_{\boldsymbol{k}}^{[0]}$.

By proposition 7.28, $\Delta_{\omega_i^j}\widetilde{w}_{\mathbf{0}} = \varrho_j.\Big(\Delta_{\omega_i^0}\varrho_{-j}.\widetilde{w}_{\mathbf{0}}\Big)$, $i = 1, 2$. Therefore, $A_0(\omega_i^j) = (-1)^j A_0(\omega_i^0)$. Remember that $\widetilde{w}_{\mathbf{0}}$ is even, thus $\widetilde{w}_{\mathbf{0}} = \varrho_{-1/2}.\widetilde{w}_{\mathbf{0}}$, while $\widetilde{w}_{\boldsymbol{e}_2} = \varrho_{1/2}.\widetilde{w}_{\boldsymbol{e}_1}$. By proposition 7.28 again, $\Delta_{\omega_2^j}\widetilde{w}_{\mathbf{0}} = \varrho_{1/2}.\Big(\Delta_{\omega_1^j}\varrho_{-1/2}.\widetilde{w}_{\mathbf{0}}\Big)$ and we deduce that $A_0(\omega_2^j) = -\mathrm{i}A_0(\omega_1^j)$.

Now for any $k_1 \in \mathbb{N}^\star$,

$$\begin{aligned}\Delta_{k_0\omega_1^j}\widetilde{W}_{k_1\boldsymbol{e}_1} &= (k_1 + k_0)A_0(k_0\omega_1^j)\widetilde{W}_{(k_1+k_0)\boldsymbol{e}_1} \\ \Delta_{k_0\omega_2^j}\widetilde{W}_{k_1\boldsymbol{e}_1} &= k_0 A_0(k_0\omega_2^j)\widetilde{W}_{k_1\boldsymbol{e}_1+k_0\boldsymbol{e}_2} + (k_1 - 1)B_1(k_0\omega_2^j)\widetilde{W}_{k_1\boldsymbol{e}_1+k_0\boldsymbol{e}_2-\mathbf{1}}.\end{aligned} \tag{8.7}$$

and $B_1(k_0\omega_2^j) = 0$ when $k_0 \geq 2$. We have in particular $\Delta_{\omega_2^j}\widetilde{W}_{\boldsymbol{e}_1} = A_0(\omega_2^j)\widetilde{W}_{\mathbf{1}}$, thus $\Delta_{\omega_2^j}\widetilde{w}_{\boldsymbol{e}_1} = A_0(\omega_2^j)z^{3/2}\widetilde{w}_{\mathbf{1}}$. Also, for $k_1 \geq 2$,

$$\Delta_{\omega_2^j}\widetilde{W}_{k_1\boldsymbol{e}_1} = A_0(\omega_2^j)\widetilde{W}_{(k_1-1)\boldsymbol{e}_1+\mathbf{1}} + (k_1 - 1)B_1(\omega_2^j)\widetilde{W}_{(k_1-1)\boldsymbol{e}_1}$$

and using (8.2),

$$\begin{aligned}\Delta_{\omega_2^j}\widetilde{w}_{k_1\boldsymbol{e}_1} = {} & A_0(\omega_2^j)\Big((k_1 - 1)\varkappa_1\log(z)z^{-3/2}\widetilde{w}_{(k_1-1)\boldsymbol{e}_1} + z^{3/2}\widetilde{w}^{[0]}_{(k_1-1)\boldsymbol{e}_1+\mathbf{1}}\Big) \\ & + (k_1 - 1)B_1(\omega_2^j)z^{-3/2}\widetilde{w}_{(k_1-1)\boldsymbol{e}_1}.\end{aligned}$$

By proposition 7.28, $\Delta_{\omega_2^j}\widetilde{w}_{2\boldsymbol{e}_1} = \varrho_j.\Big(\Delta_{\omega_i^0}\varrho_{-j}.\widetilde{w}_{2\boldsymbol{e}_1}\Big)$, therefore

$$\begin{aligned}\Delta_{\omega_2^j}\widetilde{w}_{2\boldsymbol{e}_1} = {} & (-1)^j A_0(\omega_2^0)\Big(\varkappa_1\log(z + 2\mathrm{i}\pi j)z^{-3/2}\widetilde{w}_{\boldsymbol{e}_1} + z^{3/2}\widetilde{w}^{[0]}_{\boldsymbol{e}_1+\mathbf{1}}\Big) \\ & + (-1)^j B_1(\omega_2^0)z^{-3/2}\widetilde{w}_{\boldsymbol{e}_1}\end{aligned}$$

and one deduces: $B_1(\omega_2^j) = (-1)^j\Big(B_1(\omega_2^0) + 2\mathrm{i}\pi j\varkappa_1 A_0(\omega_2^0)\Big)$. Of course, by symmetry: $B_1(\omega_1^j) = (-1)^j\Big(B_1(\omega_1^0) + 2\mathrm{i}\pi j\varkappa_2 A_0(\omega_1^0)\Big)$.

In the same way, $\Delta_{\omega_2^j}\widetilde{w}_{2\boldsymbol{e}_1} = \varrho_{1/2}.\Big(\Delta_{\omega_1^j}\varrho_{-1/2}.\widetilde{w}_{2\boldsymbol{e}_1}\Big)$ and we know that $\varrho_{-1/2}.\widetilde{w}_{2\boldsymbol{e}_1} = \widetilde{w}_{2\boldsymbol{e}_2}$, $\varrho_{1/2}.\widetilde{w}_{\boldsymbol{e}_2} = \widetilde{w}_{\boldsymbol{e}_1}$, $\varrho_{1/2}.\widetilde{w}^{[0]}_{\boldsymbol{e}_2+\mathbf{1}} = \widetilde{w}^{[0]}_{\boldsymbol{e}_1+\mathbf{1}}$. Thus,

$$\begin{aligned}\Delta_{\omega_2^j}\widetilde{w}_{2\boldsymbol{e}_1} = {} & -\mathrm{i}A_0(\omega_1^j)\Big(-\varkappa_2\log(z + \mathrm{i}\pi)z^{-3/2}\widetilde{w}_{\boldsymbol{e}_1} + z^{3/2}\widetilde{w}^{[0]}_{\boldsymbol{e}_1+\mathbf{1}}\Big) \\ & + \mathrm{i}B_1(\omega_1^j)z^{-3/2}\widetilde{w}_{\boldsymbol{e}_1}\end{aligned}$$

and $B_1(\omega_2^j) = \mathrm{i}\Big(B_1(\omega_1^j) + \mathrm{i}\pi\varkappa_2 A_0(\omega_1^j)\Big)$.

Definition 8.1. The coefficents $A_n(\omega)$ and $B_n(\omega)$ given by theorem 8.1 are called the **resurgence cofficients** for the first Painlevé equation. The coefficient $A_0(\omega_1^0)$ and $A_0(\omega_2^0)$ are the **Stokes cofficients**.

As a rule and apart from some integrable equations, the resurgence cofficients are seldom known by closed formulas but can be calculed numerically : see for instance [9] and specifically [23] for hyperasymptotic methods, see also [1]. For the first Painlevé equation and its Stokes cofficients, an explicit expression has been obtained by Kapaev [17, 18] using isomonodromic methods, see also [31] for an exact WKB offspring. This result has also been founded by Costin *et al.* [8] by means of resurgent analysis and we give this expression.

Proposition 8.2. *In theorem 8.1, the Stokes coefficients are* $A_0(\omega_1^0) = -\mathrm{i}\sqrt{\dfrac{6}{5\pi}}$ *and* $A_0(\omega_2^0) = -\sqrt{\dfrac{6}{5\pi}}$.

The Stokes coefficients are also known for the second Painlevé equations. See [15] and references therein. It is likely that the method of Costin *et al.* [8] can be used to get the other resurgence cofficients for the first Painlevé equation.

8.1.2.1 Resurgence coefficients and analytic classification

We saw with corollary 5.2 that the formal integral can be interpreted as a formal transformation $\widetilde{w} = \widetilde{\Phi}(z, \mathbf{u})$, $\widetilde{\Phi}(z, \mathbf{u}) = \sum_{\boldsymbol{k}\in\mathbb{N}^2} \mathbf{u}^{\boldsymbol{k}} \widetilde{w}_{\boldsymbol{k}}^{[0]}(z) \in \mathbb{C}[[z^{-1}, \mathbf{u}]]$ that formally conjugates the prepared equation (3.6) to its normal form (5.66). We mentionned (without proof) in Sect. 6.3 that this formal transformation gives rise to analytic transformations through Borel-Laplace summation. In other words, equation (3.6) and the normal form (5.66) are *analytically conjugated.*

It can be shown (see for instance the arguments given in [3]) that for any two differential equations that are formally conjugated to (5.66), then these differential equations are analytically conjugated if and only if their resurgence coefficients are the same. Therefore in this way, the resurgence coefficients are also called the **holomorphic invariants** of Ecalle. See [11] for further details.

8.2 Painlevé I and the Stokes phenomenon

Knowing the Stokes coefficients $A_0(\omega)$ provides a complete description for the Stokes phenomena. In what follows, we use the notations of theorem 8.1 and we denote $\theta_i^j = \arg(\omega_i^j)$, $i = 1, 2$, $j \in \mathbb{Z}$. We simply refer to [28] for the notion of "symbolic Stokes automorphism" $\Delta^+_{\theta_i^j}$ and of "symbolic Stokes infinitesimal generator" $\Delta_{\theta_i^j}$, for a given direction θ. We only recall their expressions and relationships, in our frame:

$$\begin{aligned}
&\Delta^+_{\theta_i^j}, \Delta_{\theta_i^j} : \bigoplus_{k\in\mathbb{N}} \mathrm{e}^{-k\lambda_i z}\widetilde{\mathscr{R}}_{\mathbb{Z}}^{\mathrm{ram}} \to \bigoplus_{k\in\mathbb{N}} \mathrm{e}^{-k\lambda_i z}\widetilde{\mathscr{R}}_{\mathbb{Z}}^{\mathrm{ram}},\\
&\Delta^+_{\theta_i^j} = \mathrm{Id} + \sum_{k_0\in\mathbb{N}^\star} \dot{\Delta}^+_{k_0\omega_i^j}, \quad \Delta_{\theta_i^j} = \sum_{k_0\in\mathbb{N}^\star} \dot{\Delta}_{k_0\omega_i^j} \qquad (8.8)\\
&\Delta^+_{\theta_i^j} = \exp\left(\Delta_{\theta_i^j}\right) = \mathrm{Id} + \sum_{\ell\in\mathbb{N}^\star} \frac{\mathrm{e}^{-k_0\lambda_i z}}{\ell!} \sum_{\substack{k_1+\cdots+k_\ell=k_0\\ k_i\geq 1}} \Delta_{k_\ell\omega_i^j}\circ\cdots\circ\Delta_{k_1\omega_i^j}.
\end{aligned}$$

Let us consider the formal series $\widetilde{w}_{\mathbf{0}}$. From theorem 8.1, one sees that

$$\Delta^+_{\theta_i^j}\widetilde{w}_{\mathbf{0}} = \widetilde{w}_{\mathbf{0}} + \sum_{k\in\mathbb{N}^\star} A_0(\omega_i^j)^k \mathrm{e}^{-k\lambda_i z}\widetilde{W}_{k\boldsymbol{e}_i} \tag{8.9}$$

where, on the right-hand side, on recognizes the transseries solutions. This action allows to compare left and right Borel-Laplace summation: in their intersection domain of convergence,

$$\mathscr{S}^{\theta_i^{j-}}\widetilde{w}_{\mathbf{0}} = \mathscr{S}^{\theta_i^{j+}}\widetilde{w}_{\mathbf{0}} + \sum_{k\in\mathbb{N}^\star} A_0(\omega_i^j)^k \mathrm{e}^{-k\lambda_i z}\mathscr{S}^{\theta_i^{j+}}\widetilde{W}_{k\boldsymbol{e}_i}. \tag{8.10}$$

This allows, in particular to analytically continue the sum $\mathscr{S}^{\theta_i^{j-}}\widetilde{w}_{\mathbf{0}}$, thus the tritruncated solutions, onto a wider domain.

The same calculation can be made for the (convenient) transseries as well, and one easily gets, for $i = 1, 2$:

$$\mathscr{S}^{\theta_i^{j-}}\left(\widetilde{w}_{\mathbf{0}} + \sum_{k\in\mathbb{N}^\star} U_i^k \mathrm{e}^{-k\lambda_i z}\widetilde{W}_{k\boldsymbol{e}_i}\right) = \mathscr{S}^{\theta_i^{j+}}\left(\widetilde{w}_{\mathbf{0}} + \sum_{k\in\mathbb{N}^\star}\left(U_i + A_0(\omega_i^j)\right)^k \mathrm{e}^{-k\lambda_i z}\widetilde{W}_{k\boldsymbol{e}_i}\right). \tag{8.11}$$

Once again, this provides analytic continuations of the the truncated solutions onto a wider domain.

It is a good place to mention medianization, since the k-th series $\widetilde{w}_{k\boldsymbol{e}_i}$ are all *real* formal series. For instance, since $\widetilde{w}_{\mathbf{0}}$ belongs to $\mathbb{R}[[z^{-1}]]$, its left and right Borel-Laplace sum are complex conjugate: $\overline{\mathscr{S}^{\theta_1^{0+}}\widetilde{w}_{\mathbf{0}}(z)} = \mathscr{S}^{\theta_1^{0-}}\widetilde{w}_{\mathbf{0}}(\overline{z})$. However, neither $\mathscr{S}^{\theta_1^{0+}}\widetilde{w}_{\mathbf{0}}$ nor $\mathscr{S}^{\theta_1^{0-}}\widetilde{w}_{\mathbf{0}}$ are real analytic functions, because of the Stokes phenomenon. The question is therefore the following one : can we construct from $\widetilde{w}_{\mathbf{0}}$ a real analytic function by a suitable morphism of differential algebras ?

The naive idea of taking their mean does not work (why ?).

The answer is "yes", by medianization or good averages, and is not unique. We refer to [21, 14] for this question and its subtleties, see also [6].

Remark 8.2. The fact that the Stokes coefficient $A_0(\omega_1^0)$ is nonzero can be deduced from the identity (8.10) : if $A_0(\omega_1^0) = 0$, then necessary the associated trituncated solution is holomorphic on $\mathbb{C}\setminus K$ where K is a compact domain. This would mean that this trituncated solution has only a finite number of poles and that contradicts theorem 2.2. The fact that $A_0(\omega_1^0)$ is pure imaginary can be seen also from (8.10) and from the realness of $\widetilde{w}_{\mathbf{0}}$. For $\arg(z) = 0$ and $|z|$ large enough, one can write

$$\overline{\mathscr{S}^{\theta_1^{0+}}\widetilde{w}_{\mathbf{0}}(z)} = \mathscr{S}^{\theta_1^{0+}}\widetilde{w}_{\mathbf{0}}(z) + \sum_{k\in\mathbb{N}^\star} A_0(\omega_1^0)^k \mathrm{e}^{-k\lambda_i z}\mathscr{S}^{\theta_1^{0+}}\widetilde{W}_{k\boldsymbol{e}_i}(z), \tag{8.12}$$

and $\overline{A_0(\omega_1^0)} = -A_0(\omega_1^0)$ comes as an upshot.

8.3 The alien derivatives for the seen singularities

The idea that leads to theorem 8.1 relies on the following observations. We know by proposition 8.1 that the singularity $\overset{\triangledown}{W}_{\mathbf{k}}$ belongs to $\mathrm{RES}^{(1)}$, for any $\boldsymbol{k} \in \mathbb{N}^2$, and we can apply corollary 7.2 : for any $\omega \in \underset{\bullet}{\mathbb{C}}$ so that $\overset{\bullet}{\omega} = \pm 1$ (the so-called seen singularities), the alien derivative $\Delta_\omega \overset{\triangledown}{W}_{\mathbf{k}}$ is well-defined. If these alien derivatives belong to $\mathrm{RES}^{(1)}$, then we see with corollary 7.3 that the singularities $\overset{\triangledown}{W}_{\mathbf{k}}$ belongs to $\mathrm{RES}^{(2)}$. A reasoning by induction allow to conclude.

In this section, we explain how to calculate these alien derivatives with various methods and we direct our efforts towards $\widetilde{w}_{\mathbf{0}}$.

8.3.1 *Preparations*

The formal series $\widetilde{w}_{\mathbf{0}}$ being solution of the equation (8.3), we introduce by proposition 7.4 the singularities $\overset{\triangledown}{w}_{\mathbf{0}} = {}^\flat\widehat{w}_{\mathbf{0}}$, $\overset{\triangledown}{f}_0 = {}^\flat\widehat{f}_0$, $\overset{\triangledown}{f}_1 = {}^\flat\widehat{f}_1$ and $\overset{\triangledown}{f}_2 = {}^\flat\widehat{f}_2$. Notice that $\overset{\triangledown}{f}_0$, $\overset{\triangledown}{f}_1$ and $\overset{\triangledown}{f}_2$ obviously belong to $\overset{\triangledown}{\mathrm{CONS}}$.
Equation (8.3) translates into the fact that $\overset{\triangledown}{w}_{\mathbf{0}}$ satisfies the following convolution equation:

$$\begin{aligned} P(\partial)\, \overset{\triangledown}{w}_{\mathbf{0}} + \overset{\triangledown}{I}_1 * \big[Q(\partial)\, \overset{\triangledown}{w}_{\mathbf{0}}\,\big] &= \overset{\triangledown}{F}\,(\zeta, \overset{\triangledown}{w}_{\mathbf{0}}) \\ &= \overset{\triangledown}{f}_0 + \overset{\triangledown}{f}_1 * \overset{\triangledown}{w}_{\mathbf{0}} + \overset{\triangledown}{f}_2 * \overset{\triangledown}{w}_{\mathbf{0}}^{*2}\,. \end{aligned} \tag{8.13}$$

One can rather introduce the asymptotic class $\overset{\vartriangle}{w}_{\mathbf{0}} = {}^\natural\widetilde{w}_{\mathbf{0}} \in \mathrm{ASYMP}^{\mathrm{simp}}$ (cf. Definition 7.27) and equation (8.3) becomes:

$$\begin{aligned} P(\partial)\, \overset{\vartriangle}{w}_{\mathbf{0}} + \frac{1}{z} Q(\partial)\, \overset{\vartriangle}{w}_{\mathbf{0}} &= F(z, \overset{\vartriangle}{w}_{\mathbf{0}}) \\ &= f_0 + f_1\, \overset{\vartriangle}{w}_{\mathbf{0}} + f_2\, \overset{\vartriangle}{w}_{\mathbf{0}}^{2} \end{aligned} \tag{8.14}$$

As already said, we know that $\overset{\triangledown}{w}_{\mathbf{0}}$ belongs to $\mathrm{RES}^{(1)}$, <u>resp</u>. $\overset{\vartriangle}{w}_{\mathbf{0}}$ belongs to $\widetilde{\mathrm{RES}}^{(1)}$, and corollary 7.2 can be applied : with the notations of theorem 8.1, $\overset{\triangledown}{W} = \Delta_{\omega_1^0}\, \overset{\triangledown}{w}_{\mathbf{0}}$ is a well-defined singularity of $\mathrm{SING}_{0,\pi}$, <u>resp</u>. $\overset{\vartriangle}{W} = \Delta_{\omega_1^0}\, \overset{\vartriangle}{w}_{\mathbf{0}}$ is a well-defined asymptotic class of $\mathrm{ASYMP}_{0,\pi}$.

The singularities $\overset{\triangledown}{f}_0$, $\overset{\triangledown}{f}_1$, $\overset{\triangledown}{f}_2$ and $\overset{\triangledown}{I}_1$ are all constant of resurgence. Therefore, they vanish under the action of any alien derivation. Adding to this remark the fact that $\Delta_{\omega_1^0}$ satisfies the Leibniz rule and the commutation rule $[\Delta_{\omega_1^0}, \partial] = -\Delta_{\omega_1^0}$ (corollary 7.2 and remember that $\overset{\bullet}{\omega}{}_1^0 = 1$), one deduces from (8.13) that $\overset{\triangledown}{W}$ solves in $\mathrm{SING}_{0,\pi}$ the following associated linear convolution equation:

$$P(\partial-1)\overset{\triangledown}{W}+\overset{\triangledown}{I}_1*\big[Q(\partial-1)\overset{\triangledown}{W}\big]=\frac{\partial\overset{\triangledown}{F}(\zeta,\overset{\vartriangle}{w}_{\mathbf{0}})}{\partial w}*\overset{\triangledown}{W} \tag{8.15}$$
$$=\left(\overset{\triangledown}{f}_1+2\overset{\triangledown}{f}_2*\overset{\triangledown}{w}_{\mathbf{0}}\right)*\overset{\triangledown}{W}\,.$$

For the same reasons, the asymptotic class $\overset{\vartriangle}{W}$ is solution in $\mathrm{ASYMP}_{0,\pi}$ of a linear ODE:

$$P(\partial-1)\overset{\vartriangle}{W}+\frac{1}{z}Q(\partial-1)\overset{\vartriangle}{W}=\frac{\partial F(z,\overset{\vartriangle}{w}_{\mathbf{0}})}{\partial w}\overset{\vartriangle}{W}\,. \tag{8.16}$$

Of course, (8.16) can be deduced also from (8.15) by formal Laplace transform (definition 7.25 and proposition 7.10).

The differential equation (8.16) is nothing but the equation

$$\mathfrak{P}_{\boldsymbol{e}_1}(\overset{\vartriangle}{w}_{\mathbf{0}})\overset{\vartriangle}{W}=0 \tag{8.17}$$

where $\mathfrak{P}_{\boldsymbol{e}_1}$ is the linear operator recalled in (8.4). We know by lemma 5.4 that the differential equation $\mathfrak{P}_{\boldsymbol{e}_1}(\widetilde{w}_{\mathbf{0}})\widetilde{\mathcal{W}}=0$, that is (8.17) through the Taylor map, has its general formal solution that belongs to the direct sum $\widetilde{\mathrm{Nils}}_1\oplus e^{2z}\widetilde{\mathrm{Nils}}_1$, under the form

$$\begin{aligned}\widetilde{\mathcal{W}}(z)&=C_1z^{\frac{3}{2}}\widetilde{w}_{\boldsymbol{e}_1}(z)+C_2\mathrm{e}^{2z}z^{\frac{3}{2}}\widetilde{w}_{\boldsymbol{e}_2}(z)\\&=C_1\widetilde{W}_{\boldsymbol{e}_1}(z)+C_2\mathrm{e}^{2z}\widetilde{W}_{\boldsymbol{e}_2}(z),\end{aligned} \tag{8.18}$$

where $\widetilde{W}_{\boldsymbol{e}_1}$ and $\widetilde{W}_{\boldsymbol{e}_2}$ belong to the space $\widetilde{\mathrm{Nils}}_1$ of 1-Gevrey Nilsson series.
One should precise what we mean by "general formal solution". The linear operator $\mathfrak{P}_{\boldsymbol{e}_1}$ is of order 2 in z and the particular solutions $\widetilde{W}_{\boldsymbol{e}_1}$ and $\mathrm{e}^{2z}\widetilde{W}_{\boldsymbol{e}_2}$ are two independent formal solutions : their wronskian is $\begin{vmatrix}\widetilde{W}_{\boldsymbol{e}_1} & \mathrm{e}^{2z}\widetilde{W}_{\boldsymbol{e}_2}\\ \partial\widetilde{W}_{\boldsymbol{e}_1} & \partial(\mathrm{e}^{2z}\widetilde{W}_{\boldsymbol{e}_2})\end{vmatrix}=2z^3\mathrm{e}^{2z}$.
Thus, if $\widetilde{\mathcal{W}}$ belongs to a differential algebra that contains $\widetilde{\mathrm{Nils}}_1\oplus e^{2z}\widetilde{\mathrm{Nils}}_1$ as sub-vector space, for instance the direct sum $\prod\limits_{k\in\mathbb{Z}}e^{-kz}\widetilde{\mathrm{Nils}}_1$ and if $\mathfrak{P}_{\boldsymbol{e}_1}(\widetilde{w}_{\mathbf{0}})\widetilde{W}=0$, then $\widetilde{W}$ is of the form (8.18) with $C_1,C_2\in\mathbb{C}$ given by the Kramer's formulas: $C_2=-\frac{z^{-3}\mathrm{e}^{-2z}}{2}\begin{vmatrix}\widetilde{W} & \widetilde{W}_{\boldsymbol{e}_1}\\ \partial\widetilde{W} & \partial\widetilde{W}_{\boldsymbol{e}_1}\end{vmatrix}$, $C_1=\frac{z^{-3}\mathrm{e}^{-2z}}{2}\begin{vmatrix}\widetilde{W} & \mathrm{e}^{2z}\widetilde{W}_{\boldsymbol{e}_2}\\ \partial\widetilde{W} & \partial(\mathrm{e}^{2z}\widetilde{W}_{\boldsymbol{e}_2})\end{vmatrix}$.
We claim that the general solution of equation (8.17) in $\prod_{k\in\mathbb{Z}}e^{-kz}\mathrm{ASYMP}_{0,\pi}$ is a linear combination of $\overset{\vartriangle}{W}_{\boldsymbol{e}_1}\in\mathrm{ASYMP}^{\mathrm{ram}}$ and $\mathrm{e}^{2z}\overset{\vartriangle}{W}_{\boldsymbol{e}_2}\in\mathrm{e}^{2z}\mathrm{ASYMP}^{\mathrm{ram}}$ with $\overset{\vartriangle}{W}_{\boldsymbol{e}_i}=\natural^{\mathrm{ram}}\widetilde{W}_{\boldsymbol{e}_i}$. Consequently:

Lemma 8.1. *There exists $A_0(\omega_1^0)\in\mathbb{C}$ such that the singularity $\Delta_{\omega_1^0}\overset{\triangledown}{w}_{\mathbf{0}}\in\mathrm{SING}_{0,\pi}$ is of the form*

$$\Delta_{\omega_1^0}\overset{\triangledown}{w}_{\mathbf{0}}=A_0(\omega_1^0)\overset{\triangledown}{I}_{-\frac{3}{2}}*\overset{\triangledown}{w}_{\boldsymbol{e}_1}=A_0(\omega_1^0)\overset{\triangledown}{W}_{\boldsymbol{e}_1},$$

thus can be extended uniquely to an element of SING. *In other equivalent words, $\Delta_{\omega_1^0}\overset{\vartriangle}{w}_{\mathbf{0}}=A_0(\omega_1^0)\overset{\vartriangle}{W}_{\boldsymbol{e}_1}\in\mathrm{ASYMP}^{\mathrm{ram}}$, $\Delta_{\omega_1^0}\widetilde{w}_{\mathbf{0}}=A_0(\omega_1^0)\widetilde{W}_{\boldsymbol{e}_1}\in\widetilde{\mathrm{Nils}}_1$.*

As promised, we show proposition 8.1 by two different approaches in the sequel.

8.3.2 Alien derivations, first approach

We follow here ideas developed in [16, 24], see also [27, 26, 22].

We start with the following results that come from general nonsense in 1-Gevrey theory and its proof is saved for an exercise.

Lemma 8.2. *Let $\widetilde{w} \in z^{-1}\mathbb{C}[[z^{-1}]]_1$ be a 1-Gevrey series with vanishing constant term, and $\widehat{w} \in \mathcal{O}_0$ its minor. The following properties are satisfied.*

1. *The formal series $(1+\widetilde{w}) \in \mathbb{C}[[z^{-1}]]_1$ is invertible. Its inverse $(1+\widetilde{w})^{-1}$ is 1-Gevrey and has a formal Borel transform $\tilde{\mathcal{B}}(1+\widetilde{w})^{-1} \in \mathbb{C}\delta \oplus \mathcal{O}_0$ of the form $(\delta+\widehat{w})^{*-1} = \delta + \sum_{n\geq 1}(-1)^n \widehat{w}^{*n}$.*
2. *The formal series $\log(1+\widetilde{w}) = \sum_{n\geq 1} \frac{(-1)^{n+1}}{n}\widetilde{w}^n$ is a 1-Gevrey with vanishing constant term, whose minor is given by $\log_*(\delta+\widehat{w}) := \sum_{n\geq 1}\frac{(-1)^{n+1}}{n}\widehat{w}^{*n}$.*
3. *The formal series $\widetilde{w}$ is **exponentiable** in the sense that its exponential $\mathrm{e}^{\widetilde{w}} = \sum_{n\geq 1}\frac{1}{n!}\widetilde{w}^n$ is a 1-Gevrey series, whose minor is of the form $\exp_*(\widehat{w}) := \delta + \sum_{n\geq 1}\frac{1}{n!}\widehat{w}^{*n}$. Moreover, $\log\circ\exp = \exp\circ\log = \mathrm{Id}$.*

Remark 8.3. More general results along that line in resurgence theory can be obtained, see [3, 28] and specially [30].

We are now ready to calculate the alien derivative $\overset{\triangledown}{W} = \Delta_{\omega_1^0}\overset{\triangledown}{w}_{\mathbf{0}} \in \mathrm{SING}_{0,\pi}$. We consider the 1-Gevrey Nilsson series $\widetilde{W}_{\boldsymbol{e}_1} = z^{3/2}\widetilde{w}_{\boldsymbol{e}_1} \in \widetilde{\mathrm{Nils}}_1$ solution of (8.16) (more precisely its transform through the Taylor map), and its associated singularity $\overset{\triangledown}{W}_{\boldsymbol{e}_1} = \overset{\triangledown}{I}_{-\frac{3}{2}} * \overset{\triangledown}{w}_{\boldsymbol{e}_1} \in \mathrm{SING}$, where $\overset{\triangledown}{w}_{\boldsymbol{e}_1} = \delta + {}^\flat\widehat{w}_{\boldsymbol{e}_1}$. (Remember that $\widetilde{w}_{\boldsymbol{e}_1}$ has 1 for its constant term). Since $\widetilde{w}_{\boldsymbol{e}_1}$ is invertible in $\mathbb{C}[[z^{-1}]]$, so does $\overset{\triangledown}{w}_{\boldsymbol{e}_1}$ in SING, its inverse being given by $\overset{\triangledown}{w}_{\boldsymbol{e}_1}^{*-1} = \delta + {}^\flat\left(\sum_{n\geq 1}(-1)^n\widehat{w}_{\boldsymbol{e}_1}^{*n}\right)$. Accordingly, $\overset{\triangledown}{W}_{\boldsymbol{e}_1}$ is invertible in SING and $\overset{\triangledown}{W}_{\boldsymbol{e}_1}^{*-1} = \overset{\triangledown}{I}_{\frac{3}{2}} * \overset{\triangledown}{w}_{\boldsymbol{e}_1}^{*-1}$. We now introduce the singularity $\overset{\triangledown}{S} \in \mathrm{SING}_{0,\pi}$ defined by

$$\overset{\triangledown}{W} = \overset{\triangledown}{S} * \overset{\triangledown}{W}_{\boldsymbol{e}_1} \tag{8.19}$$

and we want to show that $\overset{\triangledown}{S} = A_0(\omega_1^0)\delta$ for some $A_0(\omega_1^0) \in \mathbb{C}$. Plugging (8.19) into (8.15), using the property that ∂ is a derivation in $\mathrm{SING}_{0,\pi}$ (cf. proposition 7.6) and that $\overset{\triangledown}{W}_{\boldsymbol{e}_1}$ solves (8.15), one easily gets for $\overset{\triangledown}{S}$ the following equation:

$$\left((\partial^2-\partial)\overset{\triangledown}{S}\right) * \overset{\triangledown}{W}_{\boldsymbol{e}_1} + 2\left(\partial\overset{\triangledown}{S}\right) * \left(\partial\overset{\triangledown}{W}_{\boldsymbol{e}_1}\right) - 3\overset{\triangledown}{I}_1 * \left(\partial\overset{\triangledown}{S}\right) * \overset{\triangledown}{W}_{\boldsymbol{e}_1} = 0. \tag{8.20}$$

Since $\partial\overset{\triangledown}{W}_{\boldsymbol{e}_1} = \frac{3}{2}\overset{\triangledown}{I}_{-\frac{1}{2}} * \overset{\triangledown}{w}_{\boldsymbol{e}_1} + \overset{\triangledown}{I}_{-\frac{3}{2}} * \left(\partial\overset{\triangledown}{w}_{\boldsymbol{e}_1}\right)$, equation (8.20) reduces to the equation

$$\partial^2\overset{\triangledown}{S} = \left[\delta - 2\overset{\triangledown}{\chi}\right] * \partial\overset{\triangledown}{S}, \qquad \overset{\triangledown}{\chi} = \overset{\triangledown}{w}_{\boldsymbol{e}_1}^{*-1} * \left(\partial\overset{\triangledown}{w}_{\boldsymbol{e}_1}\right), \tag{8.21}$$

where $\overset{\triangledown}{\chi} = {}^\flat\widehat{\chi}$ is the singularity associated with the minor $\widehat{\chi}(\zeta)$ of

$$\widetilde{\chi}(z) = \frac{\partial \widetilde{w}_{\boldsymbol{e}_1}}{\widetilde{w}_{\boldsymbol{e}_1}} \in z^{-2}\mathbb{C}[[z^{-1}]]_1.$$

The formal series $\widetilde{\chi}$ has a unique primitive $\widetilde{\chi}_0(z) = \partial^{-1}\widetilde{\chi}(z) = \log\big(\widetilde{w}_{\boldsymbol{e}_1}(z)\big)$ in the maximal ideal $z^{-1}\mathbb{C}[[z^{-1}]]_1$ of $\mathbb{C}[[z^{-1}]]_1$ and, thus, $\widehat{\chi}_0$ as well as its associated singularity $\overset{\triangledown}{\chi}_0$ is exponentiable in SING. (Lemma 8.2)

More simply, $\exp_*(\overset{\triangledown}{\chi}_0) = \delta + {}^\flat\widehat{w}_{\boldsymbol{e}_1}$, thus $\exp_*(2\overset{\triangledown}{\chi}_0) = \delta + {}^\flat(2\widehat{w}_{\boldsymbol{e}_1} + \widehat{w}_{\boldsymbol{e}_1}^{*2})$.

We introduce $\overset{\triangledown}{S}_0 \in \mathrm{SING}_{0,\pi}$ given by the identity:

$$\partial \overset{\triangledown}{S} = \overset{\triangledown}{S}_0 * \exp_*(-2\overset{\triangledown}{\chi}_0). \tag{8.22}$$

By construction, $\partial \exp_*(-2\overset{\triangledown}{\chi}_0) = -2\overset{\triangledown}{\chi} * \exp_*(-2\overset{\triangledown}{\chi}_0)$. One deduces from (8.21) that $\overset{\triangledown}{S}_0$ solves the convolution equation $\partial\overset{\triangledown}{S}_0 - \overset{\triangledown}{S}_0 = 0$. This translates into the fact that $(\zeta+1)\overset{\vee}{S}_0$ is holomorphic near $\zeta = 0$, where $\overset{\vee}{S}_0$ stands for any major of $\overset{\triangledown}{S}_0$. Therefore $\overset{\vee}{S}_0$ is holomorphic as well near $\zeta = 0$, thus $\overset{\triangledown}{S}_0 = 0$. From (8.22), this means that $\partial\overset{\triangledown}{S} = 0$, that is $\zeta\overset{\vee}{S}(\zeta)$ is holomorphic near $\zeta = 0$ for any major $\overset{\vee}{S}$ of $\overset{\triangledown}{S}$. This allows to conclude that there exists a constant $A_0(\omega_1^0) \in \mathbb{C}$ such that $\overset{\triangledown}{S} = A_0(\omega_1^0)\delta$. Thus, $\Delta_{\omega_1^0}\overset{\triangledown}{w}_{\mathbf{0}} = A_0(\omega_1^0)\overset{\triangledown}{W}_{\boldsymbol{e}_1}$ which implies that $\Delta_{\omega_1^0}\overset{\triangledown}{w}_{\mathbf{0}}$ can be continued to an element of SING. This ends the proof of proposition 8.1 with the first appoach.

8.3.3 Alien derivations, second approach

The second approach We now propose another approach, based on the notion of asymptotic classes, that uses tools akin to Gevrey and 1-summability theories.
We know that $\overset{\vartriangle}{W} = \Delta_{\omega_1^0}\overset{\vartriangle}{w}_{\mathbf{0}} \in \mathrm{ASYMP}_{0,\pi}$ satisfies the condition $\mathfrak{P}_{\boldsymbol{e}_1}(\overset{\vartriangle}{w}_{\mathbf{0}})\overset{\vartriangle}{W} = 0$.
We look at the equation $\mathfrak{P}_{\boldsymbol{e}_1}(\widetilde{w}_{\mathbf{0}})\widetilde{W} = 0$. The operator $\mathfrak{P}_{\boldsymbol{e}_1}(\widetilde{w}_{\mathbf{0}})$ is of order two in z and has two linearly independent formal solutions $\widetilde{W}_{\boldsymbol{e}_1} = z^{\frac{3}{2}}\widetilde{w}_{\boldsymbol{e}_1} \in \widetilde{\mathrm{Nils}}_1$ and $\mathrm{e}^{2z}\widetilde{W}_{\boldsymbol{e}_2} = \mathrm{e}^{2z}z^{\frac{3}{2}}\widetilde{w}_{\boldsymbol{e}_2} \in \mathrm{e}^{2z}\widetilde{\mathrm{Nils}}_1$.

Let us represent the asymptotic classes $\overset{\vartriangle}{w}_{\mathbf{0}} = {}^\natural\widetilde{w}_{\mathbf{0}}$, $\overset{\vartriangle}{w}_{\boldsymbol{e}_1} = {}^\natural\widetilde{w}_{\boldsymbol{e}_1}$ and $\overset{\vartriangle}{w}_{\boldsymbol{e}_2} = {}^\natural\widetilde{w}_{\boldsymbol{e}_2}$ on restriction to $\mathrm{ASYMP}_{0,\pi}$. We pick a (good) open covering (I_i) of $J^\star =]-3\pi/2, 3\pi/2[$ with open arcs I_i of aperture less than π. We use the Borel-Ritt theorem for 1-Gevrey asymptotics to get, for each subscript i: $w_{\mathbf{0},i}, w_{\boldsymbol{e}_1,i}, w_{\boldsymbol{e}_2,i}, \in \overline{\mathcal{A}}_1(I_i)$ whose 1-Gevrey asymptotics is given by $\widetilde{w}_{\mathbf{0}}, \widetilde{w}_{\boldsymbol{e}_1}, \widetilde{w}_{\boldsymbol{e}_2}$ respectively. We know that each of these 1-Gevrey germ is uniquely defined up to 1-exponentially flat germs, that is up to elements of $\overline{\mathcal{A}}^{\leq -1}(I_i)$. As a consequence, the collections $(w_{\mathbf{0},i})$, $(w_{\boldsymbol{e}_1,i})$, $(w_{\boldsymbol{e}_2,i})$ represent the asymptotic classes we have in mind.
For each subscript i, observe that

$$T_1(I_i)\Big(\mathfrak{D}_{\boldsymbol{e}_1}(w_{\mathbf{0},i})w_{\boldsymbol{e}_1,i}\Big) = \mathfrak{D}_{\boldsymbol{e}_1}(\widetilde{w}_{\mathbf{0}})\widetilde{w}_{\boldsymbol{e}_1} = 0$$

with $\mathfrak{D}_{\boldsymbol{e}_1}$ the linear operator given by definition 5.5, because the 1-Gevrey Taylor map $T_1(I_i)$ is a morphism of differential algebras. This ensures that $\mathfrak{D}_{\boldsymbol{e}_1}(w_{\mathbf{0},i})w_{\boldsymbol{e}_1,i}$ belongs to $\overline{\mathcal{A}}^{\leq -1}(I_i)$.

We draw a first conclusion : $\mathfrak{D}_{\boldsymbol{e}_1}(\overset{\triangle}{w}_{\mathbf{0}})\ \overset{\triangle}{w}_{\boldsymbol{e}_1}= 0$ in $\mathrm{ASYMP}_{0,\pi}$ and thus, $\mathfrak{P}_{\boldsymbol{e}_1}(\overset{\triangle}{w}_{\mathbf{0}})\ \overset{\triangle}{W}_{\boldsymbol{e}_1}= 0$ as well with $\overset{\triangle}{W}_{\boldsymbol{e}_1}= z^{3/2}\ \overset{\triangle}{w}_{\boldsymbol{e}_1}\in \mathrm{ASYMP}_{0,\pi}$.

We add a property that ensues from an analogue of the M.A.E.T. (theorem 3.1) and for which we refer to [14, 16]: one can even find $h_{i,\boldsymbol{e}_1} \in \overline{\mathcal{A}}^{\leq -1}(I_i)$ so that $\mathfrak{D}_{\boldsymbol{e}_1}(w_{\mathbf{0},i})(w_{\boldsymbol{e}_1,i} - h_{\boldsymbol{e}_1,i})$ vanishes exactly, for each subscript i. Thus, one can find a representative $w_{\boldsymbol{e}_1,i} \in \overline{\mathcal{A}}_1(I_i)$ of $\overset{\triangle}{w}_{\boldsymbol{e}_1}$ so that $\mathfrak{D}_{\boldsymbol{e}_1}(w_{\mathbf{0},i})w_{\boldsymbol{e}_1,i} = 0$ and thus, $\mathfrak{P}_{\boldsymbol{e}_1}(w_{\mathbf{0},i})W_{\boldsymbol{e}_1,i} = 0$ as well with $W_{\boldsymbol{e}_1,i} = z^{\frac{3}{2}}w_{\boldsymbol{e}_1,i}$.

The same reasoning yields: one can find a representative $w_{\boldsymbol{e}_2,i} \in \overline{\mathcal{A}}_1(I_i)$ of $\overset{\triangle}{w}_{\boldsymbol{e}_2}$ so that $\mathfrak{D}_{\boldsymbol{e}_2}(w_{i,\mathbf{0}})w_{\boldsymbol{e}_2,i} = 0$, thus $\mathfrak{P}_{\boldsymbol{e}_1}(w_{\mathbf{0},i})\mathrm{e}^{2z}w_{\boldsymbol{e}_2,i} = 0$ with $W_{\boldsymbol{e}_2,i} = z^{\frac{3}{2}}w_{\boldsymbol{e}_2,i}$.

Therefore $\mathfrak{D}_{\boldsymbol{e}_2}(\overset{\triangle}{w}_{\mathbf{0}})\ \overset{\triangle}{w}_{\boldsymbol{e}_2}= 0$ in $\mathrm{ASYMP}_{0,\pi}$ and thus $\mathfrak{P}_{\boldsymbol{e}_1}(\overset{\triangle}{w}_{\mathbf{0}})\mathrm{e}^{2z}\ \overset{\triangle}{W}_{\boldsymbol{e}_2}= 0$. The key point if that $\mathrm{e}^{2z}\ \overset{\triangle}{W}_{\boldsymbol{e}_2}$ belongs to $\mathrm{e}^{2z}\mathrm{ASYMP}_{0,\pi}$ which is a vector space in direct sum with $\mathrm{ASYMP}_{0,\pi}$.

Putting things together, keeping the same notations, we see that the kernel of the linear differential operator $\mathfrak{P}_{\boldsymbol{e}_1}(w_{i,\mathbf{0}})$ in the space of sectorial germs of direction I_i is spanned by $W_{\boldsymbol{e}_1,i}$ and $\mathrm{e}^{2z}W_{\boldsymbol{e}_2,i}$.

We now go back to the asymptotic class $\overset{\triangle}{W}\in \mathrm{ASYMP}_{0,\pi}$ that satisfies $\mathfrak{P}_{\boldsymbol{e}_1}(\overset{\triangle}{w}_{\mathbf{0}})\ \overset{\triangle}{W}= 0$. Considering a refinement of (I_i) if necessary, one can find for each subscript i a representative $W_i \in \overline{\mathcal{A}}^{\leq 0}(I_i)$ of $\overset{\triangle}{W}$ and a 1-exponentially flat germ $b_i \in \overline{\mathcal{A}}^{\leq -1}(I_i)$ such that $\mathfrak{P}_{\boldsymbol{e}_1}(w_{\mathbf{0},i})W_i = b_i$. To get W_i, we apply the usual variation of constants method. One gets W_i under the form

$$W_i = B_i(z) + C_1W_{\boldsymbol{e}_1,i} + C_2\mathrm{e}^{2z}W_{\boldsymbol{e}_2,i}, \quad C_1, C_2 \in \mathbb{C}, \tag{8.23}$$
$$2B_i(z) = W_{\boldsymbol{e}_2,i}\int z^{-3}W_{\boldsymbol{e}_1,i}.b_i - W_{\boldsymbol{e}_i,i}\int z^{-3}W_{\boldsymbol{e}_2,i}.b_i.$$

It is a simple exercise to show that B_i belongs to $\overline{\mathcal{A}}^{\leq -1}(I_i)$ and one easily concludes that W_i has to be equal to $C_1W_{\boldsymbol{e}_1,i}$ modulo $\overline{\mathcal{A}}^{\leq -1}(I_i)$.

Depending on the arc, the term $C_2\mathrm{e}^{2z}W_{\boldsymbol{e}_2,i}$ either belongs to $\overline{\mathcal{A}}^{\leq -1}(I_i)$ (so one can take $C_2 = 0$) or escape from $W_i \in \overline{\mathcal{A}}^{\leq 0}(I_i)$ (thus one has to impose $C_2 = 0$).

This ends the second proof of proposition 8.1: the general solution of the linear equation $\mathfrak{P}_{\boldsymbol{e}_1}(\overset{\triangle}{w}_{\mathbf{0}})\ \overset{\triangle}{W}= 0$ in $\mathrm{ASYMP}_{0,\pi}$ is $C_1\ \overset{\triangle}{W}_{\boldsymbol{e}_1}$ and, consequently, there exists a constant $A_0(\omega_1^0) \in \mathbb{C}$ so that $\Delta_{\omega_1^0}\ \overset{\triangle}{w}_{\mathbf{0}}= A_0(\omega_1^0)\ \overset{\triangle}{W}_{\boldsymbol{e}_1}$ in $\mathrm{ASYMP}_{0,\pi}$. Thus, $\Delta_{\omega_1^0}\ \overset{\triangle}{w}_{\mathbf{0}}$ can be uniquely continued to an element of ASYMP.

Conclusion What we have shown amounts to the following upshot. The solutions of the equation $\mathfrak{P}_{\boldsymbol{e}_1}(\widetilde{w}_{\mathbf{0}})\widetilde{\mathcal{W}} = 0$ in the differential algebra $\prod_{k\in\mathbb{Z}} e^{-kz}\widetilde{\mathrm{Nils}}_1$ are spanned by the independent solutions $\widetilde{W}_{\boldsymbol{e}_1} \in \widetilde{\mathrm{Nils}}_1$ and $\mathrm{e}^{2z}\widetilde{W}_{\boldsymbol{e}_2} \in \mathrm{e}^{2z}\widetilde{\mathrm{Nils}}_1$. This implies that the solutions of the equation $\mathfrak{P}_{\boldsymbol{e}_1}(\overset{\triangle}{w}_{\mathbf{0}})\ \overset{\triangle}{\mathcal{W}}= 0$ in the differential algebra $\prod_{k\in\mathbb{Z}} e^{-kz}\mathrm{ASYMP}$, resp. $\prod_{k\in\mathbb{Z}} e^{-kz}\mathrm{ASYMP}_{0,\pi}$, are spanned by the independent solutions $\overset{\triangle}{W}_{\boldsymbol{e}_1}\in \mathrm{ASYMP}$ and $\mathrm{e}^{2z}\ \overset{\triangle}{W}_{\boldsymbol{e}_2}\in \mathrm{e}^{2z}\mathrm{ASYMP}$, resp. their restrictions. in $\mathrm{ASYMP}_{0,\pi}$ and $\mathrm{e}^{2z}\mathrm{ASYMP}_{0,\pi}$ respectively. This result can be generalized as follows.

Lemma 8.3. *For $k \in \mathbb{N}^2$, we denote by $\overset{\triangle}{W}_{\boldsymbol{k}} \in \mathrm{ASYMP}^{\mathrm{ram}}$ the asympotic class defined by $\overset{\triangle}{W}_{\boldsymbol{k}} = {}^{\natural\,\mathrm{ram}}\widetilde{W}_{\boldsymbol{k}}$ where $\widetilde{W}_{\boldsymbol{k}} \in \widetilde{\mathrm{Nils}}_1$ satisfies (8.4). Let $\theta \in \underset{\bullet}{\mathbb{S}}$ be any direction, $\alpha > 0$ and $k \in \mathbb{N}^2 \setminus \{\mathbf{0}\}$.*
If $\overset{\triangle}{\mathcal{W}} \in \prod_{\ell \in \mathbb{Z}} e^{-\ell z}\mathrm{ASYMP}_{\theta,\alpha}$ solves the linear differential equation

$$\mathfrak{P}_{\boldsymbol{k}} \overset{\triangle}{\mathcal{W}} = \sum_{\substack{\boldsymbol{k}_1+\boldsymbol{k}_2=\boldsymbol{k} \\ |\boldsymbol{k}_i|\geq 1}} \frac{\overset{\triangle}{W}_{\boldsymbol{k}_1}\overset{\triangle}{W}_{\boldsymbol{k}_2}}{2!}\frac{\partial^2 F(z,\overset{\triangle}{w}_{\mathbf{0}})}{\partial w^2}, \quad \mathfrak{P}_{\boldsymbol{k}} = \mathfrak{P}_{\boldsymbol{k}}(\overset{\triangle}{w}_{\mathbf{0}}), \tag{8.24}$$

then there exist uniquely determined constants $C_1, C_2 \in \mathbb{C}$ so that

$$\overset{\triangle}{\mathcal{W}} = \overset{\triangle}{W}_{\boldsymbol{k}} + \mathrm{e}^{\boldsymbol{\lambda}.\boldsymbol{k}z}\left(C_1\mathrm{e}^{-\lambda_1 z}\,\overset{\triangle}{W}_{\boldsymbol{e}_1} + C_2\mathrm{e}^{-\lambda_2 z}\,\overset{\triangle}{W}_{\boldsymbol{e}_2}\right). \tag{8.25}$$

Proof. The general formal solution for the equation (8.4) is of the form $\widetilde{\mathcal{W}} = \widetilde{W}_{\boldsymbol{k}} + \mathrm{e}^{\boldsymbol{\lambda}.\boldsymbol{k}z}\left(C_1\mathrm{e}^{-\lambda_1 z}\widetilde{W}_{\boldsymbol{e}_1} + C_2\mathrm{e}^{-\lambda_2 z}\widetilde{W}_{\boldsymbol{e}_2}\right)$. We already know that $\mathrm{e}^{\boldsymbol{\lambda}.\boldsymbol{k}z}\left(C_1\mathrm{e}^{-\lambda_1 z}\,\overset{\triangle}{W}_{\boldsymbol{e}_1} + C_2\mathrm{e}^{-\lambda_2 z}\,\overset{\triangle}{W}_{\boldsymbol{e}_2}\right)$ provides the general solution for the homogeneous equation $\mathfrak{P}_{\boldsymbol{k}}(\overset{\triangle}{w}_{\mathbf{0}})\,\overset{\triangle}{\mathcal{W}} = 0$ in $\prod_{\ell\in\mathbb{Z}} e^{-\ell z}\mathrm{ASYMP}_{\theta,\alpha}$. This asymptotic class $\overset{\triangle}{W}_{\boldsymbol{k}}$ is of the form (8.2), namely

$$\overset{\triangle}{W}_{\boldsymbol{k}} = \sum_{l} \frac{1}{l!}\,(\varkappa.\boldsymbol{k})^l \log^l(z) z^{-\boldsymbol{\tau}.\boldsymbol{k}}\,\overset{\triangle}{w}_{\boldsymbol{k}}^{[0]}, \quad \overset{\triangle}{w}_{\boldsymbol{k}}^{[0]} = {}^{\natural}\,\widetilde{w}_{\boldsymbol{k}}^{[0]},$$

with $\widetilde{w}_{\boldsymbol{k}}^{[0]} \in \mathbb{C}[[z^{-1}]]_1$ satisfying as linear differential equation given in corollary 5.1. This allows to conclude that $\overset{\triangle}{W}_{\boldsymbol{k}}$ is a particular solution for the equation (8.24) and one ends the proof in the same way.

8.3.4 A step further

What have been previously done works as well for the other alien derivatives $\Delta_{\omega_i^j}\,\overset{\triangledown}{w}_{\mathbf{0}} = A_0(\omega_i^j)\,\overset{\triangledown}{W}_{\boldsymbol{e}_i}$, resp. $\Delta_{\omega_i^j}\,\overset{\triangle}{w}_{\mathbf{0}} = A_0(\omega_i^j)\,\overset{\triangle}{W}_{\boldsymbol{e}_i}$, for any $i = 1,2$ and $j \in \mathbb{Z}$. Since $\overset{\triangle}{W}_{\boldsymbol{e}_1}$ and $\overset{\triangle}{W}_{\boldsymbol{e}_2}$ belong to $\widetilde{\mathrm{RES}}^{(1)}$ (proposition 8.1), one infers from corollary 7.3 that $\widetilde{w}_{\mathbf{0}}$ belongs to $\widetilde{\mathscr{R}}^{(2)}$. In particular, the alien derivatives $\Delta_{2\omega_1^j}\,\overset{\triangledown}{w}_{\mathbf{0}} \in \mathrm{SING}_{2\pi j,\pi}$, resp. $\Delta_{2\omega_1^j}\,\overset{\triangle}{w}_{\mathbf{0}} \in \mathrm{ASYMP}_{2\pi j,\pi}$ and $\Delta_{2\omega_2^j}\,\overset{\triangledown}{w}_{\mathbf{0}} \in \mathrm{SING}_{2\pi(j+1/2),\pi}$, resp. $\Delta_{2\omega_2^j}\,\overset{\triangle}{w}_{\mathbf{0}} \in \mathrm{ASYMP}_{2\pi(j+1/2),\pi}$, are well-defined. As a matter of fact, these alien derivatives are quite simple !

Lemma 8.4. *For any $\omega \in \underset{\bullet}{\mathbb{C}}$ so that $\overset{\bullet}{\omega} = \pm 2$, one has $\Delta_\omega\,\overset{\triangledown}{w}_{\mathbf{0}} = 0$. Equivalently, $\Delta_\omega\,\overset{\triangle}{w}_{\mathbf{0}} = 0$, $\Delta_\omega \widetilde{w}_{\mathbf{0}} = 0$.*

Proof. We only calculate $\overset{\vartriangle}{W}= \Delta_{2\omega_1^j} \overset{\vartriangle}{w}_{\mathbf{0}}$. Through the alien derivation $\Delta_{2\omega_1^j}$, equation (8.3) is transformed into the linear ODE

$$P(\partial-2)\overset{\vartriangle}{W} + \frac{1}{z}Q(\partial-2)\overset{\vartriangle}{W} = \frac{\partial F(z,\overset{\vartriangle}{w}_{\mathbf{0}})}{\partial w}\overset{\vartriangle}{W} \tag{8.26}$$

as a consequence of corollary 7.2. We recognize the equation $\mathfrak{P}_{2\boldsymbol{e}_1}(\overset{\vartriangle}{w}_{\mathbf{0}})\overset{\vartriangle}{W}= 0$. By lemma 5.4, the general formal solution for the linear equation $\mathfrak{P}_{2\boldsymbol{e}_1}(\widetilde{w}_{\mathbf{0}})\widetilde{\mathcal{W}} = 0$ is of the form $C_1\mathrm{e}^z\widetilde{W}_{\boldsymbol{e}_1} + C_2\mathrm{e}^{3z}\widetilde{W}_{\boldsymbol{e}_2}$ and we either conclude with the reasoning made in Sect. 8.3.2 (still write $\overset{\triangledown}{W}$ under the form $\overset{\triangledown}{W}=\overset{\triangledown}{S} * \overset{\triangledown}{W}_{\boldsymbol{e}_1}$ and show that $\overset{\triangledown}{S}= 0$) or rather directly with lemma 8.3 : the solutions of the equation $\mathfrak{P}_{2\boldsymbol{e}_1}(\overset{\vartriangle}{w}_{\mathbf{0}})\overset{\vartriangle}{\mathcal{W}}= 0$ in $\prod_{k\in\mathbb{Z}} e^{-kz}\mathrm{ASYMP}$ is $C_1\mathrm{e}^z\overset{\triangledown}{W}_{\boldsymbol{e}_1} + C_2\mathrm{e}^{3z}\overset{\triangledown}{W}_{\boldsymbol{e}_2}$ and one concludes that $\Delta_{2\omega_1^j}\overset{\vartriangle}{w}_{\mathbf{0}}= 0$ since the alien derivative belongs to $\mathrm{ASYMP}_{2\pi j,\pi}$. $\square$

We can keep on that way to get the complete resurgent structure for $\widetilde{w}_{\mathbf{0}}$ and, at the same time, to analytically continued its minor $\widehat{w}_{\mathbf{0}}$. Let us see what happens a step further.

To show that $\widetilde{w}_{\mathbf{0}}$ belongs to $\widetilde{\mathscr{R}}^{(3)}$, we have to complete the informations given by lemma 8.4. Following corollary 7.3, we would like to show that $\Delta_{\omega_2}\circ\Delta_{\omega_1}\overset{\vartriangle}{w}_{\mathbf{0}}$ belongs to $\widetilde{\mathrm{RES}}^{(1)}$ for any $\omega_1,\omega_2 \in \underset{\bullet}{\mathbb{C}}$ so that $\overset{\bullet}{\omega}_1 = \pm 1$ and $\overset{\bullet}{\omega}_2 = \pm 1$. From what we know, this amount to showing that the alien derivatives $\Delta_{\omega_2}\overset{\vartriangle}{W}_{\boldsymbol{e}_i}$ belong to $\widetilde{\mathrm{RES}}^{(1)}$.

Let us look at $\overset{\vartriangle}{W}= \Delta_{\omega_1^0}\overset{\vartriangle}{W}_{\boldsymbol{e}_1}\in \mathrm{ASYMP}_{0,\pi}$. From the identity $\mathfrak{P}_{\boldsymbol{e}_1}(\overset{\vartriangle}{w}_{\mathbf{0}})\overset{\vartriangle}{W}_{\boldsymbol{e}_1}= 0$ (equation (8.16)) and corollary 7.2, we draw:

$$P(\partial-2)\overset{\vartriangle}{W} + \frac{1}{z}Q(\partial-2)\overset{\vartriangle}{W} = \frac{\partial F(z,\overset{\vartriangle}{w}_{\mathbf{0}})}{\partial w}\overset{\vartriangle}{W} + \overset{\vartriangle}{W}_{\boldsymbol{e}_1}\Delta_{\omega_1^0}\overset{\vartriangle}{w}_{\mathbf{0}}\frac{\partial^2 F(z,\overset{\vartriangle}{w}_{\mathbf{0}})}{\partial w^2},$$

that is

$$\mathfrak{P}_{2\boldsymbol{e}_1}(\overset{\vartriangle}{w}_{\mathbf{0}})\overset{\vartriangle}{W} = A_0(\omega_1^0)\overset{\vartriangle}{W}{}^2_{\boldsymbol{e}_1}\frac{\partial^2 F(z,\overset{\vartriangle}{w}_{\mathbf{0}})}{\partial w^2}. \tag{8.27}$$

where $A_0(\omega_1^0)$ is the resurgent constant given in lemma 8.1. Observe that the general formal solution for the equation $\mathfrak{P}_{2\boldsymbol{e}_1}(\widetilde{w}_{\mathbf{0}})\widetilde{\mathcal{W}} = A_0(\omega_1^0)\widetilde{W}^2_{\boldsymbol{e}_1}\frac{\partial^2 F(z,\widetilde{w}_{\mathbf{0}})}{\partial w^2}$, deduced from (8.27) through the Taylor map, reads:

$$\widetilde{\mathcal{W}} = 2A_0(\omega_1^0)\widetilde{W}_{2\boldsymbol{e}_1} + C_1\mathrm{e}^z\widetilde{W}_{\boldsymbol{e}_1} + C_2\mathrm{e}^{3z}\widetilde{W}_{\boldsymbol{e}_2} \in \prod_{k\in\mathbb{Z}} e^{-kz}\widetilde{\mathrm{Nils}}_1$$

with $C_1, C_2 \in \mathbb{C}$. By lemma 8.3 one gets $\Delta_{\omega_1^0}\overset{\vartriangle}{W}_{\boldsymbol{e}_1}= 2A_0(\omega_1^0)\overset{\vartriangle}{W}_{2\boldsymbol{e}_1}$, which thus belongs to $\widetilde{\mathrm{RES}}^{(1)}$ by proposition 8.1.

Of course, one can keep on that way, by induction. However, a lesson has to be learned from what precedes : the resurgent structure is closely coupled with the formal integral and it is much time to introduce the bridge equation.

8.4 The bridge equation and proof of the main theorem

We go back to the formal integral

$$\widetilde{w}(z,\boldsymbol{U}) = \sum_{\boldsymbol{k}\in\mathbb{N}^2} \boldsymbol{U}^{\boldsymbol{k}} \mathrm{e}^{-\boldsymbol{\lambda}.\boldsymbol{k}z} \widetilde{W}_{\boldsymbol{k}} \quad \in \prod_{k\in\mathbb{Z}} e^{-kz}\widetilde{\mathrm{Nils}}_1[[\boldsymbol{U}]] \tag{8.28}$$

and we consider its derivatives with respect to the indeterminate U_i, $i=1,2$:

$$\begin{aligned}\frac{\partial\widetilde{w}}{\partial U_i}(z,\boldsymbol{U}) &= \sum_{\boldsymbol{k}\in\mathbb{N}^2} \boldsymbol{k}.\boldsymbol{e}_i \boldsymbol{U}^{\boldsymbol{k}-\boldsymbol{e}_i} \mathrm{e}^{-\boldsymbol{\lambda}.\boldsymbol{k}z} \widetilde{W}_{\boldsymbol{k}} \quad \in \prod_{k\in\mathbb{Z}} e^{-kz}\widetilde{\mathrm{Nils}}_1[[\boldsymbol{U}]] \\ &= \widetilde{W}_{\boldsymbol{e}_i} + O(U_1,U_2).\end{aligned} \tag{8.29}$$

Since the formal integral $\widetilde{w}$ solves the differential equation $P(\partial)\widetilde{w} + \frac{1}{z}Q(\partial)\widetilde{w} = F(z,\widetilde{w})$, one deduces that the following identity holds for $i=1,2$:

$$\left(P(\partial) + \frac{1}{z}Q(\partial) - \frac{\partial F(z,\widetilde{w})}{\partial w}\right)\frac{\partial\widetilde{w}}{\partial U_i} = 0, \;\text{ i.e. }\; \mathfrak{P}_{\mathbf{0}}(\widetilde{w})\frac{\partial\widetilde{w}}{\partial U_i} = 0. \tag{8.30}$$

The formal solutions for the equation $\mathfrak{P}_{\mathbf{0}}(\widetilde{w}_{\mathbf{0}})\widetilde{\mathcal{W}} = 0$ is spanned by $\mathrm{e}^{-\lambda_1 z}\widetilde{W}_{\boldsymbol{e}_1}$ and $\mathrm{e}^{-\lambda_2 z}\widetilde{W}_{\boldsymbol{e}_2}$. Therefore, $\dfrac{\partial\widetilde{w}}{\partial U_1}$ and $\dfrac{\partial\widetilde{w}}{\partial U_2}$ are two linearly independent solutions for the order two linear differential equation $\mathfrak{P}_{\mathbf{0}}(\widetilde{w})\widetilde{\mathcal{W}} = 0$, explicitly (wronsk stands for the wronskian):

$$\mathrm{wronsk}\left(\frac{\partial\widetilde{w}}{\partial U_1}, \frac{\partial\widetilde{w}}{\partial U_2}\right) = \mathrm{wronsk}\left(\mathrm{e}^{-\lambda_1 z}\widetilde{W}_{\boldsymbol{e}_1}, \mathrm{e}^{-\lambda_2 z}\widetilde{W}_{\boldsymbol{e}_2}\right) = 2z^3.$$

Lemma 8.3 translates into the fact that for any series of the form

$$\overset{\triangle}{\mathcal{W}}(z,\boldsymbol{U}) = \sum_{\boldsymbol{k}\in\mathbb{N}^2} \boldsymbol{U}^{\boldsymbol{k}}\, \overset{\triangle}{\mathcal{W}}_{\boldsymbol{k}}, \quad \overset{\triangle}{\mathcal{W}}_{\boldsymbol{k}} \in \prod_{k\in\mathbb{Z}} e^{-kz}\mathrm{ASYMP}_{\theta,\alpha},$$

that satisfies the second order equation $\mathfrak{P}_{\mathbf{0}}(\overset{\triangle}{w})\,\overset{\triangle}{W} = 0$, there exist uniquely determined constants $A(\omega,\boldsymbol{U}) \in \mathbb{C}[[\boldsymbol{U}]]$ and $B(\omega,\boldsymbol{U}) \in \mathbb{C}[[\boldsymbol{U}]]$ such that

$$\overset{\triangle}{\mathcal{W}}(z,\boldsymbol{U}) = A(\omega,\boldsymbol{U})\frac{\partial\overset{\triangle}{w}}{\partial U_1} + B(\omega,\boldsymbol{U})\frac{\partial\overset{\triangle}{w}}{\partial U_2}, \quad \frac{\partial\overset{\triangle}{w}}{\partial U_i} =^{\natural\,\mathrm{ram}} \frac{\partial\widetilde{w}}{\partial U_i} \tag{8.31}$$

To the formal integral $\widetilde{w}(z,\boldsymbol{U})$, one associates its analogue through the mapping $^{\natural\,\mathrm{ram}}$:

$$\overset{\triangle}{w}(z,\boldsymbol{U}) = \sum_{\boldsymbol{k}\in\mathbb{N}^2} \boldsymbol{U}^{\boldsymbol{k}} \mathrm{e}^{-\boldsymbol{\lambda}.\boldsymbol{k}z}\, \overset{\triangle}{W}_{\boldsymbol{k}}, \quad \overset{\triangle}{W}_{\boldsymbol{k}} =^{\natural\,\mathrm{ram}} \widetilde{W}_{\mathbf{k}}. \tag{8.32}$$

We take $\omega \in \underset{\bullet}{\mathbb{C}}$ and we assume for the moment that $\overset{\bullet}{\omega} = \pm 1$. By proposition 8.1 and corollary 7.2, the alien derivation Δ_ω acts on the formal integral $\overset{\triangle}{w}(z,\boldsymbol{U})$. As a matter of fact, it will be easier to use the **dotted alien derivation**, $\overset{\bullet}{\Delta}_\omega = \mathrm{e}^{-\omega z}\Delta_\omega$ which has the virtue of commuting with the derivation ∂. Therefore,

$$\overset{\bullet}{\Delta}_\omega \overset{\triangle}{w}(z,\boldsymbol{U}) = \sum_{\boldsymbol{k}\in\mathbb{N}^2} \boldsymbol{U}^{\boldsymbol{k}} \mathrm{e}^{-\boldsymbol{\lambda}.\boldsymbol{k}z} \overset{\bullet}{\Delta}_\omega \overset{\triangle}{W}_{\boldsymbol{k}}, \quad \overset{\bullet}{\Delta}_\omega \overset{\triangle}{W}_{\boldsymbol{k}} \in \mathrm{e}^{-\omega z}\mathrm{ASYMP}_{\arg(\omega),\pi}$$

and

$$\mathfrak{P}_{\boldsymbol{0}}(\overset{\triangle}{w})\overset{\bullet}{\Delta}_\omega \overset{\triangle}{w} = 0.$$

We deduce that the decomposition (8.31) holds for $\overset{\bullet}{\Delta}_\omega \overset{\triangle}{w}$. This decomposition $\overset{\bullet}{\Delta}_\omega \overset{\triangle}{w} = A(\omega,\boldsymbol{U})\dfrac{\partial \overset{\triangle}{w}}{\partial U_1} + B(\omega,\boldsymbol{U})\dfrac{\partial \overset{\triangle}{w}}{\partial U_2}$ is the so-called **bridge equation** of Ecalle, that is a link between alien derivatives and the usual partial derivatives.

Let $\Xi \subset \mathbb{N}^2$ be the set defined by $\Xi = \Xi_0 = \{k\boldsymbol{e}_1, k\boldsymbol{e}_2 \,|\, k \in \mathbb{N}\}$ and set $\Xi_n = \mathbf{n} + \Xi$ for any $n \in \mathbb{N}^\star$. With these notations, the formal integral can be written as follows:

$$\widetilde{w}(z,\boldsymbol{U}) = \sum_{n=0}^{\infty}\sum_{\boldsymbol{k}\in\Xi_n} \boldsymbol{U}^{\boldsymbol{k}}\mathrm{e}^{-\boldsymbol{\lambda}.\boldsymbol{k}z}\widetilde{W}_{\boldsymbol{k}}(z) = \sum_{n=0}^{\infty}\sum_{\boldsymbol{k}\in\Xi} \boldsymbol{U}^{\boldsymbol{k}+\mathbf{n}}\mathrm{e}^{-\boldsymbol{\lambda}.\boldsymbol{k}z}\widetilde{W}_{\boldsymbol{k}+\mathbf{n}}(z) \tag{8.33}$$

To fix the idea, suppose that $\overset{\bullet}{\omega} = k_0\lambda_1$ with $k_0 = 1$ at the moment. We get from the decomposition (8.31) the identity:

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{\boldsymbol{k}\in\Xi} \boldsymbol{U}^{\boldsymbol{k}+\mathbf{n}}\mathrm{e}^{-\boldsymbol{\lambda}.(\boldsymbol{k}+k_0\boldsymbol{e}_1)z}\Delta_\omega \overset{\triangle}{W}_{\boldsymbol{k}+\mathbf{n}} = \\
&\qquad A(\omega)\sum_{n=0}^{\infty}\sum_{\boldsymbol{k}\in\Xi}(\boldsymbol{k}+\mathbf{n}).\boldsymbol{e}_1\boldsymbol{U}^{\boldsymbol{k}+\mathbf{n}-\boldsymbol{e}_1}\mathrm{e}^{-\boldsymbol{\lambda}.\boldsymbol{k}z}\overset{\triangle}{W}_{\boldsymbol{k}+\mathbf{n}} \\
&\qquad\quad +B(\omega)\sum_{n=0}^{\infty}\sum_{\boldsymbol{k}\in\Xi}(\boldsymbol{k}+\mathbf{n}).\boldsymbol{e}_2\boldsymbol{U}^{\boldsymbol{k}+\mathbf{n}-\boldsymbol{e}_2}\mathrm{e}^{-\boldsymbol{\lambda}.\boldsymbol{k}z}\overset{\triangle}{W}_{\boldsymbol{k}+\mathbf{n}}
\end{aligned} \tag{8.34}$$

Each component $\boldsymbol{U}^{\boldsymbol{k}+\mathbf{n}}\mathrm{e}^{-\boldsymbol{\lambda}.(\boldsymbol{k}+k_0\boldsymbol{e}_1)z}\Delta_\omega \overset{\triangle}{W}_{\boldsymbol{k}+\mathbf{n}} \in \mathrm{e}^{-\boldsymbol{\lambda}.(\boldsymbol{k}+k_0\boldsymbol{e}_1)z}\mathrm{ASYMP}_{\arg(\omega),\pi}$ has its counterpart on the right-hand side of the equality. Necessarily,

$$\begin{aligned}
A(\omega,\boldsymbol{U}) &= \boldsymbol{U}^{(1-k_0)\boldsymbol{e}_1}\sum_{n\geq 0}A_n(\omega)\boldsymbol{U}^{\mathbf{n}} \\
B(\omega,\boldsymbol{U}) &= \boldsymbol{U}^{\boldsymbol{e}_2-k_0\boldsymbol{e}_1}\sum_{n\geq 0}B_n(\omega)\boldsymbol{U}^{\mathbf{n}}.
\end{aligned} \tag{8.35}$$

This implies on the one hand hand that $A_n(\omega) = 0$ when $|\omega| \geq n+2$ while $B_n(\omega) = 0$ when $|\omega| \geq n+1$. On the other hand,

$$\begin{aligned}
\Delta_\omega \overset{\triangle}{W}_{\boldsymbol{k}+\mathbf{n}} = &\sum_{m=-1}^{n} A_{n-m}(\omega)(\boldsymbol{k}+\mathbf{m}+k_0\boldsymbol{e}_1).\boldsymbol{e}_1 \overset{\triangle}{W}_{\boldsymbol{k}+\mathbf{m}+k_0\boldsymbol{e}_1} \\
&+ \sum_{m=-1}^{n} B_{n-m}(\omega)(\boldsymbol{k}+\mathbf{m}+k_0\boldsymbol{e}_1).\boldsymbol{e}_2 \overset{\triangle}{W}_{\boldsymbol{k}+\mathbf{m}+k_0\boldsymbol{e}_1}
\end{aligned} \tag{8.36}$$

with the convention use in theorem 8.1. The case $\overset{\bullet}{\omega} = k_0\lambda_2$ with $k_0 = 1$ is obtained by symmetry.

This result implies that the asymptotic class $\overset{\triangle}{W}_{\mathbf{k}} = {}^{\natural\,\mathrm{ram}}\widetilde{W}_{\mathbf{k}}$ belongs to $\widetilde{\mathrm{RES}}^{(2)}$, as a consequence of corollary 7.3. An easy induction on $k_0 \in \mathbb{N}^\star$ allows then to conclude that the $\widetilde{W}_{\mathbf{k}}$ belong to $\widetilde{\mathscr{R}}_{\mathbb{Z}}^{\mathrm{ram}}$. The rest of the theorem is shown by arguments used in remark 8.1. This ends the proof of theorem 8.1.

8.5 Comments

For differential systems of level 1 of the type (5.67), the resurgent study of the Stokes phenomenon and of the action of the symbolic Stokes automorphism Δ_θ^+ on transseries solutions were first done by Costin [4], under some conditions. This work was later extended to more general differential equations (with no resonance), and also for difference equations of the type (5.68), in particular by Braaksma and his students (see [2, 19]). These works make use of (so-called) "staircase distributions" [4, 6] and do not make appeal to alien derivations. The method explained in this chapter is closer to the ideas of Ecalle, leading to the bridge equation. Also, as we saw on the particular example of the first Painlevé equation, this method provides (theoretically) the whole set of holomophic invariants of Ecalle and passes the resonance cases under some conditions (no quasi-resonance, no nihilence [11]).

References

1. I. Aniceto, R. Schiappa, M. Vonk, *The resurgence of instantons in string theory.* Commun. Number Theory Phys. **6** (2012), no. 2, 339-496.
2. B.L.J. Braaksma, *Transseries for a class of nonlinear difference equations. In memory of W. A. Harris, Jr.* J. Differ. Equations Appl. **7** (2001), no. 5, 717-750.
3. B. Candelpergher, C. Nosmas, F. Pham, *Approche de la résurgence*, Actualités mathématiques, Hermann, Paris (1993).
4. O. Costin, *On Borel summation and Stokes phenomena for rank-*1 *nonlinear systems of ordinary differential equations.* Duke Math. J. **93** (1998), no. 2, 289-344.
5. O. Costin, *Correlation between pole location and asymptotic behavior for Painlevé I solutions.* Comm. Pure Appl. Math. **52** (1999), no. 4, 461-478.
6. O. Costin, *Asymptotics and Borel summability*, Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, 141. CRC Press, Boca Raton, FL, 2009.
7. O. Costin, R. Costin, *On the formation of singularities of solutions of nonlinear differential systems in antistokes directions.* Invent. Math. **145** (2001), no. 3, 425-485.
8. O. Costin, M. Huang, S. Tanveer, *Proof of the Dubrovin conjecture and analysis of the tritronquée solutions of* P_I, Duke Math. J. **163** (2014), no. 4, 665-704.
9. E. Delabaere, C. J. Howls, *Global asymptotics for multiple integrals with boundaries.* Duke Math. J. **112** (2002), 2, 199-264.
10. J. Écalle, *Les algèbres de fonctions résurgentes*, Publ. Math. d'Orsay, Université Paris-Sud, 1981.05 (1981).
11. J. Ecalle, *Les fonctions résurgentes. Tome III : l'équation du pont et la classification analytique des objets locaux.* Publ. Math. d'Orsay, Université Paris-Sud, 1985.05 (1985).
12. J. Écalle, *Fonctions analysables et preuve constructive de la conjecture de Dulac.* Actualités mathématiques, Hermann, Paris (1992).
13. J. Écalle, *Six lectures on transseries, Analysable functions and the Constructive proof of Dulac's conjecture*, Bifurcations and periodic orbits of vector fields (Montreal, PQ, 1992), 75-184, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 408, Kluwer Acad. Publ., Dordrecht, 1993.
14. C. Even, *Degrés de liberté des moyennes de convolution préservant la réalité.* Ann. Fac. Sci. Toulouse Math. (6) 13 (2004), no. 3, 377-420.

15. A.S. Fokas, A.R. Its, A.A. Kapaev, V.Y. Novokshenov, *Painlevé transcendents. The Riemann-Hilbert approach.* Mathematical Surveys and Monographs, 128. American Mathematical Society, Providence, RI, 2006.
16. V. Gelfreich, D. Sauzin, *Borel summation and splitting of separatrices for the Hénon map*, Ann. Inst. Fourier (Grenoble) **51** (2001), no 2, 513-567.
17. A. Kapaev, *Asymptotic behavior of the solutions of the Painlevé equation of the first kind.* Differentsial'nye Uravneniya **24** (1988), no. 10, 1684-1695, 1835; translation in Differential Equations **24** (1988), no. 10, 1107-1115 (1989).
18. A. Kapaev, A.V. Kitaev, *Connection formulae for the first Painlevé transcendent in the complex domain.* Lett. Math. Phys. **27** (1993), no. 4, 243-252.
19. G.R. Kuik, *Transseries in Difference and Differential Equations.* PhD thesis, Rijksuniversiteit Groningen (2003).
20. M. Loday-Richaud, *Divergent series and differential equations.* To appear.
21. F. Menous, *Les bonnes moyennes uniformisantes et une application à la resommation réelle.* Ann. Fac. Sci. Toulouse Math. (6) 8 (1999), no. 4, 579-628.
22. M. Loday-Richaud, P. Remy, *Resurgence, Stokes phenomenon and alien derivatives for level-one linear differential systems.* J. Differential Equations **250** (2011), no. 3, 1591-1630.
23. A.B. Olde Daalhuis, *Hyperasymptotics for nonlinear ODEs. II. The first Painlevé equation and a second-order Riccati equation.* Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **461** (2005), no. 2062, 3005-3021.
24. C. Olivé, D. Sauzin, T. Seara, *Resurgence in a Hamilton-Jacobi equation.* Proceedings of the International Conference in Honor of Frédéric Pham (Nice, 2002). Ann. Inst. Fourier (Grenoble) **53** (2003), no. 4, 1185-1235.
25. Y. Ou, *Sur la stabilité par produit de convolution d'algèbres de résurgence.* PhD thesis, Université d'Angers (2012).
26. J.-M. Rasoamanana, *Résurgence-sommabilité de séries formelles ramifiées dépendant d'un paramètre et solutions d'équations différentielles linéaires.* Ann. Fac. Sci. Toulouse Math. **(6)** 19 (2010), no. 2, 303-343.
27. D. Sauzin, *Resurgent functions and splitting problems*, RIMS Kokyuroku 1493 (31/05/2006) 48-117
28. D. Sauzin, *Introduction to 1-summability and the resurgence theory.* To appear.
29. D. Sauzin, *On the stability under convolution of resurgent functions.* Funkcial. Ekvac. **56** (2013), no. 3, 397-413.
30. D. Sauzin, *Nonlinear analysis with resurgent functions.* Preprint 2013, 35 pages. http://hal.archives-ouvertes.fr/hal-00766749
31. Y. Takei, *An explicit description of the connection formula for the first Painlevé equation.* In Toward the exact WKB analysis of differential equations, linear or non-linear (Kyoto, 1998), 204, 271-296, Kyoto Univ. Press, Kyoto, 2000.