On theories of growing bodies

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ABSTRACT

A general setting for a continuum kinematics and force theory for bodies whose material structure may evolve is presented. The body object of continuum mechanics is replaced by two distinct objects: the growing body and the material manifold. The configuration space of the growing body is given the structure of an infinite dimensional fiber bundle over the manifold containing the collection of material structures that the growing body may possess. A connection on this fiber bundle allows the decomposition of generalized velocities and forces into components pertaining to growth and mechanical power. A number of examples of growing body theories is given.

1. INTRODUCTION

This work considers a continuum mechanical framework for the description of growing bodies and gives a number of examples for growing body theories.

In terms of continuum mechanics, growing bodies are different from the usual in that their material structure is not conserved, i.e., material points and subbodies may be added or removed from the body. The theories of growing bodies are intended to model two basic classes of phenomena: addition (removal) of material to a body during a phase transition or a chemical reaction (see Eshelby [1] and Gurtin [2, 3, 4]), and growth processes in biological systems (see Skalak et al. [5] and Taber [6]).

For a continuum mechanical treatment of a growing body means that one has to generalize some of the notions of continuum mechanics so that bodies of a variable material structure may be considered. In particular, the geometry of growth should be studied in the framework of kinematics of continua. Furthermore, following the ideas of Eshelby [1] and Gurtin [2, 3, 4], one may associate generalized forces and stresses with changes in geometry due to growth processes just as the usual mechanical forces and stresses are associated with changes of the geometry due to deformation.

The framework presented here, described generally in the following Section, makes a distinction between the *growing body*, a mathematical object, and the material it contains at each growth stage, which is a subset of another mathematical object—the *material manifold*.

We use a global geometric framework to continuum mechanics (see [7, 8]) in which the collection of configurations of a system is given the structure of a differentiable manifold, whose tangent bundle contains the generalized velocities and whose cotangent bundle contains the generalized forces. Here, the collection of various material contents of the growing body (various material structures)—the *content space*—is an infinite dimensional manifold. Thus, the generalized forces associated with the growth processes are elements of the cotangent bundle of the content space. Similarly, the configuration space of the growing body contains all the configurations in space of all the material contents that the body may possess.

The basic mathematical objects, those describing the growing body, material manifold, a material content, and a configuration of the growing body in space, are defined in Section 2. Section 3 considers the content space and the configuration space of the growing body. In particular, the configuration space of the growing body is assumed to have the structure of a fiber bundle. A desirable feature in a theory of growing bodies is the possibility to decompose a generalized velocity of the growing body into two components: a rate of change of the material content and a material velocity field. Such a decomposition will induce a similar decomposition of forces into components associated with growth and components associated with mechanical power. This decomposition, whose geometrical counterpart is a connection on the configuration space (when regarded as a fiber bundle), is considered in Section 4.

Section 5, based on [9], models the growth of *organisms*, i.e., growing bodies that have identifiable elements. This theory is analogous to the theory of *volumetric growth* (see [5, 6]). Section 6, based on [10], presents a theory capable of modeling the growth of bodies due to phase transition. This theory is obtained from the theory of organisms by the requirement of invariance under the action of the group of diffeomorphisms of the growing body. Section 7 presents an example where the material manifold is a not a Euclidean space. The situation described in Section 7 can be interpreted as the growth of wood in the cross section of a tree trunk. Section 8 formalizes the surface growth of Skalak et al. [5] used to model growth in bones, horns, spiral shells and trees. Section 9 generalized the situation of Section 8 to allow variable age of the growing bodies.

Stress theory is not discussed here; however, the relevant results of [8, 11, 12, 13] concerning the representation of forces by stresses may be used to construct stress theories for the various examples considered here.

The constructions of the various infinite dimensional manifold structures use results of Kijowski & Komorowski [14], Komorowski [15], Binz & Fischer [16], Michor [17] and Hamilton [18].

2. GROWING BODIES

In this Section we describe the basic mathematical objects used in the formulation of the theories of growing bodies that we present in the following.

The notion of a growing body is used in order to extend continuum mechanics that deals traditionally with bodies of a fixed material structure so it can be used in situations in which the material structure of the body varies. This changing of material structure reflects the *growth* of the body.

While traditional continuum mechanics considers the body and the physical space as basic objects, theories of growing bodies replace the "body" by two objects: the *growing body* $\mathcal{B}$and the *material manifold* M. We will use the term *simple body* when we refer to a regular body B of continuum mechanics.

It is assumed that during the process of growth, in all the various growth stages, the growing body retains certain identifiable properties. For example, one may assume that the topology of the growing body is retained during a process of growth. The mathematical object that models the growing body is chosen in such a way that it reflects those properties that remain invariant and identifiable throughout the process of growth. Thus for example, if for a certain theory the topology is the only property that remains invariant throughout processes of growth, then $\mathcal{B}$ will be a topological space. Henceforth, it is assumed that the growing body has a differentiable structure.

The material manifold M is conceived as the collection of virtual material points. This notion is somewhat analogous to the *material universe* (see Noll [19] and Truesdell [20]). Thus, a point X in M may be a part of the body in a certain growth stage or it may not. As suggested by the terminology, it is assumed that M is a smooth manifold.

A growth stage of the body is described mathematically by a mapping $c\colon \mathcal{B} \to M$. The image of the mapping c is the collection of material points that the body contains at the growth stage described by c and accordingly, we will refer to c as a *content* of the growing body. Thus, the image of a content is a simple body. The class of mappings that are admitted as contents should reflect the invariant properties of the growing body. For the example of a body whose topology is conserved in all its growth stages, a content will be a continuous mapping. In the following we will be more restrictive than this example and it will is assumed henceforth that the images of contents and simple bodies are smooth compact manifolds with boundaries.

A simple interpretation of these notions may be offered if one identifies the material manifold with the space manifold and the images of contents of the growing body are identified with reference configurations of simple bodies in space. This interpretation can be carried one step further if we identify the growing body $\mathcal{B}$ with a simple body B and consider the images of the contents of $\mathcal{B}$ as various reference configurations of the simple body B.

Since for each content c of the growing body, $B = c\{\mathcal{B}\}$ is a simple body, we may consider configurations of B in the physical space. For simplicity, we consider here only the case where the physical space is modeled by $\Re^3$, and in accordance with most treatments of continuum mechanics, we will model configurations of simple bodies in space by embeddings. Thus, we define a *configuration of a growing body* κ to be a configuration in space of the image of some content c of the growing body.

With the interpretation of a content of the growing body as a reference configuration of a simple body, a configuration of the growing body may be interpreted as a deformation superimposed on the reference configuration.

It is customary in continuum mechanics to consider the action of groups on simple bodies and on the space manifold. The action of a group on a body is usually related to symmetry of the material properties of the body and the action of a group on the space manifold—usually, the action of the Euclidean group on $\Re^3$—is generates balance laws. In the case of theories of growing bodies, the actions of groups on the growing body and material manifold should be considered.

In the following, we will present an example for the action of a group $G_{\mathcal{B}}$ on the growing body manifold. We will denote this action by $\Psi_{\mathcal{B}}$ so that

$$\Psi_{\mathcal{B}} \colon G_{\mathcal{B}} \times \mathcal{B} \to \mathcal{B}$$

is a smooth mapping. Similarly to the case of a group action on a simple body, the action $\Psi_{\mathcal{B}}$ describes invariance of the "properties" of the growing body points under class of certain transformations. In addition, one may consider the action

$$\Psi_M : G_M \times M \to M$$

of a group G_M on the material manifold.

Theories of growing bodies may be classified according to the invariance requirements that are satisfied. That is under what group action $\Psi_{\mathcal{B}}$ on the growing body and under what group action Ψ_M on the material manifold is the theory invariant. For example, if one assumes that M has a metric structure, it is natural to require that contents that differ by an isometry of M be identified. (With the interpretation of a content as a reference configuration, this requirement means identifying reference configurations of the body that differ by rigid body motions.) Such a requirement of invariance will lead to balance of content forces.

3. THE CONTENT SPACE AND THE CONFIGURATION SPACE OF A GROWING BODY

We will study the various theories from a global geometric point of view and so we make the following definitions. The *content space* $\mathcal{C}$ is the collection of all contents of the growing body in the material manifold, i.e.,

$$\mathcal{C} = \{c \,|\, c : \mathcal{B} \to M \text{ is a content}\}.$$

It is assumed that the content space is a differentiable manifold. Clearly, as a manifold of mappings, $\mathcal{C}$ is infinite dimensional. When considering various theories of growing bodies, one has to show for each instance that $\mathcal{C}$ possesses indeed a structure of a manifold.

Given a content c, one may consider the configuration space $Q_{c\{\mathcal{B}\}}$ of its image, the simple body $B = c\{\mathcal{B}\}$, in the physical space. That is,

$$Q_{c\{\mathcal{B}\}} = \mathrm{Emb}(B, \Re^3),$$

where, “Emb” denotes the collection of embeddings of the first argument in the second.

Thus, the *configuration space of the growing body* is

$$Q_{\mathcal{B}} = \bigcup_{c \in \mathcal{C}} Q_{c\{\mathcal{B}\}}.$$

We note that $Q_{\mathcal{B}}$ has a natural projection mapping $\pi_{\mathcal{C}} : Q_{\mathcal{B}} \to \mathcal{C}$ that assigns c to every κ in $Q_{c\{\mathcal{B}\}}$.

The following results will be of use when considering the structure of the various configuration spaces. The collection $\mathrm{Emb}(\mathcal{X}, \mathcal{Y})$ is an open subset of the Frechet manifold $C^\infty(\mathcal{X}, \mathcal{Y})$ of smooth mappings of $\mathcal{X}$ in $\mathcal{Y}$, for a compact $\mathcal{X}$ (see [17, 18]). Thus, the tangent space $T_i\mathrm{Emb}(\mathcal{X}, \mathcal{Y})$ to $\mathrm{Emb}(\mathcal{X}, \mathcal{Y})$ at the embedding i can be identified with the tangent space to $C^\infty(\mathcal{X},\mathcal{Y})$ at i, i.e., with $\{u \in C^\infty(\mathcal{X}, T\mathcal{Y}) \,|\, \tau_{\mathcal{Y}} \circ u = i\}$, where τ denotes the tangent bundle projection of the corresponding manifold. Since, $\Re^3$ is a trivial manifold, it follows that for any configuration κ of $c\{\mathcal{B}\}$ in space, $T_\kappa Q_{c\{\mathcal{B}\}}$ is isomorphic with $C^\infty(c\{\mathcal{B}\},\Re^3)$. We conclude that for every c in $\mathcal{C}$, the *fiber* $\pi_{\mathcal{C}}^{-1}(c)$ is a manifold.

In [7, 8, 11] the kinematics and force theory of simple bodies was presented from a global geometrical point of view. The configuration space of a simple body in space was presented and generalized velocities were defined as elements of the tangent bundle. The results quoted in the previous paragraph indicate that as expected, such generalized velocities are the virtual velocity fields of continuum mechanics. Generalized forces, defined as elements of the cotangent bundle, were shown to generalize forces on simple bodies in continuum mechanics. In particular, under a natural choice of topology for the configuration space, forces where shown to be represented by stresses. This discussion explains the terminology *material velocity field* for an element $\mathbf{v}$ of $T_\kappa Q_{c\{\mathcal{B}\}}$ and a *simple body force* for an element f_m in $T^*_\kappa Q_{c\{\mathcal{B}\}}$, where c is a given content.

We will consider only theories of growing bodies for which the configuration space possesses a differentiable structure. For such theories it is possible to apply the framework of [7, 8, 11] so that generalized velocities are defined as elements of the tangent bundle of the configuration space and generalized forces are defined as elements of the cotangent bundle. Thus, one may formulate kinematics and force theory for growing bodies.

Similarly, since $\mathcal{C}$ was assumed to have a smooth structure, generalized velocities and forces pertaining to the growth of the body may be defined as elements of the tangent and cotangent bundles of $\mathcal{C}$ respectively. Thus, we will refer to an element $\dot{c} \in T_c\mathcal{C}$ as a *content rate* and to an element $f_c \in T^*_c\mathcal{C}$ as a *content force*. Clearly, the content rate and content force are velocity and force only in the generalized sense of the word. The use of such forces is motivated by the works of Eshelby, [1], and Gurtin (e.g., [2, 3, 4]) who uses the terms *accretive forces* and *configurational forces* in his works on multiphase bodies.

The mapping $\pi_{\mathcal{C}}$, our assumptions concerning the availability of differential structure for both $Q_{\mathcal{B}}$ and $\mathcal{C}$, and the earlier discussion concerning the structure of $\pi_{\mathcal{C}}^{-1}(c)$, $c \in \mathcal{C}$, will be supplemented by the requirement that $Q_{\mathcal{B}}$ has the structure of a fiber bundle. The availability of a connection on the fiber bundle will be related to our ability to decompose generalized velocities and forces pertaining to the growing body into the corresponding variables pertaining to growth and simple bodies. We will describe the problem in some detail in the next section.

Note that with the foregoing framework, the interpretation of contents as reference configurations of a simple body may be formulated mathematically as follows. We assume that we have a given section $s\colon \mathcal{C} \to Q_{\mathcal{B}}$ of the bundle structure of $Q_{\mathcal{B}}$. Such a section associates with every content c of the growing body a configuration $s(c)$—the "reference configuration"—of the growing body in space.

4. CONNECTION-INDUCED DECOMPOSITIONS

A desirable feature that one would like a theory of growing bodies to have is the possibility to decompose a generalized velocity into a growth rate and a velocity of a simple body. Similarly, it is desirable that forces be decomposable into content forces and simple body forces. Such decompositions are made possible by a connection on the bundle $\pi_{\mathcal{C}}\colon Q_{\mathcal{B}} \to \mathcal{C}$. This Section describes some of the details pertaining to such decompositions.

The mapping

$$\tau_{Q_{\mathcal{B}}}\colon TQ_{\mathcal{B}} \to Q_{\mathcal{B}}$$

assigns the configuration κ of the growing body to every generalized velocity of the growing body at that configuration. In addition, we note that the tangent mapping

$$T\pi_{\mathcal{C}}\colon TQ_{\mathcal{B}} \to T\mathcal{C}$$

assigns to every generalized velocity the corresponding content rate. Moreover, we can identify generalized velocity fields that are actually material velocity fields by the condition that the content rate associated with them vanishes, i.e., generalized velocity fields $\dot{\kappa}$ that satisfy the condition

$$T\pi_{\mathcal{C}}(\dot{\kappa}) = 0.$$

Geometrically, such purely material generalized velocity fields are termed *vertical vectors*, the subspace $V_{\kappa}Q_{\mathcal{B}} \subset T_{\kappa}Q_{\mathcal{B}}$ containing the vertical vectors is usually called the *vertical subspace* and the collection of all vertical subspaces is the *vertical subbundle* $VQ_{\mathcal{B}}$ of $TQ_{\mathcal{B}}$. Nevertheless, we do not have an invariant mapping that assigns material velocity fields to generalized velocities.

A *connection* on $Q_{\mathcal{B}}$ is specified by means of a family of linear injections

$$\Gamma_{\kappa}\colon T_c\mathcal{C} \to T_{\kappa}Q_{\mathcal{B}}, \quad \kappa \in Q_{\mathcal{B}},\ c = \pi_{\mathcal{C}}(\kappa)$$

satisfying, $T\pi_{\mathcal{C}} \circ \Gamma_{\kappa} = \mathit{1}$, where $\mathit{1}$ is the identity mapping. The mapping Γ_{κ} will be referred to as the connection mapping. The connection mapping induces a mapping

$$\Delta_\kappa \colon T_\kappa Q_{\mathcal{B}} \to V_\kappa Q_{\mathcal{B}}$$

for every $\kappa \in Q_{\mathcal{B}}$ by $\Delta_\kappa = \mathit{1} - \mathit{\Gamma}_\kappa \circ T\pi_{\mathcal{C}}$. We will refer to Δ_κ as the *vertical projection* and, indeed, one can verify that the image of Δ_κ consists of the vertical vectors. Similarly, we will refer to $\text{Image}\{\mathit{\Gamma}_\kappa\} = \text{Kernel}(\Delta_\kappa)$ as the *horizontal distribution*. Using ι to denote the inclusion mapping, the situation is illustrated in the following commutative diagram.

$$\begin{tikzcd}
V_\kappa Q_{\mathcal{B}} \arrow[r, "\iota"] \arrow[d, equal] & T_\kappa Q_{\mathcal{B}} \arrow[r, "T_\kappa \pi_{\mathcal{C}}"] \arrow[d, equal] & T_c\mathcal{C} \arrow[d, equal] \\
V_\kappa Q_{\mathcal{B}} & T_\kappa Q_{\mathcal{B}} \arrow[l, "\Delta_\kappa"'] & T_c\mathcal{C} \arrow[l, "\mathit{\Gamma}_\kappa"']
\end{tikzcd}$$

Thus, $(T\pi_{\mathcal{C}}, \Delta_\kappa)$ provide the required decomposition of generalized velocities.

Once a connection is given, one can take duals of the foregoing mappings and decompose forces by $(\mathit{\Gamma}_\kappa{}^*, \iota^*)$. The situation is described by the following commutative diagram.

$$\begin{tikzcd}
(V_\kappa Q_{\mathcal{B}})^* \arrow[d, equal] & (T_\kappa Q_{\mathcal{B}})^* \arrow[l, "\iota^*"'] \arrow[d, equal] & (T_c\mathcal{C})^* \arrow[l, "(T_\kappa \pi_{\mathcal{C}})^*"'] \arrow[d, equal] \\
(V_\kappa Q_{\mathcal{B}})^* \arrow[r, "(\Delta_\kappa)^*"] & (T_\kappa Q_{\mathcal{B}})^* \arrow[r, "(\mathit{\Gamma}_\kappa)^*"] & (T_c\mathcal{C})^*
\end{tikzcd}$$

5. ORGANISMS

The theory of organisms describes growing bodies whose elements are identifiable throughout the growth. It is natural to assume that for a living organism we are able to identify various parts of the organism irrespective of the growth. In accordance with continuum mechanics we require that this ability does not end at a certain scale or size. This requirement is reflected mathematically in the following definitions.

Definition 5.1. An *organism* $\mathcal{B}$ is a three dimensional compact manifold with boundary that can be embedded in $\Re^3$.

Definition 5.2. The *material manifold* of an organism is a three dimensional Euclidean space M with tangent space $\mathbf{V}$.

Definition 5.3. A *content* c of an organism is an embedding of the organism in the material manifold.

Clearly, the requirement that contents are embeddings means that not only the growing body points are identifiable but in addition there is a differentiable structure that remains invariant throughout the growth.

Proposition 5.1. The configuration space of an organism has the structure of a trivializable fiber bundle.

Proof. Given a configuration κ in $\pi_{\mathcal{C}}^{-1}(c)$ for $c \in \mathcal{C}$, set, $e \colon \mathcal{B} \to \Re^3$ by $e = \kappa \circ c$. We will refer to e as the *extent* of the growing body at the configuration κ. Consider the mapping

$$\Phi \colon Q_{\mathcal{B}} \to \mathcal{C} \times \mathrm{Emb}(\mathcal{B}, \Re^3)$$

given by

$$\Phi(\kappa) = (\pi_{\mathcal{C}}(\kappa), \kappa \circ \pi_{\mathcal{C}}(\kappa)) = (c, e).$$

This natural mapping defines a global fiber bundle chart on $Q_{\mathcal{B}}$. □

Under this global chart a generalized velocity is represented by $(c, e, \dot{c}, \dot{e})$ where $\dot{c} \in T_c\mathcal{C}$ is a vector field on $\mathcal{B}$ valued in the tangent space $\mathbf{V}$ to the material manifold M, and $\dot{e} \in T_e\mathrm{Emb}(\mathcal{B}, \Re^3)$ is a vector field on $\mathcal{B}$ valued in $\Re^3$.

The natural global fiber bundle chart Φ induces immediately a decomposition of generalized velocities for which the connection mapping $\mathit{\Gamma}_\kappa$ is given by $T_c\Phi(\mathit{\Gamma}_\kappa(\dot{c})) = (\dot{c}, 0)$. Here, $T_c\Phi$ denotes the tangent of Φ at c and it follows that $\Delta_\kappa(T_c\Phi^{-1}(\dot{c}, \dot{e})) = T_c\Phi^{-1}(0, \dot{e})$. In other words, the representative of the vertical projection Δ_κ acts as the projection on the second component in the product $\mathcal{C} \times \mathrm{Emb}(\mathcal{B}, \Re^3)$. From the definition of Φ it follows that $\Delta_\kappa(\dot{\kappa}) = \dot{e} \circ c^{-1}$. With this connection a generalized velocity $\dot{\kappa}$ belongs to the horizontal distribution if $\dot{e} = 0$ so that the growing body points remain momentarily stationary in space. This means that if the body grows but at the same time it is deformed so that the identifiable parts remain in the same location in space the corresponding generalized velocity will be horizontal.

A different connection can be defined on $Q_{\mathcal{B}}$ such that the generalized velocity corresponding to the situation describe above will no longer be horizontal and the vertical projection will have a clear kinematical meaning.

Proposition 5.2. The mapping $\mathit{\Gamma}_\kappa$ given by

$$\Phi(\mathit{\Gamma}_\kappa(\dot{c})) = \mathrm{D}\kappa \circ c(\dot{c})$$

defines a connection on $Q_{\mathcal{B}}$ for which the vertical component of a generalized velocity is the material velocity field

$$\mathbf{v}(X) = \dot{e} \circ c^{-1}(X) - \mathrm{D}\kappa(X)(\dot{c} \circ c^{-1}(X)), \quad X \in c\{\mathcal{B}\}.$$

Proof. One can verify that the last equation holds by differentiating the relation $\kappa(t)(X) = e(t) \circ c^{-1}(t)(X)$, $X \in c\{\mathcal{B}\}$, with respect to the t parameter. In addition, using the definition of Δ_κ in terms of Γ_κ one can show that the expression for $\mathbf{v}$ is indeed the result of the vertical projection. □

If $f_\mathcal{B} \colon T_\kappa Q_\mathcal{B} \to \Re$ is a force on the growing body, the global chart Φ allows a natural representation of $f_\mathcal{B}$ in terms of a pair of functionals

$$(f_c, f_e) \in C^\infty(\mathcal{B}, \mathbf{V})^* \times C^\infty(\mathcal{B}, \Re^3)^*,$$

in the form

$$f_\mathcal{B}(\dot{\kappa}) = f_c(\dot{c}) + f_e(\dot{e}),$$

where $\dot{c}$ and $\dot{e}$ are the representatives of the generalized velocity $\dot{\kappa}$. It is natural to refer to f_c as a *content force* and to f_e as an *extent force*.

Corresponding to the decomposition of generalized velocities into vertical and horizontal components, one has a decomposition of generalized forces. The decomposition induced by the first connection mentioned is virtually identical to the decomposition using the natural chart except for the fact that the domain of definition of the vertical component is $c\{\mathcal{B}\}$ and not $\mathcal{B}$. We consider therefore the decomposition induced by the material velocity connection. Thus, a force $f_\mathcal{B}$ acting on the growing body at the configuration κ may be represented by

$$(f_a, f_m) \in C^\infty(\mathcal{B}, \mathbf{V})^* \times C^\infty(c\{\mathcal{B}\}, \Re^3)^*,$$

in the form

$$f_\mathcal{B}(\dot{\kappa}) = f_a(\dot{c}) + f_m(\mathbf{v}),$$

where, $\mathbf{v}$ is the material velocity field corresponding to $\dot{\kappa}$. Naturally, we will refer to f_m—a simple body force—as the *material* component of the force. Note that although f_a belongs to the same space as f_c they are different. The power expanded by $f_\mathcal{B}$ on $\dot{\kappa}$ will be equal to the power expanded by f_c on $\dot{c}$ if $\dot{e}$ vanishes. Thus, intuitively we can say that f_c performs work on both the content rate and the rate in which the material deforms.

The relation between the components of forces are given by the following equations.

$$\begin{aligned} f_a &= f_c + f_e \circ (\mathrm{D}\kappa \circ c) \\ f_m &= f_e \circ c^*, \end{aligned}$$

where, c^* is the pullback mapping of the vector fields defined on $c\{\mathcal{B}\}$ into vector fields defined on $\mathcal{B}$, so for example, $c^*(\mathbf{v})(\xi) = \mathbf{v}(c(\xi))$, $\mathbf{v} \colon c\{\mathcal{B}\} \to \Re^3$, $c^*(\mathbf{v}) \colon \mathcal{B} \to \Re^3$, $\xi \in \mathcal{B}$.

6. INVARIANCE UNDER THE GROUP OF DIFFEOMORPHISMS

In this section we use the same setting as in the case of organisms but now we consider a theory that is invariant under the action of the group of diffeomorphisms of $\mathcal{B}$, i.e., we consider the case where $G_{\mathcal{B}}$ is the group of diffeomorphisms of $\mathcal{B}$, $\mathrm{Diff}(\mathcal{B})$. The mechanical situation that such a theory describes is completely different than the theory of organisms. Here, the invariance under the group of diffeomorphisms implies that the growing body points are no longer identifiable. Thus, the only property that remains invariant throughout growth is the differential topological structure of the growing body. As such, this theory is aimed at describing phenomena such as phase transition for the case where a single phase is considered.

Definition 6.1. We say that two embeddings c_1, c_2 have the same *shape* if there is a diffeomorphism $\mathfrak{d} \in \mathrm{Diff}(\mathcal{B})$ such that $c_2 = c_1 \circ \mathfrak{d}$.

Clearly, the condition that two embeddings have the same shape defines an equivalence relation ϱ on $\mathrm{Emb}(\mathcal{B}, M)$. The equivalence classes are the shapes of the growing body and two embeddings have the same shape if and only if they have the same image. Thus, the image of a shape is well defined and in the sequel we will often identify a shape with its image. This motivates the term "shape" and the definition above. We conclude that if we wish to construct a theory of growing bodies that is invariant under the action of the group of diffeomophisms on $\mathcal{B}$ we have to admit shapes of $\mathcal{B}$ in M as contents rather then embeddings as in the previous Section. We will use χ to denote a typical shape and we will use $\mathcal{S}$ to denote the collection of shapes.

Definition 6.2. The content space for the $\mathrm{Diff}(\mathcal{B})$-invariant growing body theory is

$$\mathcal{S} = \mathrm{Emb}(\mathcal{B}, M)/\varrho.$$

The configuration space for the $\mathrm{Diff}(\mathcal{B})$-invariant growing body theory is

$$Q_{\mathcal{B}} = \bigcup_{\chi \in \mathcal{S}} \mathrm{Emb}(\mathrm{Image}(\chi), \Re^3).$$

In order that this theory satisfies the postulates of Section 3 we have to show that the collection of shapes has a differentiable structure. This result, given first in [14] can be also found in [16, 17, 18].

Proposition 6.1. The collection of shapes $\mathcal{S}$ has a structure of a Frechet manifold. The tangent space $T_\chi \mathcal{S}$ can be identified with $C^\infty(\partial\chi)$, the collection of smooth functions defined on the boundary of χ.

Proof. We will only outline the construction of charts on the space of shapes. The rest of the proof is available in the references cited above. Let n be the unit normal field to $\partial\chi$ and for u in $C^\infty(\partial\chi)$, consider the mapping $s_u\colon \partial\chi \to M$ given by

$$s_u(X) = X + u(X)n.$$

It may be shown that for a sufficiently small u, the image of s_u is a two dimensional submanifold of M that is diffeomorphic to $\partial\chi$. Thus, the image of s_u is the boundary of a shape χ'. In addition, for two distinct u_1, u_2, Image$(s_{u_1}) \neq$ Image(s_{u_2}). In other words, once χ is given with its normal field, there is a neighborhood U of χ such that the mapping

$$\phi\colon U \to C^\infty(\partial\chi),$$

so that $\phi(\chi')$ is the unique function in $C^\infty(\partial\chi)$ with Image$(s_u) = \chi'$, is a chart on $\mathcal{S}$. We will refer to this chart as the *chart centered at* χ. □

Remark 6.1. We note that the chart in $\mathcal{S}$ constructed above and the identification of $T_\chi\mathcal{S}$ with $C^\infty(\partial\chi)$ depended on our choice of the normal vector field rather than any other vector field that is transversal to $\partial\chi$. While the choice of the normal is natural here, in general, any other vector field may be used to construct a chart. Moreover, it is not necessary to use straight lines in the construction. Any family of parametrized curves, transversal to $\partial\chi$—a *tubular neighborhood*—may be used. Such constructions may be needed if the material manifold M does not have a metric structure. In such a general case, the basic principle of the construction still applies but the the tangent space $T_\chi\mathcal{S}$ is identified with the space of smooth sections $C^\infty\big(TM/T(\partial\chi)\big)$, where $TM/T(\partial\chi)$ is the quotient vector bundle—*normal bundle*.

The next result that we quote pertains to the manifold structure of $Q_\mathcal{B}$.

Proposition 6.2. The configuration space for the Diff$(\mathcal{B})$-invariant growing body theory has a structure of a fiber bundle

$$\pi_\mathcal{C}\colon Q_\mathcal{B} \to \mathcal{C},$$

whose fiber

$$\pi_\mathcal{C}^{-1}\{\chi\} = \mathrm{Emb}(\chi, \Re^3)$$

at any shape χ can be identified with Emb$(\mathcal{B}, \Re^3)$—the space of extents.

Proof. We will outline the construction of a fiber bundle chart in a neighborhood of a content $\chi \in \mathcal{S}$. We construct a diffeomorphism between Emb$(\chi', \Re^3)$ and Emb$(\chi, \Re^3)$ for χ' in a neighborhood of χ. Such a diffeomorphism can be constructed by artificially deforming

χ to the shape χ'. Mathematically this is obtained by generating a diffeomorphism

$$\delta_{\chi'} : \chi \to \chi'$$

that depends smoothly on χ'. Such a mapping δ will be referred to as a *dragging of the domain* χ, and its existence in a neighborhood of χ can be proved. Thus, $\pi_c^{-1}\{\chi'\} = \mathrm{Emb}(\chi', \Re^3)$ can be identified with $\pi_c^{-1}\{\chi\} = \mathrm{Emb}(\chi, \Re^3)$ by

$$\kappa' \mapsto \kappa = \kappa' \circ \delta_{\chi'}, \quad \kappa' : \chi' \to \Re^3, \ \kappa : \chi \to \Re^3.$$

□

Remark 6.2. The manifold $\mathrm{Emb}(\mathcal{B}, \Re^3)$ is diffeomorphic to $\mathrm{Emb}(\chi, \Re^3)$. However, unlike the theory of organisms, there is no natural identification of $\pi_c^{-1}\{\chi\}$ with $\mathrm{Emb}(\mathcal{B}, \Re^3)$ as χ is not a particular embedding but an equivalence class.

The tangent space $T_\kappa Q_{\mathcal{B}}$ can be identified using a fiber bundle chart with $C^\infty(\partial\chi) \times C^\infty(\mathcal{B}, \Re^3)$. This way, a generalized velocity $\dot{\kappa}$ is represented by a real function on the boundary of χ and an extent rate. The first component $\dot{\chi}$ is interpreted as the *normal velocity field of growth*. We recall that the first component is the representative of $T\pi_c(\dot{\kappa})$ and it depends on our choice of the unit normal field, or in the general case, choice of a particular tubular neighborhood. The second component, the extent rate, depends in addition on our choice of dragging of the domain χ and the particular shape χ of $\mathcal{B}$.

The next step in the construction of a $\mathrm{Diff}(\mathcal{B})$-invariant growing body theory is the consideration of a connection on $Q_{\mathcal{B}}$.

Proposition 6.3. Using the unit normal field, there is a connection on $Q_{\mathcal{B}}$ such that the vertical component of a generalized velocity $\dot{\kappa}$ is the material velocity field $\mathbf{v}$.

We omit the proof of the Proposition. For the details, see Segev, Fried & deBotton [13] and Hamilton [18]. If $X \in \mathrm{Interior}(\chi)$ and $\mathcal{B}(t)$ is the motion in $Q_{\mathcal{B}}$ whose tangent at $t = 0$ is the generalized velocity $\dot{\kappa}$, then

$$\mathbf{v}(X) = \frac{d}{dt}\kappa(t)(X)\bigg|_{t=0}$$

makes sense because $X \in \pi_c \circ \kappa(t)$ for t in a neighborhood of $t = 0$. The values of $\mathbf{v}$ on the boundary of χ are obtained as limits. An additional connection is also suggested by Hamilton [18] whose mechanical application we do not consider here.

Forces on a $\mathrm{Diff}(\mathcal{B})$-invariant growing body can be represented using a fiber bundle chart by means of a pair $(f_{\mathcal{S}}, f_e)$, where $f_{\mathcal{S}} \in C^\infty(\partial\chi)^*$ is interpreted as a force normal to the boundary of χ and $f_e \in C^\infty(\mathcal{B}, \Re^3)$ an extent force. Thus,

$$f_{\mathcal{B}}(\dot{\kappa}) = f_{\mathcal{S}}(\dot{\chi}) + f_e(\dot{e}),$$

where $(\dot{\chi}, \dot{e})$ are the representatives of $\dot{\kappa}$.

Using the decomposition induced by the connection, we have a representation of forces in the form

$$f_{\mathcal{B}}(\dot{\kappa}) = f_{\mathcal{S}}(\dot{\chi}) + f_m(\mathbf{v}), \quad f_{\mathcal{S}} \in C^\infty(\partial\chi),\ f_m \in C^\infty(\chi, \Re^3)^*,$$

where, $\dot{\chi} = T\pi_{\mathcal{C}}(\dot{\kappa})$, and $\mathbf{v}$ is the material velocity field.

7. A NON-EUCLIDEAN MATERIAL MANIFOLD

In this Section we present an example in which the material manifold M is not a Euclidean space. For the sake of illustration we consider a two dimensional example but the extension to three dimensions is straightforward.

Consider the case where $\mathcal{B}$ is a two dimensional annulus. We denote by $\partial\mathcal{B}_1$ and $\partial\mathcal{B}_2$ the inner and outer components of the boundary of the annulus, respectively. We set M to be the two dimensional cylinder $S^1 \times L$, where S^1 denotes the circle and L denotes a one dimensional oriented Euclidean space—a line—together with a specific orientation. We will treat M as a one dimensional vector bundle $\pi\colon M \to S^1$ over the circle, and much of the following immediately generalizes to the case of a general vector bundle.

Note that an embedding of the annulus into the cylinder can be in one of two forms. For the first form, the boundary of the annulus can be contracted to a point on the cylinder and in the second form it is impossible to contract the image of the boundary to a point, and $\partial\mathcal{B}_1$, $\partial\mathcal{B}_2$ are the images of two sections s_1 and s_2 of π, respectively. Note also that for an embedding of the second kind, there are two possible orientations of the image of $\mathcal{B}$ in M. For one orientation, $s_1(X) > s_2(X)$ for each X on S^1 and for the second orientation $s_1(X) < s_2(X)$. Simply put, we define a content to be the image of a mapping that wraps the annulus on the cylinder like a sleeve so that the outer boundary of the annulus remains always on one particular side of the image.

Definition 7.1. A content of the growing body is a shape χ of embeddings $\mathcal{B} \to M$ such that the images of the boundary components $\partial\mathcal{B}_1$ and $\partial\mathcal{B}_2$ correspond to sections s_1 and s_2 of π, respectively, with $s_2(X) > s_1(X)$, for all X on S^1.

Remark 7.1. Note that we can also set $M = S^1 \times L^2$ with the bundle structure
$$\pi\colon S^1 \times L^2 \longrightarrow S^1,$$
induced by the projection on the first factor, so that a content can be identified with a section of π with $s_2 > s_1$.

Definition 7.2. A configuration of the growing body $\mathcal{B}$ at the content χ is an embedding $\kappa\colon \chi \longrightarrow \Re^2$ that preserves the orientation of the annulus.

The next Proposition follows immediately from the definition of a content and the fact that the pullback of the vector bundle $\pi\colon M \to S^1$ onto the images of the two components of the boundary serve as tubular neighborhoods.

Proposition 7.1. The content space
$$\mathcal{C} = \{(s_1, s_2)|s_j \in C^\infty(\pi),\ j = 1, 2,\ s_2(X) > s_1(X),\ \text{for all } X \in S^1\},$$
where, $C^\infty(\pi)$ is the Frechet space of smooth sections of π, is an open subset of $C^\infty(\pi)^2$. Given a content χ by s_1 and s_2 any shape $\chi' = (s'_1, s'_2)$ in a neighborhood of χ in $\mathcal{C}$ may be represented by $(s'_1 - s_1, s'_2 - s_2)$ which is in a neighborhood of the zero element of $C^\infty(S^1)^2$. Here, we identify the tangent space $T_X L$ with $\Re$ so the difference of two smooth sections can be identified with an element of $C^\infty(S^1)$.

Clearly, the method of dragging of the domain can be used in order to provide $Q_{\mathcal{B}}$ with a differentiable structure. A configuration of the growing body κ will be represented under a fiber bundle chart on $Q_{\mathcal{B}}$ by a pair (χ, e), where, $\chi = (s_1, s_2) \in C^\infty(S^1)^2$, and $e\colon \mathcal{B} \to \Re^2$ is an orientation preserving embedding. A generalized velocity $\dot\kappa$ is represented under a chart by the pair $(\dot\chi, \dot e)$, $\dot\chi = (\dot s_1, \dot s_2) \in C^\infty(S^1)^2$, $\dot e \in C^\infty(\mathcal{B}, \Re^2)$. Thus, a force $f_{\mathcal{B}}$ is represented under a chart by a pair $(f_{\mathcal{S}}, f_e)$, where $f_{\mathcal{S}} = (f_{shps_1}, f_{\mathcal{S}_2}, f_{\mathcal{S}_j 2}$ being real distributions over the circle, and $f_e \in C^\infty(\mathcal{B}, \Re^2)^*$.

The fact that M is not the Euclidean plane does not affect the construction of the connection outlined in the previous Section using the material velocity field as the vertical component of the generalized velocity. Using the connection, generalized velocities decompose by
$$\dot\kappa \mapsto (\dot\chi, \mathbf{v}), \quad \dot\chi \in C^\infty(S^1)^2,\ \mathbf{v} \in C^\infty(\chi, \Re^2),$$
generalized forces decompose by
$$f_{\mathcal{B}} \mapsto (f_{\mathcal{S}}, f_m), \quad f_{\mathcal{S}} = (f_{\mathcal{S}_1}, f_{\mathcal{S}_2}),\ f_{\mathcal{S}_j} \in C^\infty(S^1)^*,\ f_m \in C^\infty(\chi, \Re^2)^*,$$
and finally
$$f_{\mathcal{B}}(\dot\kappa) = f_{\mathcal{S}_1}(\dot\chi_1) + f_{\mathcal{S}_2}(\dot\chi_2) + f_m(\mathbf{v}).$$

Remark 7.2. In this example, motivated by the notion of *surface growth* of Skalak et al. [5], using a cylinder rather than $\Re^2$ to model the material manifold allows unbounded growth on both components of the boundary of the annulus, even at the inner boundary. If one uses $\Re^2$ or $\Re^2 - \{(0,0)\}$ to model M, the metric properties of the material manifold will not be represented adequately.

Remark 7.3. The addition of material at the inner boundary of the annulus is meant to model the growth of the bark of a tree trunk at the cambium, while there is no generation of material at s_2. In order to model the generation of wood at the inner side of the cambium one should consider contents that have two components $c = (\chi_1, \chi_2)$. Here, χ_1 describing the outer growth is defined as above and χ_2 is a shape of hte circle bounded by $\partial\mathcal{B}_1$ in $\Re^2$—an $\Re^+$ valued function on $\partial\mathcal{B}_1$. Alternatively, one should add an additional dimension to the cylinder representing M. For example, we can use the fiber bundle

$$\pi\colon S^1 \times L^2 \times \Re^+ \to S^1$$

to model the material manifold. Shapes of the annulus in $S^1 \times L^2 \times \Re^+$ that are images of sections of the type (s_1, s_2, s_3), $s_2 > s_1$, of this fiber bundle will model contents. The value of the third component represents the growth of wood—the circle bounded by $\partial\mathcal{B}_1$. A description of surface growth that is closer to the treatment of [5] is given in the following Section.

8. SURFACE GROWTH

Skalak et al. [5] use the notion of surface growth to model growth patterns in bones (particularly the skull), horns, spiral shells, and trees. The characteristics of surface growth in the work of Skalak et al. are as follows. The material points in the image of a content are parametrized by points on a surface and a parameter τ. For a particular X in the image of a content, the parameter τ represents the time elapsed since X was added to the body. The parameter τ varies between zero and a maximum value t, the age of the growing body. Growth may occur on both sides of the surface, possibly in two different directions.

We now turn a formal geometrical description of surface growth. We first consider growth on one side of the surface only and extend the situation to growth on two sides growth later in the Section.

Definition 8.1. The growing body for the theory of surface growth is

$$\mathcal{B} = \Sigma \times [0, 1],$$

where Σ is a compact two dimensional submanifold without boundary of $\Re^3$ to which we will refer as the *growth surface*. The material manifold M is a three dimensional Euclidean space with tangent space $\mathbf{V}$. A content of the growing body is an isotopy of the growth surface in M, i.e., an embedding $c\colon \mathcal{B} = \Sigma \times [0,1] \to M$.

Remark 8.1. By the definition, the material points in the images of contents have identifiable surface and "time" parameters, and as such, it is an example of an organism with a somewhat different interpretation of the growing body points. The fact that we consider the fixed nondimensional "time" interval $[0,1]$ means that the various contents of the growing body are interpreted as growth stages of the growing body at one fixed "age".

Remark 8.2. Consider a two dimensional example (M is 2-dimensional) in which $\Sigma = S^1$. We interpret Σ as a cross section of the cambium of a tree and for a content c in $\mathcal{C}$, we interpret Image(c) as the cross section of a tree trunk having a fixed nondimensional age one. Growth of wood on one side of the cambium is considered only, i.e., the growth of wood rather than the bark. We note that with this model, the trunk has a cavity because of the fact that Image(c) is an annulus. The rings of the trunk are the surfaces $c\{\Sigma \times \{\tau\}\}$, $\tau \in [0,1]$, and so, the points on the cross section are parametrized by the "time" they were created. The content space is interpreted as the collection of sections of tree trunks of a fixed age corresponding possibly to various kinds of trees and various growth conditions.

We note that in the previous example, the growth of the trunk in the various radial dimensions need not be identical because of the invariant parametrization by the points on Σ. If we require that the theory is invariant under the action of the diffeomorphisms or rotations on S^1, we obtain growth patterns that are symmetrical.

The fact that we consider a special case of the theory of organisms implies that all the properties of that theory hold here, in particular, the differentiable structures and the connection.

In order to model growth on two sides of the growing surface, the previous framework should be modified as follows.

Definition 8.2. The growing body for the theory of two sided surface growth is

$$\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2, \quad \mathcal{B}_1 = \Sigma \times [-1,0], \quad \mathcal{B}_2 = \Sigma \times [0,1],$$

where Σ is a growth surface. A content of the growing body in M is a continuous injection $c\colon \mathcal{B} \to M$ such that the restrictions

$$c_1 = c|_{\mathcal{B}_1} : \mathcal{B}_1 \to M, \quad c_2 = c|_{\mathcal{B}_2} : \mathcal{B}_2 \to M,$$

are smooth embeddings.

This definition makes contents for the theory of two sided surface growth identical in structure to *composite body configurations* discussed in [13]. The mathematical structure of $\mathcal{B}$ here is that of a composite body with the two phases $\mathcal{B}_j$, and interface $\Sigma \times \{0\}$. Roughly, contents are "embeddings" of $\mathcal{B}$ in M whose derivative suffers a jump across the interface. The configuration space of a composite body is considered in [13] and the results apply to two sided surface growth as follows.

Proposition 8.1. The content space $\mathcal{C}$ for the theory of two sided surface growth is an open subset of

$$\mathbf{F}_M = \{w \in C^0(\mathcal{B}, M) \mid w_j = w_{|\mathcal{B}_j} \in C^\infty(\mathcal{B}_j, M),\ j = 1, 2\}.$$

Thus, $\mathcal{C}$ is a differentiable manifold with tangent space

$$\mathbf{F}_{\mathbf{V}} = \{u \in C^0(\mathcal{B}, \mathbf{V}) \mid u_j = u_{|\mathcal{B}_j} \in C^\infty(\mathcal{B}_j, \mathbf{V}),\ j = 1, 2\}.$$

We note that the jump in the derivative of contents at $\tau = 0$ implies that the differential structure of $\mathcal{B}$ is different than that of $B = c\{\mathcal{B}\}$. Thus, while a configuration of the growing body in space is by definition a smooth embedding $\kappa\colon c\{\mathcal{B}\} \to \Re^3$, for some $c \in \mathcal{C}$, the corresponding extent $e = \kappa \circ c\colon \mathcal{B} \to \Re^3$ is no longer smooth but has a jump in the derivative at $\tau = 0$, in contrast with the situation for the theory of organisms used for one sided surface growth. The structure of the configuration space should be modified accordingly.

Proposition 8.2. The configuration space for a two sided surface growth has a differentiable structure of a trivializable fiber bundle, such that

$$\Phi\colon Q_{\mathcal{B}} \to \mathcal{C} \times \mathbf{F}^3_{\Re},$$

given by

$$\Phi(\kappa) = (\pi_{\mathcal{C}}(\kappa), \kappa \circ \pi_{\mathcal{C}}(\kappa)) = (c, e),$$

where,

$$\mathbf{F}^3_{\Re} = \{u \in C^0(\mathcal{B}, \Re^3) \mid u_j = u_{|\mathcal{B}_j} \in C^\infty(\mathcal{B}_j, \Re^3),\ j = 1, 2\},$$

is a global natural trivialization.

One would also like to use the material velocity field in order to construct a connection on the configuration space as in the case of the theory of organisms. However, recalling the expression

$$\mathbf{v}(X) = \dot{e} \circ c^{-1}(X) - \mathrm{D}\kappa(X)(\dot{c} \circ c^{-1}(X)), \quad X \in c\{\mathcal{B}\},$$

for the material velocity field, we note that because of the discontinuity in the derivative $\mathrm{D}\kappa$, the velocity field suffers a jump. Hence, the velocity field is not a member of the vertical tangent subspace

$$T_\kappa(\pi_{\mathcal{C}}^{-1}(c)) = C^\infty(c\{\mathcal{B}\}, \Re^3)$$

and the construction of a connection cannot be followed. This situation is analogous to the theory of multiphase bodies (see [13]).

9. SURFACE GROWTH WITH A VARIABLE AGE

The mathematical framework suggested in the previous Section was intended to model surface growth in the case where all bodies had a fixed age—that corresponding to the dimensionless time parameter $\tau = 1$. While a material point was parametrized by the τ-parameter specifying the time it was created, the maximum value of the time parameter—the *age* of the body—was fixed for all bodies. In this Section we wish to extend the framework suggested for surface growth so that the contents of growing bodies may have different identifiable ages. For the sake of simplicity, we restrict the discussion to one sided growth.

Definition 9.1. For a growth surface Σ, the *t-old generation* is

$$\mathcal{C}_t = \mathrm{Emb}(\Sigma \times [0, t], M), \quad t \in \Re^+.$$

The content space for a variable age surface growth theory is

$$\mathcal{C} = \bigcup_{t \in \Re^+} \mathcal{C}_t.$$

The mapping $\pi_t \colon \mathcal{C} \to \Re^+$ that assigns t to every element of $\mathcal{C}_t$ will be referred to as the *age projection*.

The construction here is somewhat similar to the situation with the configuration space over the manifold of shapes in a $\mathrm{Diff}(\mathcal{B})$-invariant theory and similar tools should be used for the construction of the bundle structure.

Proposition 9.1. The content space for a variable age surface growth theory has the structure of a trivializable fiber bundle $\pi_t \colon \mathcal{C} \to \Re^+$.

Proof. To construct a global fiber bundle chart we construct a diffeomorphism of $\mathcal{C}_t$ with $\mathcal{C}_1$ for $t \in \Re^+$. The dragging of the domain

$$\delta_t \colon \Sigma \times [0, t] \to \Sigma \times [0, 1],$$

takes the very simple form

$$(\xi, \tau) \mapsto (\xi, \frac{\tau}{t}),$$

so that

$$c' \mapsto c = c' \circ \delta_t^{-1}, \quad c \in \mathcal{C}_t,\ c \in \mathcal{C}_1,$$

is the required diffeomorphism. □

Once the content space has been defined, the structure of the configuration space

$$Q_{\mathcal{B}} = \bigcup_{c \in \mathcal{C}} \mathrm{Emb}(\mathrm{Image}(c), \Re^3)$$

should be considered. Note that we can write

$$Q_{\mathcal{B}} = \bigcup_{t \in \Re^+} \Big(\bigcup_{c \in \mathcal{C}_t} \mathrm{Emb}\big(\mathrm{Image}(c), \Re^3\big) \Big).$$

Thus, setting

$$Q_{\mathcal{B}t} = \bigcup_{c \in \mathcal{C}_t} \mathrm{Emb}\big(\mathrm{Image}(c), \Re^3\big)$$

to be the collection of configuration in space of the t-old generation, the configuration space can be written as the union of configuration spaces of fixed age surface growth theories,

$$Q_{\mathcal{B}} = \bigcup_{t \in \Re^+} Q_{\mathcal{B}t}.$$

Proposition 9.2. The configuration space for a variable age surface growth theory has two fiber bundle structures. It is a trivializable fiber bundle

$$\pi_{\mathcal{C}} \colon Q_{\mathcal{B}} \to \mathcal{C},$$

and a trivializable fiber bundle

$$\pi_t \circ \pi_{\mathcal{C}} \colon Q_{\mathcal{B}} \to \Re^+$$

where the fiber over t in $\Re^+$ is $Q_{\mathcal{B}t}$.

Proof. We first consider fiber bundle charts for $\pi_t \circ \pi_{\mathcal{C}}$. Let $t \in \Re^+$, then, the fiber bundle structure of π_t implies that there is a diffeomorphism $\mathcal{C}_t$ to $\mathcal{C}_1$ given by $c \mapsto c \circ \delta_t^{-1}$. Hence,

$$Q_{\mathcal{B}t} = \bigcup_{c' \in \mathcal{C}_t} \mathrm{Emb}\big(\mathrm{Image}(c'), \Re^3\big)$$

is diffeomorphic to $Q_{\mathcal{B}1}$ because for $c' \in \mathcal{C}_t$, $c \in \mathcal{C}_1$, $c = c' \circ \delta_t^{-1}$ we have $\text{Image}(c') = \text{Image}(c)$.

Next consider c in $\mathcal{C}_t$. We construct a diffeomorphisms of $Q_{\text{Image}(c)}$ with $\text{Emb}(\Sigma \times [0,1], \Re^3)$. Here, we only have to generalize the extent $e = \kappa \circ c$ used for organisms (or surface growth with a fixed age), into

$$e = \kappa \circ c \circ \delta_t^{-1} \in \text{Emb}(\Sigma \times [0,1], \Re^3), \quad c \in \mathcal{C}_t,\ \kappa \in Q_{\text{Image}(c)}.$$

$\square$

Thus, surface growth with variable age provides an example of an iterated fiber bundle structure for the configuration space.

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