On volumetric growth and material frames

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ABSTRACT. A theory of volumetric growth on differentiable manifolds is presented. A balance law for an extensive property, that includes a source term, induces material structure and bodies. Transformation rules for the basic variables that represent growth, under a change of frame in the physical event space, are examined and the material frame, where the volumetric growth has a simple canonical form, is defined. Finally, we give a frame-invariant variational version of volumetric growth.
Keywords. Continuum mechanics, growth, balance laws, flux, material frames.

1. INTRODUCTION

Material points and bodies are among the fundamental notions of continuum mechanics. It is generally assumed that bodies are sets whose elements, the material points, are identifiable in all configurations of the body. This basic assumption allows the definition of a configuration of a body in the physical space as an injective mapping of the body into the set representing space. In addition, the velocity field is defined as the vector field whose value at a point is the tangent to the trajectory of the body element that occupies this point. Traditionally, the existence of such invariant material structure has been associated with conservation of mass. In contrast, when one wishes to study growth of bodies, the material structure of bodies should be reexamined. In such theories, mass in not conserved and material may be added to the body or removed from it. Continuum theories of growing bodies, formulated from a global point of view using configuration spaces of growth, were presented in [7], [11], and [8].

The term continuum mechanics is used in the sense that in most cases the body and space sets are assumed to possess smooth structures. In most traditional formulations the underlying geometric structure is that of a three dimensional Euclidean space. Some attempts have been made to formulate parts of the theories on differentiable manifolds having some additional structure such as a metric, a connection, etc. (see for example [3], [5], [6]).

Here we suggest a framework where material points are derived quantities rather then assumed a-priori. Following [2], the authors defined in [13] material points as integral lines of a flux field, obtained from the balance of an arbitrary property that need not be conserved. Thus, mass is separated from the material structure. The framework presented here generalizes [13] by using a setting on general differentiable manifolds rather than a three dimensional Euclidean space as an ambient space. Furthermore, the variance properties of the relevant variables with respect to a change of frame in the physical event space are considered. It is assumed that the event space has the structure of a fiber bundle over the time axis and a frame is a trivialization of that bundle. In particular, a material frame is constructed where the volumetric growth has a simple "canonical" form.

Section 2 describes the basic notation, definitions, assumptions and relevant results to be used in the sequel. In particular, using a frame dependent definition, volumetric growth is presented as a combination of density rate of change and boundary interaction for some extensive property. The generalized Cauchy postulates and theorem are presented together with the differential version of the balance law. Section 3 specializes the foregoing results to the case where a volume element is given. In Section 4, the material structure is defined and Section 5 introduces the material frames where there is no boundary interaction term in the expression for the volumetric growth. The transformation rules for the various quantities associated with volumetric growth are discussed in Section 6. In particular, it turns out that the volume term and boundary interaction term used in Section 2 to introduce volumetric growth are not frame invariant independently. Finally, the variational version of the balance law is introduced in Section 7. This version is the one suitable for a frame invariant definition of volumetric growth.

2. Volumetric Growth and the Generalized Cauchy theory

The geometric model for the physical event space that we assume here has an absolute time coordinate. The time axis is assumed to be a one dimensional manifold $\mathcal{T}$ that is identified with $\mathbb{R}$. The collection of physical events $\mathcal{E}$ is the total space of a trivializable fiber bundle

$$\pi\colon \mathcal{E} \to \mathcal{T}.$$

The typical fiber of the bundle is an oriented manifold $\mathcal{S}$ to which we do not attribute at the moment any additional structure. We will refer to $\mathcal{S}$ as the *space manifold* and use m to denote its dimension.

A *frame* F is a (global) trivialization of the bundle, i.e., a diffeomorphism

$$F\colon \mathcal{E} \to \mathcal{S} \times \mathcal{T}.$$

In the rest of this section it is assumed that a frame F is given. For a given event e we will loosely use the notation

$$x = x(e) = F_1(e), \quad t = t(e) = F_2(e), \quad e \in \mathcal{E},$$

where $F_1,\ F_2$ are the two components of F.

Definition 2.1. A *control volume* is a compact m-dimensional submanifold with corners of $\mathcal{S}$.

Definition 2.2. A *volumetric growth* is an assignment of an m-form $\beta_{\mathcal{R}}$ on $\mathcal{R}$ and an $(m-1)$-form $\tau_{\mathcal{R}}$ on $\partial\mathcal{R}$, for each control volume $\mathcal{R}$. Thus, for any control volume we may set

$$I_{\mathcal{R}} = \int\limits_{\mathcal{R}} \beta_{\mathcal{R}} + \int\limits_{\partial\mathcal{R}} \tau_{\mathcal{R}}$$

as a value of a real valued set function defined on the collection of control volumes.

The set function $\mathcal{R} \mapsto I_{\mathcal{R}}$ is interpreted physically as the production of a certain property measured in $\mathcal{R}$. The first integral is interpreted as the growth rate of the property contained in $\mathcal{R}$ so the form $\beta_{\mathcal{R}}$ is interpreted as the rate of change of the density of the property in $\mathcal{R}$. The second integral is interpreted as the rate at which the property leaves $\mathcal{R}$ through the boundary and $\tau_{\mathcal{R}}$ is referred to as the flux density.

Cauchy's postulates are concerned with the dependence of the forms $\beta_{\mathcal{R}}$ and $\tau_{\mathcal{R}}$ on the control volume $\mathcal{R}$ under consideration. Usually, Cauchy's postulates and the resulting Cauchy theorem are formulated for a three-dimensional Euclidean space (see for example [4]). Marsden and Hughes [5] gave a formulation of the theory in the setting of a three dimensional metric manifold. Here, using the following results of [9], we present a generalized theory for the case where $\mathcal{S}$ is an m-dimensional oriented manifold as stated above.

Assumption 2.3 (**Generalized Cauchy's Postulates**)**.** The volumetric growth

$$\{(\beta_{\mathcal{R}}, \tau_{\mathcal{R}})\}, \text{ for all control volumes } \mathcal{R},$$

satisfies the following conditions.

(i) For every $x \in \mathcal{S}$ and control volume $\mathcal{R}$, $\beta_{\mathcal{R}}(x) = \beta(x)$, i.e., the value is the same for all control volumes containing x. Accordingly, one can omit the subscript $\mathcal{R}$.

(ii) For every $x \in \mathcal{S}$ and control volume $\mathcal{R}$ such that $x \in \partial\mathcal{R}$, $\tau_{\mathcal{R}}(x)$ depends only on the oriented annihilator $f \in T^*_x\mathcal{S} \setminus \{0\}$ of $T_x\partial\mathcal{R}$, i.e., $f(v) > 0$, for each element $v \in T_x\mathcal{S}$ pointing "outwards". Thus, denoting by $T^*\mathcal{S}^+$ the bundle obtained from $T^*\mathcal{S}$ by the removal of the zero element on every fiber, we have a section

$$\tau \colon T^*\mathcal{S}^+ \to \bigwedge^{m-1} T^*\mathcal{S}^+,$$

giving $\tau_{\mathcal{R}}(x)$ when evaluated on a form f representing $T_x\partial\mathcal{R}$, and satisfying $\tau(f) = \tau(af)$ for every positive number a. Here, $\bigwedge^{m-1} T^*\mathcal{S}^+$ is the vector bundle over $T^*\mathcal{S}^+$ whose fiber over the form f at x is the vector space of $(m-1)$-forms on the oriented hyperplane ($(m-1)$-dimensional subspace of $T_x\mathcal{S}$) determined by f.

(iii) The section τ is smooth.

(*iv*) There is an m-differential form ς on $\mathcal{S}$, such that

$$I_{\mathcal{R}} = \int_{\mathcal{R}} \varsigma.$$

The form ς is interpreted as a source (production density) term for the property under consideration and it reflects the fact that the property is not conserved. In the sequel we will refer to this assumption as the *balance law.*

Let $\mathcal{I}_H \colon H \to T_x\mathcal{S}$ be the inclusion of an oriented subspace H. The basic result is

Proposition 2.4 (**Generalized Cauchy Theorem**)**.** There is a unique $(m-1)$-odd form σ on $\mathcal{S}$ such that at every point $x \in T_x\mathcal{S}$

$$\tau(f) = \mathcal{I}_H^*(\sigma),$$

for any oriented hyperplane $H \subset T_x\mathcal{S}$, where f represents H and $\mathcal{I}_H^*$ denotes the pull-back of forms induced by $\mathcal{I}_H$.

Remark 2.5. The fact that σ is odd is written traditionally (where a metric is available) as $t(-\mathbf{n}) = -t(\mathbf{n})$ and in our notation $\tau(-f) = -\tau(f)$.

Remark 2.6. In the sequel we will refer to σ as the *kinetic flux* field.

Using the kinetic flux field, the balance law may be rewritten in the form of a differential equation as follows.

Proposition 2.7 (**The differential version of the balance law**)**.** Given the kinetic flux field form σ, the balance law (Assumption 2.3(*iv*)), is equivalent to

$$d\sigma + \beta = \varsigma$$

in $\mathcal{S}$.

3. The Case of Volume Manifolds

It is now assumed that $\mathcal{S}$ is a volume manifold so, in addition to an orientation, a volume element ρ is given on $\mathcal{S}$. We recall that if ρ is represented locally as

$$r(x^i)\, dx^1 \wedge \ldots \wedge dx^m,$$

then, for a vector field v represented by its coordinates v^i, the contraction $v \lrcorner \rho$ is represented by

$$\sum_{i=1}^{m} (-1)^{i+1} r v^i \, dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^m.$$

Consider a local representation for the kinetic flux σ (for the orientation induced by ρ) in the form

$$\sum_{i=1}^{m} \sigma_i dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^m.$$

Thus we have

Proposition 3.1. Given a volume element ρ and a kinetic flux form σ, there is a unique vector field v, called the *kinematic flux*, satisfying

$$v \lrcorner \rho = \sigma.$$

With the notation introduced above, the local representation of v is given by

$$v^i = \frac{(-1)^{i+1}\sigma_i}{r}.$$

Remark 3.2. We note that the kinematic flux depends upon the choice of volume element only by the particular choice of the vector field in a one dimensional sub-bundle of $T\mathcal{S}$. Thus, σ determines a unique one dimensional sub-bundle of $T\mathcal{S}$. We will refer to this sub-bundle as the *flux bundle*.

Remark 3.3. Let $\{v_1, \dots, v_{m-1}\}$ be any collection of $m-1$ vectors in a hyperplane containing the fiber of the flux bundle at $x \in \mathcal{S}$. Then, since for any non-zero element v of the flux bundle, the collection $\{v, v_1, \dots, v_{m-1}\}$ contains m linearly dependent vectors in $T\mathcal{S}$,

$$\begin{aligned}\sigma(v_1, \dots, v_{m-1}) &= v \lrcorner \rho(v_1, \dots, v_{m-1}) \\ &= \rho(v, v_1, \dots, v_{m-1}) \\ &= 0.\end{aligned}$$

We conclude that the flux density through any hyperplane containing the flux bundle vanishes.

We note that for a given volume element ρ, the differential version of the balance equation $d\sigma + \beta = \varsigma$ may also be written in terms of the kinematic flux vector field. We have

$$\begin{aligned}d\sigma &= d(v \lrcorner \rho) \\ &= \mathcal{L}_v\rho - v \lrcorner d\rho \\ &= \mathcal{L}_v\rho.\end{aligned}$$

Since $\mathcal{L}_v\rho$ is an m-form there is unique real valued function $\operatorname{div} v$, the divergence of v (see [1] p. 455), defined on $\mathcal{S}$, satisfying

$$\mathcal{L}_v\rho = \rho \operatorname{div} v.$$

Thus, using b and s to denote the unique functions such that $\beta = b\rho$ and $\varsigma = s\rho$, respectively, we have

Proposition 3.4. Given a volume element ρ on $\mathcal{S}$, the differential balance law may be written in terms of the kinematic flux in one of the following equivalent forms

(i) $$\mathcal{L}_v\rho + \beta = \varsigma,$$

(ii) $$\operatorname{div} v + b = s.$$

4. Material Elements and Bodies

The existence of a manifold B containing as elements the identifiable material points, is the standard assumption of continuum mechanics allowing the definition of a configuration of a body as an embedding

$$\kappa \colon B \to \mathcal{S}.$$

With the previous observations it is possible now to define material elements and bodies as derived notions. We note first that all the considerations of the previous sections hold for any particular time t. (The frame where the variables are defined has been kept fixed so far.) We assume now that the forms β, ς, ρ, and section τ vary smoothly with time. As a result, the kinematic flux $v(x,t)$ varies smoothly with time and as such, it determines a time dependent differential equation in space.

For $(x,t_0) \in \mathcal{S} \times \mathcal{T}$, we will use $X_{x,t_0}(t)$ to denote the value at the time t of the the integral curve passing through x at time t_0.

Definition 4.1. An integral line of v is a *body element.* The collection of all body elements is the *universal body* $\mathcal{B}$.

We note that by the theory of differential equations, for any instant $t_0 \in \mathcal{T}$ there is a neighborhood $U \subset \mathcal{T}$, $t_0 \in U$, where the flow of v

$$\phi^{t_0} \colon \mathcal{S} \times U \to \pi^{-1}(U), \quad \phi^{t_0}(x,t) = \big(X_{x,t_0}(t), t\big),$$

is a fiber bundle diffeomorphism. For the sake of simplicity we assume that ϕ^0 is a fiber bundle diffeomorphism for which $U = \mathcal{T}$. Thus, the integral lines can be parametrized by the initial conditions at $t_0 = 0$. As a result, we may identify a body element X with its unique initial condition in $\mathcal{S}$. Similarly, the universal body is diffeomorphic to $\mathcal{S}$. Traditionally, the identification of body elements with the respective initial conditions is referred to as *reference configuration.* For this reason, in the sequel we will often omit the t_0 index and use $\phi_t \colon \mathcal{S} \to \mathcal{S}$ to denote the mapping such that $\phi_t(x) = X_{x,0}(t)$.

These enable us to make the following definition.

Definition 4.2. A *body* B is a compact m-dimensional submanifold with corners of $\mathcal{B}$. A *configuration* of a body B is an embedding $B \to \mathcal{S}$. A *motion* of a body B is a fiber bundle morphism $B \times U \to \pi^{-1}(U)$, where U is open in $\mathbb{R}$, whose restriction to $B \times \{t\}$ is a configuration for all $t \in U$.

Clearly, the restrictions of ϕ_t to bodies are configurations of these bodies.

5. Time Dependent Volume Elements and Material Frames

For the fixed frame F on space-time, consider a smoothly time dependent volume element $\rho(t)$ on $\mathcal{S}$. The flow ϕ of the kinematic flux v induces another smoothly time dependent volume element

$$\rho_0(t) = \phi_t^*(\rho(t)).$$

We will refer to ρ_0 as the *reference description of the volume element* ρ. Given any body B, the definition of ρ_0 implies that

$$\int_B \rho_0(t) = \int_{\phi_t(B)} \rho(t),$$

for any instant t.

Since the integral on the left is over a fixed region, we have

$$\frac{d}{dt}\int_B \rho_0(t) = \int_B \frac{\partial \rho_0}{\partial t}(t)$$

On the other hand, a generalized version of the transport theorem (see [1] p. 471) implies that

$$\frac{d}{dt}\int_{\phi_t(B)} \rho(t) = \int_{\phi_t(B)} \left(\frac{\partial \rho}{\partial t} + \mathcal{L}_v \rho\right).$$

Thus we arrive at the following

Proposition 5.1. The reference description ρ_0 of the time dependent volume element ρ satisfies the following equivalent equations

(*i*) $$\phi_t^{-1*}\left(\frac{\partial \rho_0}{\partial t}\right) = \frac{\partial \rho}{\partial t} + \mathcal{L}_v \rho,$$

(*ii*) $$\phi_t^{-1*}\left(\frac{\partial \rho_0}{\partial t}\right) = \frac{\partial \rho}{\partial t} + \rho \operatorname{div} v,$$

(*iii*) $$\phi_t^{-1*}\left(\frac{\partial \rho_0}{\partial t}\right) = \frac{\partial \rho}{\partial t} + d\sigma.$$

So far, the interpretation of the form β as a rate of change of the density of a certain physical extensive property, as mentioned in its introduction, was not considered any further. In addition, no physical motivation was given to the introduction of the volume element ρ. The two can be combined now by assuming that ρ is the density whose rate of change is β so

$$\beta = \frac{\partial \rho}{\partial t}.$$

By comparing Proposition 5.1 with the differential version of the balance law (Proposition 2.7) it follows that

$$\varsigma = \phi_t^{-1*}\left(\frac{\partial \rho_0}{\partial t}\right),$$

and we may write for the volumetric growth of the property

$$I_{\phi_t(B)} = \int\limits_{\phi_t(B)} \beta + \int\limits_{\partial\phi_t(B)} \tau = \int\limits_{\phi_t(B)} \phi_t^{-1*}\left(\frac{\partial\rho_0}{\partial t}\right)$$
$$= \int\limits_{B} \frac{\partial\rho_0}{\partial t}.$$

The last equation is interpreted as follows. In the expression for the set function I in terms of the forms β and τ, the boundary term indicates interaction. One could ask whether there is a frame where no interaction term appears. We note that the inverse of the flow $\phi\colon \mathcal{S} \times \mathbb{R} \to \mathcal{E}$ is a global frame where B and $\phi_t(B)$ represent the same control volume and where the expression for the volumetric growth has the required property. In other words, the kinematic flux, whose existence is a result of of Cauchy's postulates, induces a canonical frame, the *material frame* where growth has a particular simple form—no interaction term is present and the source density is identical to the rate of change of the property's density.

6. Variance Properties

So far we restricted ourselves to one frame $F\colon \mathcal{S}\times\mathcal{T} \to \mathcal{E}$ and constructed the material frame for it. We did not consider the variance properties of the forms β, τ, and ς. This section discusses these issues.

Assume that in addition to the frame F we are given a frame F'. Thus,

$$F' \circ F^{-1}\colon \mathcal{S} \times \mathcal{T} \to \mathcal{S} \times \mathcal{T}$$

is a fiber bundle morphism whose base mapping $\mathcal{T} \to \mathcal{T}$ is the identity. Let

$$\psi\colon \mathcal{S} \times \mathcal{T} \to \mathcal{S},$$

be the first component of $F' \circ F^{-1}$ and denote its restriction to $\mathcal{S} \times \{t\}$ by ψ_t. The identification of $\mathcal{T}$ with $\mathbb{R}$ implies that we can regard $F' \circ F^{-1}$ as a flow whose generating vector field is $(u, 1)$, where $u = T_2\psi$ is the second partial derivative of ψ. Clearly, ψ is a flow on $\mathcal{S}$ whose generating time dependent vector field is u.

Given a control volume $\mathcal{R} \subset \mathcal{S}$ that is fixed in the frame F and an m-form ω' on $\mathcal{S}$, setting $\omega = \psi_t^*(\omega')$, we have

$$\int\limits_{\mathcal{R}} \omega = \int\limits_{\psi_t(\mathcal{R})} \omega'.$$

This transformation rule applies to the density ρ and to the production rate ς. However, although β is an m-form, being the time derivative of ρ, it transforms differently. Using the generalized transport theorem and Stokes' theorem, one has

$$
\begin{aligned}
\frac{d}{dt}\int\limits_{\psi_t(\mathcal{R})}\rho' &= \int\limits_{\psi_t(\mathcal{R})}\left(\frac{\partial\rho'}{\partial t}+\mathcal{L}_u\rho\right)\\
&= \int\limits_{\psi_t(\mathcal{R})}\left(\frac{\partial\rho'}{\partial t}+d(u\lrcorner\,\rho)\right)\\
&= \int\limits_{\psi_t(\mathcal{R})}\frac{\partial\rho'}{\partial t}+\int\limits_{\partial\psi_t(\mathcal{R})}\mathcal{I}^*(u\lrcorner\,\rho).
\end{aligned}
$$

On the other hand, since $\mathcal{R}$ is fixed in the frame F,

$$
\begin{aligned}
\frac{d}{dt}\int\limits_{\psi_t(\mathcal{R})}\rho' = \frac{d}{dt}\int\limits_{\mathcal{R}}\rho &= \int\limits_{\mathcal{R}}\frac{\partial\rho}{\partial t}\\
&= \int\limits_{\mathcal{R}}\beta.
\end{aligned}
$$

Hence,

$$
I_{\mathcal{R}} = \int\limits_{\mathcal{R}}\beta+\int\limits_{\partial\mathcal{R}}\tau = \int\limits_{\psi_t(\mathcal{R})}\frac{\partial\rho'}{\partial t}+\int\limits_{\partial\psi_t(\mathcal{R})}\left(\mathcal{I}^*(u\lrcorner\,\rho')+\psi^{-1*}(\tau)\right).
$$

It is assumed now that $I_{\mathcal{R}}$ is frame invariant (frame indifferent in the continuum mechanics terminology), i.e., that $I_{\mathcal{R}} = I_{\psi_t(\mathcal{R})}$. In addition, the expression of I for the frame F' should be the same as that for F; so there is an m-form β' on $\psi_t(\mathcal{R})$ and an $(m-1)$-form τ' on $\partial\psi_t(\mathcal{R})$, such that

$$
I_{\psi_t(\mathcal{R})} = I_{\mathcal{R}} = \int\limits_{\psi_t(\mathcal{R})}\beta'+\int\limits_{\partial\psi_t(\mathcal{R})}\tau'.
$$

It follows that

$$
\begin{aligned}
\beta' &= \frac{\partial\rho'}{\partial t} = \psi_t^{-1*}(\beta),\\
\tau' &= \mathcal{I}^*(u\lrcorner\,\rho')+\psi_t^{-1*}(\tau).
\end{aligned}
$$

It is apparent from the last equation that one cannot treat β and τ as independent frame invariant objects.

One concludes that if u satisfies the equation

$$
\mathcal{I}^*(u\lrcorner\,\rho')+\psi_t^{-1*}(\tau) = 0,
$$

then, u is a vector field generating a frame for which there is no boundary interaction. The existence of one vector field that satisfies this equation for all control volumes is a result of Cauchy's theorem.

7. The Variational Version of the Balance Law

Consider a fixed frame F on $\mathcal{E}$. Multiplying the differential version of the balance law (see Proposition 2.7) $d\sigma + \beta = \varsigma$ by a differentiable function $w\colon \mathcal{S} \to \mathbb{R}$ and integrating over a typical control volume $\mathcal{R}$, we obtain

$$\int_{\mathcal{R}} w d\sigma + \int_{\mathcal{R}} w\beta = \int_{\mathcal{R}} w\varsigma.$$

Since,

$$\begin{aligned}\int_{\mathcal{R}} w d\sigma &= \int_{\mathcal{R}} d(w\sigma) - \int_{\mathcal{R}} dw \wedge \sigma \\ &= \int_{\partial\mathcal{R}} \mathcal{I}^*(w\sigma) - \int_{\mathcal{R}} dw \wedge \sigma,\end{aligned}$$

we have using $\mathcal{I}^*(w\sigma) = w\mathcal{I}^*(\sigma) = w\tau$,

$$\int_{\mathcal{R}} w\beta + \int_{\partial\mathcal{R}} w\tau = \int_{\mathcal{R}} w\varsigma + \int_{\mathcal{R}} dw \wedge \sigma,$$

to which we will refer as the *variational form of the balance law*. The variational form of the balance law regards $I_{\mathcal{R}}$ as a linear functional, the *growth functional*, on the space of differentiable functions on $\mathcal{S}$, that is continuous with respect to the C^1 topology. Thus, we will use $I_{\mathcal{R}}(w)$ to denote either side of the last equation.

Assume that a volume element ρ is given so there is a kinematic flux field v, with $\sigma = v \lrcorner \rho$. Then,

$$0 = v \lrcorner (dw \wedge \rho) = (v \lrcorner dw) \wedge \rho - dw \wedge (v \lrcorner \rho),$$

and one concludes that in this case

$$dw \wedge \sigma = dw \wedge (v \lrcorner \rho) = dw(v)\rho.$$

Thus, for the case of volume manifolds, the growth functional may be represented in the form

$$I_{\mathcal{R}}(w) = \int_{\mathcal{R}} w\varsigma + \int_{\mathcal{R}} dw(v)\rho.$$

The significance of the variational version of the balance law is that it allows a frame invariant characterization of growth. Consider the frames F, F' with transformation mapping ψ_t, and a control volume $\mathcal{R} \subset \mathcal{S}$ at time t under F whose image under F' is $\mathcal{R}' = \psi_t(\mathcal{R})$. Then, the requirement that the evaluation of the growth functional is frame invariant, i.e., $I_{\mathcal{R}}(w) = I_{\mathcal{R}'}(w')$, $w' = \psi_t^{-1*}(w)$, implies

$$\int_{\mathcal{R}} w\varsigma + \int_{\mathcal{R}} dw \wedge \sigma = \int_{\mathcal{R}'} w'\varsigma' + \int_{\mathcal{R}'} dw' \wedge \sigma'.$$

As this should hold for an arbitrary control volume and arbitrary test function w, we conclude that ς and σ indeed transform according to the rules $\varsigma' = \psi_t^{-1*}(\varsigma)$ and $\sigma' = \psi_t^{-1*}(\sigma)$. Thus, for a given control volume in $\mathcal{E}_t = \pi^{-1}(t)$, all the quantities in the last equation are defined independently of the choice of frame. Hence we arrive at the following frame indifferent characterization of growth that includes Cauchy's postulates.

Definition 7.1. A volumetric growth at time t consists of an m-form ς and an $(m-1)$-form σ on $\mathcal{E}_t$ that represent, for each control volume $\mathcal{R} \subset \mathcal{E}_t$, a growth functional

$$I_{\mathcal{R}}(w) = \int\limits_{\mathcal{R}} w\varsigma + \int\limits_{\mathcal{R}} dw \wedge \sigma$$

on the space of differentiable real valued functions $C^1(\mathcal{R})$.

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References

[1] R. Abraham, J.E. Marsden, & R. Ratiu, *Manifolds, Tensor Analysis, and Applications*, Springer, New York, 1988.

[2] G. Capriz & P. Trebeschi, Reflections upon the axioms of continuum mechanics prompted by the study of complex materials, *Bull. Tech. Univ. Istanbul* **47** (1994) 1–12.

[3] M. Epstein & R. Segev, Differentiable manifolds and the principle of virtual work in continuum mechanics, *Journal of Mathematical Physics* **21** (1980) 1243–1245.

[4] M.E. Gurtin & W.O. Williams, An axiomatic foundation for continuum thermodynamics, *Archive for Rational Mechanics and Analysis* **26** (1967) 83–117.

[5] J.E. Marsden & T.J.R. Hughes, *Mathematical Foundations of Elasticity*, Prentice-Hall, Englewood Cliffs, NJ, 1983.

[6] R. Segev, Forces and the existence of stresses in invariant continuum mechanics, *Journal of Mathematical Physics* **27** (1986) 163–170.

[7] R. Segev, On smoothly growing bodies and the Eshelby Tensor, *Meccanica* **31** (1996) 507–518.

[8] R. Segev, On symmetrically growing bodies, *Extracta Mathematicae* **12** (1997) 261–272.

[9] R. Segev, On the geometry of Cauchy's theory for balance of scalar valued properties, submitted for publication.

[10] R. Segev & G. de Botton, On the consistency conditions for force systems, *International Journal of Nonlinear Mechanics* **26** (1991) 47–59.

[11] R. Segev & M. Epstein, On theories of growing bodies, in *Contemporary Research in the Mechanics and Mathematics of Materials* R.C. Batra & M.F. Beatty editors, CIMNE, Barcelona, 1996, 119–130.

[12] R. Segev & E. Fried, Kinematics of and forces on nonmaterial interfaces, *Mathematical Models and Methods in Applied Sciences* **6** (1995) 739–753.

[13] R. Segev & G. Rodnay, Continuum kinematics of volumetric growth, *Proceedings of the Continuous Models and Discrete Systems 9*, Istanbul, July 1998, E. Inan & K.Z. Markov eds., World Scientific, to appear.

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