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YET AGAIN ON TWO EXAMPLES BY IYAMA AND YOSHINO

DANIELE FAENZI

ABSTRACT. We give an elementary proof of Iyama-Yoshino’s classification of rigid MCM modules on Veronese embeddings in \( \mathbb{P}^9 \).

INTRODUCTION

The beautiful theory of cluster tilting in triangulated categories has been developed by Iyama and Yoshino; as an important outcome of this the authors gave in \([10, \text{Theorem 1.2 and Theorem 1.3}]\) the classification of rigid indecomposable MCM modules over two Veronese embeddings in \( \mathbb{P}^9 \), given, respectively, by plane cubics and space quadrics. Another proof, that makes use of Orlov’s singularity category, appears in \([12]\), where the link between power series Veronese rings and the graded rings of the corresponding varieties is also explained. Also, \([13]\) contains yet another argument.

The goal of this note is to present a simple proof of Iyama-Yoshino’s classification of rigid MCM modules over the aforementioned Veronese rings, making use of vector bundles and Beilinson’s theorem. This proof works over a field \( k \) which is algebraically closed or finite.

Consider the embedding of the projective space \( \mathbb{P}^n \), with \( n \geq 2 \) given by homogeneous forms of degree \( d \), i.e., the \( d \)-fold Veronese variety. A coherent sheaf \( E \) on \( \mathbb{P}^n \) is arithmetically Cohen-Macaulay (ACM) with respect to this embedding if and only if \( E \) is locally free and has no intermediate cohomology:

\[
H^i(\mathbb{P}^n, E(dt)) = 0, \quad \text{for all } t \in \mathbb{Z} \text{ and all } 0 < i < n.
\]

This is equivalent to ask that the module of global sections associated with \( E \) is MCM over the corresponding Veronese ring. For \( d \)-fold Veronese embeddings of \( \mathbb{P}^n \) in \( \mathbb{P}^9 \) (i.e., \( \{n, d\} = \{2, 3\} \)), we are going to classify ACM bundles \( E \) which are rigid, i.e., \( \text{Ext}_{\mathbb{P}^n}^1(E, E) = 0 \). We set \( \ell = \binom{n+1}{2} \).

To state the classification, we define the Fibonacci numbers \( a_{\ell, k} \) by the relations:

\[
a_{\ell, 0} = 0, \quad a_{\ell, 1} = 1 \quad \text{and} \quad a_{\ell, k+1} = \ell a_{\ell, k} - a_{\ell, k-1}.
\]

For instance, \( (a_{3, k}) \) is given by the odd values of the usual Fibonacci sequence:

\[
a_{3, k} = 0, 1, 3, 8, 21, 55, 144, \ldots \quad \text{for } k = 0, 1, 2, 3, 4, 5, 6, \ldots
\]

**Theorem 1.** Assume \( \{n, d\} = \{2, 3\} \), let \( E \) be an indecomposable bundle on \( \mathbb{P}^n \) satisfying condition \((1)\), namely \( H^i(\mathbb{P}^n, E(dt)) = 0 \) for all \( t \in \mathbb{Z} \) and all \( 0 < i < n \).

\( i) \) If \( E \) has no non-trivial endomorphism factoring through \( \mathcal{O}_{\mathbb{P}^n}(t) \), then there are \( a, b \geq 0 \) such that, up to a twist by \( \mathcal{O}_{\mathbb{P}^n}(s) \), \( E \) or \( E^\ast \) is the cokernel of an injective map:

\[
\Omega_{\mathbb{P}^n}^2(1)^b \to \mathcal{O}_{\mathbb{P}^n}(-1)^a
\]
ii) If $E$ is rigid, then there is $k \geq 1$ such that, up to tensoring with $\mathcal{O}_{\mathbb{P}^n}(s)$, $E$ or $E^*$ is the cokernel of an injective map:
$$\Omega^2_{\mathbb{P}^n}(1)^{a_{\ell,k-1}} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{a_{\ell,k}}$$
Conversely for any $k \geq 1$, there is a unique indecomposable bundle $E_k$ fitting into:
$$0 \to \Omega^2_{\mathbb{P}^n}(1)^{a_{\ell,k-1}} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{a_{\ell,k}} \to E_k \to 0.$$ 
Finally, both $E_k$ and $E_k^*$ are ACM and exceptional.

In the previous statement, it is understood that a bundle $E$ is exceptional if it is rigid, simple (i.e. $\text{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(E,E) \simeq k$) and $\text{Ext}^i_{\mathcal{O}_{\mathbb{P}^n}}(E,E) = 0$ for $i \geq 2$. We write $\Omega^p_{\mathbb{P}^n} = \wedge^p \Omega^1_{\mathbb{P}^n}$ for the bundle of differential $p$-forms on $\mathbb{P}^n$.

**Remark 2.** Part [3] of Theorem [1] is a version of Iyama-Yoshino’s general results on Veronese rings [10, Theorem 9.1 and 9.3], to the effect that for any $E$ and $\mathcal{O}_{\mathbb{P}^n}(s)$, there is a unique indecomposable bundle $E_k$ fitting into:
$$0 \to \Omega^2_{\mathbb{P}^n}(1)^{a_{\ell,k-1}} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{a_{\ell,k}} \to E_k \to 0.$$ 
Finally, both $E_k$ and $E_k^*$ are ACM and exceptional.

**Remark 3.** The rank of the bundle $E_k$ is given by the Fibonacci number between $a_{3,k-1}$ and $a_{3,k}$ in case $(n,d) = (2,3)$. In this case $E_{2k}$ (respectively, $E_{2k+1}$) is the $k$-th sheaffied syzygy occurring in the resolution of $\mathcal{O}_{\mathbb{P}^2}(1)$ (respectively, of $\mathcal{O}_{\mathbb{P}^2}(2)$) over the Veronese ring, twisted by $\mathcal{O}_{\mathbb{P}^2}(3(k-1))$. A similar result holds for $(n,d) = (3,2)$.

As for notation, we write small letters for the dimension of a space in capital letter, for instance $h^i(\mathbb{P}^n,E) = \dim_k H^i(\mathbb{P}^n,E)$. We also write $\chi(E,F) = \sum (-1)^i \text{ext}^i_{\mathcal{O}_{\mathbb{P}^n}}(E,F)$ and $\chi(E) = \chi(\mathcal{O}_{\mathbb{P}^n}, E)$. The symbol $\delta_{i,j}$ is Kronecker’s delta.

1. **Fibonacci bundles**

1.1. **Representations of the Kronecker quiver.** Fix an integer $\ell \geq 1$. Let us write $\Upsilon_\ell$ for the $\ell$-th Kronecker quiver, namely the oriented graph with two vertices $e_0$ and $e_1$, and $\ell$ arrows from $e_0$ to $e_1$. A representation $R$ of $\Upsilon_\ell$, with dimension vector $(a,b)$ is the choice of $\ell$ matrices of size $a \times b$.

$$\Upsilon_3:$$

We identify a basis of $H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}(2))$ with the set of $\ell = \binom{n+1}{2}$ arrows of $\Upsilon_\ell$. Then the derived category of finite-dimensional representations of $\Upsilon_\ell$ embeds into the derived category of $\mathcal{O}_{\mathbb{P}^n}$-modules by sending $R$ to the cone $\Phi(R)$ of the morphism $\epsilon_R$ associated with $R$ according to this identification:
$$\Phi(R)[-1] \to \mathcal{O}_{\mathbb{P}^n}(-1)^a \xrightarrow{\epsilon_R} \Omega_{\mathbb{P}^n}(1)^b,$$
where we denote by $[-1]$ the shift to the right of complexes. It is clear that for all pairs of representations $R, R'$ we have:
$$\text{Ext}^i_{\Upsilon_\ell}(\Phi(R), \Phi(R')) \simeq \text{Ext}^i_{\Upsilon_\ell}(R, R'),$$
for all $i$. 

(3)
1.2. Rigid representations and Schur roots. We will use Kac’s classification of rigid \( \mathcal{T}_\ell \)-modules as Schur roots (hence the restriction on \( k \)), which is also one of the main ingredients in Iyama-Yoshino’s proof. By [11] Theorem 4, any non-zero rigid \( \mathcal{T}_\ell \)-module is a direct sum of rigid simple representations of the form \( R_k \), for some \( k \in \mathbb{Z} \), where \( R_k \) is defined as the unique (up to isomorphism) indecomposable representation of \( \mathcal{Y}_\ell \) with dimension vector \((a_{\ell,k}, a_{\ell,-k})\) for \( k \geq 1 \), or \((a_{\ell,1-k}, a_{\ell,-k})\) for \( k \leq 0 \).

1.3. Fibonacci bundles and their cohomology. Set \( F_k = \Phi(R_k) \) for \( k \geq 1 \), and \( F_k = \Phi(R_k)[-1] \) for \( k \leq 0 \). It turns out that \( F_k \) is an exceptional locally free sheaf, called a Fibonacci bundle, cf. [4]. We rewrite the defining exact sequences of \( F_k \):

\[
0 \to \mathcal{O}_{\mathbb{P}^n}(-1)^{a_{\ell,k}-1} \to \Omega_{\mathbb{P}^n}(1)^{a_{\ell,k}} \to F_k \to 0,
\]
for \( k \geq 1 \),

\[
0 \to F_k \to \mathcal{O}_{\mathbb{P}^n}(-1)^{a_{\ell,-1-k}} \to \Omega_{\mathbb{P}^n}(1)^{a_{\ell,-k}} \to 0,
\]
for \( k \leq 0 \).

Here is a lemma on the intermediate cohomology of Fibonacci bundles. Its statement should be read with a grain of salt, namely for \( k = 1 \) also \( H^{n-1}(\mathbb{P}^n, F_1(-n)) \) vanishes.

**Lemma 4.** For \( k \geq 1 \), the only non-vanishing intermediate cohomology of \( F_k \) is:

\[
h^1(\mathbb{P}^n, F_k(-1)) = a_{\ell,k}, \quad h^{n-1}(\mathbb{P}^n, F_k(-n)) = a_{\ell,k-1}.
\]

**Proof.** All the terminology and results we need on exceptional collections in order to establish this lemma are contained in [2,3]. We consider the left and right mutation endofunctors of the derived category of coherent sheaves on \( \mathbb{P}^n \). These associate with a pair \((E,F)\) of complexes two complexes, denoted respectively by \( R_F E \) and \( L_E F \), defined as the cones of the natural evaluation maps \( f_{E,F} \) and \( g_{E,F} \):

\[
E \xrightarrow{f_{E,F}} R \operatorname{Hom}_{\mathbb{P}^n}(E,F)^* \otimes F \to R_F E, \quad L_E F \to R \operatorname{Hom}_{\mathbb{P}^n}(E,F) \otimes E \xrightarrow{g_{E,F}} F.
\]

It is well-known (cf. [4]) that the Fibonacci bundles \( F_k \) can be defined recursively from \( F_0 = \mathcal{O}_{\mathbb{P}^n}(-1) \) and \( F_1 = \Omega_{\mathbb{P}^n}(1) \) by setting:

\[
F_{k+1} = R_{F_k} F_{k-1}, \quad \text{for } k \geq 1,
\]

\[
F_{k-1} = L_{F_k} F_{k+1}, \quad \text{for } k \leq 0.
\]

This way, for any \( k \in \mathbb{Z} \) we get a natural exact sequence:

\[
0 \to F_{k-1} \to (F_k)^\ell \to F_{k+1} \to 0.
\]

Over \( \mathbb{P}^n \), we start with the standard collection:

\[
(\mathcal{O}_{\mathbb{P}^n}(1-n), \ldots, \mathcal{O}_{\mathbb{P}^n}(-1), \mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1)).
\]

By the mutation \( \mathcal{O}_{\mathbb{P}^n}(-1) \simeq L_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n}(1) \), we replace this with:

\[
(\mathcal{O}_{\mathbb{P}^n}(1-n), \ldots, \mathcal{O}_{\mathbb{P}^n}(-1), \mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}),
\]

By [5], we can replace the previous exceptional sequence with:

\[
(\mathcal{O}_{\mathbb{P}^n}(1-n), \ldots, \mathcal{O}_{\mathbb{P}^n}(-1), F_{k-1}, F_k).
\]

By [3] Theorem 4.1], the iterated mutation of \( F_k \) through this last full exceptional collection must give back \( F_k \otimes \omega_{\mathbb{P}^n} \), i.e.:

\[
L_{\mathcal{O}_{\mathbb{P}^n}(1-n)} \cdots L_{\mathcal{O}_{\mathbb{P}^n}(-1)} L_{F_{k-1}} F_k \simeq F_k(-n-1).
\]
Also, since the first sequence we started with is strongly exceptional of vector bundles, and the same happens to its Koszul dual collection, in view of [3 Theorem 8.3] we know that all objects obtained by the mutations in the previous display consist of vector bundles. This provides us with integers \( u_1, \ldots, u_n \) and with a long exact sequence:

\[
0 \to F_k(-n-1) \to \mathcal{O}_{\mathbb{P}^n}(1-n)^{u_1} \to \cdots \to \mathcal{O}_{\mathbb{P}^n}(-1)^{u_{n-1}} \to F_k^{u_n} \to 0,
\]

where each short exact sequence extracted from the long one is given by one mutation in the sequence of mutations [3]. Now by (4) we get:

\[
\tau_{\mathbb{P}^n}(\mathcal{O}_{\mathbb{P}^n}(1-n)^{u_1} \to \cdots \to \mathcal{O}_{\mathbb{P}^n}(-1)^{u_{n-1}} \to F_k^{u_n}) = 0.
\]

Indeed, the kernel \( K \) of the map \( F_k^{u_n} \to \mathcal{O}_{\mathbb{P}^n}(1-n)^{u_1} \to \cdots \to \mathcal{O}_{\mathbb{P}^n}(-1)^{u_{n-1}} \to F_k^{u_n} \to 0 \) is \( I_{F_k^{u_n}} F_k \), so that \( u_n = \ell \) and \( K \simeq F_{k-2} \) in view of (7). Then, using \( \tau_{\mathbb{P}^n}(F_{k-2}) = 0 \) for \( t \leq 0 \) (which of course holds also for \( k = 1, 2 \)) and chasing cohomology in (7) we see that \( H^{n-1}(\mathbb{P}^n, F_k(t)) = 0 \) for \( t \leq -n - 1 \).

1.4. Ext groups among Fibonacci bundles. We would like to compute now the Ext groups between pairs of Fibonacci bundles. Of course when \( F_k \) and \( F_j \) are “close”, i.e. \( |k-j| \leq 1 \), we already know what happens, as the two Fibonacci bundles then form an exceptional pair. But what if \( |k-j| \geq 2 \)? The next lemma gives the answer. Note the second formula holds also for \( k = j \) since \( F_k \) is exceptional.

**Lemma 5.** For any pair of integers \( j \geq k + 1 \) we have:

\[
\text{ext}^i_{\mathbb{P}^n}(F_j, F_k) = \delta_{1, i} a_{\ell, j-k-1}, \quad \text{ext}^i_{\mathbb{P}^n}(F_k, F_j) = \delta_{0, i} a_{\ell, j-k+1}.
\]

**Proof.** Set \( \Upsilon = \Upsilon_j \) and \( a_{\ell, j} = a_{j} \). We easily compute \( \chi(F_j, F_k) = -a_{j-k-1} \) and \( \chi(F_k, F_j) = a_{j-k+1} \) by calculating \( \chi \) of \( \Upsilon \)-modules via the Cartan form and using faithfulness of \( \Phi \).

Therefore, the second formula is proved once we show \( \text{Ext}^i_{\mathbb{P}^n}(F_k, F_j) = 0 \) for \( i \neq 0 \), and this is of course true for \( i < 0 \). For \( k \leq 0 \) and \( j \geq 1 \) we have \( F_k \simeq \Phi(R_k)[-1] \) and thus \( \text{Ext}^i_{\mathbb{P}^n}(F_k, F_j) \simeq \text{Ext}^{i+1}_{\Upsilon}(R_k, R_j) \), which is zero for \( i \neq -1, 0 \) since the category of \( \Upsilon \)-representations is hereditary. Further, if \( j, k \geq 1 \) or \( j, k \leq 0 \) then it suffices to prove \( \text{Ext}^1_{\mathbb{P}^n}(F_k, F_j) = 0 \). Using (7), this vanishing holds for \( j \) if it does for \( j - 1 \). By induction, it suffices to check \( \text{Ext}^1_{\mathbb{P}^n}(F_k, F_k) = 0 \), which in turn is obvious.

Let us now look at the first formula. To prove it, we consider the Auslander-Reiten translate \( \tau \); we refer for instance to [11 for the definition and properties of \( \tau \). This satisfies:

\[
\text{ext}^1_{\Upsilon}(R_j, \tau R_{k+2}) = \text{ext}^1_{\Upsilon}(R_{k+2}, R_j).
\]

The functor \( \tau \) operates on dimension vectors via the Coxeter transform, encoded by the matrix:

\[
\begin{pmatrix}
\ell^2 - 1 & \ell \\
-\ell & -1
\end{pmatrix}.
\]
This means that, if $R$ has dimension vector $(a, b)$, then $\tau R$ has dimension vector:

$$((\ell^2 - 1)a - \ell b, |\ell a - b|).$$

A straightforward computation involving the recursive definition of $a_k$ now says that $(a_{k+1}, a_{k+2})$ is sent to $(a_{k-1}, a_k)$. This in turn implies that $\tau R_{k+2} \simeq R_k$. Therefore, the second part of (10) follows from the first part, by combining it with (3) and (11). □

**Remark 6.** This lemma holds more generally (with the same proof) for any exceptional pair $(F_0, F_1)$ of objects on a projective $k$-variety $X$, with $\text{hom}_X(F_0, F_1) = \ell$, by defining recursively $F_k$ for all $k \in \mathbb{Z}$ by (5) and (6).

2. **Rigid ACM bundles on the third Veronese surface**

We prove here Theorem 1 in case $(n, d) = (2, 3)$.

2.1. **The Beilinson complex and the proof of (i).** Let us first prove (i). So let $E$ be an indecomposable vector bundle on $\mathbb{P}^2$ satisfying (1). Without loss of generality, we may replace $E$ with $G = E(s_0)$, where $s_0$ is the smallest integer $s$ such that $h^0(\mathbb{P}^2, E(s)) \neq 0$. Define the integers:

$$\alpha_{i,j} = h^i(\mathbb{P}^2, G(-j)).$$

Since we defined $G = E(s_0)$, we have $\alpha_{0,j} = 0$ if and only if $j \geq 1$. The Beilinson complex $F$ associated with $G$ (see for instance [B] Chapter 8) reads:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^{\alpha_{2,2}} \oplus \mathcal{O}_{\mathbb{P}^2}(1)^{\alpha_{1,1}} \oplus \mathcal{O}_{\mathbb{P}^2}(\alpha_{0,0}) \rightarrow \mathcal{O}_{\mathbb{P}^2} \rightarrow 0.$$

The term consisting of three summands in the above complex sits in degree 0 (we call it the middle term), and the cohomology of this complex is $G$. By condition (1), at least one of the $\alpha_{1,j}$ is zero, for $j = 0, 1, 2$.

If $\alpha_{1,2} = 0$, then $d_0 = 0$. By minimality of the Beilinson complex the restriction of $d_1$ to the summand $\mathcal{O}_{\mathbb{P}^2}(\alpha_{0,0})$, of the middle term is also zero. Therefore $\mathcal{O}_{\mathbb{P}^2}(\alpha_{0,0})$ is a direct summand of $G$, so $G \simeq \mathcal{O}_{\mathbb{P}^2}$ by indecomposability of $E$ (and hence of $G$).

If $\alpha_{1,1} = 0$, then the non-zero component of $d_0$ is just a map $\mathcal{O}_{\mathbb{P}^2}(-1)^{\alpha_{1,2}} \rightarrow \mathcal{O}_{\mathbb{P}^2}(\alpha_{0,0})$, and a direct summand of $G$ is the cokernel of this map. By indecomposability of $G$, in this case $G(-1)$ has a resolution of the desired form with $a = \alpha_{0,0}$ and $b = \alpha_{1,2}$.

So in the cases $\alpha_{1,2} = 0$ or $\alpha_{1,1} = 0$, the bundle $E$ has the required resolution up to twist. In the case $\alpha_{1,0} = 0$, we shall see that this holds for $E^*$. We thus call this case the “dual” one.

2.2. **The dual case and the end of the proof of (i).** It remains to look at the case $\alpha_{1,0} = 0$, actually $\alpha_{1,3k} = 0$ for all integers $k$. Note that the restriction of $d_1$ to $\mathcal{O}_{\mathbb{P}^2}(1)^{\alpha_{1,1}} \oplus \mathcal{O}_{\mathbb{P}^2}(\alpha_{0,0})$ is zero, which implies that a direct summand of $G$ (hence all of $G$ by indecomposability) has the resolution:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^{\alpha_{1,2}} \rightarrow \mathcal{O}_{\mathbb{P}^2}(1)^{\alpha_{1,1}} \oplus \mathcal{O}_{\mathbb{P}^2}(\alpha_{0,0}) \rightarrow G \rightarrow 0$$

(12)
and \( \alpha_{2,j} = 0 \) for \( j = 0,1,2 \). Recall also \( \alpha_{1,0} = 0 \), and obviously \( \alpha_{0,3} = 0 \). We compute \( \chi(G(-3)) = 3\alpha_{1,2} - 3\alpha_{1,1} + \alpha_{0,0} \), so:
\[
\alpha_{2,3} = h^0(\mathbb{P}^2, G^*) = h^2(\mathbb{P}^2, G(-3)) = 3\alpha_{1,2} - 3\alpha_{1,1} + \alpha_{0,0}.
\]
If this value is positive, then there is a non-trivial morphism \( g : G \to O_{\mathbb{P}^2} \), and since \( \alpha_{0,0} \neq 0 \) there also exists \( 0 \neq f : O_{\mathbb{P}^2} \to G \). So \( G \) (and hence \( E \)) has a non-trivial endomorphism factoring through \( O_{\mathbb{P}^2} \), a contradiction.

Summing up, we may assume \( 3\alpha_{1,2} - 3\alpha_{1,1} + \alpha_{0,0} = 0 \), in other words \( \alpha_{2,3} = 0 \). Therefore, the Beilinson complex associated with \( G(-1) \) gives a resolution:
\[
0 \to G(-1) \to \Omega_{\mathbb{P}^2}(1)^{\alpha_{1,2}} \to O_{\mathbb{P}^2}^{\alpha_{1,1}} \to 0.
\]
It it easy to convert this resolution into the form we want by the diagram:
\[
\begin{array}{ccccccccc}
0 & & 0 & & \\
\downarrow & & \downarrow & & \\
0 & \longrightarrow & G(-1) & \longrightarrow & \Omega_{\mathbb{P}^2}(1)^{\alpha_{1,2}} & \longrightarrow & O_{\mathbb{P}^2}^{\alpha_{1,1}} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & O_{\mathbb{P}^2}^{3\alpha_{1,2}-\alpha_{1,1}} & \longrightarrow & O_{\mathbb{P}^2}^{\alpha_{1,1}} & \longrightarrow & O_{\mathbb{P}^2}^{\alpha_{1,1}} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & \downarrow & & \\
O_{\mathbb{P}^2}(1)^{\alpha_{1,2}} & = & O_{\mathbb{P}^2}(1)^{\alpha_{1,2}} & & & & & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & & & & & 
\end{array}
\]
From the leftmost column, it follows that \( G^* \) has a resolution of the desired form, with \( a = 3\alpha_{1,2} - \alpha_{1,1} \) and \( b = \alpha_{1,2} \). Claim (i) is thus proved.

2.3. The proof of (ii) and the rigid representation associated with \( E \). The first step to prove (ii) is to associate with a rigid ACM sheaf \( E \) (i.e. with \( G \)) a rigid representation \( R \) of \( \Upsilon_3 \). This is obvious whenever the conclusion of (ii) holds, as \( G \) is associated with a representation \( R \) via \( \Phi \), and \( R \) is rigid since \( \Phi \) is fully faithful. However, looking back to the proof of (i), we see that the conclusion of (ii) holds if \( \alpha_{1,1} = 0 \) or \( \alpha_{1,2} = 0 \), as we did not use the condition on the endomorphisms of \( E \) for those cases. So we only have to work out the dual case, and we assume \( \alpha_{1,0} = 0 \).

Consider (12), let \( e \) be the restricted map \( e : O_{\mathbb{P}^2}(-1)^{\alpha_{1,2}} \to \Omega_{\mathbb{P}^2}(1)^{\alpha_{1,1}} \) extracted from \( d_0 \) and let \( F \) be its cone, shifted by 1:
\[
(13) \quad F \to O_{\mathbb{P}^2}(-1)^{\alpha_{1,2}} \xrightarrow{\zeta} \Omega_{\mathbb{P}^2}(1)^{\alpha_{1,1}}.
\]
This is a complex with two terms, and its cohomology is concentrated in degrees zero and one, namely \( \mathcal{H}^0 F \simeq \ker(e) \) and \( \mathcal{H}^1 F \simeq \operatorname{coker}(e) \). From (12) we easily see that \( F \) fits into a distinguished triangle:
\[
(14) \quad F \to O_{\mathbb{P}^2}^{\alpha_{0,0}} \to G.
\]
Also, by (13) there is a representation \( R \) of \( \Upsilon_3 \) such that \( \Phi(R) \simeq F[1] \).

Applying Hom\(_{\mathbb{P}^2}(O_{\mathbb{P}^2}, -) \) to (13), we get Ext\(_i\)(\( O_{\mathbb{P}^2}, F \)) = 0 for all \( i \). So (14) gives:
\[
\operatorname{Ext}^i_{\mathbb{P}^2}(F, F) \simeq \operatorname{Ext}^i_{\mathbb{P}^2}(G, F[1]), \quad \text{for all } i.
\]
Also, we have Ext\(_{\mathbb{P}^2}(G, O_{\mathbb{P}^2}) \simeq H^0(\mathbb{P}^2, G(-3))^* = 0 \), so applying Hom\(_{\mathbb{P}^2}(G, -) \) to (14) we get:
\[
\operatorname{Ext}^1_{\mathbb{P}^2}(G, G) \to \operatorname{Ext}^1_{\mathbb{P}^2}(G, F[1]) \to 0.
\]
Putting this together, we obtain a surjection:
\[ \text{Ext}^i_{p_2}(E, E) \cong \text{Ext}^i_{p_2}(G, G) \rightarrow \text{Ext}^i_{p_2}(F, F) \cong \text{Ext}^i_{F_2}(R, R). \]
We understand now that, if \( E \) is rigid, then so is \( R \).

2.4. The 4-term sequence and of the proof of [13]. If \( R \) is rigid, then by [11, §2] \( R \) is a direct sum of rigid simple representations of the form \( R_k \). Therefore, the rigid object \( F \) is a direct sum of (shifted) copies of the bundles \( F_k \) obtained from the rigid representations \( R_k \). Taking cohomology of [14], we obtain the 4-term exact sequence:
\[ 0 \rightarrow \bigoplus_{i \leq 0} F_i^r_1 \rightarrow O_{F_2}^{\alpha_0,0} \rightarrow G \rightarrow \bigoplus_{i \geq 1} F_i^r_1 \rightarrow 0, \]
for some integers \( r_i \).

If only \( R_i \) with \( i \leq 0 \) appear, this sequence says that \( G \) is globally generated. Then, whenever \( \alpha_{2,3} = h^0(\mathbb{P}^2, G^*) \neq 0 \), composing \( O_{F_2}^{\alpha_0,0} \rightarrow G \) with a non-trivial map \( G \rightarrow O_{p_2} \) we get a surjection \( O_{p_2}^{\alpha_0,0} \rightarrow O_{p_2} \) which splits a direct summand \( O_{p_2} \) of \( G \). In this case \( G \cong O_{p_2} \) by indecomposability of \( G \).

This case being settled, we may assume \( \alpha_{2,3} = h^0(\mathbb{P}^2, G^*) = 0 \), so that the end of proof of the dual case [22] works and says that \( G^* \) has a resolution of the form [2] for some integers \( a, b \). Since the representation associated with \( G^* \) is rigid, we know that there is \( k \geq 1 \) such that \( a = a_{3,k-1} \) and \( b = a_{3,k} \).

If some \( R_i \) appears with \( i \geq 1 \), we call \( I \) the (non-zero) image of the middle map in the previous exact sequence, and we show \( \text{Ext}^1_{p_2}(F_j, I) = 0 \) for all \( j \geq 1 \), which contradicts \( G \) being indecomposable. To check this, note that \( \text{Hom}_{p_2}(F_j, -) \) gives an exact sequence:
\[ \text{Ext}^1_{p_2}(F_j, O_{p_2}) \cong \text{Ext}^1_{p_2}(F_j, I) \rightarrow \bigoplus_{i \leq 0} \text{Ext}^2_{p_2}(F_j, F_i)^r_1. \]

Relying on [11, Theorem 4], we see that the last statement of Theorem [11] is clear by Lemma [4] and by exceptionality of Fibonacci bundles. The fact that \( E^* \) is also ACM is obvious by Serre duality.

Remark 7. If \( k \) is algebraically closed of characteristic zero, we may apply [3, Corollaire 7], to the effect that a rigid bundle is a direct sum of exceptional bundles. So, at the price of relying on this result, from [11] we may deduce directly [11] via Kac’s theorem.

3. ACM bundles on the second Veronese threefold

The techniques we have just seen are applied to the embedding of \( \mathbb{P}^3 \) in \( \mathbb{P}^9 \) by quadratic forms. Again we replace \( E \) with the \( G = E(s_0) \), where \( s \) is the smallest integer \( s \) such that \( E(s) \) has non-zero global sections, and set \( \alpha_{i,j} = h^i(\mathbb{P}^3, G(-j)) \). If [11] gives \( \alpha_{1,1} = \alpha_{2,1} = 0 \), then \( G(-1) \) has the desired resolution. On the other hand, if [11] tells \( \alpha_{1,0} = \alpha_{2,0} = \alpha_{1,2} = \alpha_{2,2} = 0 \), then we are left with a resolution of the form:
\[ 0 \rightarrow O_{p_3}(-1)^{\alpha_{1,3}} \xrightarrow{d_3} \Omega_{p_3}^1(1)^{\alpha_{1,1}} \oplus O_{p_3}^{\alpha_0,0} \rightarrow G \rightarrow 0. \]
We also have \( \alpha_{3,4} = 0 \) for \( i = 0, 1, 2 \) again by [11]. The fact that \( G \) has no endomorphism factoring through \( O_{p_3} \) this time gives \( \alpha_{3,4} = h^0(\mathbb{P}^3, G^*) = h^0(\mathbb{P}^3, G^*) = 0 \). So \( G(-1) \) has a resolution like:
\[ 0 \rightarrow G(-1) \rightarrow \Omega_{p_3}^2(2)^{\alpha_{1,3}} \rightarrow O_{p_3}^{\alpha_{1,1}} \rightarrow 0. \]
Then, using the same trick as in the proof of the previous theorem, we see that $G^*$ has the desired resolution, with $a = 6\alpha_{1,3} - \alpha_{1,1}$ and $b = \alpha_{1,3}$.

This proves the first statement. The rest follows by the same path. Drezet’s theorem as shortcut for $[\text{I}] \Rightarrow [\text{II}]$ may be replaced by [7].

Remark 8. It should be noted that, in [10, Theorem 1.2 and Theorem 1.3], the ACM bundle $E$ on the given Veronese variety is assumed to have a rigid module of global sections. This implies, respectively, $\text{Ext}^2_\mathbb{P}(E, E(3t)) = 0$, or $\text{Ext}^1_\mathbb{P}(E, E(2t)) = 0$, for all $t \in \mathbb{Z}$. A priori, this is a stronger requirement than just $\text{Ext}^1_\mathbb{P}(E, E) = 0$. However, our proof shows that the two conditions are equivalent for ACM bundles.

4. Rigid ACM bundles on higher Veronese surfaces

Assume $k$ algebraically closed. The next result shows that, for $d \geq 4$, the class of rigid ACM bundles on $d$-fold Veronese surfaces contains the set of exceptional bundles on $\mathbb{P}^2$, which is quite a rich class, cf. [6]. At least if $\text{char}(k) = 0$, the two classes coincide by [5, Corollaire 7].

Theorem 9. Let $F$ be an exceptional bundle on $\mathbb{P}^2$ and fix $d \geq 4$. Then there is an integer $t$ such that $E = F(t)$ satisfies (1).

Proof. It is known that $F$ is actually stable by [3]. This implies that $F$ has natural cohomology by [8], i.e. for all $t \in \mathbb{Z}$ there is at most one $i$ such that $H^i(\mathbb{P}^2, F(t)) \neq 0$. Then, $H^1(\mathbb{P}^2, F(t)) \neq 0$ if and only if $\chi(F(t)) < 0$.

Let now $r, c_1, c_2$ be the rank and the Chern classes of $F$. Riemann-Roch shows that $\chi(F(t))$ is a polynomial of degree 2 in $t$, of dominant term $r/2$, whose discriminant is:

$$\Delta = c_1^2(1 - r) + r(2c_2 + r/4) = -\chi(F,F) + 5r^2/4.$$ 

So, $\chi(F,F) = 1$ implies $\Delta = -1 + 5r^2/4$. Hence, the roots of $\chi(F(t))$ differ by:

$$\frac{2\sqrt{\Delta}}{r} = \frac{\sqrt{5r^2 - 4}}{r} < 3.$$ 

Therefore, there is an integer $t_0$ such that $\chi(F(t))$ is non-negative except when $t$ takes one of the values $t_0, t_0 + 1, t_0 + 2$. Hence only these values of $t$ may give $H^1(\mathbb{P}^2, F(t)) \neq 0$. This means that $E = F(t_0 - 1)$ satisfies (1) for any choice of $d \geq 4$. \hfill \box

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