The Mezard-Parisi equation for matchings in pseudo-dimension d>1

Justin Salez

The Mézard-Parisi equation for matchings in pseudo-dimension $d > 1$

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Abstract

We establish existence and uniqueness of the solution to the cavity equation for the random assignment problem in pseudo-dimension $d > 1$, as conjectured by Aldous and Bandyopadhyay (Annals of Applied Probability, 2005) and Wästlund (Annals of Mathematics, 2012). This fills the last remaining gap in the proof of the original Mézard-Parisi prediction for this problem (Journal de Physique Lettres, 1985).

Keywords: recursive distributional equation; random assignment problem; mean-field combinatorial optimization; cavity method.

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1 Introduction

The *random assignment problem* is a now classical problem in probabilistic combinatorial optimization. Given an $n \times n$ array $\{X_{i,j}\}_{1 \le i,j \le n}$ of IID non-negative random variables, it asks about the statistics of

$$M_n \;:=\; \min_{\sigma} \sum_{i=1}^{n} X_{i,\sigma(i)},$$

where the minimum runs over all permutations σ of $\{1, \ldots, n\}$. This corresponds to finding a minimum-length perfect matching on the complete bipartite graph $K_{n,n}$ with edge-lengths $\{X_{i,j}\}_{1 \le i,j \le n}$. Using the celebrated *replica symmetry ansatz* from statistical physics, Mézard and Parisi [10, 11, 12] made a remarkably precise prediction concerning the regime where n tends to infinity while the distribution of $X_{i,j}$ is kept fixed and satisfies

$$\mathbb{P}\left(X_{i,j} \le x\right) \;\sim\; x^d \quad \text{as} \quad x \to 0^+,$$

for some exponent $0 < d < \infty$. Specifically, they conjectured that

$$\frac{M_n}{n^{1-1/d}} \xrightarrow[n\to\infty]{\mathbb{P}} -d\int_{\mathbb{R}} f(x)\ln f(x)dx, \tag{1}$$

where the function $f\colon \mathbb{R} \to [0,1]$ solves the so-called *cavity equation*:

$$f(x) = \exp\left(-\int_{-x}^{+\infty} d(x+y)^{d-1} f(y)dy\right). \tag{2}$$

Aldous [1, 3] proved this conjecture in the special case $d = 1$, where the term $(x+y)^{d-1}$ simplifies and makes the cavity equation exactly solvable, yielding

$$f(x) = \frac{1}{1+e^x} \quad \text{and} \quad -d\int_{\mathbb{R}} f(x)\ln f(x)dx = \frac{\pi^2}{6}.$$

Since then, several alternative proofs have been found [9, 13, 15]. This stands in sharp contrast with the case $d \neq 1$, where showing that the Mézard-Parisi equation (2) admits a unique solution has until now remained an open problem [4, Open Problem 63]. Wästlund [16] circumvented this issue by considering instead the truncated equation

$$f_\lambda(x) = \exp\left(-\int_{-x}^{\lambda} d(x+y)^{d-1} f_\lambda(y)dy\right), \qquad 0 < \lambda < \infty. \tag{3}$$

Using an ingenious game-theoretical interpretation of this equation, he showed the existence of a unique, global attractive solution $f_\lambda\colon [-\lambda,\lambda] \to [0,1]$ for every $0 < \lambda < \infty$, provided $d \geq 1$. He then used this fact to establish that

$$\frac{M_n}{n^{1-1/d}} \xrightarrow[n\to\infty]{\mathbb{P}} \lim_{\lambda\to+\infty} \uparrow -d\int_{-\lambda}^{\lambda} f_\lambda(x)\ln f_\lambda(x)dx. \tag{4}$$

Wästlund [16] explicitly left open the problem of completing the proof of the original Mézard-Parisi prediction by showing (i) that the untruncated cavity equation admits a unique solution f and (ii) that $f_\lambda \to f$ as $\lambda \to \infty$. The purpose of this short paper is to establish this conjecture.

Theorem 1. *For $d > 1$, the Mézard-Parisi equation (2) admits a unique solution $f\colon \mathbb{R} \to [0,1]$. Moreover, $f_\lambda \to f$ pointwise as $\lambda \to +\infty$, and*

$$\int_{-\lambda}^{\lambda} f_\lambda(x)\ln f_\lambda(x)dx \xrightarrow[\lambda\to+\infty]{} \int_{\mathbb{R}} f(x)\ln f(x)dx.$$

Consequently, the two limits in (1) and (4) coincide.

In addition, we provide a short alternative proof of the crucial result of [16] that the truncated equation (3) admits a unique, attractive solution.

Remark 1. Very recently, a proof of uniqueness for the truncated equation (3) has been announced [8] for the case $0 < d < 1$. It would be interesting to see if the result of the present paper can be extended to this regime.

Remark 2. For a random variable Z with $\mathbb{P}(Z > x) = f(x)$, the cavity equation (2) simply expresses the fact that Z solves the distributional identity

$$Z \stackrel{d}{=} \min_{i\geq 1} \{\xi_i - Z_i\}, \tag{5}$$

where $\{\xi_i\}_{i\geq 1}$ is a Poisson point process with intensity $dx^{d-1}\partial x$ on $[0,\infty)$, and $\{Z_i\}_{i\geq 1}$ are IID with the same distribution as Z, independent of $\{\xi_i\}_{i\geq 1}$. Such *recursive distributional equations* arise naturally in a variety of models from statistical physics, and the question of existence and uniqueness of solutions plays a crucial role for the rigorous understanding of those models. We refer the interested reader to the comprehensive surveys [2, 4] for more details. In particular, [4, Section 7.4] contains a detailed discussion on equation (5), and [4, Open Problem 63] raises explicitly the uniqueness issue. We note that the refined question of *endogeny* remains a challenging open problem. Recursive distributional equations for other mean-field combinatorial optimization problems have been analysed in e.g. [5, 14, 6].

The remainder of the paper is organized as follows. Section 2 deals with the truncated equation (3) for fixed $0 < \lambda < \infty$ and is devoted to the alternative analytical proof that there is a unique, globally attractive solution f_λ. Section 3 prepares the $\lambda \to \infty$ limit by providing uniform controls on the family $\{f_\lambda \colon 0 < \lambda < \infty\}$ and by characterizing the possible limit points. This reduces the proof of Theorem 1 to establishing uniqueness in the untruncated Mézard-Parisi equation ($\lambda = \infty$), which is done in Section 4.

2 The truncated cavity equation ($\lambda < \infty$)

Fix a parameter $0 < \lambda < \infty$. On the set $\mathfrak{F}$ of non-increasing functions $f \colon [-\lambda, \lambda] \to [0,1]$, define an operator T by

$$(Tf)(x) = \exp\left(-d\int_{-x}^{\lambda} (x+y)^{d-1} f(y)dy\right). \tag{6}$$

The purpose of this section is to give a short and purely analytical proof of the following result, which was the main technical ingredient in [16] and was therein established using an ingenious game-theoretical framework.

Proposition 1. *T admits a unique fixed point f_λ and it is attractive in the sense that $|T^n f(x) - f_\lambda(x)| \xrightarrow[n\to\infty]{} 0$, uniformly in both $x \in [-\lambda, \lambda]$ and $f \in \mathfrak{F}$.*

Proof. Write $f \le g$ to mean $f(x) \le g(x)$ for all $x \in [-\lambda, \lambda]$. In particular,

$$\mathbf{0} \le f \le T\mathbf{0}$$

for every $f \in \mathfrak{F}$, where $\mathbf{0}$ denotes the constant-zero function. Note also that the operator T is non-increasing, in the sense that

$$f \le g \implies Tf \ge Tg.$$

Those two observations imply that the sequences $\{T^{2n}\mathbf{0}\}_{n\ge 0}$ and $\{T^{2n+1}\mathbf{0}\}_{n\ge 0}$ are respectively non-decreasing and non-increasing, and that their respective pointwise limits f^- and f^+ satisfy

$$f^- \;\le\; \liminf_{n\to\infty} T^n f \;\le\; \limsup_{n\to\infty} T^n f \;\le\; f^+,$$

for any $f \in \mathfrak{F}$. Moreover, the dominated convergence Theorem ensures that T is continuous with respect to pointwise convergence, allowing to pass to the limit in the identity $T^{n+1}\mathbf{0} = T(T^n\mathbf{0})$ to deduce that

$$Tf^- = f^+ \quad \text{and} \quad Tf^+ = f^-. \tag{7}$$

Therefore, the proof boils down to the identity $f^- = f^+$, which we now establish. By definition, we have for any $f \in \mathfrak{F}$,

$$(Tf)(x) \;=\; \exp\left(-d\int_{-\lambda}^{\lambda}(x+y)^{d-1}\mathbf{1}_{(x+y\ge 0)}f(y)dy\right).$$

Since $d > 1$, we may differentiate under the integral sign to obtain

$$(Tf)'(x) \;=\; -d(d-1)(Tf)(x)\int_{-\lambda}^{\lambda}(x+y)^{d-2}\mathbf{1}_{(x+y\ge 0)}f(y)dy.$$

Integrating over $[-\lambda, \lambda]$ and noting that $(Tf)(-\lambda) = 1$, we conclude that

$$1 - (Tf)(\lambda) \;=\; d(d-1)\iint_{[-\lambda,\lambda]^2}(x+y)^{d-2}\mathbf{1}_{(x+y\ge 0)}(Tf)(x)f(y)dxdy.$$

Let us now specialize to $f = f^\pm$. In both cases, the right-hand side is

$$d(d-1)\iint_{[-\lambda,\lambda]^2}(x+y)^{d-2}\mathbf{1}_{(x+y\ge 0)}f^+(x)f^-(y)dxdy,$$

by (7). Therefore, we have $(Tf^+)(\lambda) = (Tf^-)(\lambda)$, i.e.

$$\int_{-\lambda}^{\lambda} d(\lambda+y)^{d-1} f^+(y)dy \;=\; \int_{-\lambda}^{\lambda} d(\lambda+y)^{d-1} f^-(y)dy.$$

Since we already know that $f^- \le f^+$, this forces $f^- = f^+$ almost-everwhere on $[-\lambda, \lambda]$, and hence everywhere by continuity. Finally, the convergence $T^n\mathbf{0} \to f_\lambda = f^\pm$ is automatically uniform on $[-\lambda, \lambda]$, by Dini's Theorem. □

3 Relative compactness of solutions $(\lambda \to \infty)$

In order to study properties of the family $\{f_\lambda \colon 0 < \lambda < \infty\}$, we extend the domain of f_λ to $\mathbb{R}$ by setting $f_\lambda(x) = 1$ for $x \le -\lambda$ and $f_\lambda(x) = 0$ for $x > \lambda$.

Proposition 2 (Uniform bounds)**.** *For all* $0 < \lambda < \infty$ *and* $x \ge 0$*,*

$$\begin{aligned} f_\lambda(x) &\le \exp\left(-\frac{x^d}{e}\right) \\ 1 - f_\lambda(-x) &\le \exp\left(-\frac{x^d}{e}\right) \\ f_\lambda(-x)\ln\frac{1}{f_\lambda(-x)} &\le \exp\left(-\frac{x^d}{e}\right) \\ f_\lambda(x)\ln\frac{1}{f_\lambda(x)} &\le \left(1+\frac{x^d}{e}\right)\exp\left(-\frac{x^d}{e}\right). \end{aligned}$$

Proof. Let $0 < \lambda < \infty$. We may assume that $x \in [0, \lambda]$, otherwise the above bounds are trivial. By definition, we have

$$f_\lambda(x) \;=\; \exp\left(-\int_{-x}^{\lambda} d(x+y)^{d-1} f_\lambda(y)dy\right). \tag{8}$$

Now, since $x \ge 0$ and f_λ is non-increasing, we have

$$\begin{aligned} \int_{-x}^{\lambda} (x+y)^{d-1} f_\lambda(y)dy &= \int_{-x}^{0} (x+y)^{d-1} f_\lambda(y)dy + \int_{0}^{\lambda} (x+y)^{d-1} f_\lambda(y)dy \\ &\ge f_\lambda(0)\frac{x^d}{d} + \int_0^\lambda y^{d-1} f_\lambda(y)dy. \end{aligned}$$

Applying $u \mapsto \exp(-du)$ to both sides and using (8), we obtain

$$f_\lambda(x) \;\le\; f_\lambda(0)\exp(-f_\lambda(0)x^d). \tag{9}$$

In turn, this inequality implies that for all $x \geq 0$,

$$\int_x^\lambda d(y-x)^{d-1} f_\lambda(y)dy \;\leq\; f_\lambda(0)\int_x^{+\infty} dy^{d-1}e^{-f_\lambda(0)y^d}dy \;=\; \exp(-f_\lambda(0)x^d).$$

Applying $u \mapsto \exp(-u)$ to both sides, we conclude that

$$f_\lambda(-x) \;\geq\; \exp\left(-e^{-f_\lambda(0)x^d}\right). \tag{10}$$

In particular, taking $x = 0$ yields $f_\lambda(0) \geq e^{-1}$, and reinjecting this into (9) and (10) easily yields the first three claims. For the last one, observe that $u \mapsto u \ln \frac{1}{u}$ increases on $[0, e^{-1}]$ and decreases on $[e^{-1}, 1]$, with the value at $u = e^{-1}$ being precisely e^{-1}. Therefore, if $\exp(-x^d/e) \leq e^{-1}$, we may use the bound $f_\lambda(x) \leq \exp(-x^d/e)$ to deduce that

$$f_\lambda(x) \ln \frac{1}{f_\lambda(x)} \;\leq\; \frac{x^d}{e}\exp\left(-\frac{x^d}{e}\right).$$

On the other hand, if $\exp(-x^d/e) \geq e^{-1}$, then

$$f_\lambda(x) \ln \frac{1}{f_\lambda(x)} \;\leq\; e^{-1} \;\leq\; \exp\left(-\frac{x^d}{e}\right).$$

In both cases, the last inequality holds, and the proof is complete. □

Proposition 3. *The family $\{f_\lambda \colon 0 < \lambda < \infty\}$ is relatively compact with respect to the topology of uniform convergence on $\mathbb{R}$, and any sub-sequential limit as $\lambda \to \infty$ must solve the cavity equation (2).*

Proof. Let $\{\lambda_n\}_{n\geq 0}$ be any sequence of positive numbers such that $\lambda_n \to \infty$ as $n \to \infty$. By Helly's compactness principle for uniformly bounded monotone functions (see e.g. [7, Theorem 36.5]), there exists an increasing sequence $\{n_k\}_{k\geq 0}$ in $\mathbb{N}$ and a non-increasing function $f \colon \mathbb{R} \to [0,1]$ such that

$$f_{\lambda_{n_k}}(x) \;\xrightarrow[k\to\infty]{}\; f(x), \tag{11}$$

for all $x \in \mathbb{R}$. Thanks to the first inequality in Proposition 2, we may invoke dominated convergence to deduce that for each $x \in \mathbb{R}$,

$$\int_{-x}^{\lambda_{n_k}} f_{\lambda_{n_k}}(y)(x+y)^{d-1}dy \;\xrightarrow[k\to\infty]{}\; \int_{-x}^{+\infty} f(y)(x+y)^{d-1}dy.$$

Applying $u \mapsto \exp(-du)$ and recalling (8), we see that

$$f(x) \;=\; \exp\left(-d\int_{-x}^{+\infty} f(y)(x+y)^{d-1}dy\right),$$

which shows that f must solve the cavity equation (2). This identity easily implies that f is continuous. Consequently, the convergence (11) is uniform in $x \in \mathbb{R}$, by Dini's Theorem. □

4 The un-truncated cavity equation ($\lambda = \infty$)

To conclude the proof of Theorem 1, it now remains to show that the un-truncated equation

$$f(x) \quad = \quad \exp\left(-d\int_{-x}^{+\infty}(x+y)^{d-1}f(y)dy\right). \tag{12}$$

admits at most one fixed point $f\colon \mathbb{R} \to [0,1]$. Proposition 3 will then guarantee the convergence $f_\lambda \xrightarrow[\lambda\to\infty]{} f$, which will in turn imply

$$\int_{-\lambda}^{\lambda} f_\lambda(x)\ln f_\lambda(x)dx \quad \xrightarrow[\lambda\to+\infty]{} \quad \int_{\mathbb{R}} f(x)\ln f(x)dx,$$

by dominated convergence, thanks to the last inequalities in Proposition 2.

A quick inspection of the proof of Proposition 2 reveals that it remains valid when $\lambda = \infty$. In particular, any solution f to (12) must satisfy

$$\max(f(x), 1-f(-x)) \quad \leq \quad \exp\left(-\frac{x^d}{e}\right), \tag{13}$$

for all $x \geq 0$. It also clear from (12) that f must be $(0,1)-$valued and continuous. We will use those properties in the proofs below.

Lemma 1. *If f, g solve (12), then there exists $t \geq 0$ such that for all $x \in \mathbb{R}$,*

$$f(x+t) \leq g(x) \leq f(x-t).$$

Proof. (13) ensures that for any $t \in \mathbb{R}$, $y \mapsto (1+|y|)(f(y-t)-g(y))$ is integrable on $\mathbb{R}$, so that by dominated convergence,

$$\frac{1}{x^{d-1}}\int_{-x}^{+\infty}(y+x)^{d-1}\left(f(y-t)-g(y)\right)dy \quad \xrightarrow[x\to+\infty]{} \quad \Delta(t), \tag{14}$$

where

$$\Delta(t) \quad := \quad \int_{\mathbb{R}}\left(f(y-t)-g(y)\right)dy. \tag{15}$$

Observe that $t \mapsto \Delta(t)$ increases continuously from $-\infty$ to $+\infty$, as can be seen from the decomposition

$$\Delta(t) \quad = \quad \int_0^{+\infty}(1-g(-y)-g(y))dy + \int_{-t}^{+\infty} f(y)dy - \int_t^{+\infty}(1-f(-y))dy.$$

In particular, we can find $t_0 \geq 0$ such that $\Delta(-t_0) < 0 < \Delta(t_0)$. In view of (14), we deduce the existence of $a \geq 0$ such that for all $x \geq a$,

$$\int_{-x}^{+\infty} (y+x)^{d-1} g(y)dy \;\geq\; \int_{-x}^{+\infty} (y+x)^{d-1} f(y+t_0)dy \tag{16}$$

$$\int_{-x}^{+\infty} (y+x)^{d-1} g(y)dy \;\leq\; \int_{-x}^{+\infty} (y+x)^{d-1} f(y-t_0)dy. \tag{17}$$

Applying $u \mapsto \exp(-du)$, we conclude that for all $x \geq a$,

$$f(x+t_0) \;\leq\; g(x) \;\leq f(x-t_0). \tag{18}$$

In turn, this implies that (16)-(17) also hold when $x \leq -a$, so that (18) actually holds for all x outside $(-a, a)$. On the other hand, since g is $(0,1)-$valued and f has limits $0, 1$ at $\pm\infty$, we can choose $t_1 \geq 0$ large enough so that

$$f(-a+t_1) \;\leq\; g(a) \;\leq\; g(-a) \;\leq\; f(a-t_1).$$

Since f, g are non-increasing, this inequality implies that for all $x \in [-a, a]$,

$$f(x+t_1) \;\leq\; g(x) \;\leq\; f(x-t_1). \tag{19}$$

In view of (18)-(19), taking $t := \max(t_0, t_1)$ concludes the proof. $\square$

Proof of Proposition 3. Let f, g solve equation (12) and let t be the smallest non-negative number satisfying for all $x \in \mathbb{R}$,

$$f(x+t) \;\leq\; g(x) \;\leq\; f(x-t). \tag{20}$$

Note that t exists by Lemma 1 and the continuity of f. Now assume for a contradiction that $t > 0$. Clearly, each of the two inequalities in (20) must be strict at some point $x \in \mathbb{R}$ (and hence on some open interval by continuity), otherwise we would have $g \geq f$ or $g \leq f$ and (12) would then force $g = f$, contradicting the assumption that $t > 0$. Consequently, the function Δ defined in (15) must satisfy $\Delta(-t) < 0 < \Delta(t)$. By continuity of Δ, there must exists $t_0 < t$ such that $\Delta(-t_0) < 0 < \Delta(t_0)$. As we have already seen, this inequality implies

$$f(x+t_0) \;\leq\; g(x) \;\leq\; f(x-t_0), \tag{21}$$

for all x outside some compact $[-a, a]$. In particular, we now see that the inequalities in (20) must be strict for all large enough x. Thus, for all $x \in \mathbb{R}$,

$$\int_{-x}^{+\infty} (y+x)^{d-1} g(y)dy \;>\; \int_{-x}^{+\infty} (y+x)^{d-1} f(y+t)dy$$

$$\int_{-x}^{+\infty} (y+x)^{d-1} g(y)dy \;<\; \int_{-x}^{+\infty} (y+x)^{d-1} f(y-t)dy.$$

Applying $u \mapsto \exp(-du)$ now shows that the inequalities in (20) must actually be strict everywhere on $\mathbb{R}$, hence in particular on the compact $[-a, a]$. By uniform continuity, there must exists $t_1 < t$ such that

$$f(x + t_1) \;\leq\; g(x) \;\leq\; f(x - t_1), \tag{22}$$

for all $x \in [-a, a]$. In view of (21)-(22), the number $t' := \max(t_0, t_1)$ now contradicts the minimality of t. □

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