A stochastic control approach for options market making

Sofiene El Aoud, Frédéric Abergel

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Sofiene El Aoud [‡*]　　　　Frédéric Abergel [†‡]

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Abstract

This paper presents a model for the market making of options on a liquid stock. The stock price follows a generic stochastic volatility model under the real-world probability measure $\mathcal{P}$. Market participants price options on this stock under a risk-neutral pricing measure $\mathcal{Q}$, and they may misspecify the parameters controlling the dynamics of the volatility process. We first consider that there is a risk-neutral agent who is willing to make markets in an option on the stock, with the aim of maximizing the expected terminal wealth at maturity. Using standard tools in optimal stochastic control, we provide analytical expressions for the optimal bid and ask quotes of the market maker. We then assume that the agent is risk-averse, and perturb the linear utility function by adding a variance term. In this setting, analytic approximations of the optimal bid and ask quotes are obtained. In the case where the stock price process follows a Heston model, Monte Carlo simulations are used to compare the optimal strategy to a "zero-intelligence" strategy, and to highlight the effects of some parameters' misspecification on the performance of the strategy.

JEL Classification: JEL: C51 Model construction and estimation, JEL: C52 Model evaluation and testing.

Keywords: High frequency Market Making, Stochastic optimal control, Utility maximization, Mean-variance framework.

[*]sofiene.elaoud@ecp.fr

[†]frederic.abergel@ecp.fr

[‡]Chair of Quantitative Finance, Laboratoire of Mathematics Applied to Systems, CentraleSupélec, Grande Voie des Vignes, 92290 Châtenay Malabry, France

1 Introduction

Market makers act on the market by providing liquidity on specific securities. Their role consists in continuously setting bid and ask quotes on instruments of their own choosing. Market makers play a fundamental role, in that they provide liquidity to other market participants, typically to impatient agents who are willing to cross the bid-ask spread. The profit made by a market making strategy comes from the alternation of buy and sell orders, and its targeted gain per trade should depend on the accumulated inventory, a market risk potentially causing significant losses.

In the recent literature, several works focus on the problem of market making on stocks through the use of a stochastic optimization method. Inspired by the seminal paper [11], [1] addresses the problem of high frequency market making of a stock. In this model framework, the stock price is modelled as a Wiener process at the intraday timescale, and the market maker seeks the optimal quotes in order to maximize an exponential utility function. More recently, [8] deals with the problem of optimally using limit orders when liquidating a portfolio. In [9], a similar approach is used to deal with the problem of market making with inventory risk. In [3] [4], the approach proposed in [1] is generalized to the case where the process modelling the stock price has a drift term and the volatility is stochastic.

The issue of model ambiguity has been addressed in [2] or [15], to account for the fact that the dynamics followed by the stock price and/or the arrival rate of market orders may be misspecified. The conclusion is that, as the uncertainty increases, the market maker has to adapt her quotes in order to reduce her inventory.

In this paper, we address the problem of market making in **options**. Despite its importance, this subject has received little theoretical attention. The only reference we are aware of is [18], where a mean-variance framework is proposed for optimal market making in options in the case of a complete market. In our framework, the volatility of the stock is stochastic, and our goal is to determine the optimal market making strategy on options in the setting of a generic stochastic volatility model.

The paper is organized as follows: In Section 2, we specify the joint dynamics of the spot and its instantaneous volatility under both the historical and pricing measures. Section **??** presents various realistic models for the order flow and market impact function. In Section 4, the market maker is risk neutral and aims to maximize the expectation of her wealth at the maturity date of the option, whereas Section 5 tackles the more challenging case of a risk-adverse market maker in a mean-variance framework. Finally, Section 6 presents a numerical study of the performance of the optimal strategy, and of the effects of model misspecification.

2 Model setup

Consider a financial market living on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathcal{P})$, where the filtration $\mathbb{F} = \{\mathcal{F}_t, t \in \mathbf{R}^+\}$ satisfies the usual conditions, and where $\mathcal{P}$ denotes the real-world probability measure. Under $\mathcal{P}$, the spot process S has the following dynamics:

$$\frac{dS_t}{S_t} = \mu dt + \sigma(y_t)dW_t^{(1)}, \tag{2.1}$$

$$dy_t = a_R(y_t)dt + b_R(y_t)dW_t^{(2)}, \tag{2.2}$$

where $W^{(1)}$ and $W^{(2)}$ are two $(\mathcal{P},\mathcal{F})$ Wiener processes such that $d\left\langle W^{(1)}, W^{(2)}\right\rangle_t = \rho_R dt$. The functions a_R, b_R satisfy sufficient conditions to ensure the existence of a strong solution to (2.2) satisfying, $\forall T > 0$:

$$E^{\mathcal{P}}\left(\int_0^T \sigma(y_t)^2 dt\right) < +\infty,$$
$$E^{\mathcal{P}}\left(\int_0^T \left(a_R^2(y_t) + b_R^2(y_t)\right) dt\right) < +\infty.$$

Suppose that a european option with maturity T and payoff function $h(S_T)$ is traded in the option market. Let the quantity $C_{\mathcal{P}}$ be defined as:

$$C_{\mathcal{P}}(t, S_t, y_t) = E^{\mathcal{P}}\left(h(S_T)|\mathcal{F}_t\right). \tag{2.3}$$

Although $C_{\mathcal{P}}$ is not the option price - since $\mathcal{P}$ is not the pricing measure - introducing this notion will prove useful.

Market participants price options under the probability measure $\mathcal{Q}$. To make the model as general as possible, we shall allow, as discussed e.g. in [12], for misspecifications of the parameters characterizing the volatility dynamics.

Let r denote the risk-free rate. Under the pricing measure $\mathcal{Q}$, the process S evolves as follows:

$$\frac{dS_t}{S_t} = rdt + \sigma(y_t)dW_t^{*,(1)}, \tag{2.4}$$

$$dy_t = a_I(y_t)dt + b_I(y_t)dW_t^{*,(2)}, \tag{2.5}$$

where $W^{*,(1)}$ and $W^{*,(2)}$ are two $(\mathcal{Q},\mathcal{F})$ Brownian motions such that $d\left\langle W^{*,(1)}, W^{*,(2)}\right\rangle_t = \rho_I dt$. Again, the equation (2.5) is supposed to have a unique strong solution satisfying $\forall T > 0$:

$$E^{\mathcal{Q}}\left(\int_0^T \sigma(y_t)^2 dt\right) < +\infty,$$
$$E^{\mathcal{Q}}\left(\int_0^T \left(a_R^2(y_t) + b_R^2(y_t)\right) dt\right) < +\infty.$$

The consensus, and therefore, **observed** option price $C_{\mathcal{Q}}$ is then given by

$$C_{\mathcal{Q}}(t, S_t, y_t) = e^{-r(T-t)}E^{\mathcal{Q}}(h(S_T)|\mathcal{F}_t). \tag{2.6}$$

A market maker is an agent who publishes bid and ask quotes around the option mid-price $C_{\mathcal{Q}}$, meets market orders when they match his quotes, while trading continuously in the stock for delta-hedging purposes. At a given time t, this market maker sets an ask price C_t^a and a bid price C_t^b such that:

$$\begin{aligned} C_t^a &= C_{\mathcal{Q}}(t, S_t, y_t) + \delta_t^+, \\ C_t^b &= C_{\mathcal{Q}}(t, S_t, y_t) - \delta_t^-, \end{aligned}$$

where $\delta_t^-, \delta_t^+ > 0$. The way the parameters δ_t^+, δ_t^- influence the probability of being hit by a market order and therefore, the inventory, is made precise in Section 3.

Below are some notations which will be used in the rest of the paper:

- $q_{1,t}$ denotes the option inventory held by the market maker at time t.
- $q_{2,t}$ is the stock inventory held by the market maker at time t.
- X_t is the market value of the cash and stock position held by the market maker at time t.
- $C_{\mathcal{Q}}(t, S_t, y_t)$ is the mid-price of the option at time t.
- $\Delta_t = \Delta(t, S_t, y_t)$ is the option's delta at time t.
- $\mathcal{W}_t$ denotes the wealth of the market maker at time t.

In order to ease notations, X will be also called the **cash** process. Thanks to the infinite liquidity assumption made on the stock, this abuse of notations is legitimate.
The arrival of market orders matching the quotes of the market maker are modeled by two independent Poisson processes: N^+ for buy orders consuming the ask quotes, and N^- for sell orders consuming the bid quotes. Therefore, the dynamics of the process q_1 can be described as follows:

$$dq_{1,t} = dN_t^- - dN_t^+. \tag{2.7}$$

Due to the continuously adjusted inventory in stock, there holds $q_{2,t} = -q_{1,t}\Delta_t$, and therefore:

$$dq_{2,t} = -\Delta_t dq_{1,t} - q_{1,t} d\Delta_t - d\langle q_1, \Delta\rangle_t = -\Delta_t dN_t^- + \Delta_t dN_t^+ - q_{1,t} d\Delta_t. \tag{2.8}$$

The cash process X evolves according to the arrival of market orders as well as continuous trading in the stock, and has the following dynamics:

$$dX_t = (C_{\mathcal{Q}}(t, S_t, y_t) + \delta_t^+) dN_t^+ - (C_{\mathcal{Q}}(t, S_t, Y_t) - \delta_t^-) dN_t^- + q_{2,t} dS_t, \tag{2.9}$$

so that the wealth of the market maker at time t is given by

$$\mathcal{W}_t = X_t + q_{1,t} C_{\mathcal{Q}}(t, S_t, y_t). \tag{2.10}$$

The market maker's strategy is to provide liquidity in order to maximize the expected utility of the terminal wealth. In order to formulate the optimization problem, a crucial step is to choose a model for the intensities of the processes N^+ and N^-.

3 Intensity of arrivals of market orders

Recent researches have studied the market order flow in order to estimate the probability of execution of limit orders. In [16], the authors point out that the execution probability of limit orders is affected by the liquidity on the opposite side of the order book, and also the bid-ask spread. [13] uses survival analysis in order to estimate the conditional distribution of limit-order execution times as a function of different variables such as the limit price, the order size, and current market conditions (volatility, bid-ask spread). In particular, the authors proved empirically the seemingly natural fact that, the larger the distance between the mid-quote price and the limit price, the longer the expected time-to-execution.
For the sake of simplicity, we will assume that the probability of execution of a limit order depends only on its distance to the mid-price, in such a way that the instantaneous intensities λ_t^+ and λ_t^- can be expressed as decreasing functions of δ_t^+ and δ_t^- respectively. More precisely, the following hypothesis will be enforced throughout the rest of this paper:
$\forall \delta^+, \delta^- \geq 0$,

$$\lambda^+(\delta^+) \quad = \quad \frac{A}{\left(B + (\delta^+)^{\frac{1}{\beta}}\right)^{\gamma}}, \qquad \lambda^-(\delta^-) = \frac{A}{\left(B + (\delta^-)^{\frac{1}{\beta}}\right)^{\gamma}},$$

where $A, B > 0$, $\gamma > 1$, β is a positive parameter characterizing the market impact function, and $\lambda^+(\delta^+)$ (resp. $\lambda^-(\delta^-)$) denotes, with a slight abuse of notation, the intensity of N^+ (resp. N^-) conditional on $\delta_t^+ = \delta^+$ (resp. $\delta_t^- = \delta^-$).
A heuristic derivation of this functional form is provided in Appendix 8.1.

4 Optimization problem for a risk-neutral market maker

Generally speaking, the market maker seeks to maximize the expected utility of the terminal wealth at maturity T, and the optimal distances $(\delta_{L,*,t}^+, \delta_{L,*,t}^-)$ solve the following problem:

$$(\delta_{L,*,t}^+, \delta_{L,*,t}^-) \quad = \quad ArgSup_{\left\{\delta_t^+, \delta_t^-\right\}} E^{\mathcal{P}}(U(X_T + q_{1,T}h(S_T))|S_t = s, y_t = y, q_{1,t} = q_1, X_t = x).$$

where U is the utility function. In this section, the market maker is **risk-neutral**, hence the utility function U is linear:

$$U(T, S_T, y_T, q_{1,T}, X_T) \quad = \quad X_T + q_{1,T}h(S_T), \tag{4.1}$$

where

$$X_T \quad = \quad \int_0^T (C_{\mathcal{Q}}(t, S_t, y_t) + \delta_t^+) dN_t^+ - \int_0^T (C_{\mathcal{Q}}(t, S_t, Y_t) - \delta_t^-) dN_t^- + \int_0^T q_{2,t} dS_t.$$

In this section, the problem associated with the linear utility function is addressed. The cases corresponding to $\beta = \frac{1}{2}$ (square root market impact) and $\beta = 1$ (linear market impact) are studied separately, and the optimal bid and ask quotes are provided analytically in each case.

In order to solve the optimization problem, a stochastic control approach is used. The value function of the market maker can be defined in the following way:

$$u(t,s,y,q_1,x) = Sup_{\left\{(\delta_t^+,\delta_t^-)\in\mathcal{A}\right\}} E^{\mathcal{P}}(X_T + q_{1,T}h(S_T))|S_t = s, y_t = y, q_{1,t} = q_1, X_t = x),$$

where $\mathcal{A} = \mathbb{R}^+ \times \mathbb{R}^+$ denotes the set of admissible values for the controls.

Introducing the differential operators $\mathcal{L}_1, \mathcal{L}_2$

$$\begin{aligned} \mathcal{L}_1 &= \mu S_t \frac{\partial}{\partial S} + \frac{1}{2}\sigma^2(y_t) S_t^2 \frac{\partial^2}{\partial S^2} + a_R(y_t)\frac{\partial}{\partial y} + \frac{1}{2} b_R^2(y_t)\frac{\partial^2}{\partial y^2} + \rho_R b_R(y_t)\sigma(y_t) S_t \frac{\partial^2}{\partial S \partial y}, \\ \mathcal{L}_2 &= \frac{\partial}{\partial x} q_{2,t}\mu S_t + \frac{1}{2}\frac{\partial^2}{\partial x^2} q_{2,t}^2 \sigma(y_t)^2 S_t^2 + \frac{\partial^2}{\partial x \partial S} q_{2,t}\sigma^2(y_t) S_t^2 + \frac{\partial^2}{\partial x \partial y} q_{2,t}\sigma(y_t) S_t b_R(y_t)\rho_R, \end{aligned}$$

we let u_0 be the solution of the Hamilton-Jacobi-Bellman (HJB) equation:

$$(\partial_t + \mathcal{L}_1 + \mathcal{L}_2)u_0 + sup_{\{(\delta^+,\delta^-)\in\mathcal{A}\}} \left(J^+\left(\delta^+\right) + J^-\left(\delta^-\right)\right) = 0, \tag{4.2}$$

where the functions J^+ and J^- are defined as follows:

$$\begin{aligned} J^+(\delta^+) &= \lambda^+(\delta^+)\left(u_0(t,s,y,q_1-1,x+(C_{\mathcal{Q}}+\delta^+)) - u_0(t,s,y,q_1,x)\right), \\ J^-(\delta^-) &= \lambda^-(\delta^-)\left(u_0(t,s,y,q_1+1,x-(C_{\mathcal{Q}}-\delta^-)) - u_0(t,s,y,q_1,x)\right), \end{aligned}$$

and u_0 satisfies the terminal condition

$$u_0(T,s,y,q_1,x) = x + q_1 h(s).$$

A deep result in [17] implies that, if u_0 is smooth, finite and has polynomial growth at infinity, then the value function u is equal to u_0. Hence, our approach will be to solve the HJB equation, and prove that it satisfies the finiteness, smoothness and growth conditions of [17].

4.1 The case $\beta = \frac{1}{2}$ (square root market impact)

The intensities of arrivals of market orders are as follows:

$\lambda^+(\delta^+) = \frac{A}{\left(B+(\delta^+)^2\right)^\gamma}$ and $\lambda^-(\delta^-) = \frac{A}{\left(B+(\delta^-)^2\right)^\gamma}$,

where $A, B \geq 0$ and $\gamma > 1$.

4.1.1 Analytic solution

Proposition 4.1. *Let* $M_0(t,s,y) = C_{\mathcal{Q}}(t,s,y) - C_{\mathcal{P}}(t,s,y) + \mu E^{\mathcal{P}}_{t,s,y}\left(\int_t^T \Delta(u,S_u,y_u)S_u du\right)$.

The optimal controls $(\delta^+_{L,*,t}, \delta^-_{L,*,t})$ *of the market maker at time* t *are given by:*

$$\delta^+_{L,*,t} = \frac{\sqrt{\gamma^2 M_0^2(t,s,y) + B(2\gamma-1)} - \gamma M_0(t,s,y)}{2\gamma - 1}, \tag{4.3}$$

$$\delta^-_{L,*,t} = \frac{\sqrt{\gamma^2 M_0^2(t,s,y) + B(2\gamma-1)} + \gamma M_0(t,s,y)}{2\gamma-1}. \tag{4.4}$$

Moreover, the value function is given by:

$$u_0(t,s,y,q_1,x) = x + \theta_0(t,s,y) + q_1\left(C_{\mathcal{P}}(t,s,y) - \mu E^{\mathcal{P}}_{t,s,y}\left(\int_t^T \Delta(u,S_u,y_u)S_u du\right)\right), \tag{4.5}$$

where

$$\theta_0(t,s,y) = E_{t,s,y}\left(\int_t^T J_0(u,S_u,y_u)du\right)$$

and:

$$\begin{aligned} J_0(t,s,y) = \lambda^+(\delta^+_{L,*,t})&\left(\frac{\sqrt{\gamma^2 M_0^2 + B(2\gamma-1)} + M_0(\gamma-1)}{2\gamma-1}\right) \\ +\lambda^-(\delta^-_{L,*,t})&\left(\frac{\sqrt{\gamma^2 M_0^2 + B(2\gamma-1)} + M_0(1-\gamma)}{2\gamma-1}\right). \end{aligned}$$

Proof. Since the utility function is linear, the solution is sought under the form:

$$u(t,s,y,q_1,x) = x + \theta_0(t,s,y) + q_1\theta_1(t,s,y). \tag{4.6}$$

Let then $f_0^+ = J^+$. Using (4.6), the function f_0^+ can be rewritten as:

$$f_0^+(\delta^+) = \lambda^+(\delta^+)(\delta^+ + M_0(t,s,y)),$$

where $M_0(t,s,y) = C_{\mathcal{Q}}(t,s,y) - \theta_1(t,s,y)$.
In order to determine $\delta^+_{L,*,t} = ArgMax_{\{x>0\}} f_0^+(x)$, the derivative of f_0^+ is computed:

$$(f_0^+)'(\delta^+) = \frac{\lambda^+(\delta^+)}{B + (\delta^+)^2}\left(\left(\delta^+\right)^2(1-2\gamma) - 2\gamma M_0\left(t,s,y\right)\delta^+ + B\right).$$

The function $(f_0^+)'$ vanishes at the points x_1^+ and x_2^+:

$$x_1^+ = \frac{-\gamma M_0 - \sqrt{\gamma^2 M_0^2 + B(2\gamma-1)}}{2\gamma-1}, \qquad x_2^+ = \frac{-\gamma M_0 + \sqrt{\gamma^2 M_0^2 + B(2\gamma-1)}}{2\gamma-1}.$$

Since $\gamma \geq 1$ and $B > 0$, there holds:

$$\gamma|M_0| \leq \sqrt{\gamma^2 M_0^2 + B(2\gamma-1)},$$

which implies that $x_1^+ < 0$ and $x_2^+ > 0$.

Recall here that $1-2\gamma<0$. Therefore, f_0^+ reaches its maximum on R^+ at x_2^+. It follows that:

$$\delta^+_{L,*,t} \quad = \quad \frac{\sqrt{\gamma^2 M_0^2 + B(2\gamma-1)} - \gamma M_0}{2\gamma-1}$$

and:

$$f_0^+(\delta^+_{L,*,t}) \quad = \quad \lambda^+(\delta^+_{L,*,t}) \left(\frac{\sqrt{\gamma^2 M_0^2 + B(2\gamma-1)} + M_0(\gamma-1)}{2\gamma-1} \right). \tag{4.7}$$

Let now $f_0^- = J^-$. As above, f_0^- can be rewritten as:

$$f_0^-(\delta^-) \quad = \quad \lambda^-(\delta^-)(\delta^- - M_0(t,s,y)).$$

and its derivative is given by:

$$(f_0^-)'(\delta^-) \quad = \quad \frac{\lambda^-(\delta^-)}{B+(\delta^-)^2} \left((1-2\gamma)\,(\delta^-)^2 + 2\gamma M_0 \delta^- + B \right).$$

The function $(f_0^-)'$ vanishes at the two points:

$$x_1^- = \frac{\gamma M_0 - \sqrt{\gamma^2 M_0^2 + B(2\gamma-1)}}{2\gamma-1}, \qquad x_2^- = \frac{\gamma M_0 + \sqrt{\gamma^2 M_0^2 + B(2\gamma-1)}}{2\gamma-1},$$

and, using the same reasoning as before, it can be proved that $x_1^- < 0$ and $x_2^- > 0$. Hence, there holds:

$$\delta^-_{L,*,t} \quad = \quad \frac{\gamma M_0 + \sqrt{\gamma^2 M_0^2 + B(2\gamma-1)}}{2\gamma-1}.$$

and

$$f_0^-(\delta^-_{L,*,t}) \quad = \quad \lambda^-(\delta^-_{L,*,t}) \left(\frac{\sqrt{\gamma^2 M_0^2 + B(2\gamma-1)} + M_0(1-\gamma)}{2\gamma-1} \right). \tag{4.8}$$

Finally, using (4.7) and (4.8), $J_0(t,s,y) = f_0^+(\delta^+_{L,*,t}) + f_0^-(\delta^-_{L,*,t})$ can be written as follows:

$$\begin{aligned} J_0(t,s,y) = {} & \lambda^+(\delta^+_{L,*,t}) \left(\frac{\sqrt{\gamma^2 M_0^2 + B(2\gamma-1)} + M_0(\gamma-1)}{2\gamma-1} \right) \\ & + \lambda^-(\delta^-_{L,*,t}) \left(\frac{\sqrt{\gamma^2 M_0^2 + B(2\gamma-1)} + M_0(1-\gamma)}{2\gamma-1} \right). \end{aligned}$$

and it is straightforward to see that J_0 is independent of q_1.

Equation (4.2) becomes:

$$(\partial_t + \mathcal{L}_1 + \mathcal{L}_2)u + J_0(t,s,y) \quad = \quad 0.$$

This equation is solved by separately cancelling its constant and linear part in terms of q_1, leading to the following equations:

$$\begin{array}{rcl} (0) & : & (\partial_t + \mathcal{L}_1)\theta_0 + J_0(t,s,y) = 0, \\ (1) & : & (\partial_t + \mathcal{L}_1)\theta_1 - \Delta_t \mu S_t = 0. \end{array}$$

The function θ_1, satisfying the terminal condition $\theta_1(T,s,y) = h(s)$, is given by the the Feynman-Kac formula

$$\theta_1(t,s,y) = E^{\mathcal{P}}_{t,s,y}(h(S_T)) - \mu E^{\mathcal{P}}_{t,s,y}(\int_t^T \Delta_u S_u du).$$

The quantity $M_0(t,s,y)$ can now be computed:

$$M_0(t,s,y) \quad = \quad C_{\mathcal{Q}}(t,s,y) - C_{\mathcal{P}}(t,s,y) + \mu E^{\mathcal{P}}_{t,s,y}(\int_t^T \Delta_u S_u du).$$

As for θ_0, given the terminal condition $\theta_0(T,s,y) = 0$, it is given using Feynman-Kac formula, by:

$$\theta_0(t,s,y) = E_{t,s,y}\left(\int_t^T J_0(u,S_u,y_u)du\right).$$

As a conclusion, $u_0(t,s,y,q_1,x) = x + \theta_0(t,s,y) + q_1\left(C_{\mathcal{P}}(t,s,y) - \mu E^{\mathcal{P}}_{t,s,y}(\int_t^T \Delta_u S_u du)\right)$ is the solution of the HJB equation (4.2). Finally, proceeding as for Theorem 3.5.2 in [17], we prove that the verification theorem holds (see Appendix 8.4 for details), so that u_0 is indeed equal to the value function. □

4.1.2 Interpretation of the strategy

The quantity $C_{\mathcal{P}}(t,s,y)$ represents the expected payoff of the option under the historical probability measure $\mathcal{P}$. In addition, the quantity:

$$\mu E^{\mathcal{P}}_{t,s,y}\left(\int_t^T \Delta(u,S_u,y_u)S_u du\right) \equiv E^{\mathcal{P}}_{t,s,y}\left(\int_t^T \Delta(u,S_u,y_u)dS_u\right)$$

represents the cost of delta-hedging the option when there is a trend in the dynamics of the underlying. Thus, the indifference price for the option under the probability measure $\mathcal{P}$ is $C_{\mathcal{P}}(t,s,y) - \mu E^{\mathcal{P}}_{t,s,y}\left(\int_t^T \Delta(u,S_u,y_u)S_u du\right)$, and the quantity:

$$M_0(t,s,y) = C_{\mathcal{Q}}(t,s,y) - \left(C_{\mathcal{P}}(t,s,y) - \mu E^{\mathcal{P}}_{t,s,y}\left(\int_t^T \Delta(u,S_u,y_u)S_u du\right)\right)$$

is naturally interpreted as the difference between the option price under the risk-neutral probability $\mathcal{Q}$ and its indifference price under the historic probability $\mathcal{P}$.

Let the functions f, g be defined by:

$$f(x) = \frac{\gamma x + \sqrt{\gamma^2 x^2 + B(2\gamma - 1)}}{2\gamma - 1}, \quad g(x) = \frac{\sqrt{\gamma^2 x^2 + B(2\gamma - 1)} - \gamma x}{2\gamma - 1}.$$

By simple differentiation, it can be shown:

$$\begin{aligned}
\frac{\partial \delta^-_{L,*,t}}{\partial M_0} &\equiv f'(M_0) = \frac{\gamma}{2\gamma - 1} \frac{\sqrt{\gamma^2 M_0^2 + B(2\gamma - 1)} + \gamma M_0}{\sqrt{\gamma^2 M_0^2 + B(2\gamma - 1)}} \geq 0, \\
\frac{\partial \delta^+_{L,*,t}}{\partial M_0} &\equiv g'(M_0) = \frac{\gamma}{2\gamma - 1} \frac{\gamma M_0 - \sqrt{\gamma^2 M_0^2 + B(2\gamma - 1)}}{\sqrt{\gamma^2 M_0^2 + B(2\gamma - 1)}} \leq 0.
\end{aligned}$$

It follows that the bid distance $\delta^-_{L,*,t} = f(M_0)$ is an increasing function of M_0 while the ask distance $\delta^+_{L,*,t} = g(M_0)$ is a decreasing function of M_0. Indeed, if M_0 increases, it becomes more rational for the market maker to sell the option. Thus, she decreases both her bid quote ($\delta^-_{L,*,t}$ increases) and her ask quote ($\delta^+_{L,*,t}$ decreases). In this way, the ask quote is more likely, and the bid quote, less likely to be executed.
Following the same reasoning, if M_0 decreases, it becomes more profitable to buy the option. Thus, the market maker increases both her bid quote ($\delta^-_{L,*,t}$ decreases) and her ask quote ($\delta^+_{L,*,t}$ increases).
In the particular case where $M_0 = 0$, the market price of the option is equal to its indifference price under $\mathcal{P}$. In this case $\delta^-_{L,*,t} = \delta^+_{L,*,t} = \frac{B}{\sqrt{2\gamma - 1}}$. This means that the bid quote C^b_t and the ask quote C^a_t are symmetric around the mid-price $C_{\mathcal{Q}}$, and the market maker makes no directional bets.

It is also interesting to study the impact of the parameters on the bid-ask spread. Recall that the bid-ask spread $\mathcal{S}_{L,*,t}$ is simply $\delta^-_{L,*,t} + \delta^+_{L,*,t}$, so that $\mathcal{S}_{L,*,t} = 2\frac{\sqrt{\gamma^2 M_0^2 + B(2\gamma - 1)}}{2\gamma - 1}$. Differentiating with respect to the variable γ yields:

$$\frac{\partial \mathcal{S}_{L,*,t}}{\partial \gamma} = -\frac{2\left(B\left(2\gamma - 1\right) + \gamma M_0^2\right)}{\left(2\gamma - 1\right)^2 \sqrt{\gamma^2 M_0^2 + B\left(2\gamma - 1\right)}} < 0.$$

This means that the bid-ask spread $\mathcal{S}_{L,*,t}$ is a decreasing function of the parameter γ. Indeed when γ increases, the intensity of arrivals of market orders decreases and the probability of execution of a quote at a distance δ from the mid-price decreases. Consequently, the market maker contracts her bid-ask spread and places her quotes closer to the mid price. Moreover, the spread $\mathcal{S}_{L,*,t}$ increases with the parameter B, and does not depend on A. It can also be noticed that $\mathcal{S}_{L,*,t}$ is an increasing function of $|M_0|$, which means that the market maker widens her spread when the gap between the indifference price and the market price becomes significant.

4.2 The case $\beta = 1$ (linear market impact)

The intensities of arrivals of market orders are now given by:

$\lambda^+(\delta^+) = \frac{A}{(B+\delta^+)^\gamma}$ and $\lambda^-(\delta^-) = \frac{A}{(B+\delta^-)^\gamma}$.
The optimization problem is solved similarly to the previous case.

4.2.1 Analytic solution

Proposition 4.2. *Let* $\mathcal{S} = \frac{B}{\gamma}$, *and* $M_0(t,s,y) = C_{\mathcal{Q}}(t,s,y) - C_{\mathcal{P}}(t,s,y) + \mu E^{\mathcal{P}}_{t,s,y}\left(\int_t^T \Delta_u S_u du\right)$.

The optimal controls $(\delta^+_{L,*}, \delta^-_{L,*})$ *are:*

$$\delta^+_{L,*,t} = \left(\frac{B - \gamma M_0(t,s,y)}{\gamma - 1}\right)^+, \tag{4.9}$$

$$\delta^-_{L,*,t} = \left(\frac{B + \gamma M_0(t,s,y)}{\gamma - 1}\right)^+, \tag{4.10}$$

and the value function is:

$$u_0(t,s,y,q_1,x) = x + \theta_0(t,s,y) + q_1\left(C_{\mathcal{P}}(t,s,y) - \mu E^{\mathcal{P}}_{t,s,y}\left(\int_t^T \Delta_u S_u du\right)\right), \tag{4.11}$$

where

$$\theta_0(t,s,y) = E_{t,s,y}\left(\int_t^T J_0(u,S_u,y_u)du\right)$$

and:

$$J_0(t,s,y) = \begin{cases} \frac{A(\gamma-1)^{\gamma-1}}{\gamma^\gamma(B-M_0(t,s,y))^{\gamma-1}} + \frac{A}{(B-C)^\gamma}(-M_0(t,s,y)) & \text{if} \quad M_0(t,s,y) \in]-\infty, -\mathcal{S}], \\ \frac{A(\gamma-1)^{\gamma-1}}{\gamma^\gamma(B-M_0(t,s,y))^{\gamma-1}} - \frac{A(1-\gamma)^{\gamma-1}}{(-\gamma)^\gamma(B+M_0(t,s,y))^{\gamma-1}} & \text{if} \quad M_0(t,s,y) \in [-\mathcal{S}, \mathcal{S}], \\ -\frac{A(1-\gamma)^{\gamma-1}}{(-\gamma)^\gamma(B+M_0(t,s,y))^{\gamma-1}} + \frac{A}{B^\gamma}(M_0(t,s,y)) & \text{if} \quad M_0(t,s,y) \in [\mathcal{S}, +\infty[. \end{cases}$$

The proof of Proposition 4.2 is essentially identical to that of Proposition 4.1. For the sake of completeness, it is given in Appendix (8.2).

4.2.2 Interpretation of the strategy

Similarly to the approach used in 4.2, the following derivatives are computed:

$$\begin{aligned} \frac{\partial \delta^+_{L,*,t}}{\partial M_0} &= -\frac{\gamma}{\gamma-1} 1_{\{M_0 \le \mathcal{S}\}} \le 0, \\ \frac{\partial \delta^-_{L,*,t}}{\partial M_0} &= \frac{\gamma}{\gamma-1} 1_{\{M_0 \ge -\mathcal{S}\}} \ge 0. \end{aligned}$$

The results are qualitatively identical to the case $\beta = \frac{1}{2}$, that is, the distance $\delta^+_{L,*,t}$ of the ask-quote to the mid-price is a decreasing function of the mispricing term M_0, and the distance $\delta^-_{L,*,t}$ of the bid quote to the mid-price is an increasing function of the mispricing term M_0.

Now, the expression for the spread $\mathcal{S}_{L,*,t} = \delta^+_{L,*,t} + \delta^-_{L,*,t}$ is:

$$\mathcal{S}_{L,*,t} = \begin{cases} \frac{B-\gamma M_0(t,s,y)}{\gamma-1} & \text{if} \quad M_0(t,s,y) \leq -\mathcal{S}, \\ \frac{2B}{\gamma-1} & \text{if} \quad M_0(t,s,y) \in [-\mathcal{S}, \mathcal{S}], \\ \frac{B+\gamma M_0(t,s,y)}{\gamma-1} & \text{if} \quad M_0(t,s,y) \geq \mathcal{S}, \end{cases}$$

so that

$$\frac{\partial \mathcal{S}_{L,*,t}}{\partial \gamma} = \begin{cases} \frac{M_0(t,s,y)-B}{(\gamma-1)^2} & \text{if} \quad M_0(t,s,y) \leq -\mathcal{S}, \\ -\frac{2B}{(\gamma-1)^2} & \text{if} \quad M_0(t,s,y) \in [-\mathcal{S}, \mathcal{S}], \\ -\frac{M_0(t,s,y)+B}{(\gamma-1)^2} & \text{if} \quad M_0(t,s,y) \geq \mathcal{S}. \end{cases}$$

Again, $\mathcal{S}_{L,*,t}$ is a decreasing function of the parameter γ, as in the case $\beta = \frac{1}{2}$.

5 Optimization problem for a risk-averse market maker

In order to risk-manage her strategy, the market maker may want to solve a different optimization problem for the optimal distances $\left(\delta^-_{t,*}, \delta^+_{t,*}\right)$. Typically, a mean-variance approach would seek to maximize the expected wealth at maturity while keeping its variance under control.
In this section, the variance of final wealth is going to be approximated in such a way that the HJB equation retain its analytical tractability.
Under the assumption that $\delta^+, \delta^- \ll C_{\mathcal{Q}}$, we can write:

$$X_T \sim \int_t^T C_{\mathcal{Q}}(u, S_u, y_u) dN^+_u - \int_t^T C_{\mathcal{Q}}(u, S_u, Y_u) dN^-_u + \int_0^T q_{2,u} dS_u.$$

so that the conditional variance at time t of X_T is:

$$V(X_T|\mathcal{F}_t) \sim E^{\mathcal{P}}\left(\int_t^T C^2_{\mathcal{Q}}(u, S_u, y_u)\left(\lambda^+_u + \lambda^-_u\right) du + \int_t^T q^2_{1,u} \Delta^2_u \sigma^2(y_u) S^2_u du|\mathcal{F}_t\right).$$

Moreover, there holds (using similar notations)

$$V(q_{1,T} h(S_T)|\mathcal{F}_t) \sim E^{\mathcal{P}}\left(\int_t^T E^{\mathcal{P}}\left(h^2(S_T)|\mathcal{F}_u\right)\left(\lambda^+_u + \lambda^-_u\right) du|\mathcal{F}_t\right),$$

and:

$$Cov(X_T, q_{1,T} h(S_T)|\mathcal{F}_t) \sim -E^{\mathcal{P}}\left(\int_t^T C_{\mathcal{Q}}(u, S_u, y_u) C_{\mathcal{P}}(u, S_u, y_u)\left(\lambda^+_u + \lambda^-_u\right) du|\mathcal{F}_t\right).$$

As a consequence, $Var\left(X_T + q_{1,T} h(S_T)|\mathcal{F}_t\right)$ can be approximated by $E^{\mathcal{P}}\left(\int_t^T \left(q^2_{1,u} V_u + Z_u\right) du|\mathcal{F}_t\right)$ where

$$Z_t = \left(C^2_{\mathcal{Q}} + E^{\mathcal{P}}\left(h^2(S_T)|\mathcal{F}_t\right) - 2C_{\mathcal{Q}} C_{\mathcal{P}}\right)\left(\lambda^+_{L,t} + \lambda^-_{L,t}\right), \tag{5.1}$$

$$V_t = \Delta^2_t \sigma^2(y_t) S^2_t. \tag{5.2}$$

A final approximation consists in replacing the quantities λ_u^+ and λ_u^-, which depend on δ_u^+ and δ_u^-, by $\lambda_{L,u}^+ = \lambda^+(\delta_{L,*,u}^+)$ and $\lambda_{L,u}^- = \lambda^-(\delta_{L,*,u}^-)$.
The stochastic control problem we consider is now introduced. The value function u^ϵ of the market maker is defined as follows:

$$u^\epsilon(t,s,y,q_1,x) \quad = \quad Sup_{\left\{(\delta_t^-,\delta_t^+)\in\mathcal{A}\right\}} E^{\mathcal{P}}_{t,s,y,q_1,x}\left(H^\epsilon(t,T,S_T,y_T,q_{1,T},X_T)\right),$$

with H^ϵ given by:

$$H^\epsilon(t,T,S_T,y_T,q_{1,T},X_T) \quad = \quad X_T + q_{1,T}h(S_T) - \epsilon\int_t^T \left(q_{1,u}^2 V_u + Z_u\right) du.$$

Let u_0^ϵ be the solution of the following HJB equation:

$$\left(\partial_t + \mathcal{L}\right) u_0^\epsilon + sup_{\{(\delta^-,\delta^+)\in\mathcal{A}\}} J^\epsilon(\delta^-,\delta^+) \quad = \quad \epsilon\left(q_1^2 V + Z\right), \tag{5.3}$$

where as in Section 4 $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$ and $J^\epsilon = J^{-,\epsilon} + J^{+,\epsilon}$ with

$$\begin{aligned} J^{+,\epsilon}(\delta^+) &= \lambda^+(\delta^+)\left(u_0^\epsilon(t,s,y,q_1-1,x+(C_{\mathcal{Q}}+\delta^+)) - u_0^\epsilon(t,s,y,q_1,x)\right), \\ J^{-,\epsilon}(\delta^-) &= \lambda^-(\delta^-)\left(u_0^\epsilon(t,s,y,q_1+1,x-(C_{\mathcal{Q}}-\delta^-)) - u_0^\epsilon(t,s,y,q_1,x)\right). \end{aligned}$$

Also note that in the RHS of Equation (5.3), we have denoted by V, Z the function of the state variables defiend by (5.1)(5.2).
Again, if u_0^ϵ is smooth, finite and has polynomial growth, it coincides with the value function u^ϵ.
The optimal spreads $(\delta_{*,t}^+, \delta_{*,t}^-)$ solve the optimal control problem:

$$(\delta_{*,t}^+, \delta_{*,t}^-) \quad = \quad Argsup_{\left\{\delta_t^+,\delta_t^-\right\}} E^{\mathcal{P}}\left(H^\epsilon(t,T,S_T,y_T,q_{1,T},X_T) | S_t = s, y_t = y, q_{1,t} = q_1, X_t = x\right),$$

and the cases $\beta = \frac{1}{2}$ and $\beta = 1$ are now studied.

5.1 The case $\beta = \frac{1}{2}$ (square-root market impact)

5.1.1 Analytic approximation

We state and prove the

Proposition 5.1. *The optimal controls $(\delta_{*,t}^+, \delta_{*,t}^-)$ can be approximated at the first order in the penalization parameter ϵ by $(\hat{\delta}_{*,t}^-, \hat{\delta}_{*,t}^+)$ defined as*

$$\hat{\delta}_{*,t}^+ \quad = \quad \frac{-\gamma M^+ + \sqrt{\gamma^2(M^+)^2 + B(2\gamma-1)}}{2\gamma-1}, \tag{5.4}$$

$$\hat{\delta}_{*,t}^- \quad = \quad \frac{\gamma M^- + \sqrt{\gamma^2(M^-)^2 + B(2\gamma-1)}}{2\gamma-1}, \tag{5.5}$$

where the quantities $M^+(t,s,y,q_1)$ *and* $M^-(t,s,y,q_1)$ *are given by*

$$\begin{aligned} M^+(t,s,y,q_1) &= M_0(t,s,y) + \epsilon M_1(t,s,y,q_1), \\ M^-(t,s,y,q_1) &= M_0(t,s,y) + \epsilon M_2(t,s,y,q_1), \end{aligned}$$

and

$$\begin{aligned} M_1(t,s,y,q_1) &= -\theta_3(t,s,y) + (1-2q_1)\theta_4(t,s,y), \\ M_2(t,s,y,q_1) &= -\theta_3(t,s,y) - (1+2q_1)\theta_4(t,s,y), \end{aligned}$$

with

$$\begin{aligned} \theta_4(t,s,y) &= -E^{\mathcal{P}}_{t,s,y}\left(\int_t^T V_u du\right), \\ \theta_3(t,s,y) &= -2E^{\mathcal{P}}_{t,s,y}\left(\int_t^T \theta_4(u,s_u,y_u)\left(\lambda^+(\delta^+_{L,*,u}) - \lambda^-(\delta^-_{L,*,u})\right) du\right). \end{aligned}$$

Moreover, there holds

$$|\delta^+_{*,t} - \hat{\delta}^+_{*,t}| = O\big(\epsilon^2\big),$$
$$|\delta^-_{*,t} - \hat{\delta}^-_{*,t}| = O\big(\epsilon^2\big).$$

Proof. Let u_0^ϵ be the solution of the HJB equation (5.3). Under the assumption that $\epsilon \sim 0$, a singular perturbation technique can be performed with respect to the parameter ϵ:

$$u_0^\epsilon(t,s,y,q_1,x) = x + \sum_{k=0}^{+\infty} \epsilon^k v_k(t,s,y,q_1). \tag{5.6}$$

If $\epsilon = 0$, then the utility is obviously linear. This implies that $u_0^0(t,s,y,q_1,x) = x + v_0(t,s,y,q_1) = u_0(t,s,y,q_1,x)$ where u_0 is the function defined in (4.5). Therefore, it is assumed that v_0 has the following form:

$$v_0(t,s,y,q_1) = \theta_0(t,s,y) + q_1\theta_1(t,s,y).$$

Since the utility function contains a constraint on the square of the option inventory q_1, v_1 is assumed to have the following form:

$$v_1(t,s,y,q_1) = \theta_2(t,s,y) + q_1\theta_3(t,s,y) + q_1^2\theta_4(t,s,y).$$

In order to solve the HJB equation, the jump terms $J^{+,\epsilon}$ and $J^{-,\epsilon}$ have to be calculated.

Let $f^+ = J^{+,\epsilon}$. The function f^+ writes

$$\begin{aligned} f^+(\delta^+) &= \lambda^+(\delta^+)(u(t,s,y,q_1-1,x+(c+\delta^+)) - u(t,s,y,q_1,x)), \\ &= \lambda^+(\delta^+)\left(\delta^+ + M_0(t,s,y) + \epsilon M_1(t,s,y,q_1) + \epsilon^2 R^+(t,s,y,q_1)\right), \end{aligned}$$

where:

$$
\begin{aligned}
M_1(t,s,y,q_1) &= v_1(t,s,y,q_1-1) - v_1(t,s,y,q_1),\\
&= -\theta_3(t,s,y) + (1-2q_1)\theta_4(t,s,y),
\end{aligned}
$$

and:

$$
R^+(t,s,y,q_1) = \sum_{k=2}^{+\infty} \epsilon^{k-2}\left(v_k\left(t,s,y,q_1-1\right) - v_k\left(t,s,y,q_1\right)\right).
$$

Let $M^+(t,s,y,q_1) = M_0(t,s,y)+\epsilon M_1(t,s,y,q_1)$. In order to determine $\delta^+_{*,t} = ArgSup_{\{x\geq 0\}} f^+(x)$, the derivative $(f^+)'$ is computed:

$$
(f^+)'(\delta^+) = \frac{\lambda^+(\delta^+)}{B+\left(\delta^+\right)^2}\Big(\left(\delta^+\right)^2(1-2\gamma) - 2\gamma\left(M^+(t,s,y) + \epsilon^2 R^+(t,s,y,q_1)\right)\delta^+ + B\Big).
$$

The function $(f^+)'$ has two zeros x_1^+ and x_2^+:

$$
\begin{aligned}
x_1^+ &= \frac{-\gamma\left(M^+ + \epsilon^2 R^+\right) - \sqrt{\gamma^2\left(M^+ + \epsilon^2 R^+\right)^2 + B(2\gamma-1)}}{2\gamma-1} < 0,\\
x_2^+ &= \frac{-\gamma\left(M^+ + \epsilon^2 R^+\right) + \sqrt{\gamma^2\left(M^+ + \epsilon^2 R^+\right)^2 + B(2\gamma-1)}}{2\gamma-1} > 0.
\end{aligned}
$$

Hence, $\delta^+_{*,t} = x_2^+$, and a Taylor expansion yields:

$$
\delta^+_{*,t} = \frac{-\gamma M^+ + \sqrt{\gamma^2(M^+)^2 + B(2\gamma-1)}}{2\gamma-1} + O\big(\epsilon^2\big)
$$

or else

$$
\delta^+_{*,t} = \delta^+_{L,*,t} + \epsilon\left(-\frac{\gamma}{2\gamma-1}M_1 + \frac{\gamma^2 M_0 M_1}{\sqrt{\gamma^2 M_0^2 + B(2\gamma-1)}}\right) + O\big(\epsilon^2\big).
$$

It will be useful to write $f^+(\delta^+_{*,t})$ as the sum of $f_0^+(\delta^+_{L,*,t})$ plus a perturbation term: noticing that $f^+(x) = f_0^+(x)+\epsilon\lambda^+(x)M_1(t,s,y,q_1)+O(\epsilon^2)$ and using a Taylor expansion, the following obtains:

$$
\begin{aligned}
f^+(\delta^+_{*,t}) &= f^+(\delta^+_{L,*,t}) + (f^+)'(\delta^+_{L,*,t})(\delta^+_{*,t} - \delta^+_{L,*,t}) + O\big((\delta^+_{*,t} - \delta^+_{L,*,t})^2\big),\\
&= f_0^+(\delta^+_{L,*,t}) + \epsilon\lambda^+(\delta^+_{L,*,t})M_1 + \epsilon\left(-\frac{\gamma}{2\gamma-1}M_1 + \frac{\gamma^2 M_0 M_1}{\sqrt{\gamma^2 M_0^2 + B(2\gamma-1)}}\right)(f^+)'(\delta^+_{L,*,t}) + O\big(\epsilon^2\big).
\end{aligned}
$$

Using that $(f^+)'(x) = (f_0^+)'(x) + O(\epsilon)$ and $(f_0^+)'(\delta^+_{L,*,t}) = 0$, there holds:

$$
f^+(\delta^+_{*,t}) = f_0^+(\delta^+_{L,*,t}) + \epsilon\lambda^+(\delta^+_{L,*,t})M_1 + O\big(\epsilon^2\big).
$$

Similarly, let now $f^- = J^{-,\epsilon}$. Using the form of the value function suggested in (5.6), the function f^- can be written as

$$\begin{aligned} f^-(\delta^-) &= \lambda^-(\delta^-)(u(t,s,y,q_1+1,x-(c-\delta^-)) - u(t,s,y,q_1,x)), \\ &= \lambda^-(\delta^-)\left(\delta^- - \left(M_0(t,s,y) + \epsilon M_2(t,s,y,q_1) + \epsilon^2 R^-(t,s,y,q_1)\right)\right) \end{aligned}$$

where

$$\begin{aligned} M_2(t,s,y,q_1) &= -\left(v_1(t,s,y,q_1+1) - v_1(t,s,y,q_1)\right), \\ &= -\theta_3(t,s,y) - (1+2q_1)\theta_4(t,s,y), \end{aligned}$$

and

$$R^-(t,s,y,q_1) = -\sum_{k=2}^{+\infty} \epsilon^{k-2}\left(v_k(t,s,y,q_1+1) - v_k(t,s,y,q_1)\right).$$

The quantity $M^-(t,s,y,q_1) = M_0(t,s,y) + \epsilon M_2(t,s,y,q_1)$ is introduced to simplify the notations. Standard computations yield that

$$(f^-)'(\delta^-) = \frac{\lambda^-(\delta^-)}{B + (\delta^-)^2}\left((1-2\gamma)(\delta^-)^2 + 2\gamma\left(M^- + \epsilon^2 R^-\right)\delta^- + B\right).$$

Thus, $(f^-)'$ vanishes at the points

$$\begin{aligned} x_1^- &= \frac{\gamma\left(M^- + \epsilon^2 R^-\right) - \sqrt{\gamma^2\left(M^- + \epsilon^2 R^-\right)^2 + B(2\gamma-1)}}{2\gamma-1}, \\ x_2^- &= \frac{\gamma\left(M^- + \epsilon^2 R^-\right) + \sqrt{\gamma^2\left(M^- + \epsilon^2 R^-\right)^2 + B(2\gamma-1)}}{2\gamma-1}. \end{aligned}$$

Since $x_1^- < 0$, $x_2^- > 0$ and $\gamma > 1$, then $\delta_{*,t}^- = ArgSup_{\{x\geq 0\}} f^-(x) = x_2^-$. It follows that:

$$\delta_{*,t}^- = \frac{\gamma M^- + \sqrt{\gamma^2(M^-)^2 + B(2\gamma-1)}}{2\gamma-1} + O\big(\epsilon^2\big).$$

Using again a Taylor expansion with respect to ϵ, the last relation yields:

$$\delta_{*,t}^- = \delta_{L,*,t}^- + \epsilon\left(\frac{\gamma}{2\gamma-1}M_2 + \frac{\gamma^2 M_0 M_2}{\sqrt{\gamma^2 M_0^2 + B(2\gamma-1)}}\right) + O\big(\epsilon^2\big).$$

The quantity $f^-(\delta_{*,t}^-)$ can be written as the sum of $f_0^-(\delta_{L,*,t}^-)$ plus a perturbation term. Indeed, since:

$$f^-(x) = f_0^-(x) - \epsilon\lambda^-(x)M_2(t,s,y,q_1) + O\big(\epsilon^2\big),$$

it follows that

$$\begin{aligned} f^-(\delta^-_{*,t}) &= f^-(\delta^-_{L,*,t}) + (f^-)'(\delta^-_{L,*,t})(\delta^-_{*,t} - \delta^-_{L,*,t}) + O\big((\delta^-_{*,t} - \delta^-_{L,*,t})^2\big)\,, \\ &= f_0^-(\delta^-_{L,*,t}) - \epsilon\lambda^-(\delta^-_{L,*,t})M_2 + \epsilon\left(\frac{\gamma}{2\gamma-1}M_2 + \frac{\gamma^2 M_0 M_2}{\sqrt{\gamma^2 M_0^2 + B(2\gamma-1)}}\right)(f^-)'(\delta^-_{L,*,t}) + O\big(\epsilon^2\big)\,. \end{aligned}$$

Recalling that $(f^-)'(x) = (f_0^-)'(x) + O(\epsilon)$ and $(f_0^-)'(\delta^-_{L,*,t}) = 0$, the previous equation becomes:

$$f^-(\delta^-_{*,t}) = f_0^-(\delta^-_{L,*,t}) - \epsilon\lambda^-(\delta^-_{L,*,t})M_2 + O\big(\epsilon^2\big)\,.$$

Using the above approximations for $f^+(\delta^+_{*,t})$ and $f^-(\delta^-_{*,t})$, the quantity $J^\epsilon(\delta^-_{*,t},\delta^+_{*,t}) \equiv f^-(\delta^-_{*,t}) + f^+(\delta^+_{*,t})$ can be written as

$$\begin{aligned} J^\epsilon(\delta^-_{*,t},\delta^+_{*,t}) &= f_0^+(\delta^+_{L,*,t}) + \epsilon M_1\lambda^+(\delta^+_{L,*,t}) + f_0^-(\delta^-_{L,*,t}) - \epsilon M_2\lambda^-(\delta^-_{L,*,t}) + O\big(\epsilon^2\big)\,, \\ &= J_0(\delta^-_{L,*,t},\delta^+_{L,*,t}) + \epsilon\left(M_1\lambda^+(\delta^+_{L,*,t}) - M_2\lambda^-(\delta^-_{L,*,t})\right) + O\big(\epsilon^2\big)\,. \end{aligned}$$

Rearranging the expression for $J^\epsilon(\delta^-_{*,t},\delta^+_{*,t})$ in powers of ϵ yields

$$J^\epsilon(\delta^-_{*,t},\delta^+_{*,t}) = J_0(t,s,y) + \epsilon J_1(t,s,y,q_1) + O\big(\epsilon^2\big)\,,$$

where $J_1(t,s,y,q_1) = J_{1,0}(t,s,y) + q_1 J_{1,1}(t,s,y)$ and

$$\begin{aligned} J_{1,0}(t,s,y) &= \lambda^+(\delta^+_{L,*,t})(-\theta_3+\theta_4) - \lambda^-(\delta^-_{L,*,t})(-\theta_3-\theta_4), \\ J_{1,1}(t,s,y) &= -2\theta_4\left(\lambda^+(\delta^+_{L,*,t}) - \lambda^-(\delta^-_{L,*,t})\right)\,. \end{aligned}$$

The HJB equation (5.3) is now rewritten in increasing powers of ϵ. The zeroth order term leads to the equation:

$$(\partial_t + \mathcal{L}_1 + \mathcal{L}_2)(x + \theta_0 + q_1\theta_1) + J_0(t,s,y) = 0$$

with the final conditions

$$\theta_0(T,s,y) = 0, \quad \theta_1(T,s,y) = h(s).$$

The functions θ_0, θ_1 are then easily calculated:

$$\begin{aligned} \theta_1(t,s,y) &= C_{\mathcal{P}}(t,s,y) - \mu E^{\mathcal{P}}_{t,s,y}\left(\int_t^T \Delta(u,S_u,y_u)S_u du\right), \\ \theta_0(t,s,y) &= E_{t,s,y}\left(\int_t^T J_0(u,S_u,y_u)du\right). \end{aligned}$$

The first order term yields the following equation:

$$(\partial_t + \mathcal{L})(\theta_2 + q_1\theta_3 + q_1^2\theta_4) + J_1(t,s,y) = q^2 V + Z,$$

and the terms of this equation are now sorted by their orders in q_1. The second order term in q_1 provides the following equation for θ_4:

$$(\partial_t + \mathcal{L}_1)\theta_4(t, s, y) \;\; = \;\; V.$$

Using the final condition $\theta_4(T, s, y) = 0$, it can be deduced that:

$$\theta_4(t, s, y) \;\; = \;\; -E^{\mathcal{P}}_{t,s,y}\left(\int_t^T V_u du\right).$$

The function θ_3 is calculated using the first order term in q_1:

$$(\partial_t + \mathcal{L}_1)\theta_3(t, s, y) + J_{1,1}(t, s, y) \;\; = \;\; 0,$$

and since it also satisfies the final condition $\theta_3(T, s, y) = 0$, it follows that:

$$\theta_3(t, s, y) \;\; = \;\; E^{\mathcal{P}}_{t,s,y}\left(\int_t^T J_{1,1}(u, s_u, y_u)du\right).$$

Finally, the function θ_2 solves:

$$(\partial_t + \mathcal{L}_1)\theta_2(t, s, y) + J_{1,0}(t, s, y) = Z$$

subject to the final condition $\theta_2(T, s, y) = 0$. Using again the Feynman-Kac formula, the following is obtained:

$$\theta_2(t, s, y) \;\; = \;\; E^{\mathcal{P}}_{t,s,y}\left(\int_t^T \left(J_{1,0} - T\right)(u, s_u, y_u)du\right).$$

As before, a verification theorem can be used, once it is proven that u_0^ϵ is smooth, finite and satisfies a polynomial growth condition. □

5.1.2 Interpretation of the strategy

The effect of the mispricing term M_0 on the optimal bid and ask distances remains the same as in the case without inventory constraints ($\epsilon = 0$), since

$$\begin{aligned}
\frac{\partial \hat{\delta}^+_{*,t}}{\partial M_0} &= \frac{\partial \hat{\delta}^+_{*,t}}{\partial M^+}\frac{\partial M^+}{\partial M_0} = g'(M^+) < 0, \\
\frac{\partial \hat{\delta}^-_{*,t}}{\partial M_0} &= \frac{\partial \hat{\delta}^-_{*,t}}{\partial M^-}\frac{\partial M^-}{\partial M_0} = f'(M^-) > 0.
\end{aligned}$$

The new feature here is the dependence of the distances $\hat{\delta}^-_{*,t}$ and $\hat{\delta}^+_{*,t}$ on the inventory q_1. In order to study this dependence, the partial derivatives of $\hat{\delta}^-_{*,t}$ and $\hat{\delta}^+_{*,t}$ with respect to the variable q_1 are computed:

$$\begin{aligned}
\frac{\partial \hat{\delta}^+_{*,t}}{\partial q_1} &= -2\epsilon\theta_4(t, s, y)\frac{\gamma}{2\gamma - 1}g'(M^+) < 0, \\
\frac{\partial \hat{\delta}^-_{*,t}}{\partial q_1} &= -2\epsilon\theta_4(t, s, y)\frac{\gamma}{2\gamma - 1}f'(M^-) > 0,
\end{aligned}$$

It can be deduced from the expressions above that, when the option inventory q_1 increases, the market maker posts more aggressive ask quotes and more conservative bid quotes in order to reduce her inventory. Thus, the market maker, being risk-averse, adjusts her quoting policy in order not to accumulate a large inventory, thereby avoiding being too exposed to market moves.

The bid-ask spread of the market maker is $\mathcal{S}_{*,t} = \hat{\delta}^{-}_{*,t} + \hat{\delta}^{+}_{*,t}$. Using a Taylor expansion, $\mathcal{S}_{*,t}$ can be approximated as follows:

$$\begin{aligned}
\mathcal{S}_{*,t} &= 2\frac{\sqrt{\gamma^2 M_0^2 + B(2\gamma-1)}}{2\gamma-1} + \epsilon\frac{\gamma}{2\gamma-1}\left[(M_2 - M_1) + \frac{M_0\gamma}{\sqrt{\gamma^2 M_0^2 + B(2\gamma-1)}}(M_1 + M_2)\right] \\
&= \mathcal{S}_{L,*,t} + \epsilon\frac{\gamma}{2\gamma-1}\left[-2\theta_4 + 2\frac{M_0\gamma}{\sqrt{\gamma^2 M_0^2 + B(2\gamma-1)}}(-\theta_3 - 2q_1\theta_4)\right].
\end{aligned}$$

The term $(M_2 - M_1)$ is a positive quantity that increases the bid-ask spread. Indeed, having a constraint on the variance of her final wealth, the risk-averse market maker requires a bigger margin in order to compensate the risk. Therefore, the bid-ask spread is widened. The next term depends on q_1 and M_0, its effect on $\mathcal{S}_{*,t}$ is not straightforward to interpret. It is easier to see the effects of q_1 directly on $\hat{\delta}^{-}_{*,t}$ and $\hat{\delta}^{+}_{*,t}$.

5.2 The case $\beta = 1$ (linear market impact)

5.2.1 Analytic approximation

Proposition 5.2. *The optimal controls $(\delta^{+}_{*,t}, \delta^{-}_{*,t})$ of the market maker can be approximated at the first order in ϵ by $(\hat{\delta}^{-}_{*,t}, \hat{\delta}^{+}_{*,t})$ defined as*

$$\hat{\delta}^{+}_{*,t} = \begin{cases} \frac{B - \gamma M^{+}(t,s,y,q_1)}{\gamma - 1} & \text{if } M^{+}(t,s,y) \le \mathcal{S}, \\ 0 & \text{if } M^{+}(t,s,y) \ge \mathcal{S}, \end{cases} \tag{5.7}$$

$$\hat{\delta}^{-}_{*,t} = \begin{cases} 0 & \text{if } M^{-}(t,s,y) \le -\mathcal{S}, \\ \frac{B + \gamma M^{-}(t,s,y,q_1)}{\gamma - 1} & \text{if } M^{-}(t,s,y) \ge -\mathcal{S} \end{cases} \tag{5.8}$$

and the approximation error is of order 2 in ϵ:

$$\begin{aligned}
|\hat{\delta}^{+}_{*,t} - \delta^{+}_{*,t}| &= O\big(\epsilon^2\big), \\
|\hat{\delta}^{-}_{*,t} - \delta^{-}_{*,t}| &= O\big(\epsilon^2\big).
\end{aligned}$$

The proof, similar to that of Proposition 5.1, is omitted.

5.2.2 Interpretation of the strategy

The effect of the mispricing term M_0 on the optimal bid and ask distances remains the same as in the case without inventory constraints ($\epsilon = 0$), indeed:

$$\begin{aligned}\frac{\partial \hat{\delta}^{+}_{*,t}}{\partial M_0} &= -\frac{\gamma}{\gamma - 1} 1_{\{M^+ \leq \mathcal{S}\}} \leq 0, \\ \frac{\partial \hat{\delta}^{-}_{*,t}}{\partial M_0} &= \frac{\gamma}{\gamma - 1} 1_{\{M^- \geq -\mathcal{S}\}} \geq 0.\end{aligned}$$

The proxies $\hat{\delta}^{-}_{*,t}$ and $\hat{\delta}^{+}_{*,t}$ of the optimal distances depend on the inventory q_1. Their first derivatives with respect to the variable q_1 can be computed explicitly:

$$\begin{aligned}\frac{\partial \hat{\delta}^{+}_{*,t}}{\partial q_1} &= 2\epsilon\theta_4(t,s,y)\frac{\gamma}{\gamma - 1} 1_{\{M^+ \leq \mathcal{S}\}} \leq 0, \\ \frac{\partial \hat{\delta}^{-}_{*,t}}{\partial q_1} &= -2\epsilon\theta_4(t,s,y)\frac{\gamma}{\gamma - 1} 1_{\{M^- \geq -\mathcal{S}\}} \geq 0.\end{aligned}$$

The first derivatives highlights the effect of the option inventory q_1 on the distances $\hat{\delta}^{-}_{*,t}$ and $\hat{\delta}^{+}_{*,t}$. Indeed, if q_1 increases, the market maker lowers her bid and ask quotes, seeking to cut down her option inventory.

6 Numerical Simulations

In this section, Monte Carlo simulations are performed in order to test the performance of the market making strategies previously characterized. It is supposed in this section that the spot process follows a Heston model under $\mathcal{P}$:

$$\begin{aligned}\frac{dS_t}{S_t} &= \mu dt + \sqrt{y_t} dW^{(1)}_t \\ dy_t &= k_R(\theta_R - y_t)dt + \eta_R \sqrt{y_t} dW^{(2)}_t\end{aligned}$$

where $d\left\langle W^{(1)}, W^{(2)}\right\rangle_t = \rho_R dt$.

Market participant evaluate options using $\mathcal{Q}$, under which the spot process S has the following dynamics:

$$\begin{aligned}\frac{dS_t}{S_t} &= r dt + \sqrt{y_t} dW^{*,(1)}_t \\ dy_t &= k_I(\theta_I - y_t)dt + \eta_I \sqrt{y_t} dW^{*,(2)}_t\end{aligned}$$

where $d\left\langle W^{*,(1)}, W^{*,(2)}\right\rangle_t = \rho_I dt$.
The functions a_R, b_R and σ are:

$$\begin{aligned}a_R(y_t) &= k_R\left(\theta_R - y_t\right), \\ b_R(y_t) &= \eta_R \sqrt{y_t}, \\ \sigma(y_t) &= \sqrt{y_t},\end{aligned}$$

The term $C_{\mathcal{Q}}(t,s,y)$ is the option price in the Heston model (see [10]). The term $C_{\mathcal{P}} = E^{\mathcal{P}}((S_T - K)^+|\mathcal{F}_t)$ can be computed explicitly as explained in Appendix (**??**).

The first part of the numerical study is devoted to the comparison between the optimal strategy and a zero-intelligence strategy, in the case of a linear utility function. In the second part, the effect of the misspecification of the parameters is studied.

The numerical simulations are specified as follows: the traded option has a maturity equal to 3 Months ($T = 0.25$) and a strike equal to 100 ($K = 100$). We simulate 1000 Monte Carlo paths of the spot and instantaneous variance $(S_t, v_t)_{\{0\leq t\leq T\}}$ starting from $S_0 = 100$ and $v_0 = 0.04$. It is assumed that there are 6 trading hours per day, and that the market maker refreshes the quotes every 5 minutes. This means that there are $12 \times 6 = 72$ points per day. Since there are approximately 64 business days in a 3 months period, this amounts to $64 \times 72 = 4608$ points per simulated path. At each point, the quantities $C_{\mathcal{Q}}(t,s,y)$ and $C_{\mathcal{P}}(t,s,y)$ are computed using a Fast Fourier Transform method. This simulation task is numerically consuming and was performed using the computing cluster[1] at École Centrale Paris.

For each simulated path, the optimal market making strategy is used from the inception date ($t = 0$) until the maturity date of the option ($t = T$).

6.1 Comparison with a zero-intelligence agent

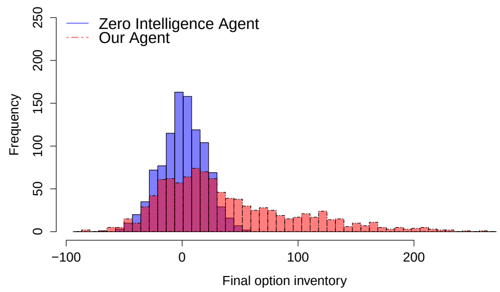
The implied volatility $\Sigma_t(K,T)$ of the call option is defined as usual by $C_{\mathcal{Q}}(t, S_t, y_t) = P_{BS}(t, S_t, \Sigma_t(K,T))$, where P_{BS} denotes the Black-Scholes call price formula. In addition, let $\vartheta_{BS} = \frac{\partial P_{BS}(t,S_t,\Sigma_t(K,T))}{\partial \Sigma_t}$ be the Black-Scholes vega of the option. In this subsection, we suppose that there is an agent who places equidistant bid and ask quotes denoted by $C^b_{ZI,t}$ and $C^a_{ZI,t}$ respectively:

$$\begin{aligned} C^a_{ZI,t} &= C_{\mathcal{Q}}(t, S_t, y_t) + \delta_{ZI,t}, \\ C^b_{ZI,t} &= C_{\mathcal{Q}}(t, S_t, y_t) - \delta_{ZI,t}, \end{aligned}$$

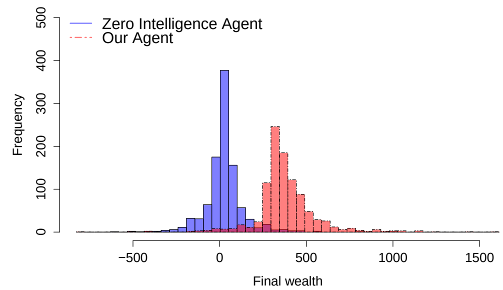
where $\delta_{ZI,t} = 0.005 \times \vartheta_{BS}$. In other words, the "zero-intelligence" agent attempts to earn $0.005\vartheta_{BS}$ for each trade.

The assumption here is that $(k_R, \theta_R, \eta_R, \rho_R) = (k_I, \theta_I, \eta_I, \rho_I) = (4, 0.04, 0.5, -0.4)$ (no misspecification). Besides, the historical drift μ and risk-free rate r are both zero. The results of the numerical simulations are summarized below:

[1] http://www.mesocentre.ecp.fr

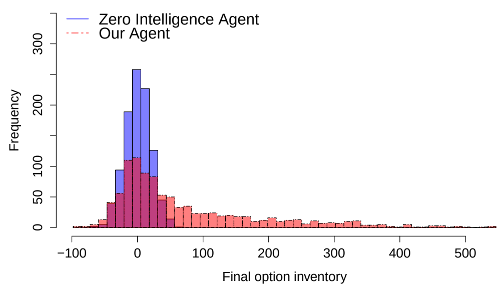


(a) Final value of q_1

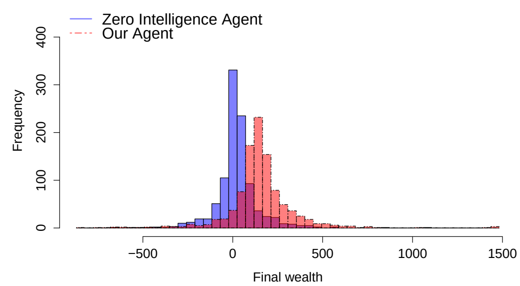


(b) Final value of Wealth

Figure 1: Statistics for $\beta = 0.5$



(a) Final value of q_1



(b) Final value of Wealth

Figure 2: Statistics for $\beta = 1$

Cases	Median	Mean	Std	Skewness	Kurtosis
$\beta = \frac{1}{2}$ and with optimal quotes	23.00	38.42	58.66	0.98	0.63
$\beta = \frac{1}{2}$ and with zero-intelligence	1.00	0.80	19.05	-0.14	-0.02
$\beta = 1$ and with optimal quotes	29.00	72.94	114.14	1.52	1.94
$\beta = 1$ and with zero-intelligence	1.00	1.13	19.90	-0.07	0.00

Table 1: Statistics on final option inventory $q_{1,T}$

Cases	Median	Mean	Std	Skewness	Kurtosis
$\beta = \frac{1}{2}$ and with optimal quotes	356.89	379.64	176.13	0.72	9.38
$\beta = \frac{1}{2}$ and with zero-intelligence	21.34	25.01	116.44	0.82	11.05
$\beta = 1$ and with optimal quotes	144.54	154.90	187.97	0.75	12.24
$\beta = 1$ and with zero-intelligence	17.60	23.32	121.75	0.86	11.51

Table 2: Statistics on final wealth $\mathcal{W}_T$

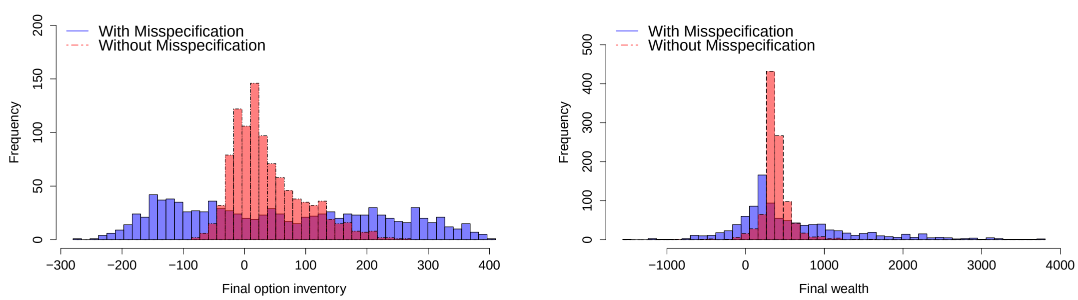
The numerical simulations show that the strategy using the optimal quotes performs better than the zero-intelligence strategy.

6.2 Effect of the misspecification of parameters

In this section, the effect of parameter misspecification in the case of a linear utility function is addressed.

6.2.1 Misspecification of the parameter ρ

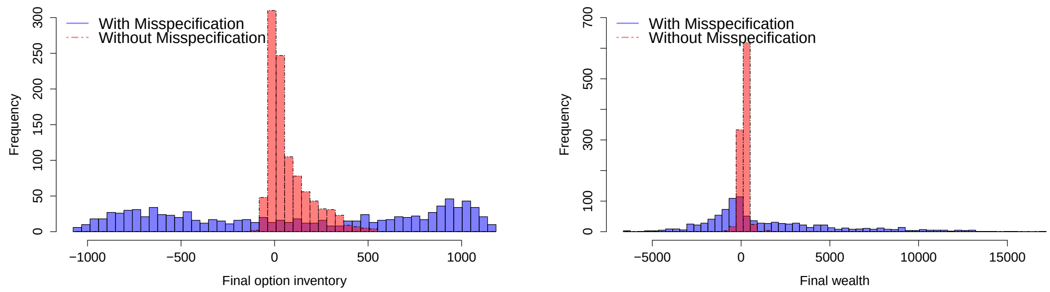
Monte Carlo simulations are performed using the following parameters: $(\mu, k_R, \theta_R, \eta_R, \rho_R) = (0, 4, 0.04, 0.5, -0.4)$ and $(r, k_I, \theta_I, \eta_I, \rho_I) = (0, 4, 0.04, 0.5, -0.9)$. The statistics of 1000 simulations are given below:



(a) Final value of q_1

(b) Final value of Wealth

Figure 3: Statistics for $\beta = 0.5$



(a) Final value of q_1 (b) Final value of Wealth

Figure 4: Statistics for $\beta = 1$

Cases	Median	Mean	Std	Skewness	Kurtosis
$\beta = \frac{1}{2}$ and $\rho_R = \rho_I$	23.00	38.41	58.66	0.97	0.62
$\beta = \frac{1}{2}$ and $\rho_R \neq \rho_I$	42.50	57.30	165.14	0.19	-1.16
$\beta = 1$ and $\rho_R = \rho_I$	29.00	72.942	114.13	1.51	1.93
$\beta = 1$ and $\rho_R \neq \rho_I$	105.50	112.38	699.43	-0.049	-1.48

Table 3: Statistics on final option inventory $q_{1,T}$

In order to interpret these results, it is convenient to use [5], where an approximation of the implied volatility for the Heston model is derived:

$$\Sigma(K,T) \quad = \quad \sqrt{y_0}\left(1 + \frac{1}{4}\frac{\rho\eta}{y_0}\log(\frac{K}{S_0}) + \left(\frac{1}{24} - \frac{5}{48}\rho^2\right)\frac{\eta^2}{y_0^2}\log(\frac{K}{S_0})^2 + O\left(\log(\frac{K}{S_0})^3\right)\right) \tag{6.1}$$

The approximation (6.1), valid for small maturities and log-moneyness, points out the effect of the parameter ρ on the option price when $T \sim 0$. Indeed, at the first order in $\log(\frac{K}{S_t})$, the implied volatility $\Sigma_t(K,T)$ is an increasing function of ρ if $K > S_t$ and a decreasing function if $K < S_t$. Consequently, in each simulation, the evolution of the moneyness $\left(\log(\frac{K}{S_t})\right)$ of the option determines the quantity $C_{\mathcal{P}}(t,s,y) - C_{\mathcal{Q}}(t,s,y)$ and influences the aggressiveness of the quotes on either the bid or the ask side. It can be noticed that the distribution of the option inventory $q_1(T)$ at the maturity date T is more spread out in the case where $\rho_R \neq \rho_I$.

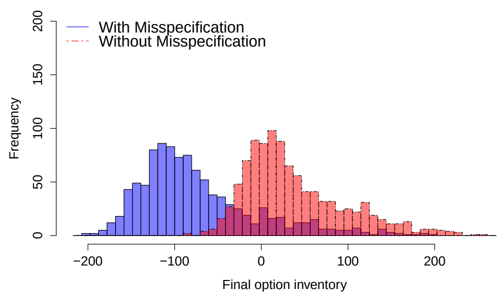
Cases	Median	Mean	Std	Skewness	Kurtosis
$\beta = \frac{1}{2}$ and $\rho_R = \rho_I$	356.89	379.63	176.13	0.72	9.38
$\beta = \frac{1}{2}$ and $\rho_R \neq \rho_I$	331.58	584.06	772.01	1.29	2.01
$\beta = 1$ and $\rho_R = \rho_I$	144.54	154.90	187.97	0.74	12.24
$\beta = 1$ and $\rho_R \neq \rho_I$	102.22	1485.31	3780.57	1.20	1.26

Table 4: Statistics on final wealth $\mathcal{W}_T$.

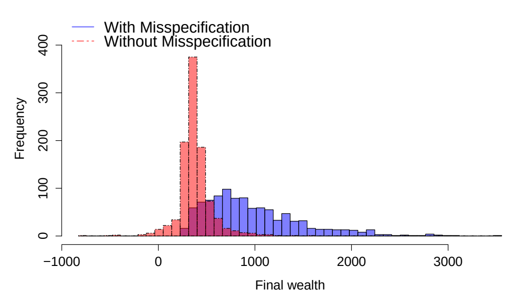
The statistics show that the final wealth in the case where $\rho_R = -0.4$ and $\rho_I = -0.9$ is on average higher than in the case where $\rho_R = \rho_I = -0.4$. Indeed, the quoting policy of the market maker is adapted in order to benefit from the misspecification of the parameter ρ.

6.2.2 Misspecification of the parameter θ

Monte Carlo simulations are performed using the following parameters: $(\mu, k_R, \theta_R, \eta_R, \rho_R) = (0, 4, 0.04, 0.5, -0.4)$ and $(r, k_I, \theta_I, \eta_I, \rho_I) = (0, 4, 0.0625, 0.5, -0.4)$.

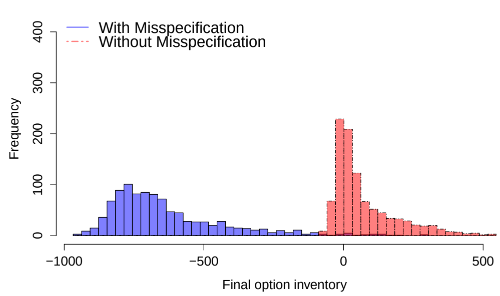


(a) Final value of q_1

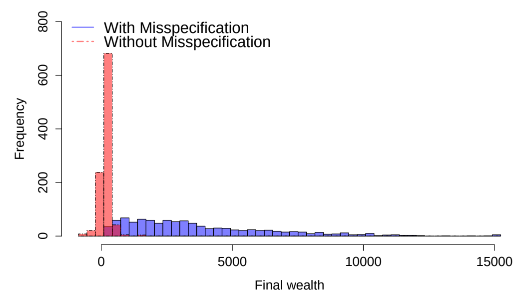


(b) Final value of Wealth

Figure 5: Statistics for $\beta = 0.5$



(a) Final value of q_1



(b) Final value of Wealth

Figure 6: Statistics for $\beta = 1$

Cases	Median	Mean	Std	Skewness	Kurtosis
$\beta = \frac{1}{2}$ and $\theta_R = \theta_I$	23.00	38.42	58.66	0.98	0.63
$\beta = \frac{1}{2}$ and $\theta_R \neq \theta_I$	-90.50	-73.81	68.76	1.22	1.45
$\beta = 1$ and $\theta_R = \theta_I$	29.00	72.94	114.14	1.52	1.94
$\beta = 1$ and $\theta_R \neq \theta_I$	-693.00	-636.17	204.57	1.51	2.52

Table 5: Statistics on final option inventory $q_{1,T}$.

Since $\theta_R < \theta_I$, we have $C_{\mathcal{P}}(t, S_t, y_t) < C_{\mathcal{Q}}(t, S_t, y_t)$. The market maker posts aggressive ask quotes and conservative bid quotes. Therefore, she should be more likely to finish with a short option position, a fact that is confirmed by the simulations.

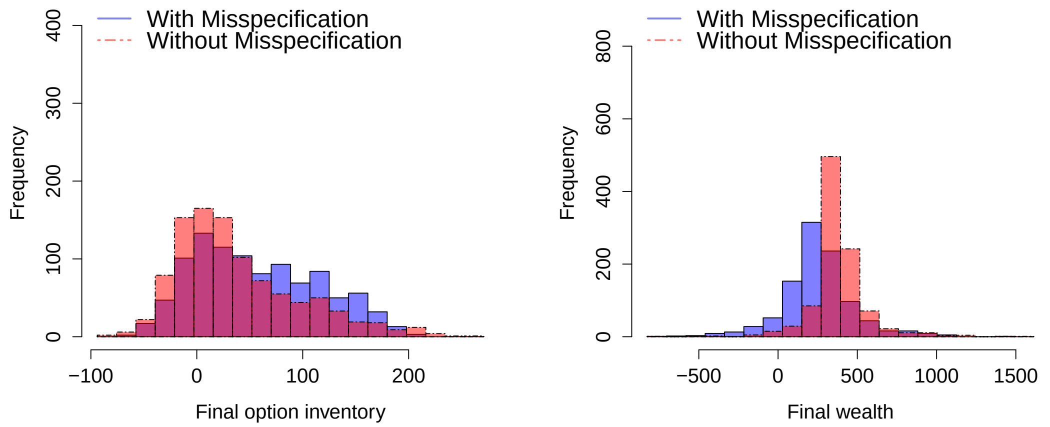
Cases	Median	Mean	Std	Skewness	Kurtosis
$\beta = \frac{1}{2}$ and $\theta_R = \theta_I$	356.89	379.64	176.13	0.72	9.38
$\beta = \frac{1}{2}$ and $\theta_R \neq \theta_I$	856.03	972.96	504.59	1.26	1.91
$\beta = 1$ and $\theta_R = \theta_I$	144.54	154.90	187.97	0.75	12.24
$\beta = 1$ and $\theta_R \neq \theta_I$	3029.61	3737.93	2834.59	1.18	1.26

Table 6: Statistics on final wealth $\mathcal{W}_T$

The final wealth in the case where the parameter θ does not have the same values under $\mathcal{P}$ and $\mathcal{Q}$ ($\theta_R = 0.04, \theta_I = 0.0625$) is in average higher than in the default case ($\theta_R = \theta_I = 0.04$). The numerical simulations support the results of the theoretical study and show experimentally how the optimal strategy enables the market maker to take advantage from the parameter misspecification.

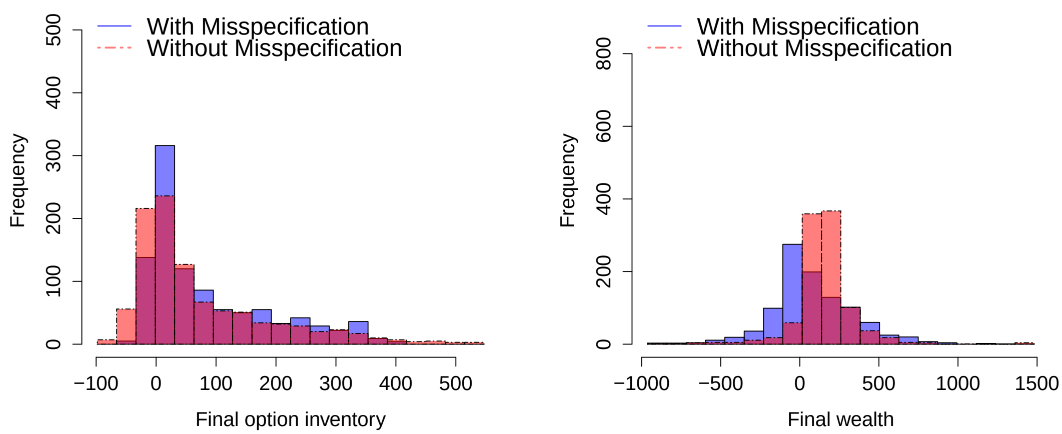
6.2.3 Misspecification of the parameter η

In this subsection, Monte Carlo simulations are performed using the following parameters: $(\mu, k_R, \theta_R, \eta_R, \rho_R) = (0, 4, 0.04, 0.5, -0.4)$ and $(r, k_I, \theta_I, \eta_I, \rho_I) = (0, 4, 0.04, 0.7, -0.4)$. The statistics of 1000 simulations are given below:



(a) Final value of q_1 (b) Final value of Wealth

Figure 7: Statistics for $\beta = 0.5$



(a) Final value of q_1 (b) Final value of Wealth

Figure 8: Statistics for $\beta = 1$

It can be seen through the approximation (6.1) that the effect of the parameter η on the implied volatility $\Sigma(K, T)$ of the option depends on its moneyness. Thus, at the first order in $\log(\frac{K}{S_t})$, if $\rho < 0$, an increase of the parameter η increases the option price if $K < S_t$, and decreases it if $K > S_t$. Therefore, the aggressiveness of the quotes of the market maker on either the bid or the ask side of the order book depends on the path followed by S in each simulation.

Cases	Median	Mean	Std	Skewness	Kurtosis
$\beta = \frac{1}{2}$ and $\eta_R = \eta_I$	23.00	38.42	58.66	0.98	0.63
$\beta = \frac{1}{2}$ and $\eta_R \neq \eta_I$	48.00	57.00	58.72	0.34	-0.83
$\beta = 1$ and $\eta_R = \eta_I$	29.00	72.94	114.14	1.52	1.94
$\beta = 1$ and $\eta_R \neq \eta_I$	38.50	86.20	105.06	1.13	0.19

Table 7: Statistics on final option inventory $q_{1,T}$.

Cases	Median	Mean	Std	Skewness	Kurtosis
$\beta = \frac{1}{2}$ and $\eta_R = \eta_I$	356.89	379.64	176.13	0.72	9.38
$\beta = \frac{1}{2}$ and $\eta_R \neq \eta_I$	251.15	250.49	227.15	0.22	2.98
$\beta = 1$ and $\eta_R = \eta_I$	144.54	154.90	187.97	0.75	12.24
$\beta = 1$ and $\eta_R \neq \eta_I$	34.57	81.90	269.94	0.44	2.27

Table 8: Statistics on final wealth $\mathcal{W}_T$.

The interesting point here is that the final wealth $\mathcal{W}_T$ in the case $\eta_R \neq \eta_I$ has a lower average and a higher standard deviation than in the case $\eta_R = \eta_I$ (for both $\beta = \frac{1}{2}$ and $\beta = 1$). This observation may seem, at first glance, counter-intuitive, since the misspecification of a parameter should give the opportunity to benefit from price inefficiency. However, there is a simple explanation to this result: increasing the parameter η under the pricing measure also increases the volatility risk of the option. Since this risk is not hedged out, the variance of the final wealth $\mathcal{W}_T$ increases, thereby curbing the profit of the market maker.

7 Conclusion

In this paper, we have proposed a framework for the study of option market making. Using a stochastic control approach, we have derived analytic expressions for optimal bid and ask quotes in the case of a risk-neutral market maker. The risk-adverse case is also considered, in a mean-variance framework, and we have used a singular perturbation technique in order to provide approximations for the optimal quotes. Finally, using Monte Carlo simulations, we have provided some numerical evidence for our theoretical findings, and studied the impact of parameters misspecification.

8 Appendices

8.1 Intensity of arrivals of market orders

In order to determine the form of the functions λ^+ and λ^-, we need to specify the distribution function of the size of market orders and also the market impact following the execution of a market order.

Let $f_{\mathcal{V}}$ denote the density distribution of the size of market orders in absolute value of their cash amount. Several studies proved that this density decays as a power law (see [1]). We will suppose here that $f_{\mathcal{V}}$ can be well fitted by a power law density:

$$\forall x > 0, f_{\mathcal{V}}(x) = \frac{\gamma L^\gamma}{(L+x)^{\gamma+1}}. \tag{8.1}$$

From a practical point of view, there is a strictly positive lower bound for x which corresponds to the option price x_{Min}. Nevertheless, it is supposed here that the density $f_{\mathcal{V}}$ is positive for $0 \leq x \leq x_{Min}$.

On the other hand, market impact has been studied by different authors in the econophysics literature, and it is widely accepted that the change in price ΔP following a market order of size $\mathcal{V}$ can be written as follows:

$$\Delta P = K\mathcal{V}^\beta. \tag{8.2}$$

There are two values of β which are supported by different researchers: $\beta = 1$ which corresponds to a linear market impact and $\beta = \frac{1}{2}$ which corresponds to a square root market impact.

The probability that a bid quote (respectively ask quote) placed at a distance δ^- (respectively δ^+) from the mid price gets executed is equal to the probability that a sell market order (respectively buy market order) triggers a market impact which is higher or equal to δ^- (respectively δ^+). Suppose as in [1] that the arrival rate of market orders is constant and equal to F. Then, for $\delta \geq 0$:

$$\begin{aligned}
\lambda(\delta) &= F \times P(\Delta P \geq \delta), \\
&= F \times P(\mathcal{V}^\beta \geq \frac{\delta}{K}), \\
&= F \times P\left(\mathcal{V} \geq \left(\frac{\delta}{K}\right)^{\frac{1}{\beta}}\right), \\
&= F \int_{\left(\frac{\delta}{K}\right)^{\frac{1}{\beta}}}^{+\infty} f_{\mathcal{V}}(x)dx, \\
&= F \frac{L^\gamma}{(L + \left(\frac{\delta}{K}\right)^{\frac{1}{\beta}})^\gamma}.
\end{aligned}$$

Therefore, the intensity of order arrivals can be written as $\lambda(\delta) = \frac{A}{(B+\delta^C)^\gamma}$ where $A = FK^{\frac{\gamma}{\beta}}L^\gamma$, $B = LK^{\frac{1}{\beta}}$ and $C = \frac{1}{\beta}$.

As regards the parameter γ, several empirical studies show that $\gamma > 1$, see e.g. [7] ($\gamma = 1.53$), [14] ($\gamma = 1.4$) or [6] ($\gamma = 1.5$).

8.2 Solution of the HJB equation with linear impact and linear utility

The utility function U given in (4.1) is linear. Therefore, we make the following Ansatz for u:

$$u(t,s,y,q_1,x) \quad = \quad x + \theta_0(t,s,y) + q_1\theta_1(t,s,y), \tag{8.3}$$

Let $f_0^+ = J^+$, this function represents the jump part due to the execution of the ask quote. Using (8.3), the function f_0^+ can be written explicitly as:

$$f_0^+(\delta^+) \quad = \quad \lambda^+(\delta^+)\left(\delta^+ + M_0(t,s,y)\right),$$

where $M_0(t,s,y) = C_{\mathcal{Q}}(t,s,y) - \theta_1(t,s,y)$.

In order to determine $\delta^+_{L,*,t} = ArgMax_{\{x\geq 0\}} f_0^+(x)$, the derivative of the function f_0^+ is computed:

$$(f_0^+)'(\delta^+) = \frac{\lambda^+(\delta^+)}{B+\delta^+}\left(\delta^+(1-\gamma) + B - \gamma M_0(t,s,y)\right),$$

If $M_0(t,s,y) \geq \frac{B}{\gamma}$, then $\forall\ \delta^+ \geq 0$, $(f_0^+)'(\delta^+) \leq 0$, and then $\delta^+_{L,*,t} = 0$.
If $M_0(t,s,y) \leq \frac{B}{\gamma}$, the function $(f_0^+)'$ changes sign on $[0,+\infty[$ and gets null at $x^+ = \frac{B-\gamma M_0(t,s,y)}{\gamma-1}$. Using the sign of $(f_0^+)'$ on $[0,x^+]$ and $[x^+,+\infty[$, it can be deduced that $\delta^+_{L,*,t} = x^+$.
In conclusion, $\delta^+_{L,*,t}$ can be determined as follows:

$$\delta^+_{L,*,t} \quad = \quad \left(\frac{B-\gamma M_0(t,s,y)}{\gamma-1}\right)^+,$$

and:

$$f_0^+(\delta^+_{L,*,t}) = \begin{cases} \frac{A(\gamma-1)^{\gamma-1}}{\gamma^\gamma(B-M_0(t,s,y))^{\gamma-1}} & \text{if} \quad M_0(t,s,y) \leq \frac{B}{\gamma} \\ \frac{A}{B^\gamma}M_0(t,s,y) & \text{if} \quad M_0(t,s,y) \geq \frac{B}{\gamma} \end{cases} \tag{8.4}$$

The same approach can be applied to the function $f_0^- = J^-$. Indeed, using the form of the value function suggested in (8.3), the function f_0^- can be written as follows:

$$f_0^-(\delta^-) \quad = \quad \lambda^-(\delta^-)(\delta^- - M_0(t,s,y)).$$

The derivative of f_0^- writes:

$$(f_0^-)'(\delta^-) = \frac{\lambda^-(\delta^-)}{B+\delta^-}((1-\gamma)\delta^- + (B+\gamma M_0(t,s,y))).$$

If $M_0(t,s,y) \leq -\frac{B}{\gamma}$, then $\forall\ \delta^- \geq 0$, $(f_0^-)'(\delta^-) < 0$ and therefore $\delta^-_{L,*,t} = 0$. On the other hand, if $M_0(t,s,y) \geq -\frac{B}{\gamma}$, it can be proved that $\delta^-_{L,*,t} = x^- = \frac{B+\gamma M_0(t,s,y)}{\gamma-1}$.
Consequently, $\delta^-_{L,*,t}$ writes:

$$\delta^-_{L,*,t} = \left(\frac{B+\gamma M_0(t,s,y)}{\gamma-1}\right)^+,$$

and:

$$f_0^-(\delta^-_{L,*,t}) = \begin{cases} -\frac{A(1-\gamma)^{\gamma-1}}{(-\gamma)^\gamma(B+M_0(t,s,y))^{\gamma-1}} & \text{if} \quad M_0(t,s,y) \geq -\frac{B}{\gamma} \\ -\frac{A}{B^\gamma}M_0(t,s,y) & \text{if} \quad M_0(t,s,y) \leq -\frac{B}{\gamma} \end{cases} \tag{8.5}$$

In order to simplify the notations, the following quantities are introduced: $\mathcal{S} = \frac{B}{\gamma}$ and $J_0(t,s,y) = f_0^+(\delta^+_{L,*,t}) + f_0^-(\delta^-_{L,*,t})$.
The equation (4.2) becomes:

$$\mathcal{H} : (\partial_t + \mathcal{L}_1 + \mathcal{L}_2)\left(x + \theta_0(t,s,y) + q_1\theta_1(t,s,y)\right) + J_0(t,s,y) = 0$$

The terms of the HJB equation are sorted by powers of q_1:

$$\begin{aligned} (0) &: (\partial_t + \mathcal{L}_1)\theta_0 + J_0(t,s,y) = 0, \\ (1) &: (\partial_t + \mathcal{L}_1)\theta_1 - \mu\Delta s = 0, \end{aligned}$$

Using the final conditions and applying the Feynman-Kac formula yields:

$$\begin{aligned} \theta_1(t,s,y) &= C_{\mathcal{P}}(t,s,y) - \mu E^{\mathcal{P}}_{t,s,y}\left(\int_t^T \Delta(u,S_u,y_u)S_u du\right), \\ \theta_0(t,s,y) &= E_{t,s,y}\left(\int_t^T J_0(u,S_u,y_u)du\right). \end{aligned}$$

Based on these results, the quantity $M_0(t,s,y)$ can be deduced:

$$M_0(t,s,y) = C_{\mathcal{Q}}(t,s,y) - C_{\mathcal{P}}(t,s,y) + \mu E^{\mathcal{P}}_{t,s,y}\left(\int_t^T \Delta(u,S_u,y_u)S_u du\right),$$

and $u_0(t,s,y,q_1,x) = x + \theta_0(t,s,y) + q_1\left(C_{\mathcal{P}}(t,s,y) - \mu E^{\mathcal{P}}_{t,s,y}\left(\int_t^T \Delta(u,S_u,y_u)S_u du\right)\right)$ is the solution of the HJB equation (4.2).

The function u_0 coincides with the value function if it is smooth, finite and has a quadratic growth.
In order to prove the quadratic growth of u_0, we start by studying the function θ_0:

- If $M_0(t,s,y) \in [-\mathcal{S},\mathcal{S}]$, then:

$$\begin{aligned} &(\frac{\gamma+1}{\gamma}B)^{-(\gamma-1)} \leq (B - M_0(t,s,y))^{-(\gamma-1)} \leq (\frac{\gamma-1}{\gamma}B)^{-(\gamma-1)}, \\ &(\frac{\gamma+1}{\gamma}B)^{-(\gamma-1)} \leq (B + M_0(t,s,y))^{-(\gamma-1)} \leq (\frac{\gamma-1}{\gamma}B)^{-(\gamma-1)}, \end{aligned}$$

  Then $\exists M > 0$ such that $|J_0(t,s,y)| \leq M$.

- If $M_0(t,s,y) \geq \mathcal{S}$, then:

$$E^{\mathcal{P}}_{t,s,y}\left(|J_0(u,S_u,y_u)|\right) \leq \frac{A(\gamma-1)^{\gamma-1}}{\gamma^\gamma (B+\mathcal{S})^{\gamma-1}} + \frac{A}{B^\gamma} E^{\mathcal{P}}_{t,s,y}\left(|M_0(u,S_u,y_u)|\right),$$

- If $M_0(t,s,y) \leq -\mathcal{S}$ then:

$$E^{\mathcal{P}}_{t,s,y}\left(|J_0(u,S_u,y_u)|\right) \leq \frac{A(\gamma-1)^{\gamma-1}}{\gamma^\gamma (B-\mathcal{S})^{\gamma-1}} + \frac{A}{B^\gamma} E\left(|M_0(u,S_u,y_u)|\right),$$

We assume here that the traded option is a call or a put, so that there exists $C_2' > 0$ with $E\left(|M_0(t,s,y)|\right) \leq C_2'(1+s)$. It follows that there exists $C_2'' > 0$, such that $|\theta_0(t,s,y)| \leq C_2''(1+s)$. Using Young's inequality $\forall a,b \in \mathbb{R}$, $ab \leq \frac{a^2+b^2}{2}$, we can find $C_2''' > 0$ such that:

$$|u_0(t,s,y,q_1,x)| \leq C_2'''\left(1+x^2+s^2+y^2+q_1^2\right),$$

which implies that u_0 is finite and has a quadratic growth.
Recall here that θ_0 is the solution of the equation $(\partial_t + \mathcal{L}_1)\theta_0 + J_0(t,s,y) = 0$ with the final condition $\theta_0(T,s,y) = 0$. Since the function J_0 is continuous (J_0 is at least $C^{0,0,0}$), then θ_0 is smooth. Consequently, the function u_0 is also smooth and it coincides with the value function.

8.3 Solution of the HJB equation with linear impact in the mean-variance framework

Let u^ϵ be the solution of the HJB equation (5.3). Under the assumption that $\epsilon \sim 0$, a singular perturbation technique is performed with respect to the parameter ϵ:

$$u^\epsilon(t,s,y,q_1,x) = x + \sum_{k=0}^{+\infty} \epsilon^k v_k(t,s,y,q_1),$$

Given the form of the utility function, the following Ansatz on v_0 and v_1 is made:

$$\begin{aligned} v_0(t,s,y,q_1) &= \theta_0(t,s,y) + q_1\theta_1(t,s,y), \\ v_1(t,s,y,q_1) &= \theta_2(t,s,y) + q_1\theta_3(t,s,y) + q_1^2\theta_4(t,s,y), \end{aligned}$$

In order to solve the HJB equation, the jump terms $J^{+,\epsilon}$ and $J^{-,\epsilon}$ have to be calculated.
Let $f^+ = J^{+,\epsilon}$, the function f^+ writes:

$$\begin{aligned} f^+(\delta^+) &= \lambda^+(\delta^+)(u(t,s,y,q_1-1,x+(c+\delta^+)) - u(t,s,y,q_1,x)), \\ &= \lambda^+(\delta^+)\left(\delta^+ + M_0(t,s,y) + \epsilon M_1(t,s,y,q_1) + \epsilon^2 R^+(t,s,y,q_1)\right), \end{aligned}$$

Let $M^+(t,s,y,q_1) = M_0(t,s,y,q_1) + \epsilon M_1(t,s,y,q_1)$. By differentiating f^+, it can be shown that:

$$(f^+)'(\delta^+) = \frac{\lambda^+(\delta^+)}{B+\delta^+}\left(\delta^+(1-\gamma) + B - \gamma M^+(t,s,y,q_1) - \gamma\epsilon^2 R^+(t,s,y,q_1)\right),$$

In order to determine $\delta^+_{*,t} = ArgMax_{\{x\geq 0\}} f^+(x)$, two cases should be distinguished:

- $M^+(t,s,y,q_1)+\epsilon^2 R^+(t,s,y,q_1)\geq \mathcal{S}$: in this case, $\forall\ \delta^+\geq 0$, $(f^+)'(\delta^+)\leq 0$, then the function f^+ is decreasing on the interval $[0,+\infty[$ and $\delta^+_{*,t}=0$.
- $M^+(t,s,y,q_1)+\epsilon^2 R^+(t,s,y,q_1)\leq \mathcal{S}$: the function $(f^+)'$ changes its sign on $[0,+\infty[$ and vanishes at x^+:

$$x^+ \quad = \quad \frac{B-\gamma\left(M^+(t,s,y,q_1)+\epsilon^2 R^+(t,s,y,q_1)\right)}{\gamma-1}.$$

  Since $\gamma>1$ then $\delta^+_{*,t}=x^+$.

In conclusion, $\delta^+_{*,t}$ writes:

$$\delta^+_{*,t} \quad = \quad \left(\frac{B-\gamma\left(M^+(t,s,y,q_1)+\epsilon^2 R^+(t,s,y,q_1)\right)}{\gamma-1}\right)^+.$$

Using a Taylor expansion, it follows:

$$\delta^+_{*,t}=\delta^+_{L,*,t}-\epsilon\frac{\gamma}{\gamma-1}M_1(t,s,y,q_1)1_{\{M^++\epsilon^2R^+\leq\mathcal{S}\}}+O\big(\epsilon^2\big)\,.$$

In order to solve the HJB equation, it is useful to write $f^+(\delta^+_{*,t})$ as the sum of $f^+_0(\delta^+_{L,*,t})$ plus a correction that depends on the parameter ϵ. Indeed, we have:

$$f^+(x) \quad = \quad f^+_0(x)+\epsilon\lambda^+(x)M_1(t,s,y,q_1)+O\big(\epsilon^2\big)\,,$$

and therefore:

$$\begin{aligned}
f^+(\delta^+_{*,t}) \quad &= \quad f^+(\delta^+_{L,*,t})+(f^+)'(\delta^+_{L,*,t})(\delta^+_{*,t}-\delta^+_{L,*,t})+O\big((\delta^+_{*,t}-\delta^+_{L,*,t})^2\big)\,,\\
&= \quad f^+_0(\delta^+_{L,*,t})+\epsilon\lambda^+(\delta^+_{L,*,t})M_1-\epsilon(f^+)'(\delta^+_{L,*,t})\frac{\gamma}{\gamma-1}M_1 1_{\{M^++\epsilon^2R^+\leq\mathcal{S}\}}+O\big(\epsilon^2\big)\,.
\end{aligned}$$

Since $(f^+)'(x)=(f^+_0)'(x)+O(\epsilon)$, the last equation becomes:

$$f^+(\delta^+_{*,t}) \quad = \quad f^+_0(\delta^+_{L,*,t})+\epsilon M_1\left(\lambda^+(\delta^+_{L,*,t})-\frac{\gamma}{\gamma-1}(f^+_0)'(\delta^+_{L,*,t})1_{\{M^++\epsilon^2R^+\leq\mathcal{S}\}}\right)+O\big(\epsilon^2\big)\,,$$

It can be recalled at this stage that $(f^+_0)'(\delta^+_{L,*,t})=\frac{B-\gamma M_0}{B}\lambda^+(0)1_{\{M_0>\mathcal{S}\}}$.
Notice that if $M_0\in[min\left(\mathcal{S},\mathcal{S}-\epsilon M_1-\epsilon^2\mathcal{R}^+\right),max\left(\mathcal{S},\mathcal{S}-\epsilon M_1-\epsilon^2\mathcal{R}^+\right)]$, then $|\frac{B-\gamma M_0}{B}\lambda^+(0)|=O(\epsilon)$. This means:

$$(f^+_0)'(\delta^+_{L,*,t}) \quad = \quad \frac{B-\gamma M_0}{B}\lambda^+(0)1_{\{M^++\epsilon^2R^+>\mathcal{S}\}}+O(\epsilon)\,,$$

and then:

$$f^+(\delta^+_{*,t}) \quad = \quad f^+_0(\delta^+_{L,*,t})+\epsilon M_1(t,s,y,q_1)\lambda^+(\delta^+_{L,*,t})+O\big(\epsilon^2\big)\,.$$

On the other hand, the optimal bid distance $\delta^-_{*,t}$ can be determined similarly. Let $f^- = J^{-,\epsilon}$, then the function f^- writes:

$$
\begin{aligned}
f^-(\delta^-) &= \lambda^-(\delta^-)(u(t,s,y,q_1+1,x-(c-\delta^-)) - u(t,s,y,q_1,x)), \\
&= \lambda^-(\delta^-)\left(\delta^- - \left(M_0(t,s,y) + \epsilon M_2(t,s,y,q_1) + \epsilon^2 R^-(t,s,y,q_1)\right)\right).
\end{aligned}
$$

Let $M^-(t,s,y,q_1) = M_0(t,s,y) + \epsilon M_2(t,s,y,q_1)$. Differentiating f^- yields:

$$
(f^-)'(\delta^-) = \frac{\lambda^-(\delta^-)}{B+\delta^-}\left(\delta^-(1-\gamma) + B + \gamma M^-(t,s,y,q_1) + \gamma\epsilon^2 R^-(t,s,y,q_1)\right).
$$

Afterward, it can be seen that if $M^- + \epsilon^2 R^- < -\mathcal{S}$, then $\delta^-_{*,t} = 0$, whereas if $M^- + \epsilon^2 R^- \geq -\mathcal{S}$, then $(f^-)'$ changes its sign on $[0,+\infty[$ and gets null in $x^- = \frac{B+\gamma M^-(t,s,y,q_1)+\gamma\epsilon^2 R^-(t,s,y,q_1)}{\gamma-1} = \delta^-_{*,t}$. So, the quantity $\delta^-_{*,t}$ writes:

$$
\delta^-_{*,t} = \left(\frac{B + \gamma\left(M^-(t,s,y,q_1) + \epsilon^2 R^-(t,s,y,q_1)\right)}{\gamma - 1}\right)^+.
$$

Using again the Taylor expansion, we obtain:

$$
\delta^-_{*,t} = \delta^-_{L,*,t} + \epsilon\frac{\gamma}{\gamma-1}M_2(t,s,y,q_1)1_{\{M^-+\epsilon^2R^-\geq -\mathcal{S}\}} + O\big(\epsilon^2\big).
$$

Once again, we aim to write the quantity $f^-(\delta^-_{*,t})$ as the sum of $f^-_0(\delta^-_{L,*,t})$ plus a correction term due to the parameter ϵ. First, we use the relation:

$$
f^-(x) = f^-_0(x) - \epsilon\lambda^-(x)M_2(t,s,y,q_1) + O\big(\epsilon^2\big),
$$

and then, based on Taylor's expansion, we obtain:

$$
\begin{aligned}
f^-(\delta^-_{*,t}) &= f^-(\delta^-_{L,*,t}) + (f^-)'(\delta^-_{L,*,t})(\delta^-_{*,t} - \delta^-_{L,*,t}) + O\big((\delta^-_{*,t} - \delta^-_{L,*,t})^2\big), \\
&= f^-_0(\delta^-_{L,*,t}) - \epsilon\lambda^-(\delta^-_{L,*,t})M_2 + \epsilon(f^-)'(\delta^-_{L,*,t})\frac{\gamma}{\gamma-1}M_2 1_{\{M^-+\epsilon^2R^-\geq -\mathcal{S}\}} + O\big(\epsilon^2\big).
\end{aligned}
$$

The relation $(f^-)'(x) = (f^-_0)'(x) + O(\epsilon)$ implies:

$$
f^-(\delta^-_{*,t}) = f^-_0(\delta^-_{L,*,t}) + \epsilon M_2(t,s,y,q_1)\left(-\lambda^-(\delta^-_{L,*,t}) + \frac{\gamma}{\gamma-1}(f^-_0)'(\delta^-_{L,*,t})1_{\{M^-+\epsilon^2R^-\geq -\mathcal{S}\}}\right) + O\big(\epsilon^2\big).
$$

We have also $(f^-_0)'(\delta^-_{L,*,t}) = \left(\frac{B+\gamma M_0}{B}\lambda^-(0)\right)1_{\{M_0<-\mathcal{S}\}}$. Following the same method, it can be shown that if $M_0 \in [min(-\mathcal{S}, -\mathcal{S} - \epsilon M_2 - \epsilon^2\mathcal{R}^-), max(-\mathcal{S}, -\mathcal{S} - \epsilon M_2 - \epsilon^2\mathcal{R}^-)]$, then $|\frac{B+\gamma M_0}{B}\lambda^-(0)| = O(\epsilon)$. Therefore, it can be deduced that:

$$
(f^-_0)'(\delta^-_{L,*,t}) = \frac{B+\gamma M_0}{B}\lambda^-(0)1_{\{M^-+\epsilon^2R^-<-\mathcal{S}\}} + O(\epsilon).
$$

and:

$$
f^-(\delta^-_{*,t}) = f^-_0(\delta^-_{L,*,t}) - \epsilon M_2(t,s,y,q_1)\lambda^-(\delta^-_{L,*,t}) + O\big(\epsilon^2\big),
$$

Now that the terms $f^+(\delta^+_{*,t})$ and $f^-(\delta^-_{*,t})$ are computed separately, the term $J^\epsilon(\delta^-_{*,t},\delta^+_{*,t}) = f^+(\delta^+_{*,t}) + f^-(\delta^-_{*,t})$ is deduced:

$$\begin{aligned} J(\delta^-_{*,t},\delta^+_{*,t}) &= f_0^+(\delta^+_{L,*,t}) + \epsilon M_1 \lambda^+(\delta^+_{L,*,t}) + f_0^-(\delta^-_{L,*,t}) - \epsilon M_2 \lambda^-(\delta^-_{L,*,t}) + O\big(\epsilon^2\big)\,, \\ &= J_0(\delta^-_{L,*,t},\delta^+_{L,*,t}) + \epsilon M_1(t,s,y,q_1)\lambda^+(\delta^+_{L,*,t}) - \epsilon M_2(t,s,y,q_1)\lambda^-(\delta^-_{L,*,t}) + O\big(\epsilon^2\big)\,, \end{aligned}$$

The terms of $J(\delta^-_{*,t},\delta^+_{*,t})$ are classified according to their power in ϵ:

$$J(\delta^-_{*,t},\delta^+_{*,t}) = J_0(t,s,y) + \epsilon J_1(t,s,y,q_1) + O\big(\epsilon^2\big)\,,$$

where $J_1(t,s,y,q_1) = J_{1,0}(t,s,y) + q_1 J_{1,1}(t,s,y)$ and:

$$\begin{aligned} J_{1,0}(t,s,y) &= \lambda^+(\delta^+_{L,*,t})(-\theta_3+\theta_4) - \lambda^-(\delta^-_{L,*,t})(-\theta_3-\theta_4), \\ J_{1,1}(t,s,y) &= -2\theta_4\left(\lambda^+(\delta^+_{L,*,t}) - \lambda^-(\delta^-_{L,*,t})\right), \end{aligned}$$

The HJB equation can be separated into several terms according to the order of the parameter ϵ. By cancelling the term of order 0 in ϵ, it can be obtained that:

$$(\partial_t + \mathcal{L}_1 + \mathcal{L}_2)(x + \theta_0 + q_1\theta_1) + J_0 = 0,$$

with the final conditions:

$$\theta_0(T,s,y) = 0, \quad \theta_1(T,s,y) = h(s).$$

The functions θ_0 and θ_1 are equivalent to those found in the case of a linear utility function without inventory constraints, thus:

$$\begin{aligned} \theta_1(t,s,y) &= C_{\mathcal{P}}(t,s,y) - \mu E_{t,s,y}\left(\int_t^T \Delta_u S_u du\right) \\ \theta_0(t,s,y) &= E_{t,s,y}\left(\int_t^T J_0(u,S_u,y_u)du\right), \end{aligned}$$

The term of order 1 in ϵ leads to the following equation:

$$(\partial_t + \mathcal{L}_1 + \mathcal{L}_2)(\theta_2 + q_1\theta_3 + q_1^2\theta_4) + J_1(t,s,y) = q_1^2 V + T,$$

with the final conditions:

$$\theta_2(T,s,y) = 0, \quad \theta_3(T,s,y) = 0, \quad \theta_4(T,s,y) = 0.$$

The functions θ_2, θ_3 and θ_4 are:

$$\begin{aligned} \theta_2(t,s,y) &= E^{\mathcal{P}}_{t,s,y}\left(\int_t^T (J_{1,0} - T)(u,s_u,y_u)du\right), \\ \theta_3(t,s,y) &= E^{\mathcal{P}}_{t,s,y}(\int_t^T J_{1,1}(u,s_u,y_u)du), \\ \theta_4(t,s,y) &= -E^{\mathcal{P}}_{t,s,y}(\int_t^T V_u du). \end{aligned}$$

It can be demonstrated, as it was done in the case $\beta = \frac{1}{2}$, that the function u^ϵ is smooth, finite and has polynomial growth. Besides, u^ϵ can be approximated at order 1 in ϵ by $\tilde{u}^\epsilon(t,s,y,q_1,x) = x + v_0(t,s,y) + \epsilon v_1(t,s,y,q_1)$.
Remark:
The optimal ask quote $\delta^+_{*,t}$ can be approximated at order 1 in ϵ by $\hat{\delta}^+_{*,t}$:

$$\hat{\delta}^+_{*,t} \;=\; \left(\frac{B - \gamma M^+(t,s,y,q_1)}{\gamma - 1}\right)^+ .$$

Indeed, if $M^+(t,s,y) \in [Min\left(\mathcal{S} - \epsilon^2 R^+, \mathcal{S}\right), Max\left(\mathcal{S} - \epsilon^2 R^+, \mathcal{S}\right)]$, then $|\frac{B-\gamma M^+(t,s,y,q_1)}{\gamma-1}| = O(\epsilon^2)$ and consequently:

$$|\hat{\delta}^+_{*,t} - \delta^+_{*,t}| = O\big(\epsilon^2\big)\,,$$

Similarly, it can be seen that if $M^- \in [Min\left(-\mathcal{S}, -\mathcal{S} - \epsilon^2 R^-\right), Max\left(-\mathcal{S}, -\mathcal{S} - \epsilon^2 R^-\right)]$, then $|\frac{B+\gamma M^-}{\gamma-1}| = O(\epsilon^2)$. Thus, the optimal bid quote $\delta^-_{*,t}$ can be approximated at order 1 in ϵ by $\hat{\delta}^-_{*,t}$:

$$\hat{\delta}^-_{*,t} \;=\; \left(\frac{B + \gamma M^-}{\gamma - 1}\right)^+ ,$$

and the approximation error is at order 2 in ϵ:

$$|\hat{\delta}^-_{*,t} - \delta^-_{*,t}| = O\big(\epsilon^2\big)$$

8.4 Verification theorem in the case of linear utility

It is sufficient to prove that u_0, the solution to the HJB equation (4.2), is finite, smooth and has a polynomial growth.
By using the concavity of the square root function, we obtain:

$$|J_0(t,s,y)| \leq \frac{2A}{B^\gamma} \frac{(2\gamma - 1)\,|M_0| + \sqrt{B(2\gamma - 1)}}{2\gamma - 1}.$$

Under the simplifying assumption that the traded option is a call or a put, there exists $C_1 > 0$ such that:

$$max\left(C_{\mathcal{P}}(t,s,y), C_{\mathcal{Q}}(t,s,y)\right) \leq C_1(1 + S_t)$$

and we also have $|\Delta(t,s,y)| \leq 1$. It follows that $\exists M_1 > 0$, $\forall u < T$, $|M_0(u,S_u,y_u)| \leq M_1(1 + S_u)$. Using this result, we can state that:

$$\begin{aligned} |E_{t,s,y}\left(J_0(u,S_u,y_u)\right)| &\leq E_{t,s,y}\,|J_0(u,S_u,y_u)|\,, \\ &\leq \frac{2A}{B^\gamma(2\gamma - 1)}\Big((2\gamma - 1)\left(M_1 + M_1 S_t e^{\mu(u-t)}\right) + \sqrt{B(2\gamma - 1)}\Big)\,, \end{aligned}$$

and then:

$$|\theta_0(t,s,y)| \le \frac{2A}{B^\gamma}\left(\left(M_1+\sqrt{\frac{B}{2\gamma-1}}\right)(T-t)+M_1S_t\frac{e^{\mu(T-t)}-1}{\mu}\right).$$

Since obviously

$$|u_0(t,s,y,q_1,x)| \le |x|+|\theta_0(t,s,y)|+|q_1|\left(C_{\mathcal{P}}(t,s,y)+S_te^{\mu(T-t)}\right),$$

then $\exists C_2>0$ such that

$$|u_0(t,s,y,q_1,x)| \le C_2\left(1+x^2+s^2+y^2+q_1^2\right),$$

which proves that u_0 is finite and has a quadratic growth.
Moreover, u_0 is smooth since it is a combination of smooth functions.
Let us now prove that u_0 coincides with the value function. Let then $\delta=(\delta^-,\delta^+)$ be an admissible control process, and consider the following processes:

$$\begin{aligned}
dq_{1,t} &= dN_t^- - dN_t^+,\\
dX_t &= (C_{\mathcal{Q}}(t,S_t,y_t)+\delta_t^+)dN_t^+ - (C_{\mathcal{Q}}(t,S_t,Y_t)-\delta_t^-)dN_t^- + q_{2,t}dS_t,\\
\frac{dS_t}{S_t} &= \mu dt+\sigma(y_t)dW_t^{(1)},\\
dy_t &= a_R(y_t)dt+b_R(y_t)dW_t^{(2)},
\end{aligned}$$

where N^- and N^+ are Poisson processes with intensities λ^- and λ^+ respectively, where $\forall\tau\ge 0$:

$$\lambda_\tau^+=\lambda^+(\delta_\tau^+),\quad \lambda_\tau^-=\lambda^-(\delta_\tau^-).$$

Let $t_N = T\wedge\{\tau>t, |S_\tau-s|\ge n\}\wedge\{|y_\tau-y|\ge n\}\wedge\left\{\left|N_\tau^+-N_t^+\right|\ge n\right\}\wedge\left\{\left|N_\tau^--N_t^-\right|\ge n\right\}$.
Since u_0 is smooth, we have:

$$\begin{aligned}
u_0(t_n,S_{t_n},y_{t_n},q_{1,t_n},X_{t_n}) &= u_0(t,s,y,q_1,x)+\int_{\tau=t}^{t_n}\left(\partial_t+\mathcal{L}_1+\mathcal{L}_2\right)u_0(\tau,s_\tau,y_\tau,q_{1,\tau},x_\tau)d\tau\\
&+\int_{\tau=t}^{t_n}\frac{\partial u}{\partial s}S_\tau\sigma(y_\tau)dW_\tau^{(1)}+\frac{\partial u_0}{\partial y}b_R(y_\tau)dW_\tau^{(2)}-\frac{\partial u_0}{\partial x}q_{1,\tau}\Delta_\tau S_\tau\sigma(y_\tau)dW_\tau^{(1)}\\
&+\int_{\tau=t}^{t_n}\lambda^+(\delta^+)\left(u_0(\tau,s_\tau,y_\tau,q_{1,\tau^-}-1,x_{\tau^-}+C_{\mathcal{Q}}+\delta^+)-u_0(\tau,s_\tau,y_\tau,q_{1,\tau^-},x_{\tau^-})\right)d\tau\\
&+\int_{\tau=t}^{t_n}\lambda^-(\delta^-)\left(u_0(\tau,s_\tau,y_\tau,q_{1,\tau^-}+1,x_{\tau^-}-(C_{\mathcal{Q}}-\delta^-))-u_0(\tau,s_\tau,y_\tau,q_{1,\tau^-},x_{\tau^-})\right)d\tau\\
&+\int_{\tau=t}^{t_n}\lambda^+(\delta^+)\left(u_0(\tau,s_\tau,y_\tau,q_{1,\tau^-}-1,x_{\tau^-}+C_{\mathcal{Q}}+\delta^+)-u_0(\tau,s_\tau,y_\tau,q_{1,\tau^-},x_{\tau^-})\right)dM_\tau^+\\
&+\int_{\tau=t}^{t_n}\lambda^-(\delta^-)\left(u(\tau,s_\tau,y_\tau,q_{1,\tau^-}+1,x_{\tau^-}-(C_{\mathcal{Q}}-\delta^-))-u(\tau,s_\tau,y_\tau,q_{1,\tau^-},x_{\tau^-})\right)dM_\tau^-
\end{aligned}$$

where M^+ and M^- are the compensated processes associated to N^+ and N^- respectively. Using the polynomial growth of u and the fact that the functions λ^+ and λ^- are bounded, one can argue as in [17] that the local martingales in the previous equation are martingales. Thus, by taking expectations on both sides of the last equation, we obtain:

$$E\left(u_0(t_n, S_{t_n}, y_{t_n}, q_{1,t_n}, X_{t_n})\right) = u_0(t,s,y,q_1,x) + E\left(\int_{\tau=t}^{t_n} \left(\partial_t + \mathcal{L}_1 + \mathcal{L}_2\right) u_0(\tau, s_\tau, y_\tau, q_{1,\tau}, x_\tau) d\tau\right)$$
$$+ E\left(\int_{\tau=t}^{t_n} \lambda^+(\delta^+)\left(u_0(\tau, s_\tau, y_\tau, q_{1,\tau^-} - 1, x_{\tau^-} + C_{\mathcal{Q}} + \delta^+) - u_0(\tau, s_\tau, y_\tau, q_{1,\tau^-}, x_{\tau^-})\right) d\tau\right)$$
$$+ E\left(\int_{\tau=t}^{t_n} \lambda^-(\delta^-)\left(u_0(\tau, s_\tau, y_\tau, q_{1,\tau^-} + 1, x_{\tau^-} - (C_{\mathcal{Q}} - \delta^-)) - u_0(\tau, s_\tau, y_\tau, q_{1,\tau^-}, x_{\tau^-})\right) d\tau\right)$$

Using again the polynomial growth of u_0, we can deduce that u_0 is integrable on $[0, t_n]$, and the dominated convergence theorem yields that

$$lim_{n\to+\infty} E\left(u_0(t_n, S_{t_n}, y_{t_n}, q_{1,t_n}, X_{t_n})\right) = E\left(u_0(T, S_T, y_T, q_{1,T}, X_T)\right),$$

and the equation becomes:

$$E\left(u_0(T, S_T, y_T, q_{1,T}, X_T)\right) = u_0(t,s,y,q_1,x) + E\left(\int_{\tau=t}^{T} \left(\partial_\tau + \mathcal{L}_1 + \mathcal{L}_2\right) u_0(\tau, s_\tau, y_\tau, q_{1,\tau}, x_\tau) d\tau\right)$$
$$+ E\left(\int_{\tau=t}^{T} \lambda^+(\delta^+)\left(u_0(\tau, s_\tau, y_\tau, q_{1,\tau^-} - 1, x_{\tau^-} + C_{\mathcal{Q}} + \delta^+) - u_0(\tau, s_\tau, y_\tau, q_{1,\tau^-}, x_{\tau^-})\right) d\tau\right)$$
$$+ E\left(\int_{\tau=t}^{T} \lambda^-(\delta^-)\left(u_0(\tau, s_\tau, y_\tau, q_{1,\tau^-} + 1, x_{\tau^-} - (C_{\mathcal{Q}} - \delta^-)) - u_0(\tau, s_\tau, y_\tau, q_{1,\tau^-}, x_{\tau^-})\right) d\tau\right).$$

Recalling that u_0 is the solution of the HJB equation (4.2), we have for $(\delta_t^-, \delta_t^+) \in \mathcal{A}$:

$$\left(\partial_t + \mathcal{L}_1 + \mathcal{L}_2\right) u_0 + \lambda^+(\delta^+)\left(u_0(\tau, s_\tau, y_\tau, q_{1,\tau^-} - 1, x_{\tau^-} + C_{\mathcal{Q}} + \delta_t^+) - u_0(\tau, s_\tau, y_\tau, q_{1,\tau^-}, x_{\tau^-})\right)$$
$$+ \lambda^-(\delta^-)\left(u_0(\tau, s_\tau, y_\tau, q_{1,\tau^-} + 1, x_{\tau^-} - (C_{\mathcal{Q}} - \delta_t^-)) - u_0(\tau, s_\tau, y_\tau, q_{1,\tau^-}, x_{\tau^-})\right) \leq 0,$$

which means that $E\left(U(T, S_T, y_T, q_{1,T}, X_T)\right) \leq u_0(t,s,y,q_1,x)$ and then $u(t,s,y,q_1,x) = Sup_{(\delta^-,\delta^+)\in\mathcal{A}} E\left(U(T, S_T, y_T, q_{1,T}, X_T)\right) \leq u_0(t,s,y,q_1,x)$.

In addition, since u_0 solves (4.2), then for $(\delta_t^-, \delta_t^+) = (\delta_{t,*}^-, \delta_{t,*}^+)$ we have:

$$E\left(U(T, S_T^{(\delta_{t,*}^-,\delta_{t,*}^+)}, y_T^{(\delta_{t,*}^-,\delta_{t,*}^+)}, q_{1,T}^{(\delta_{t,*}^-,\delta_{t,*}^+)}, X_T^{(\delta_{t,*}^-,\delta_{t,*}^+)})\right) = u_0(t,s,y,q_1,x),$$

and then $u_0(t,s,y,q_1,x) \leq Sup_{(\delta_t^-,\delta_t^+)\in\mathcal{A}} E\left(U(T, S_T^{(\delta_t^-,\delta_t^+)}, y_T^{(\delta_t^-,\delta_t^+)}, q_{1,T}^{(\delta_t^-,\delta_t^+)}, X_T^{(\delta_t^-,\delta_t^+)})\right) = u(t,s,y,q_1,x)$.

Finally, we conclude that $u_0(t,s,y,q_1,x) = u(t,s,y,q_1,x)$.

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