A high gain observer with updated gain for a class of MIMO non-triangular systems

Monia Oueder, Mondher Farza, Ridha Ben Abdennour, Mohammed M'Saad

A high gain observer with updated gain for a class of MIMO non-triangular systems

M. Oueder [a,b], M. Farza [a,*], R. Ben Abdennour [b], M. M'Saad [a]

[a] GREYC, UMR 6072 CNRS, Université de Caen, ENSICAEN, 6 Boulevard Maréchal Juin, 14050 Caen Cedex, France
[b] Unité de Recherche CONPRI, ENIG Gabès, Rue Omar Ibn El Khattab, 6029 Gabès, Tunisie

A B S T R A C T

A high gain like observer with updated gain is proposed for a class of cascade nonlinear and non triangular systems that are observable for any input. The objective of the gain adaptation is to perform an admissible tradeoff between state reconstruction dynamics on the one hand versus noise amplification on the other hand. To this end, the gain of the proposed observer is tuned through a single scalar time-varying parameter governed by an adequate differential Riccati equation. The involved adaptation process is mainly driven by the power of the output observation error norm computed on a moving horizon window. Simulation results are given to show the effectiveness of the proposed observer, namely its exponential convergence and its insensitivity with respect to noise measurements.

Keywords:
Nonlinear system
Non-triangular structure
High gain observer
Updated gain
Lyapunov equation
Riccati equation
Exponential convergence

1. Introduction

A remarkable research activity has been devoted to the high gain observer design for uniformly observable nonlinear systems which are diffeomorphic to the well known triangular form (see for instance [1–3]). Recently, a high gain observer design has been proposed in [4] for a class of non triangular systems. The latter is particularly composed of cascade subsystems where each subsystem is associated with a subset of the output variables, exhibits a triangular dependence on its own state variables and may depend on the state variables of all other subsystems. This characterizes a large class of MIMO systems that are observable for any input and includes the previous canonical forms that have been used in the high gain observer design (see for instance [5,6]). The appealing feature of these designs consists in their simplicity as the observer gain is calibrated through the choice of a fixed scalar parameter which is referred to as the gain parameter. This parameter is commonly chosen high enough with respect to the Lipschitz constant of the system nonlinearities leading thereby to noise amplification.

An important effort has been particularly devoted to overcome this noise sensitivity problem for single output nonlinear systems by appropriately updating the gain parameter of the proposed observers [7–10]. In the seminal contribution [10], the gain parameter is updated under the guise of a scalar differential equation. The initial value of the parameter is set to low values and it is updated to continuously grows until the observation error convergence is achieved. This parameter is not allowed to decrease if required to properly deal with the noise sensitivity problem. In the contribution [7], the gain parameter updating consists in properly switching between relatively low and high values according to a noise power determined from the output observation error. The achieved observer performances heavily depends on the involved gain parameter specifications. The gain parameter adaptation proposed in [8] is performed through a differential equation involving some constant design parameters depending on the system nonlinearities. In [9], the gain parameter is updated such that the resulting values vary between the value one and some relatively high value. The adaptation process is performed using a differential equation driven by an innovation sequence that is proportional to the power of the output observation error. Unlike in the above mentioned contributions where the considered systems are single output ones, a high gain

* Corresponding author.
E-mail addresses: oueder.monya@gmail.com (M. Oueder), mondher.farza@unicaen.fr, mfarza@greyc.ensicaen.fr (M. Farza), Ridha.benabdennour@enig.rnu.tn (R. Ben Abdennour), mohammed.msaad@ensicaen.fr (M. M'Saad).

observer with an updated gain has been proposed in [11] for a class of multiple output systems that are characterized by a triangular canonical form. The gain parameter adaptation is performed using an appropriate scalar Riccati differential equation involving a design parameter that depends on the Lipschitz constant of the system nonlinearities and an innovation sequence similar to that considered in [9].

It is worth noticing that the noise sensitivity issue is relatively important for the observer proposed in [4] for a class of cascade nonlinear and non triangular systems as the scalar design parameter involved in the observer gain intervenes with integer powers that are as high as the dimension of the subsystems are. In this paper, one aims at reviving this high gain observer design bearing in mind the performance enhancements provided by the gain parameter adaptation for triangular systems in [11]. The tuning of the resulting observer is hence achieved according to an appropriate scalar Riccati differential equation driven by the output error dynamics. Though the involved observer design is a genuine combination of the designs already proposed in [4,11], it is by no means a trivial exercise since it requires a suitable adaptation of these designs to deal with the underlying convergence analysis. Such a feature will be more appreciated throughout the constructive Lyapunov approach that has been adopted to this end, namely it will be shown that the observer is able to provide accurate and almost free-noise estimates with low values of the gain parameter that can be specified a priori.

The paper is organized as follows. The problem formulation is given in Section 2 with a particular emphasis on the class of nonlinear systems which will be the subject of the observer design. Useful notations and preliminaries are given in Section 3. Section 4 is devoted to the observer design together with a convergence analysis. An academic observer design problem is addressed in Section 5 for illustration purposes.

2. Problem formulation

As it has been mentioned above, one aims at designing an observer with parameter gain updating for a class of cascade nonlinear and non triangular systems which are diffeomorphic to the following form:

$$\begin{cases} \dot{x} = Ax + \varphi(u, x) \\ y = Cx \end{cases} \tag{1}$$

where x, u, y and $\varphi(\cdots)$ denote respectively the state, the input, the output and the nonlinear function field of the system. More specifically

- the system output is particularly composed as follows

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_q \end{pmatrix} \in \mathbb{R}^p$$

where $y_k \in \mathbb{R}^{p_k}$ for $k = 1, \ldots, q$ and hence

$$\sum_{k=1}^{q} p_k = p$$

- the system state is composed as follows

$$x = \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^q \end{pmatrix} \in \mathbb{R}^n \quad \text{with } x^k = \begin{pmatrix} x_1^k \\ x_2^k \\ \vdots \\ x_{\lambda_k}^k \end{pmatrix} \in \mathbb{R}^{n_k}$$

where $x_i^k \in \mathbb{R}^{p_k}$ for $i = 1, \ldots, \lambda_k$ and $k = 1, \ldots, q$. This leads to the following dimensional property

$$\sum_{k=1}^{q} n_k = \sum_{k=1}^{q} p_k \lambda_k = n \quad \text{with } p_k \geq 1 \text{ and } \lambda_k \geq 2$$

- the matrices A and C are respectively given by

$$A = \begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_q \end{bmatrix} \quad \text{with}$$

$$A_k = \begin{bmatrix} 0 & I_{p_k} & & 0 \\ \vdots & & \ddots & \\ 0 & \ldots & 0 & I_{p_k} \\ 0 & \ldots & 0 & 0 \end{bmatrix} \tag{2}$$

and

$$C = \begin{bmatrix} C_1 & & \\ & \ddots & \\ & & C_q \end{bmatrix} \quad \text{with } C_k = \begin{bmatrix} I_{p_k} & 0 & \cdots & 0 \end{bmatrix} \tag{3}$$

- the nonlinear function field is composed as follows

$$\varphi(u, x) = \begin{pmatrix} \varphi^1(u, x) \\ \varphi^2(u, x) \\ \vdots \\ \varphi^q(u, x) \end{pmatrix} \in \mathbb{R}^n \quad \text{with}$$

$$\varphi^k(u, x) = \begin{pmatrix} \varphi_1^k(u, x) \\ \varphi_2^k(u, x) \\ \vdots \\ \varphi_{\lambda_k}^k(u, x) \end{pmatrix} \in \mathbb{R}^{n_k}$$

where the function $\varphi_i^k(u, x) \in \mathbb{R}^{p_k}$ is differentiable with respect to x and assumes the following structural dependence on the state variables.

- for $1 \leq i \leq \lambda_k - 1$:

$$\varphi_i^k(u, x) = \varphi_i^k(u, x^1, x^2, \ldots, x^{k-1}, x_1^k, x_2^k, \ldots, x_i^k, x_1^{k+1}, x_1^{k+2}, \ldots, x_1^q) \tag{4}$$

- for $i = \lambda_k$:

$$\varphi_{\lambda_k}^k(u, x) = \varphi_{\lambda_k}^k(u, x^1, x^2, \ldots, x^q). \tag{5}$$

Recall that a high gain observer has been already designed for system (1) in [4] where the underlying gain parameter intervenes with integer powers that are as high as the dimension of the subsystems. This leads to poor performance issues in the presence of unavoidable noise measurements. In the following, one will revive the observer design given in [4] to perform an admissible compromise between the accuracy and noise sensitivity using an adequate design parameter adaptation that borrows from the updating gain based observer proposed in [11]. As generally assumed in the high gain observer design, one considers the following Lipschitz assumption.

Assumption 1. $\varphi(u, x)$ is a globally Lipschitz nonlinear function with respect to x uniformly in u.

It is worth noticing that though the Lipschitz conditions would be too restrictive since they are in general locally satisfied, they can be particularly relaxed in the case where the system state trajectory lies in a bounded set thanks to suitable prolongation techniques which make it possible to get global Lipschitz conditions on the whole state space.

3. Preliminaries

In the following, one introduces some variables that will be used in the observer equations together with some identities satisfied by these variables and a technical lemma that will be needed in the proof of the main result, namely the convergence of the observation error.

Let $\theta : \mathbb{R} \to \mathbb{R}, t \mapsto \theta(t)$ be a real-valued function and for $k = 1, \ldots, q$, let $\Delta_k(\theta)$ be the diagonal matrix defined by

$$\Delta_k(\theta) = \operatorname{diag}\left(\frac{1}{\theta^{\delta_k}} I_{p_k}, \frac{1}{\theta^{2\delta_k}} I_{p_k}, \ldots, \frac{1}{\theta^{\lambda_k \delta_k}} I_{p_k}\right) \tag{6}$$

where $\{\delta_k\}$ is sequence which indicates the power of θ and is defined as follows:

$$\begin{cases} \delta_k = 2^{q-k}\left(\prod_{i=k+1}^{q}\left(\lambda_i - \frac{3}{2}\right)\right) & \text{for } k = 1, \ldots, q-1 \\ \delta_q = 1. \end{cases} \tag{7}$$

Notice that for any $k = 1, \ldots, q-1$, one has

$$\frac{\delta_k}{2} = \left(\lambda_{k+1} - \frac{3}{2}\right)\delta_{k+1}. \tag{8}$$

And since $\lambda_k \geq 2$, $\{\delta_k\}$ constitute a non increasing sequence of positive real numbers, i.e.

$$\delta_1 \geq \delta_2 \geq \cdots \geq \delta_q = 1. \tag{9}$$

The sequence $\{\delta_k\}$ allows to provide a real sequence that reflects in some sense the interconnections between the block nonlinearities and establish the following lemma which is similar (but not identical) to that given in [4]. This result is of a fundamental interest for the forthcoming observation error convergence analysis.

Lemma 3.1. *For $k, l = 1, \ldots, q$ and $i = 1, \ldots, \lambda_k$, $j = 2, \ldots, \lambda_l$, let*

$$\chi_{l,j}^{k,i} = \begin{cases} 0 & \text{if } \dfrac{\partial \varphi_i^k}{\partial x_j^l}(u, x) \equiv 0 \\ 1 & \text{otherwise} \end{cases}$$

and consider the following sequence of reals

$$\sigma_i^k = \sigma_1^k + i\delta_k \quad \text{for } k = 1, \ldots, q \text{ and } i = 1, \ldots, \lambda_k \tag{10}$$

with

$$\sigma_1^k = -\lambda_k\delta_k + \lambda_1\delta_1 + \left(1 - \frac{1}{2^k}\right) \tag{11}$$

where the δ_k's are given by (7). Then, the terms of this sequence satisfy the following property

$$\text{if } \chi_{l,j}^{k,i} = 1 \quad \text{then } \sigma_j^l - \sigma_i^k - \frac{\delta_k}{2} - \frac{\delta_l}{2} \leq -\frac{1}{2^q}. \tag{12}$$

Though the expressions of the reals σ_i^k are slightly different from those in [4], the proof of Lemma 3.1 is similar to that given in [4] and is hence omitted. Nevertheless, one shall put forward two properties of the reals σ_i^k that will be used throughout this paper, namely

$\mathcal{P}1$. *For $k = 1, \ldots, q$* and $i = 1, \ldots, \lambda_k$, one has $\sigma_i^k > 0$

and

$\mathcal{P}2$. *For $k = 1, \ldots, q$*, one has $\sigma_{\lambda_k}^k = \sigma_{\lambda_1}^1 + \left(\frac{1}{2} - \frac{1}{2^k}\right)$.

To check the first property, one first notices that according to (11), one has $\sigma_1^1 = \frac{1}{2} > 0$. Then, one shall show that $\sigma_1^k > \sigma_1^{k-1}$ for $k = 2, \ldots, q$. Indeed,

$$\begin{aligned} \sigma_1^k - \sigma_1^{k-1} &= \lambda_{k-1}\delta_{k-1} - \lambda_k\delta_k + \left(\frac{1}{2^{k-1}} - \frac{1}{2^k}\right) \\ &> \lambda_{k-1}\delta_{k-1} - \lambda_k\delta_k \\ &= \lambda_{k-1}\delta_{k-1} - \frac{3\delta_k}{2} - \left(\lambda_k - \frac{3}{2}\right)\delta_k \\ &= \lambda_{k-1}\delta_{k-1} - \frac{3\delta_k}{2} - \frac{\delta_{k-1}}{2} \text{ according to (8)} \\ &= \left(\lambda_{k-1} - \frac{1}{2}\right)\delta_{k-1} - \frac{3\delta_k}{2} \\ &\geq \frac{3\delta_{k-1}}{2} - \frac{3\delta_k}{2} \text{ since } \lambda_{k-1} \geq 2 \\ &\geq 0 \text{ according to (9).} \end{aligned}$$

As a result one has $\sigma_1^k > \sigma_1^{k-1} > \sigma_1^1 > 0$ and henceforth $\sigma_i^k > 0$ for $i \geq 2$ since $\sigma_i^k = \sigma_1^k + \delta_k\lambda_k$ and $\lambda_k\delta_k > 0$. The second property follows from (11) as one has

$$\begin{aligned} \sigma_{\lambda_k}^k &= \sigma_1^k + \lambda_k\delta_k = \lambda_1\delta_1 + \left(1 - \frac{1}{2^k}\right) \\ &= \frac{1}{2} + \lambda_1\delta_1 + \left(\frac{1}{2} - \frac{1}{2^k}\right) = \sigma_{\lambda_1}^1 + \left(\frac{1}{2} - \frac{1}{2^k}\right). \end{aligned} \tag{13}$$

Now, let us consider the following diagonal matrix

$$\Lambda_k(\theta) = \theta^{-\sigma_1^k}\Delta_k(\theta) \tag{14}$$

where the σ_1^k's are the positive reals given by Lemma 3.1 and $\Delta_k(\theta)$ is defined by (6). Taking into account the structure of the matrices A_k and C_k respectively given by (2) and (3), one can show that the following identities hold

$$\begin{aligned} &\Lambda_k(\theta)A_k\Lambda_k^{-1}(\theta) = \Delta_k(\theta)A_k\Delta_k^{-1}(\theta) = \theta^{\delta_k}A_k \\ &C_k\Lambda_k^{-1}(\theta) = \theta^{\sigma_1^k + \delta_k}C_k. \end{aligned} \tag{15}$$

Moreover, one can check that

$$\dot{\Lambda}_k(\theta) \triangleq \frac{d}{dt}\Lambda_k(\theta) = -\frac{\dot{\theta}(t)}{\theta(t)}\left(\sigma_1^k I_k + \delta_k D_k\right)\Lambda_k(\theta) \tag{16}$$

where I_k is the $k \times k$ identity matrix and D_k is the following $n_k \times n_k$ diagonal matrix

$$D_k = \operatorname{diag}(1, 2, \ldots, n_k).$$

Furthermore, for $k = 1, \ldots, q$, let

$$K_k = \begin{bmatrix} K_{k,1} & K_{k,2} & \cdots & K_{k,n_k} \end{bmatrix}^T \text{ where } K_{k,i} \in R^{p_k \times p_k} \tag{17}$$

be the $n_k \times p_k$ matrix such that $\tilde{A}_k \triangleq A_k - K_kC_k$ is Hurwitz. Then, there exist a strictly positive real number $a > 0$ and q symmetric positive definite $n_k \times n_k$ matrices S_k such that [12,13]

$$\tilde{A}_k^T S_k + S_k\tilde{A}_k \leq -aS_k \quad \text{and} \quad D_kS_k + S_kD_k \geq 0. \tag{18}$$

This allows to define some matrices that will be used throughout the observation error convergence analysis, namely

$$\begin{aligned} Q_k &= S_kD_k + D_kS_k, \\ \Omega_k &= \left(\sigma_1^kS_k + \delta_kS_kD_k\right) + \left(\sigma_1^kS_k + \delta_kD_kS_k\right) \\ &= \delta_kQ_k + 2\sigma_1^kS_k \\ \Omega &= \operatorname{diag}(\Omega_1, \Omega_2, \ldots, \Omega_q) \\ S &= \operatorname{diag}(S_1, S_2, \ldots, S_q). \end{aligned} \tag{19}$$

Taking into account the facts that $Q_k = Q_k^T \geq 0$, $\sigma_1^k > 0$ and $S_k = S_k^T > 0$, one can deduce that Ω_k is symmetric positive definite and so is Ω.

Finally, one shall denote by $\lambda_M(\cdot)$ and $\lambda_m(\cdot)$ the largest and the smallest eigenvalue of $(\cdot)$, respectively. And the conditioning number of $(\cdot)$ shall be denoted by $\mu(\cdot)$, i.e. $\mu(\cdot) = \frac{\lambda_M(\cdot)}{\lambda_m(\cdot)}$.

4. Observer design

As mentioned above, the main motivation of the paper consists in providing a suitable high gain observer with an appropriate updating of the underlying design parameter for the class of cascade nonlinear systems (1). In the following, one will propose the observer equations with the underlying fundamental convergence result and provide a full observation error convergence analysis. Some insights are given about the design parameter specification at the end of the section.

4.1. Observer equations

The candidate observer borrows from the high gain observers proposed in [4,11] up to an appropriate change as pointed out by the following equations.

$$\begin{cases} \dot{\hat{x}}^k(t) = A_k\hat{x}^k(t) + \varphi^k(u(t), \underline{\hat{x}}(t)) - \Delta_k^{-1}(\theta(t))K_kC_ke^k(t) \\ \quad \text{for } k = 1, \ldots, q \\ \dot{\theta}(t) = -\dfrac{1}{2}\mu_1\theta(t)\left(a(\theta(t)-1) - g(t)\gamma\left(\|\tilde{y}(t)\|\right)\right) \quad \text{with} \\ \quad \theta(0) \geq 1 \\ g(t) = \dfrac{M}{1 + \min\left(\rho, \frac{1}{T}\int_{\max(0,t-T)}^{t} \|\tilde{y}(\tau)\|^2 d\tau\right)} \end{cases} \tag{20}$$

where u and y are respectively the inputs and outputs of system (1), $\hat{x} \in \mathbb{R}^n$ denotes the state estimates given by

$$\hat{x} = \begin{pmatrix} \hat{x}^1 \\ \hat{x}^2 \\ \vdots \\ \hat{x}^q \end{pmatrix} \in \mathbb{R}^n \quad \text{with } \hat{x}^k = \begin{pmatrix} \hat{x}_1^k \\ \hat{x}_2^k \\ \vdots \\ \hat{x}_{\lambda_k}^k \end{pmatrix} \in \mathbb{R}^{n_k}$$

where $\hat{x}_i^k \in \mathbb{R}^{p_k}$ for $i = 1, \ldots, \lambda_k$ and $k = 1, \ldots, q$ and hence $\sum_{k=1}^q n_k = n$, $\underline{\hat{x}}_i^k \in \mathbb{R}^{p_k}$ denotes a state estimate up to an output injection as follows for $k = 1, \ldots, q$.

$$\underline{\hat{x}}_i^k = \begin{cases} x_1^k & \text{for i} = 1 \\ \hat{x}_i^k & \text{for } i = 2, \ldots, \lambda_k \end{cases}$$

$e^k \in \mathbb{R}^{n_k}$ and $\tilde{y} \in \mathbb{R}^p$ are the k'th subcomponent of the observation error and output observation error respectively given by

$$e^k = \hat{x}^k - x^k \quad \text{and} \quad \tilde{y} = Ce = C\left(\hat{x} - x\right)$$

ρ, T and M are positive scalars which represent the observer design parameters, $\mu_1 = \frac{\lambda_m(S)}{\lambda_M(\Omega)}$ where S and Ω are given by (19) and finally $\gamma : \|\tilde{y}\| \mapsto \gamma(\|\tilde{y}\|)$ is a real-valued non negative, non decreasing and bounded function satisfying $\gamma(0) = 0$.

Remark 4.1. The Riccati differential equation governing the dynamics of the gain parameter $\theta(t)$ is similar to that given in [11]. The choice of the parameter ρ is not crucial as it is mainly introduced to saturate the integral term $\frac{1}{T}\int_{\max(0,t-T)}^{t} \|\tilde{y}(\tau)\|^2 d\tau$ and can be set to arbitrarily high values. However, the design parameters M and T have to be specified carefully as pointed out in [11]. This shall be discussed later to make the paper self containing.

The following fundamental result provides the properties of the observer under consideration.

Theorem 4.1. *Under Assumption 1, the trajectories of observer (20) converge exponentially to those of system (1) for relatively high values of the parameter M.*

4.2. Convergence analysis

The proof of Theorem 4.1 is carried out step by step as follows. First of all, one shall prove the boundedness of the design parameter $\theta(t)$ while providing expressions of the corresponding lower and upper bounds. In particular, one shall show that for all $t \geq 0$, $\theta(t) \geq 1$. Next, one shall derive the dynamics of the observation error. This allows to establish the boundedness of the output observation error $\tilde{y}(t)$ and subsequently the boundedness of the observation error $e(t) = \hat{x}(t) - x(t)$. Finally, one successively prove the exponential convergence to zero of the output and state observation errors.

4.2.1. Boundedness of $\theta(t)$

To this end, one shall consider two cases depending on the sign of $\dot{\theta}(t)$. Before considering these cases, one notices that the expression of $\dot{\theta}$ in (20) is such that

$$\theta = 1 \Longrightarrow \dot{\theta} \geq 0.$$

As a result, one has $\theta(t) \geq 1$ for all $t \geq 0$ as soon as $\theta(0) \geq 1$. One also notices that

$$\forall t \geq 0 : \ 0 < g(t) \leq M.$$

Let us now discuss both cases mentioned above.

- The case $\dot{\theta}(t) \geq 0$. Since $\theta(t) \geq 1$ and according to (20), one has $a(\theta(t) - 1) - g(t)\gamma(\|\tilde{y}\|) \leq 0$ which implies that

$$\theta(t) \leq 1 + \frac{g(t)\gamma(\|\tilde{y}\|)}{a} \leq 1 + \frac{M}{a}\gamma(\|\tilde{y}\|).$$

And henceforth, one has

$$\forall t \geq 0 : \ 1 \leq \theta(t) \leq 1 + \frac{M\gamma_{\max}}{a} \tag{21}$$

where $\gamma_{\max}$ is the upper bound of γ.

- The case $\dot{\theta}(t) \leq 0$. Since $\theta(t) \geq 1$ and from (20), one has

$$\begin{aligned} \dot{\theta}(t) &\leq -\frac{\mu_1}{2}\left(a(\theta(t) - 1) - g(t)\gamma(\|\tilde{y}\|)\right) \\ &= -\frac{a\mu_1}{2}\theta(t) + \frac{a\mu_1}{2}\left(1 + \frac{M\gamma_{\max}}{a}\right). \end{aligned} \tag{22}$$

Integrating (22) from some $t_0 < t$ to t yields

$$\begin{aligned} \theta(t) &\leq e^{-\frac{a\mu_1}{2}(t-t_0)}\theta(t_0) + \left(1 + \frac{M\gamma_{\max}}{a}\right) \\ &\leq \theta(t_0) + \left(1 + \frac{M\gamma_{\max}}{a}\right). \end{aligned} \tag{23}$$

Now, the time t_o may be 0 and in this case $\theta(0) \geq 1$ is arbitrary, or the final time of an interval on which $\dot{\theta}(t) \geq 0$ and according to (21), one has $\theta(t_0) \leq 1 + \frac{M\gamma_{\max}}{a}$. As a result, for any t_0, one has

$$\theta(t_0) \leq \max\left(\theta(0), 1 + \frac{M\gamma_{\max}}{a}\right). \tag{24}$$

Using (24), inequality (23) becomes

$$\theta(t) \leq \max\left(\theta(0), 1 + \frac{M\gamma_{\max}}{a}\right) + \left(1 + \frac{M\gamma_{\max}}{a}\right). \tag{25}$$

To summarize and according to (21) and (25), $\theta(t)$ is bounded and satisfies

$$\begin{aligned} \forall t \geq 0 : \ \theta(t) \leq \theta_{\max} &\triangleq \max\left(\theta(0), 1 + \frac{M\gamma_{\max}}{a}\right) \\ &+ \left(1 + \frac{M\gamma_{\max}}{a}\right). \end{aligned} \tag{26}$$

4.2.2. Observation error dynamics

For writing convenience and as long as there is no ambiguity, one shall omit the time t for each variable. The observation error equation is given by

$$\dot{e}^k = A_k e^k + \varphi^k(u, \hat{\underline{x}}) - \varphi^k(u, x) - \Delta_k^{-1}(\theta) K_k C_k e^k \tag{27}$$

where u is an admissible control such that $\|u\|_\infty \leq \eta$ with η being a positive scalar.

For $k = 1, \ldots, q$, let

$$\bar{e}^k = \Lambda_k(\theta) e^k \tag{28}$$

where $\Lambda_k(\theta)$ is given by (14). From Eq. (27) and using (15) and (16), one gets

$$\begin{aligned}
\dot{\bar{e}}^k &= \Lambda_k(\theta)\dot{e}^k + \dot{\Lambda}_k(\theta)\Lambda_k^{-1}(\theta)\bar{e}^k \\
&= \Lambda_k(\theta) A_k \Lambda_k(\theta)^{-1}\bar{e}^k + \Lambda_k(\theta)(\varphi^k(u, \hat{\underline{x}}) - \varphi^k(u, x)) \\
&\quad - \Lambda_k(\theta)\Delta_k^{-1}(\theta) K_k C_k \Lambda_k^{-1}(\theta)\bar{e}^k - \frac{\dot{\theta}}{\theta}(\sigma_1^k I_k + \delta_k D_k)\bar{e}^k \\
&= \theta^{\delta_k} A_k \bar{e}^k - \theta^{\delta_k} K_k C_k \bar{e}^k + \Lambda_k(\theta)(\varphi^k(u, \hat{\underline{x}}) - \varphi^k(u, x)) \\
&\quad - \frac{\dot{\theta}}{\theta}(\sigma_1^k I_k + \delta_k D_k)\bar{e}^k \\
&= \theta^{\delta_k}\tilde{A}_k \bar{e}^k + \Lambda_k(\theta)(\varphi^k(u, \hat{\underline{x}}) - \varphi^k(u, x)) \\
&\quad - \frac{\dot{\theta}}{\theta}(\sigma_1^k I_k + \delta_k D_k)\bar{e}^k.
\end{aligned}$$

Set

$$V_k(\bar{e}^k) = \bar{e}^{k^T} S_k \bar{e}^k \tag{29}$$

and let $V(\bar{e}) = \sum_{k=1}^q V_k(\bar{e}^k) = \bar{e}^T S \bar{e}$ where S is given by (19), be the candidate Lyapunov function. Notice that according to (28) and (29) and from the fact $\sigma_i^k > 0$, one has for $\theta(t) \geq 1$:

$$\forall t \geq 0: \quad \|e^k(t)\|^2 \leq \|\bar{e}^k(t)\|^2 \leq \frac{1}{\lambda_m(S_k)} V_k(\bar{e}^k(t))$$

and this yields

$$\forall t \geq 0: \ \|e(t)\| \leq \|\bar{e}(t)\| \leq \frac{1}{\sqrt{\lambda_m(S)}}\sqrt{V(\bar{e}(t))}. \tag{30}$$

Now, one has

$$\begin{aligned}
\dot{V}_k &= 2\bar{e}^{k^T} S_k \dot{\bar{e}}^k \\
&= 2\theta^{\delta_k}\bar{e}^{k^T} S_k \tilde{A}_k \bar{e}^k + 2\bar{e}^{k^T} S_k \Lambda_k(\theta)(\varphi^k(u, \hat{\underline{x}}) - \varphi^k(u, x)) \\
&\quad - 2\frac{\dot{\theta}}{\theta}\left(\sigma_1^k \bar{e}^{k^T} S_k \bar{e}^k + \delta_k \bar{e}^{k^T} S_k D_k \bar{e}^k\right) \\
&= 2\theta^{\delta_k}\bar{e}^{k^T} S_k \tilde{A}_k \bar{e}^k + 2\bar{e}^{k^T} S_k \Lambda_k(\theta)(\varphi^k(u, \hat{\underline{x}}) - \varphi^k(u, x)) \\
&\quad - \frac{\dot{\theta}}{\theta}\bar{e}^{k^T}\Omega_k \bar{e}^k
\end{aligned}$$

where the symmetric positive definite matrix Ω_k is given by (19). And using (18), one gets

$$\begin{aligned}
\dot{V}_k &\leq -a\theta^{\delta_k}\bar{e}^{k^T} S_k \bar{e}^k + 2\bar{e}^{k^T} S_k \Lambda_k(\theta)(\varphi^k(u, \hat{\underline{x}}) - \varphi^k(u, x)) \\
&\quad - \frac{\dot{\theta}}{\theta}\bar{e}^{k^T}\Omega_k \bar{e}^k \\
&\leq -a\theta^{\delta_k} V_k + 2\|S_k \bar{e}^k\| \|\Lambda_k(\theta)(\varphi^k(u, \hat{\underline{x}}) - \varphi^k(u, x))\| \\
&\quad - \frac{\dot{\theta}}{\theta}\bar{e}^{k^T}\Omega_k \bar{e}^k \\
&\leq -a\theta^{\delta_k} V_k + 2\sqrt{\lambda_M(S)}\sqrt{V_k} \\
&\quad \times \sum_{i=1}^{\lambda_k} \frac{1}{\theta^{\sigma_i^k}}\|\varphi_i^k(u, \hat{\underline{x}}) - \varphi_i^k(u, x)\| - \frac{\dot{\theta}}{\theta}\bar{e}^{k^T}\Omega_k \bar{e}^k
\end{aligned}$$

where σ_i^k and σ_j^l are as given in (11). Therefore,

$$\begin{aligned}
\dot{V}_k &\leq -a\theta^{\delta_k} V_k + 2\rho_k\sqrt{\lambda_M(S)}\sqrt{V_k} \\
&\quad \times \sum_{i=1}^{\lambda_k}\sum_{l=1}^{q}\sum_{j=2}^{\lambda_l} \chi_{l,j}^{k,i}\theta^{-\sigma_i^k}\|e_j^l\| - \frac{\dot{\theta}}{\theta}\bar{e}^{k^T}\Omega_k \bar{e}^k
\end{aligned}$$

where $\rho_k = \sup\left\{\left\|\frac{\partial\varphi_i^k}{\partial x_j^l}(u, x)\right\|; x \in \mathbb{R}^n \text{ and } \|u\|_\infty \leq \eta\right\}$; the $\chi_{l,j}^{k,i}$'s have the same definition as in Lemma 3.1.

Hence

$$\begin{aligned}
\dot{V}_k &\leq -a\theta^{\delta_k} V_k + 2\rho_k\sqrt{\lambda_M(S)}\sqrt{V_k} \\
&\quad \times \sum_{i=1}^{\lambda_k}\sum_{l=1}^{q}\sum_{j=2}^{\lambda_l} \chi_{l,j}^{k,i}\theta^{\sigma_j^l - \sigma_i^k}\|\bar{e}_j^l\| - \frac{\dot{\theta}}{\theta}\bar{e}^{k^T}\Omega_k \bar{e}^k.
\end{aligned}$$

As a result, one obtains

$$\begin{aligned}
\dot{V}_k &\leq -a\theta^{\delta_k} V_k + 2\rho_k\sqrt{\lambda_M(S)}\sqrt{V_k} \\
&\quad \times \sum_{i=1}^{\lambda_k}\sum_{l=1}^{q}\sum_{j=2}^{\lambda_l} \chi_{l,j}^{k,i}\theta^{\sigma_j^l - \sigma_i^k}\frac{\sqrt{V_l}}{\sqrt{\lambda_m(S)}} - \frac{\dot{\theta}}{\theta}\bar{e}^{k^T}\Omega_k \bar{e}^k
\end{aligned}$$

and hence

$$\begin{aligned}
\dot{V}_k &\leq -a\theta^{\delta_k} V_k + 2\rho_k\mu(S)\sqrt{\theta^{\delta_k} V_k} \\
&\quad \times \sum_{i=1}^{\lambda_k}\sum_{l=1}^{q}\sum_{j=2}^{\lambda_l} \chi_{l,j}^{k,i}\theta^{\sigma_j^l - \sigma_i^k - \frac{\delta_k}{2} - \frac{\delta_l}{2}}\sqrt{\theta^{\delta_l} V_l} - \frac{\dot{\theta}}{\theta}\bar{e}^{k^T}\Omega_k \bar{e}^k.
\end{aligned} \tag{31}$$

Now, since $\theta(t) \geq 1$ and according to Lemma 3.1, one has $\theta^{\sigma_j^l - \sigma_i^k - \frac{\delta_k}{2} - \frac{\delta_l}{2}} \leq \theta^{-\frac{1}{2q}}$.

Inequality (31) becomes

$$\begin{aligned}
\dot{V}_k &\leq -a\theta^{\delta_k} V_k + 2\lambda_k\rho_k\mu(S)\theta^{-\frac{1}{2q}}\sqrt{\theta^{\delta_k} V_k} \\
&\quad \times \sum_{l=1}^{q}\sum_{j=2}^{\lambda_l}\sqrt{\theta^{\delta_l} V_l} - \frac{\dot{\theta}}{\theta}\bar{e}^{k^T}\Omega_k \bar{e}^k.
\end{aligned} \tag{32}$$

Set $W_k(\bar{e}^k) = \bar{e}^{k^T}\Omega_k \bar{e}^k$ and $W(\bar{e}) = \sum_{k=1}^q W_k(\bar{e}^k)$. One has

$$\mu_1 W_k \triangleq \frac{\lambda_m(S)}{\lambda_M(\Omega)} W_k \leq V_k \leq \frac{\lambda_M(S)}{\lambda_m(\Omega)} W_k \triangleq \mu_2 W_k.$$

Notice that $\frac{\mu_2}{\mu_1} = \mu(S)\mu(\Omega)$ and hence inequality (32) becomes

$$\begin{aligned}
\dot{V}_k &\leq -a\theta^{\delta_k} V_k + 2\lambda_k\rho_k\mu(S)\theta^{-\frac{1}{2q}}\sqrt{\theta^{\delta_k} V_k} \\
&\quad \times \sum_{l=1}^{q}\sum_{j=2}^{\lambda_l}\sqrt{\theta^{\delta_l} V_l} - \frac{\dot{\theta}}{\theta} W_k.
\end{aligned}$$

Now, for $k = 1, \ldots, q$, set $V_k^* = a\theta^{\delta_k} V_k$ and let $V^* = \sum_{k=1}^q V_k^*$. Since $\theta \geq 1$ and according to (9), one has

$$a\theta V = a\theta^{\delta_q} V \leq V^* \leq a\theta^{\delta_1} V.$$

Then

$$\begin{aligned}
\dot{V}_k &\leq -V_k^* + 2\lambda_k\frac{\rho_k}{a}\mu(S)\theta^{-\frac{1}{2q}}\sqrt{V_k^*}\sum_{l=1}^{q}\sum_{j=2}^{\lambda_l}\sqrt{V_l^*} - \frac{\dot{\theta}}{\theta} W_k \\
&\leq -V_k^* + 2\lambda_k n\frac{\rho_k}{a}\mu(S)\theta^{-\frac{1}{2q}}\sqrt{V_k^*}\sqrt{V^*} - \frac{\dot{\theta}}{\theta} W_k \\
&\leq -V_k^* + 2\lambda_k n\frac{\rho_k}{a}\mu(S)\theta^{-\frac{1}{2q}} V^* - \frac{\dot{\theta}}{\theta} W_k.
\end{aligned}$$

And henceforth

$$\dot{V} \leq -V^* + 2n^2\frac{\rho}{a}\mu(S)\theta^{-\frac{1}{2q}} V^* - \frac{\dot{\theta}}{\theta} W$$

where $\rho = \max\{\rho_k, 1 \le k \le q\}$. Substituting $\frac{\dot\theta}{\theta}$ by its expression, one gets

$$\dot V \le -V^* + 2n^2 \frac{\rho}{a}\mu(S)\theta^{-\frac{1}{2q}} V^* + \frac{\mu_1}{2}\left(a(\theta-1) - g(t)\gamma(\|\tilde y\|)\right) W. \tag{33}$$

Two cases shall be considered depending on whether $2n^2 \frac{\rho}{a}\mu(S)\theta^{-\frac{1}{2q}} < \frac{1}{4}$ or not, i.e. depending on whether or not

$$\theta > \left(8n^2 \frac{\rho}{a}\mu(S)\right)^{2q} \triangleq \theta_c. \tag{34}$$

- In the case where $\theta > \theta_c$ (or equivalently $2n^2 \frac{\rho}{a}\mu(S)\theta^{-\frac{1}{2q}} < \frac{1}{4}$), one has

$$1 - \left(2n^2 \frac{\rho}{a}\mu(S)\right)\theta^{-\frac{1}{2q}}(t) > \frac{3}{4} \tag{35}$$

and inequality (33) becomes

$$\begin{aligned}
\dot V(t) &\le -\left(1 - 2n^2 \frac{\rho}{a}\mu(S)\theta^{\frac{1}{2q-1}}\right) V^* \\
&\quad + \frac{\mu_1}{2}\left(a(\theta-1) - g(t)\gamma(\|\tilde y\|)\right) W \\
&\le -\frac{3}{4}a\theta V + \frac{\mu_1}{2}\left(a(\theta-1) - g(t)\gamma(\|\tilde y\|)\right) W \\
&= -\frac{3}{4}a\theta V + \frac{\mu_1}{2}a(\theta-1)W - \frac{\mu_1}{2}g(t)\gamma(\|\tilde y\|)W \\
&\le -\frac{3}{4}a\theta V + \frac{1}{2}a(\theta-1)V - \frac{\mu_1}{2\mu_2}g(t)\gamma(\|\tilde y\|)V \\
&= -\frac{1}{4}a(\theta+2)V(t) - \frac{\mu_1}{2\mu_2}g(t)\gamma(\|\tilde y\|)V(t) \\
&\le -\frac{1}{4}a(\theta+1)V(t) - \frac{\mu_1}{4\mu_2}g(t)\gamma(\|\tilde y\|)V(t).
\end{aligned} \tag{36}$$

- In the case where $\theta \le \theta_c$ and since $\theta(t) \ge 1$, one has $\theta^{-\frac{1}{2q}}(t) \le 1$ and inequality (33) becomes

$$\begin{aligned}
\dot V(t) &\le -V^* + \frac{\theta_c^{\frac{1}{2q}}}{4}V^* + \frac{\mu_1}{2}\left(a(\theta-1) - g(t)\gamma(\|\tilde y\|)\right) W \\
&\le -V^* + a\frac{\theta_c^{\frac{1}{2q}}}{4}\theta^{\delta_1}V + \frac{\mu_1}{2}\left(a(\theta-1) - g(t)\gamma(\|\tilde y\|)\right) W \\
&\le -V^* + a\frac{\theta_c^{\frac{1}{2q}}}{4}\theta_c^{\delta_1}V + \frac{\mu_1}{2}\left(a(\theta-1) - g(t)\gamma(\|\tilde y\|)\right) W \\
&\le -a\theta V + \frac{a\theta_c^{\delta_1+\frac{1}{2q}}}{4}V + \frac{1}{2}a(\theta-1)V \\
&\quad - \frac{\mu_1}{2\mu_2}g(t)\gamma(\|\tilde y\|)V \\
&= -\frac{1}{2}a(\theta+1)V(t) + \frac{a\theta_c^{\delta_1+\frac{1}{2q}}}{4}V - \frac{\mu_1}{2\mu_2}g(t)\gamma(\|\tilde y\|)V \\
&\le -\frac{1}{4}a(\theta+1)V(t) + \frac{a\theta_c^{\delta_1+\frac{1}{2q}}}{4}V \\
&\quad - \frac{\mu_1}{4\mu_2}g(t)\gamma(\|\tilde y\|)V.
\end{aligned} \tag{37}$$

Comparing (36) and (37), one can easily conclude that for any θ, one has

$$\dot V(t) \le -\frac{1}{4}a(\theta+1)V(t) + \frac{a\theta_c^{\delta_1+\frac{1}{2q}}}{4}V - \frac{\mu_1}{4\mu_2}g(t)\gamma(\|\tilde y\|)V. \tag{38}$$

4.2.3. Boundedness of the output observation error

Let us show that $\|\tilde y(t)\|$ is bounded. Indeed, suppose the contrary and choose ρ high enough. Then,

$$\exists N > 0;\ \exists t^\star > T;\ \forall t \ge t^\star - T : \|\tilde y(t)\| > N \quad \text{and}$$

$$\frac{1}{T}\int_{t^\star - T}^{t} \|\tilde y(\tau)\|^2 \mathrm{d}\tau > \rho. \tag{39}$$

As a result, one has

$$\forall t \ge t^\star - T : \|\tilde y(t)\| > N \quad \text{and} \quad g(t) = \frac{M}{1+\rho}.$$

Let $\gamma_N = \gamma(N)$. Since γ is a positive non decreasing function of $\|\tilde y\|$, one has

$$\forall t \ge t^\star - T : \gamma(\|\tilde y(t)\|) \ge \gamma_N.$$

Let us choose M as follows

$$M = \frac{\lambda(1+\rho)}{\gamma_N} \tag{40}$$

where λ is a positive constant and is such that $\lambda > \theta_c^{\delta_1+\frac{1}{2q}}\frac{\mu_2}{\mu_1}$. For $t \ge t^\star - T$, using (39), inequality (38) becomes

$$\begin{aligned}
\dot V(\bar e(t)) &\le -\frac{1}{4}a(\theta+1)V(\bar e(t)) + \frac{a\theta_c^{\delta_1+\frac{1}{2q}}}{4}V(\bar e(t)) \\
&\quad - \frac{\mu_1}{4\mu_2}\frac{M\gamma_N}{1+\rho}V(\bar e(t)).
\end{aligned}$$

Substituting M by its expression (40), one gets

$$\begin{aligned}
\dot V(\bar e(t)) &\le -\frac{1}{4}a(\theta+1)V(\bar e(t)) + \left(\theta_c^{\delta_1+\frac{1}{2q}} - \frac{\mu_1}{\mu_2}\lambda\right)\frac{1}{4}V(\bar e(t)) \\
&\le -\left(\frac{\mu_1}{\mu_2}\lambda - \theta_c^{\delta_1+\frac{1}{2q}}\right)\frac{1}{4}V(\bar e(t)).
\end{aligned}$$

Then, integrating both sides from $t^\star - T$ to t yields

$$\begin{aligned}
V(\bar e(t)) &\le \exp\left(-\frac{1}{4}\left(\frac{\mu_1}{\mu_2}\lambda - \theta_c^{\delta_1+\frac{1}{2q}}\right)(t - (t^\star - T))\right) \\
&\quad \times V(\bar e(t^\star - T)).
\end{aligned} \tag{41}$$

And choosing $t = 2(t^* - T)$, one gets

$$\begin{aligned}
V\left(\bar e\left(2\left(t^* - T\right)\right)\right) &\le \exp\left(-\frac{1}{4}\left(\frac{\lambda\mu_1}{\mu_2} - \theta_c^{\delta_1+\frac{1}{2q}}\right)\left(t^* - T\right)\right) \\
&\quad \times V\left(\bar e\left(t^* - T\right)\right).
\end{aligned}$$

Since $V(\bar e(t^\star - T))$ is constant, one can choose λ high enough such that

$$\begin{aligned}
V\left(\bar e\left(2\left(t^* - T\right)\right)\right) &\le \exp\left(-\frac{1}{4}\left(\frac{\mu_1}{\mu_2}\lambda - \theta_c^{\delta_1+\frac{1}{2q}}\right)\left(t^* - T\right)\right) \\
&\quad \times V\left(\bar e\left(t^* - T\right)\right) < \lambda_m(S)\frac{N^2}{4}.
\end{aligned} \tag{42}$$

Combining (41), (42) and (30), one gets for $t \ge t^\star - T$:

$$\begin{aligned}
\|\tilde y\left(2\left(t^* - T\right)\right)\|^2 &\le \|e\left(2\left(t^* - T\right)\right)\|^2 \le \|\bar e\left(2\left(t^* - T\right)\right)\|^2 \\
&\le \frac{1}{\lambda_m(S)}V\left(\bar e\left(2\left(t^* - T\right)\right)\right) < \frac{N^2}{4}
\end{aligned}$$

and henceforth

$$\|\tilde y\left(2\left(t^* - T\right)\right)\| < \frac{N}{2}.$$

This is in contradiction with (39) and $\|\tilde y(t)\|$ is then bounded. In the sequel, one shall denote by $B_{\tilde y}$ the upper bound of $\|\tilde y(t)\|$, i.e.

$$\forall t \ge 0 : \ \|\tilde y(t)\| \le B_{\tilde y}. \tag{43}$$

4.2.4. Boundedness of the state observation error

To show that $e(t)$ is bounded, one introduces the following change of coordinates

$$\tilde{e}^k = \Lambda_k(\bar{\theta})e^k, \quad k = 1, \dots, q \tag{44}$$

where $\bar{\theta}$ is a constant satisfying

$$\bar{\theta} > \max\{\theta_{\max}, \theta_c\} \tag{45}$$

where $\theta_{\max}$ and θ_c are respectively given by (26) and (34). Then, the error dynamics (27) is transformed into

$$\begin{aligned}\dot{e}^k &= A_k e^k - \Delta_k^{-1}(\bar{\theta})K_k C_k e^k + \varphi^k(u, \hat{\underline{x}}) - \varphi^k(u, x)\\ &\quad + \Delta_k^{-1}(\bar{\theta})K_k C_k e^k - \Delta_k^{-1}(\theta)K_k C_k e^k.\end{aligned} \tag{46}$$

Using (44) and (46) and proceeding as above, one gets

$$\begin{aligned}\dot{\tilde{e}}^k &= \bar{\theta}^{\delta_k}(A_k - K_k C_k)\tilde{e}^k + \Lambda(\bar{\theta})\left(\varphi^k(u, \hat{\underline{x}}) - \varphi^k(u, x)\right)\\ &\quad + \bar{\theta}^{-\sigma_1^k} K_k \tilde{y}_k - \bar{\theta}^{-\sigma_1^k}\Delta^{-1}\left(\frac{\theta}{\bar{\theta}}\right)K_k\tilde{y}_k.\end{aligned}$$

Now set $\tilde{V}_k(\tilde{e}^k) = \tilde{e}^{kT}S_k\tilde{e}^k$ and consider the Lyapunov function $\tilde{V}(\tilde{e}) = \sum_{k=1}^q \tilde{V}_k(\tilde{e}^k)$ for system (46).

Proceeding as above, one can show that (see inequality (32)):

$$\begin{aligned}\dot{\tilde{V}}_k &\le -a\bar{\theta}^{\delta_k}\tilde{V}_k + 2\lambda_k\rho_k\mu(S)\bar{\theta}^{-\frac{1}{2q}}\sqrt{\bar{\theta}^{\delta_k}\tilde{V}_k}\\ &\quad \times \sum_{l=1}^{q}\sum_{j=2}^{\lambda_l}\sqrt{\bar{\theta}^{\delta_l}\tilde{V}_l} + \bar{\theta}^{-\sigma_1^k}\tilde{e}^{kT}S_kK_k\tilde{y}_k\\ &\quad - \bar{\theta}^{-\sigma_1^k}\tilde{e}^{kT}S_k\Delta_k^{-1}\left(\frac{\theta}{\bar{\theta}}\right)K_k\tilde{y}_k.\end{aligned}$$

Now, for $k = 1, \dots, q$, set $\tilde{V}_k^* = \bar{\theta}^{\delta_k}\tilde{V}_k$ and let $\tilde{V}^* = \sum_{k=1}^q \tilde{V}_k^*$. Notice that

$$\bar{\theta}\tilde{V} \le \tilde{V}^\star \le \bar{\theta}^{\delta_1}\tilde{V}. \tag{47}$$

Then

$$\begin{aligned}\dot{\tilde{V}}_k &\le -a\tilde{V}_k^* + 2\lambda_k\rho_k\mu(S)\bar{\theta}^{-\frac{1}{2q}}\sqrt{\tilde{V}_k^\star}\\ &\quad \times \sum_{l=1}^{q}\sum_{j=1}^{\lambda_l}\sqrt{\tilde{V}_l^\star} + \bar{\theta}^{-\sigma_1^k}\tilde{e}^{kT}S_kK_k\tilde{y}_k\\ &\quad - \bar{\theta}^{-\sigma_1^k}\tilde{e}^{kT}S_k\Delta_k^{-1}\left(\frac{\theta}{\bar{\theta}}\right)K_k\tilde{y}_k\\ &\le -a\tilde{V}_k^* + 2\lambda_k n\rho_k\mu(S)\bar{\theta}^{-\frac{1}{2q}}\sqrt{\tilde{V}_k^*}\sqrt{\tilde{V}^*}\\ &\quad + \bar{\theta}^{-\sigma_1^k}\tilde{e}^{kT}S_kK_k\tilde{y}_k - \bar{\theta}^{-\sigma_1^k}\tilde{e}^{kT}S_k\Delta_k^{-1}\left(\frac{\theta}{\bar{\theta}}\right)K_k\tilde{y}_k\\ &\le -a\tilde{V}_k^* + 2\lambda_k n\rho_k\mu(S)\bar{\theta}^{-\frac{1}{2q}}\tilde{V}^*\\ &\quad + \bar{\theta}^{-\sigma_1^k}\tilde{e}^{kT}S_kK_k\tilde{y}_k - \bar{\theta}^{-\sigma_1^k}\tilde{e}^{kT}S_k\Delta_k^{-1}\left(\frac{\theta}{\bar{\theta}}\right)K_k\tilde{y}_k.\end{aligned}$$

Hence

$$\begin{aligned}\dot{\tilde{V}} &\le -a\tilde{V}^* + 2n^2\rho\mu(S)\bar{\theta}^{-\frac{1}{2q}}\tilde{V}^*\\ &\quad + \sum_{k=1}^{q}\left(\bar{\theta}^{-\sigma_1^k}\tilde{e}^{kT}S_kK_k\tilde{y}_k - \bar{\theta}^{-\sigma_1^k}\tilde{e}^{kT}S_k\Delta_k^{-1}\left(\frac{\theta}{\bar{\theta}}\right)K_k\tilde{y}_k\right).\end{aligned}$$

According to the choice of $\bar{\theta}$ and using (35), one gets

$$\begin{aligned}\dot{\tilde{V}} &\le -a\frac{3}{4}\bar{\theta}\tilde{V}\\ &\quad + \sum_{k=1}^{q}\left(\bar{\theta}^{-\sigma_1^k}\tilde{e}^{kT}S_kK_k\tilde{y}_k - \bar{\theta}^{-\sigma_1^k}\tilde{e}^{kT}S_k\Delta_k^{-1}\left(\frac{\theta}{\bar{\theta}}\right)K_k\tilde{y}_k\right).\end{aligned}$$

Now, since $\sigma_1^k > 0$, one has

$$\sum_{k=1}^{q}\bar{\theta}^{-\sigma_1^k}\tilde{e}^{kT}S_kK_k\tilde{y}_k \le \|K_M\|B_{\tilde{y}}\frac{\lambda_{\max}(S)}{\sqrt{\lambda_{\min}(S)}}\sqrt{\tilde{V}} \tag{48}$$

where $B_{\tilde{y}}$ is the upper bound of $\tilde{y}(t)$ given by (43) and $K_M = \max_{1\le k\le q}\|K_k\|$.

Similarly, since $\frac{\theta}{\bar{\theta}} < 1$ one has $\left\|\Delta_k^{-1}\left(\frac{\theta}{\bar{\theta}}\right)\right\| < 1$ and

$$\sum_{k=1}^{q}\bar{\theta}^{-\sigma_1^k}\tilde{e}^{kT}S_k\Delta_k^{-1}\left(\frac{\theta}{\bar{\theta}}\right)K_k\tilde{y}_k \le \|K_M\|B_{\tilde{y}}\frac{\lambda_{\max}(S)}{\sqrt{\lambda_{\min}(S)}}\sqrt{\tilde{V}}. \tag{49}$$

In view of the estimations above, one has

$$\dot{\tilde{V}} \le -\frac{3}{4}a\bar{\theta}\tilde{V} + 2\beta B_{\tilde{y}}\sqrt{\tilde{V}} \tag{50}$$

where $\beta = \|K_M\|\frac{\lambda_{\max}(S)}{\sqrt{\lambda_{\min}(S)}}$. This yields

$$\sqrt{\tilde{V}(\tilde{e}(t))} \le \exp\left(-\frac{3}{8}a\bar{\theta}t\right)\sqrt{\tilde{V}(\tilde{e}(0))} + \frac{8\beta}{3a\bar{\theta}}B_{\tilde{y}}. \tag{51}$$

As a result, $\tilde{V}(\tilde{e}(t))$ is bounded and so is $\tilde{e}(t)$, or equivalently $e(t)$. In the sequel, one shall denote B_e the upper bound of $\|e(t)\|$, i.e.

$$\forall\, t \ge 0: \ \|e(t)\| \le B_e. \tag{52}$$

4.2.5. Convergence of the output observation error

Let us show that $\|\tilde{y}(t)\|$ converges to zero. In fact, one shall show not only that $\tilde{y}$ converges to zero, but also it does it exponentially. This can be done by a contradiction argument. Indeed, suppose that $\tilde{y}$ does not exponentially converge to zero. This implies that

$$\forall c > 0;\ \forall\alpha > 0;\ \forall T^\star > T;\ \exists t^\star > T^\star: \|\tilde{y}(t^\star)\| > ce^{-\alpha t^\star}.$$

But since the function $t \mapsto \|\tilde{y}(t)\| - ce^{-\alpha t}$ is continuous, one also has

$$\begin{aligned}&\forall c > 0;\ \forall\alpha > 0;\ \forall T^\star > T;\ \exists T_1 > 0;\\ &\forall t \in [t^\star, t^\star + T_1]: \|\tilde{y}(t)\| > ce^{-\alpha t} \ge ce^{-\alpha(t^\star + T_1)}.\end{aligned} \tag{53}$$

Now, since $\|\tilde{y}\|$ is bounded and by choosing $\rho > B_{\tilde{y}}^2$, one has for $t \ge t^\star$:

$$g(t) \ge \frac{M}{1 + B_{\tilde{y}}^2}. \tag{54}$$

Now, choose M as follows

$$M = \frac{\lambda(1 + B_{\tilde{y}}^2)}{\gamma_m} \tag{55}$$

where $\gamma_m = \inf\{\gamma(\|\tilde{y}(t)\|);\ t \in [t^\star, t^\star + T_1]\}$ and λ is a positive constant satisfying $\lambda > \theta_c^{\delta_1 + \frac{1}{2q}}\frac{\mu_2}{a\mu_1}$.

Using (54), inequality (38) becomes

$$\dot{V}(t) \le -\frac{1}{4}a(\theta + 1)V(t) + \frac{a\theta_c^{\delta_1 + \frac{1}{2q}}}{4}V - \frac{\mu_1}{4\mu_2}\frac{M}{(1 + B_{\tilde{y}}^2)}\gamma_m V.$$

Substituting M by its expression, (55), in (38), one obtains for all $t \in [t^\star, t^\star + T_1]$:

$$\begin{aligned}\dot{V} &\le -\frac{1}{4}a(\theta + 1)V(t) + \left(a\theta_c^{\delta_1 + \frac{1}{2q}} - \frac{\mu_1}{\mu_2}\lambda\right)\frac{1}{4}V\\ &\le -\left(\frac{\mu_1}{\mu_2}\lambda - a\theta_c^{\delta_1 + \frac{1}{2q}}\right)\frac{1}{4}V.\end{aligned} \tag{56}$$

Integrating (56) on $[t^\star, t^\star + T_1]$, one gets

$$V(\bar{e}(t^\star + T_1)) \leq \exp\left(-\frac{T_1}{4}\left(\frac{\mu_1}{\mu_2}\lambda - a\theta_c^{\delta_1+\frac{1}{2q}}\right)\right) V(\bar{e}(t^\star))$$

or equivalently,

$$\|\bar{e}(t^\star + T_1)\| \leq \sqrt{\mu(S)}\exp\left(-\frac{T_1}{4}\left(\frac{\mu_1}{\mu_2}\lambda - a\theta_c^{\delta_1+\frac{1}{2q}}\right)\right)\|\bar{e}(t^\star)\|.$$

Now, according to (28) and (14) and using (10) and (13), one has for $\theta(t) \geq 1$:

$$\begin{aligned}\|\bar{e}^k(t)\| &\leq \|e^k\| \leq \theta^{\sigma_1^k+\lambda_k\delta_k}\|\bar{e}^k\| = \theta^{\sigma_{\lambda_k}^k}\|\bar{e}^k\| \\ &= \theta^{\sigma_{\lambda_1}^1+\left(\frac{1}{2}-\frac{1}{2^k}\right)}\|\bar{e}^k\| \leq \theta^{\sigma_{\lambda_1}^1+\frac{1}{2}}\|\bar{e}^k\|. \end{aligned} \tag{57}$$

Hence

$$\|\bar{e}(t)\| \leq \|e\| \leq \theta^{\sigma_{\lambda_1}^1+\frac{1}{2}}\|\bar{e}\|.$$

As a result, one has

$$\begin{aligned}\|\tilde{y}(t^\star + T_1)\| &\leq \|e(t^\star + T_1)\| \leq \theta^{\sigma_{\lambda_1}^1+\frac{1}{2}}\|\bar{e}(t^\star + T_1)\| \\ &\leq \theta_{\max}^{\sigma_{\lambda_1}^1+\frac{1}{2}}\|\bar{e}(t^\star + T_1)\|. \end{aligned} \tag{58}$$

Combining (57) and (58), one gets

$$\begin{aligned}\|\tilde{y}(t^\star + T_1)\| &\leq \sqrt{\mu(S)}\,\theta_{\max}^{\sigma_{\lambda_1}^1+\frac{1}{2}} e^{\left(-\frac{T_1}{4}\left(\frac{\mu_1}{\mu_2}\lambda - a\theta_c^{\delta_1+\frac{1}{2q}}\right)\right)}\|\bar{e}(t^\star)\| \\ &\leq \sqrt{\mu(S)}\,\theta_{\max}^{\sigma_{\lambda_1}^1+\frac{1}{2}} e^{\left(-\frac{T_1}{4}\left(\frac{\mu_1}{\mu_2}\lambda - a\theta_c^{\delta_1+\frac{1}{2q}}\right)\right)}\|e(t^\star)\| \\ &\leq \sqrt{\mu(S)}\,B_e\theta_{\max}^{\sigma_{\lambda_1}^1+\frac{1}{2}} e^{\left(-\frac{T_1}{4}\left(\frac{\mu_1}{\mu_2}\lambda - a\theta_c^{\delta_1+\frac{1}{2q}}\right)\right)} \end{aligned}$$

where B_e is the upper bound of $e(t)$ as given by (52). Now, it is clear that one can choose λ high enough such that

$$\sqrt{\mu(S)}\,B_e\theta_{\max}^{\sigma_{\lambda_1}^1+\frac{1}{2}} e^{\left(-\frac{T_1}{4}\left(\frac{\mu_1}{\mu_2}\lambda - a\theta_c^{\delta_1+\frac{1}{2q}}\right)\right)} < \frac{1}{2}ce^{-\alpha(T_1+t^\star)}. \tag{59}$$

Combining (53) and (59) leads to a contradiction, i.e.

$$0 < ce^{-\alpha(T_1+t^\star)} < \|\tilde{y}(t^\star + T_1)\| < \frac{1}{2}c\,e^{-\alpha(T_1+t^\star)}.$$

The convergence to zero of $\tilde{y}$ is proven and thus one has

$$\exists c > 0;\ \alpha > 0;\ \exists T^\star > T;\ \forall t > T^\star : \left\|\tilde{y}(t)\right\| \leq c\,e^{-\alpha t}. \tag{60}$$

4.2.6. Convergence of state observation error

Consider again the same change of variable given by (44). Proceeding as above and miming (51), one can show for $t \geq T^\star$ where $T^\star$ is given by (60) that

$$\sqrt{\tilde{V}\left(\tilde{e}(t)\right)} \leq \exp\left(-\frac{3}{8}a\bar{\theta}t\right)\sqrt{\tilde{V}\left(\tilde{e}(0)\right)} + m_1 c_1 e^{-\alpha_1 t} \tag{61}$$

where $\bar{\theta}$ is given by (45) and m_1, c_1, α_1 are real positive constants. This ends the proof of the theorem.

4.3. Design parameters specification

The equations of the observer requires to specify the function γ together with the three design parameters M, T and ρ.

Recall that $\gamma(\cdot)$ is non negative, non decreasing and bounded real-valued function with $\gamma(0) = 0$. One give here two expressions that have been used in the examples given in simulation.

$$\gamma(\xi) = \frac{\xi^2}{1+\xi^2} \quad \text{where } \xi \in \mathbb{R}_+^\star$$

and

$$\gamma(\xi) = \tanh\xi^2 \quad \text{where } \xi \in \mathbb{R}_+^\star.$$

In other respects, the function g can be written as follows

$$g(t) = \frac{M}{1 + \min\left(\rho, \mathcal{P}_T(t)\right)}$$

where

$$\mathcal{P}_T(t) = \frac{1}{T}\int_{\max(0,t-T)}^{t}\|\tilde{y}(\tau)\|^2 d\tau$$

represents the power of the output observation error determined on a moving window with a width equal to T. It is hence clear that the function g is bounded as one has

$$\forall t \geq 0 : \quad \frac{M}{1+\rho} \leq g(t) \leq M.$$

Notice that the design parameter M can be set to very high values in the absence of noise measurements. These high values allows the gain parameter $\theta(t)$ to quickly reach high values. This leads to the observation error vanishing and hence the decreasing of the gain parameter $\theta(t)$ to the predefined lower value, e.g. $\theta(t) = 1$. However, it is more advisable to avoid high values of M to reduce the sensitivity of the observer with respect to unavoidable noise measurements. The normalization of the design parameter M by the power of the output observation error on a moving window with a width equal to T can be thought as an interesting practical feature from a noise insensitivity point of view. It is worth noticing that small values of T are more advisable in the presence of noise measurements with a variance that varies significantly and relatively quickly in a continuous manner. On the contrary, relatively high values of T have to adopted when the variance of the noise measurements is constant or varies slowly.

5. Simulation example

In order to illustrate the performances of the updated high gain observer, one considers the following nonlinear system

$$\begin{cases}\dot{x}_1^1 = x_2^1 - x_1^1 + x_1^2\,u \\ \dot{x}_2^1 = x_3^1 - x_2^1 \\ \dot{x}_3^1 = -x_3^1 - \dfrac{x_2^1}{1+\left(x_2^1\right)^2} \\ \qquad - \dfrac{x_2^2}{1+\left(x_2^2\right)^2} - 10\,\cos\left(10\,x_2^1\right) + u \\ y_1 = x_1^1 \end{cases} \tag{62}$$

$$\begin{cases}\dot{x}_1^2 = x_2^2 - x_2^1x_3^2 - x_1^2 + u\,x_3^1 \\ \dot{x}_2^2 = -x_2^2 - \dfrac{x_2^2}{1+\left(x_2^2\right)^2} + \dfrac{x_3^1}{1+\left(x_3^1\right)^2} \\ \qquad -10\,x_2^1\sin\left(10\,x_2^1\right) + \nu \\ y_2 = x_1^2 \end{cases} \tag{63}$$

where $u = 5\,\sin(2\pi\,t)$ and ν is an external disturbance that takes the value 3 between $t_1 = 6$ s and $t_2 = 7$ s and zero elsewhere. The introduction of ν is motivated by illustration purposes; this allows to test the behaviour of the observer, and in particular that of $\theta(t)$, in the presence of such a disturbance. Notice that system (62)–(63) is in form (1) with $q = 2$, $p_1 = p_2 = 1$, $\lambda_1 = 3$ and $\lambda_2 = 2$ and is not included in the classes of systems considered in [5,6]. Moreover, the state trajectory of the system is bounded and the required Lipschitz assumption holds.

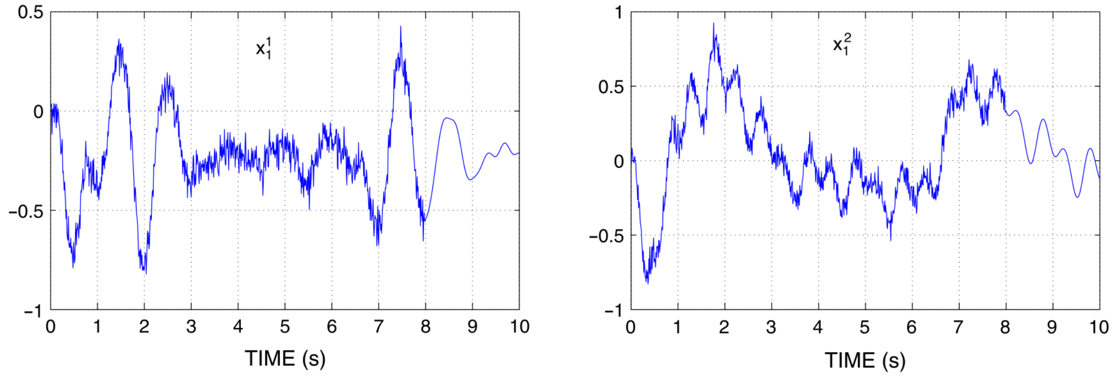

Fig. 1. Noisy measurements of x_1^1 and x_1^2.

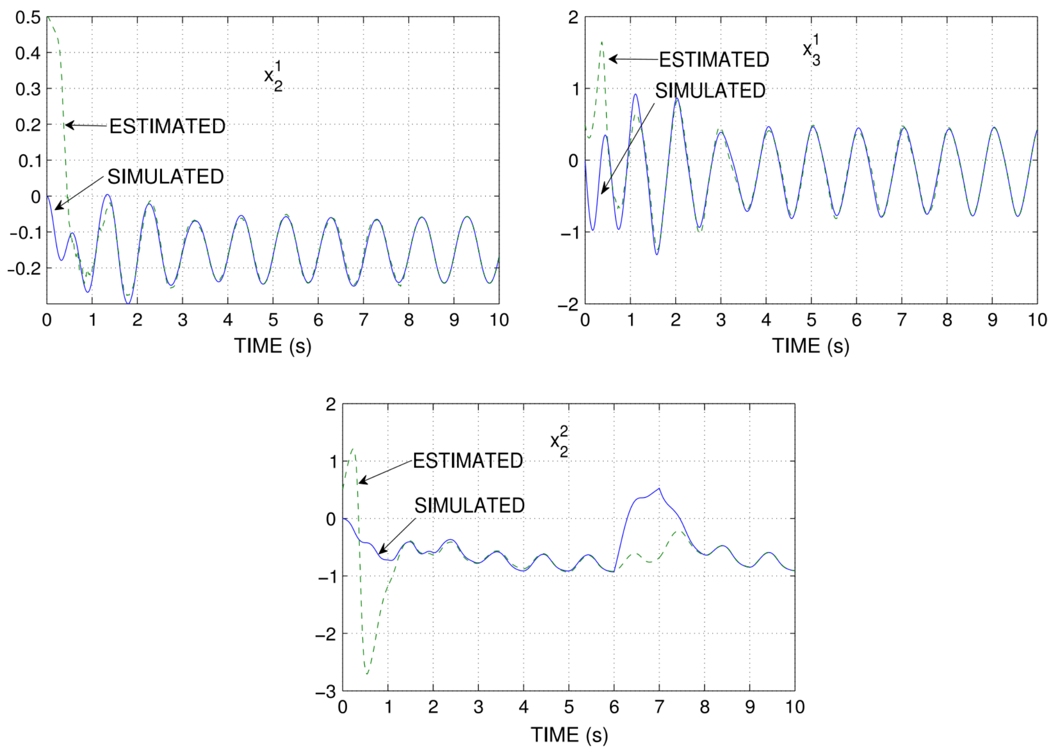

Fig. 2. Estimation of the missing states with an updated design parameter.

An observer of form (20) can be designed for system (62)–(63). The equations of this observer specialize as follows

$$\begin{pmatrix} \dot{\hat{x}}_1^1 \\ \dot{\hat{x}}_2^1 \\ \dot{\hat{x}}_3^1 \end{pmatrix} = \begin{pmatrix} \hat{x}_2^1 \\ \hat{x}_3^1 \\ 0 \end{pmatrix} + \begin{pmatrix} -x_1^1 + x_1^2 u \\ -\hat{x}_2^1 \\ -\hat{x}_3^1 - \dfrac{\hat{x}_2^1}{1+\left(\hat{x}_2^1\right)^2} - \dfrac{\hat{x}_2^2}{1+\left(\hat{x}_2^2\right)^2} - 10\cos\left(10\hat{x}_2^1\right) + u \end{pmatrix} - \begin{pmatrix} 3\theta^{\delta_1}(t) \\ 3\theta^{2\delta_1}(t) \\ \theta^{3\delta_1}(t) \end{pmatrix} \left(\hat{x}_1^1 - y_1\right) \tag{64}$$

$$\begin{pmatrix} \dot{\hat{x}}_1^2 \\ \dot{\hat{x}}_2^2 \end{pmatrix} = \begin{pmatrix} \hat{x}_2^2 \\ 0 \end{pmatrix} + \begin{pmatrix} -\hat{x}_2^1\hat{x}_3^1 - x_1^2 + u\hat{x}_3^1 \\ -\hat{x}_2^2 - \dfrac{\hat{x}_2^2}{1+\left(\hat{x}_2^2\right)^2} + \dfrac{\hat{x}_3^1}{1+\left(\hat{x}_3^1\right)^2} - 10\,\hat{x}_2^1 \sin\left(10\,\hat{x}_2^1\right) \end{pmatrix} - \begin{pmatrix} 2\theta^{\delta_2}(t) \\ \theta^{2\delta_2}(t) \end{pmatrix} \left(\hat{x}_1^2 - y_2\right) \tag{65}$$

where $\delta_1 = \delta_2 = 1$ are computed according to (7). The gain matrices K_k, with $k = 1$ and 2, of the observer are chosen such that all the poles of each matrix $\tilde{A}_k = A_k - K_k C_k$ are assigned to the value -1.

In the following, one gives simulation results involving system (62)–(63) with the following initial values:

$\hat{x}_1^1(0) = \hat{x}_2^1(0) = x_i^k(0) = 0;\quad k = 1, 2$ and

$i = 1, \ldots, \lambda_k;\qquad \hat{x}_2^1(0) = \hat{x}_3^1(0) = \hat{x}_2^2(0) = 0.5,\qquad \theta(0) = 1.$

The simulation experiment has been carried out between 0 and 10 s. Between 0 and 8 s, each measured variable has been corrupted by an additive Gaussian noise with zero mean value and a standard deviation equal to $\sqrt{0.1}$. For illustration purposes, the noise

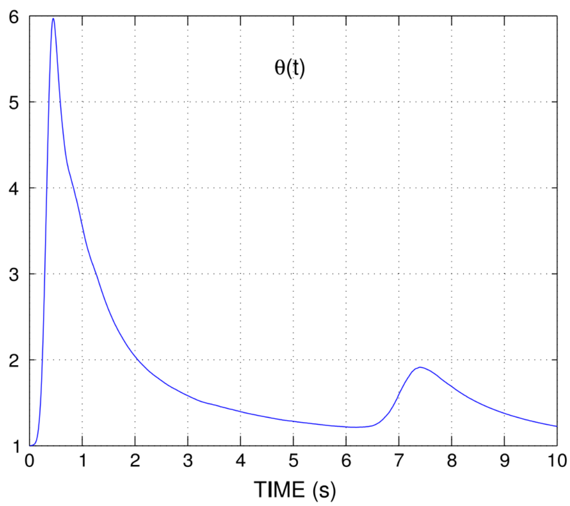

Fig. 3. Evolution of $\theta(t)$.

measurement is cut off at time $t = 8$ s and the outputs then become noise-free. The underlying noise realizations are given in Fig. 1.

The values of the parameters M, T and a used in simulation were respectively equal to 1000, 0.25 and 20. Concerning the function γ, it was specified as follows in this application:

$$\gamma(\tilde{y}) = \frac{\|\tilde{y}\|^2}{1 + \|\tilde{y}\|^2}.$$

The estimates of the missing state variables provided by the observer are given in Fig. 2 while Fig. 3 shows the evolution of the gain parameter $\theta(t)$. One notices that the value of $\theta(t)$ grows from the beginning of the simulation until time 0.5 s approximatively. At this time, all the observation errors decrease and this allows θ to decrease. Similarly, at the occurring of the external disturbance (at $t = 6$ s), the observation errors, in particular those related to x_2^2, increase and this induces an increasing of $\theta(t)$. As soon as the external disturbance is cut off, the observation error as well as the value of θ decrease. These results clearly demonstrate that as soon as the state estimates become accurate, the value of the gain parameter $\theta(t)$ decreases and is maintained at low values providing thereby accurate and almost noise-free estimates.

For comparison purposes, one has simulated the proposed observer using two constant values for the gain parameter θ, namely $\theta_1 = 1.5$ and $\theta_2 = 5$. The underlying observer is a classical high gain one [1,14]. The resulting estimates are given in Fig. 4 where they are compared to their true values issued from the simulation of system (62)–(63). Notice that the observer cannot track the unknown system trajectories when the design parameter is set to the low value θ_1 and that the value θ_2 is too high to provide smooth estimate of the state variables in the presence of a significant noise. One notices that when the output becomes noise free (at $t = 8$ s), the observer working with the constant value $\theta = 5$ performs as well as the update gain observer while that working with $\theta = 1.5$ still provide erroneous estimates due to the low value of the gain parameter θ. These results confirm the particular interest of the proposed updated gain parameter observer.

6. Conclusion

A high gain observer with an updated gain has been designed for a class of cascade non triangular systems that are observable for any input. Such a design has been particularly inspired from the observer designs proposed in the contributions [4,11]. The first contribution allowed to consider a large class of nonlinear and non triangular systems whereas the second one suggested a gain parameter adaptation process to improve the performance of high gain observers in noisy environments. A constructive Lyapunov approach has been pursued to address the convergence analysis problem using an appropriate adaptation of the approaches that have been developed in [4,11]. More specifically, the proposed observer can be viewed as an improved version of that observer

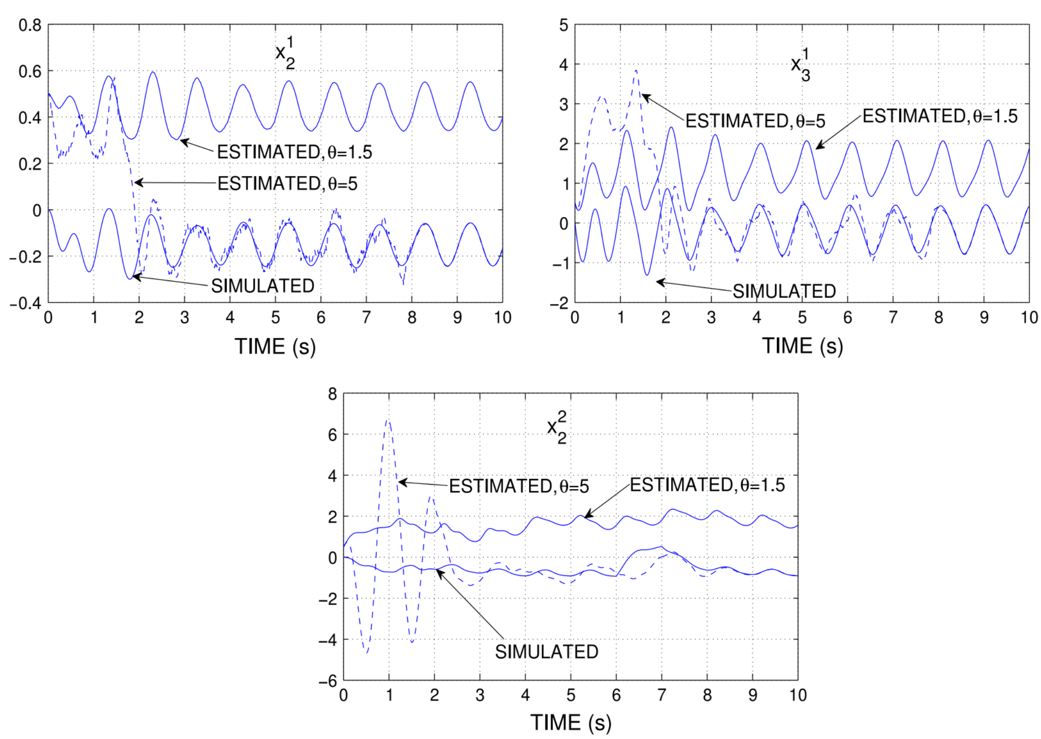

Fig. 4. Estimation of the missing states with constant values of the design parameter.

proposed in [4]. Indeed, it has been shown that the gain parameter can be maintained at *a priori* specified relatively small values leading thereby to an admissible insensitivity of the observer with respect to the unavoidable noise measurements. Simulation results have been given to demonstrate this engineering feature.

References

[1] J.P. Gauthier, H. Hammouri, S. Othman, A simple observer for nonlinear systems – application to bioreactors, IEEE Transactions on Automatic Control 37 (1992) 875–880.

[2] J.P. Gauthier, I. Kupka, Deterministic Observation Theory and Applications, Cambridge University Press, 2001.

[3] Y. Shen, X. Xia, Semi-global finite-time observers for nonlinear systems, Automatica 44 (2008) 3152–3156.

[4] M. Farza, M. M'Saad, M. Triki, T. Maatoug, High gain observer for a class of non-triangular systems, Systems & Control Letters 60 (1) (2010) 27–35.

[5] H. Hammouri, M. Farza, Nonlinear observers for locally uniformly observable systems, ESAIM Journal on Control, Optimisation and Calculus of Variations 9 (2003) 353–370.

[6] H. Shim, Y.I. Son, J.H. Seo, Semi-global observer for multi-output nonlinear systems, System Control Letters 42 (2001) 233–244.

[7] J.H. Ahrens, H. Khalil, High-gain observers in the presence of measurements noise: a switched-gain approach, Automatica 45 (2009) 936–943.

[8] V. Andrieu, L. Praly, A. Astolfi, High gain observers with updated gain and homogeneous correction terms, Automatica 45 (2009) 422–428.

[9] N. Boizot, E. Busvelle, J.P. Gauthier, An adaptive high-gain observer for nonlinear systems, Automatica 46 (2010) 1483–1488.

[10] E. Bullinger, Frank Allgöwer, An adaptive high-gain oberver for nonlinear systems, in: Proceedings of the 36th IEEE Conference on Decision and Control, San Diego, CA, 1997, pp. 4348–4353.

[11] M. Farza, M. Oueder, R. Benabdennour, M. M'Saad, High gain observer with updated gain for a class of MIMO systems, International Journal of Control 84 (2) (2011) 270–280.

[12] P. Krishnamurthy, F. Khorrami, Generalized adaptive output–feedback form with unknown parameters multipliying high output relative-degree states, in: Proceedings of the 41st IEEE Conference on Decision and Control, 2002.

[13] H. Lei, W. Lin, Universal adaptive control of nonlinear systems with unknown growth rate by output feedback, Automatica 42 (2006) 1783–1789.

[14] M. Farza, M. M'Saad, M. Sekher, A set of observers for a class of nonlinear systems. in: Proceedings of the 16th IFAC World Congress, praha, Czech Republic, July 2005, pp. 4–8.