The Divergence of Stress and the Principle of Virtual Power on Manifolds

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Stresses on manifolds may be introduced from two different points of view. For an m-dimensional material universe, the variational approach regards stresses as fields that associate m-forms, the power densities, with the first jets of generalized velocity fields. In the second approach, the Cauchy approach, stresses are covector valued $(m-1)$-forms whose odd restrictions to the the boundary of bodies give the surface forces on them. The relation between the two approaches is studied for general manifolds that are not equipped with a connection.

1 Introduction

This paper considers some aspects of force and stress theory on general differentiable manifolds. In the course of generalizing force and stress theory to differentiable manifolds, one encounters difficulties originating from the lack of metric structure used in the traditional formulation of Cauchy's theorem for the existence of stresses. In addition, since vector fields on manifolds cannot be integrated, one has to integrate the power density and define forces as functionals producing the power from generalized velocities.

For bodies that are m-dimensional manifolds, stresses may be introduced using two different approaches. The first approach, to which we will refer as the variational approach, introduces stresses as measures on bodies that produce the power from the derivatives, or more precisely jets, of the generalized velocity fields. This approach was developed in Epstein and Segev (1980) and Segev (1986). The second approach, to which we refer as the Cauchy approach, developed recently in Segev (1998) and Segev and Rodnay (1999), presents stresses as $(m-1)$ vector valued differential forms on the material manifold whose oriented restriction to the boundaries of bodies, $(m-1)$-dimensional submanifolds, provide the surface forces on them.

Some of the relations between the variational approach and the Cauchy approach is discussed in Epstein and Segev (1980) and Segev (1986) for the particular case where a connection is given on the space manifold. In this work we will study these relations further and will generalize them to the case where a connection is not specified.

The general setting is as follows. The *material manifold* or *universal body* is a manifold $\mathcal{U}$ of dimension m, and *bodies* are compact m-dimensional submanifolds with boundary of $\mathcal{U}$. For a given configuration of the universal body, a *generalized velocity field* is a vector field or a section $w : \mathcal{U} \to W$ of a vector bundle $\pi : W \to \mathcal{U}$. This vector bundle may be thought of as the pullback of the tangent bundle of the physical space manifold using the current configuration of the material manifold in the physical space. (For motivation and details see Segev (1986).) Throughout this paper it is assumed that the manifold $\mathcal{U}$ is oriented by a specific orientation. This restriction, that we make in order to simplify the notation, may be removed using odd forms (see Segev and Rodnay, 1999).

2 Generalized Cauchy Stresses

This Section reviews the generalization of the Cauchy approach for the introduction of stresses to manifolds. The Cauchy approach views stresses as means for specifying the surface forces on the various subbodies by a single field—the stress field.

2.1 Body Forces and Surface Forces

As mentioned above, forces for manifolds are defined in terms of the power they produce for a generalized velocity field. In general, force densities will be pointwise linear mappings that take generalized velocities and give the corresponding power densities—forms of order $n \leq m$ that can be integrated over n-dimensional submanifolds of $\mathcal{U}$.

Thus, a *body force* over a body $\mathcal{B}$ is a section $\beta_{\mathcal{B}}$ of $L\big(W, \bigwedge^m (T^*\mathcal{B})\big)$ and a *surface force* on $\mathcal{B}$ is a section $\tau_{\mathcal{B}}$ of $L\big(W, \bigwedge^{m-1}(T^*\partial\mathcal{B})\big)$. Using body forces and surface forces, the force (power functional) $F_{\mathcal{B}}$ is represented in the form

$$F_{\mathcal{B}}(w) = \int_{\mathcal{B}} \beta_{\mathcal{B}}(w) + \int_{\partial\mathcal{B}} \tau_{\mathcal{B}}(w)$$

We note that body forces and surface forces may be regarded as covector valued forms. For example, a surface force $\tau_{\mathcal{B}}$ may be identified with a section $\hat{\tau}_{\mathcal{B}}$ of $\bigwedge^{m-1}\big(T(\partial\mathcal{B}), W^*\big)$. The two are related by

$$\hat{\tau}_{\mathcal{B}}(v_1, \ldots, v_{m-1})(w) = \tau_{\mathcal{B}}(w)(v_1, \ldots, v_{m-1})$$

2.2 Cauchy Stresses and Their Inclined Restrictions

We use the term *(generalized) Cauchy stress* for a section of the bundle $L\big(W, \bigwedge^{m-1}(T^*\mathcal{U})\big)$. Again, a Cauchy stress may be regarded as an element of $\bigwedge^{m-1}\big(T\mathcal{U}, W^*\big)$. A Cauchy stress σ associates with an arbitrary body $\mathcal{B}$ a surface force $\tau_{\mathcal{B}}$ as follows. Consider a body $\mathcal{B}$ and a point $x \in \partial\mathcal{B}$. Let $v \in T_x\mathcal{U}$ be a vector transversal to $\partial\mathcal{B}$ and pointing outwards from $\mathcal{B}$. The *inclined restriction* $\iota^*_{\mathcal{B}}(\sigma)_x$ of $\sigma_x = \sigma(x)$ to $L\big(W, \bigwedge^{m-1}(T^*\partial\mathcal{B})\big)$ is given by the requirement that for any element $w \in W_x$

$$\iota^*_{\mathcal{B}}(\sigma)_x(w)(v_1, \ldots, v_{m-1}) = \sigma_x(w)(v_1, \ldots, v_{m-1})$$

if $\{v, v_1, \ldots, v_{m-1}\}$ are positively oriented and

$$\iota^*_{\mathcal{B}}(\sigma)_x(w)(v_1, \ldots, v_{m-1}) = -\sigma_x(w)(v_1, \ldots, v_{m-1})$$

if $\{v, v_1, \ldots, v_{m-1}\}$ are negatively oriented. (In other words, the vector valued forms associated with σ is restricted to $\partial\mathcal{B}$ with odd dependence on the outer orientation.) Thus, the surface force induced by the Cauchy stress σ is given by the *generalized Cauchy formula*

$$\tau_{\mathcal{B}} = \iota^*_{\mathcal{B}}(\sigma)$$

It is noted that in case $\mathcal{B}$ and $\mathcal{B}'$ are two bodies with $T_x\partial\mathcal{B} = T_x\partial\mathcal{B}'$ that lie on opposite sides of the common tangent space, then, $\iota^*_{\mathcal{B}}(\sigma)(x) = -\iota^*_{\mathcal{B}'}(\sigma)(x)$ as expected.

3 Generalized Variational Stresses

3.1 C^1-Force Functionals

The rational behind the generalized variational formulation of stress theory is the framework for mechanical theories where a configuration manifold is constructed for the system under consideration, generalized velocities are defined as elements of the tangent bundle to the configuration manifold, and generalized forces are defined as elements of the cotangent bundle of the configuration space. If one considers a configuration of a body in continuum mechanics as an embedding of the body manifold $\mathcal{B}$ in a space manifold $\mathcal{M}$, the natural topology for the collection of such embeddings is the C^1 topology for which the collection of embeddings is open in the collection of all C^1 mappings of the body into space. Using this topology, the tangent space to the configuration manifold at the configuration $\kappa\colon \mathcal{B} \to \mathcal{M}$ is $C^1\big(\kappa^*(T\mathcal{M})\big)$, the Banachable space of C^1 sections of the pullback $\kappa^*(T\mathcal{M})$. Thus, forces in continuum mechanics are elements of $C^1\big(\kappa^*(T\mathcal{M})\big)^*$—linear functionals on the space of differentiable vector fields equipped with the C^1 topology.

The basic representation theorem (see Segev, 1986) states that a force functional $F \in C^1\big(\kappa^*(T\mathcal{M})\big)^*$ may be represented by measures on $\mathcal{U}$—the *variational stress measures*— valued in $J^1(\kappa^*(T\mathcal{M}))^*$, the dual of the first jet bundle $J^1(\kappa^*(T\mathcal{M})) \to \mathcal{U}$.

Assuming that κ is defined on the whole of the material universe $\mathcal{U}$, we use the notation W for $\kappa^*T\mathcal{M}$. This vector bundle can be restricted to the individual bodies, and with some abuse of notation, we use the same notation for both the bundle and its restriction to the individual bodies.

3.2 Variational Stress Densities

In the smooth case, the variational stress measures are given in terms of sections of the vector bundle of linear mapping $L\big(J^1(W), \bigwedge^m(T^*\mathcal{U})\big)$. We will refer such sections as variational stress densities. If S is a variational stress density, then the power of the force F it represents over the body $\mathcal{B}$, while the the generalized velocity is w, is given by

$$F_{\mathcal{B}}(w) = \int\limits_{\mathcal{B}} S(j^1(w))$$

This expression makes sense as $S(j^1(w))$, is an m-form whose value at a point $x \in \mathcal{B}$ is $S(x)\big(j^1(w)(x)\big)$.

Assume that a vector bundle coordinate system (x^i, w^α), $i = 1, \dots, m$, $\alpha = 1, \dots, \dim(W)$ is given in $\pi^{-1}(U)$, for an open set $U \subset \mathcal{U}$. Thus, a section of W will be represented locally by the functions $\{w^\alpha(x^i)\}$ and the jet of a section is represented locally by the functions $\{w^\alpha(x^i), w^\beta_{,j}(x^k)\}$, where a subscript following a comma indicates partial differentiation. A variational stress density will be represented locally by the functions $\{S_{\alpha 1\dots m}, S^j_{\beta 1\dots m}\}$ so that the single component of the m-form $S\big(j^1(w)\big)$ in this coordinate system is

$$S\big(j^1(w)\big)_{1\dots m} = S_{\alpha 1\dots m} w^\alpha + S^j_{\beta 1\dots m} w^\beta_{,j}$$

Note that the notation distinguishes between the components of S that are dual to the values of the section and those dual to the derivatives by the number of indices only. (Here and in the sequel we abuse the notation by using the same notation for both a function and its values.) Since in the sequel we consider only the smooth case, we will use "variational stresses" to refer to the densities.

3.3 Connections and Variational Stress Tensor Densities

If a connection is given on the vector bundle W, the jet bundle is isomorphic with the Whitney sum $W \oplus_{\mathcal{U}} L\big(T\mathcal{U}, W\big)$ by $j^1(w) \mapsto (w, \nabla w)$, where ∇ denotes covariant derivative. Thus, in case a connection is given, a variational stress may be represented by sections (S_0, S_1) of

$$L\big(W, \bigwedge^m(T^*\mathcal{U})\big) \oplus_{\mathcal{U}} L\big(L\big(T\mathcal{U}, W\big), \bigwedge^m(T^*\mathcal{U})\big)$$

so the power is given by (see Segev, 1986)

$$F_{\mathcal{B}}(w) = \int\limits_{\mathcal{B}} S_0(w) + \int\limits_{\mathcal{B}} S_1(\nabla w)$$

We will refer to the section S_1 of $L\big(L\big(T\mathcal{U}, W\big), \bigwedge^m(T^*\mathcal{U})\big)$ as the *variational stress tensor*.

In Segev (1986) it was shown that with a given connection, and with appropriate definition of the divergence of the variational stress tensor S_1 the power may be written in terms of body forces and surface forces.

4 The Cauchy Stress Associated with a Variational Stress

4.1 The Vertical Sub-bundle of the Jet Bundle

We recall that a k-jet $A \in J^k(W)_x$ is an equivalence class of sections of W that have the same values of their k-tangent at x, where for $k = 0$, $J^0(W)$ is identified with W. Let $\pi^1_0 \colon J^1(W) \to W$ be the natural projection on the jet bundle that assign to any 1-jet at $x \in \mathcal{U}$ the value of the corresponding 0-jet, i.e., the value of the section at x.

We define $VJ^1(W)$, the *vertical sub-bundle* of $J^1(W)$, to be the vector bundle over $\mathcal{U}$ such that

$$VJ^1(W) = (\pi^1_0)^{-1}(0)$$

where 0 is the zero section of W. A jet $A \in J^1(W)$ is represented locally by $(x^i, A^\alpha, A^\beta_j)$, where A^α represent the value of a section and A^β_j represent a value of the derivative of a section in the particular coordinate system chosen. Thus, elements of the the vertical sub-bundle will be represented in the form $(x^i, 0, A^\beta_j)$.

Next, we show that $VJ^1(W)$ is isomorphic to the bundle of linear mappings $L\big(T\mathcal{U}, W\big)$. Consider a point $x_0 \in \mathcal{U}$ and let w be a section of W that represents an element $A \in VJ^1(W)_{x_0}$, i.e., $w(x_0) = 0$. We will show that w induces an element of $L\big(T\mathcal{U}, W\big)$ linearly in A and injectively. Consider the mapping

$$T_0 w \colon T_{x_0}\mathcal{U} \to T_{0(x_0)}W$$

given by

$$T_0 w(v) = Tw(v) - T0(v) \quad v \in T_{x_0}\mathcal{U}$$

Note that the subtraction done in TW makes sense as both vectors are tangent at the zero vector at W_{x_0}. Clearly,

$$T\pi \circ T_0 w = T\pi \circ Tw - T\pi \circ T0 = 0$$

so the image of $T_0 w$ is in the vertical sub-bundle of TW, specifically in $T_{0(x_0)}(W_{x_0})$. As a tangent space to a vector space, there is a natural isomorphism

$$i \colon T_{0(x_0)}(W_{x_0}) \to W_{x_0}$$

If in local coordinates v and w are represented by (x^i_0, v^j) and (x^i, w^α) respectively, with $w^\alpha(x_0) = 0$, then, $j^1(w)$ is represented locally by $(x^i, w^\alpha, w^\beta_{,k})$ and $Tw(v)$ is represented by $(x^i_0, 0, v^j, w^\beta_{,k} v^k)$. In addition, as $T0(v)$ is represented by $(x^i_0, 0, v^j, 0)$, $T_0 w(v)$ is represented by $(x^i_0, 0, 0, w^\beta_{,k} v^k)$ and $i \circ T_0 w(v)$ is represented by $(x^i_0, w^\beta_{,k} v^k)$. Thus, $i \circ T_0 w(x_0) \in L\big(T_{x_0}\mathcal{U}, W_{x_0}\big)$ and from the local representation, $(x^i_0, w^\beta_{,j})$, it is clear that it depends on the jet of w linearly and injectively. We conclude that the mapping that takes the J^1-equivalence class of w into $i \circ T_0 w$ is an isomorphism that we denote by

$$I^+ \colon VJ^1(W) \to L\big(T\mathcal{U}, W\big)$$

4.2 The Vertical Component of a Variational Stress

Let $\mathcal{I}_V \colon VJ^1(W) \to J^1(W)$ be the inclusion mapping of the sub-bundle. Clearly $\mathcal{I}_V$ is injective. Thus, we may consider the linear injection

$$\mathcal{I} = \mathcal{I}_V \circ (I^+)^{-1} \colon L\big(T\mathcal{U}, W\big) \to J^1(W)$$

If an element $A \in L\big(T\mathcal{U}, W\big)$ is represented locally by (x^i, A^α_j), then $(I^+)^{-1}(A)$ is represented locally by $(x^i, 0, A^\alpha_j)$ and so is $\mathcal{I}(A)$.

The foregoing allow us to define a "dual" linear surjection

$$\mathcal{I}^*\colon L\big(J^1(W), \bigwedge^m (T^*\mathcal{U})\big) \to L\big(L(T\mathcal{U}, W), \bigwedge^m (T^*\mathcal{U})\big)$$

given by $\mathcal{I}^*(S) = S \circ \mathcal{I}$.

For a variational stress S, we will refer to

$$S^+ = \mathcal{I}^*(S) \in L\big(L(T\mathcal{U}, W), \bigwedge^m (T^*\mathcal{U})\big)$$

as the *vertical component* of S. Assume that the value of the variational stress is represented locally by $(S_{\alpha 1\ldots m}, S^j_{\beta 1\ldots m})$ and $S^+ = \mathcal{I}^*(S)$ is represented by $({S^+}^i_{\alpha 1\ldots m})$. Then, for an element $A \in L\big(T\mathcal{U}, W\big)_x$ that is represented locally by (x^i, A^α_j), we have

$${S^+}^i_{\alpha 1\ldots m} A^\alpha_j = S_{\alpha 1\ldots m} 0 + S^j_{\alpha 1\ldots m} A^\alpha_j$$

Hence, the collection of components $\{{S^+}^i_{\alpha 1\ldots m} = S^i_{\alpha 1\ldots m}\}$ has an invariant meaning.

Clearly, one cannot define invariantly (without a connection) a "horizontal" component to the stress.

4.3 Variational Fluxes

Since the jet of a real valued function φ on $\mathcal{U}$ can be identified with a pair $(\varphi, d\varphi)$ in the trivial case where $W = \mathcal{U} \times \mathbb{R}$, the jet bundle can be identified with the Whitney sum $W \oplus_{\mathcal{U}} T^*\mathcal{U}$. Thus, $VJ^1(W)$ can be identified with $T^*\mathcal{U}$ and the vertical component of the variational stress is valued in $L\big(T^*\mathcal{U}, \bigwedge^m (T^*\mathcal{U})\big)$. We will refer to sections of $L\big(T^*\mathcal{U}, \bigwedge^m (T^*\mathcal{U})\big)$ as *variational fluxes*.

We first note that the mapping

$$i_\wedge\colon \bigwedge^{m-1} (T^*\mathcal{U}) \to L\big(T^*\mathcal{U}, \bigwedge^m (T^*\mathcal{U})\big)$$

given by $i_\wedge(\omega)(\phi) = \phi \wedge \omega$ is an isomorphism. (Clearly, the dimensions of the two spaces are equal and $i_\wedge(\omega) = 0$ implies that $\omega = 0$.) Let $\omega \in \bigwedge^{m-1}(T^*\mathcal{U})$ be given locally by $\omega_{1\ldots\hat{\imath}\ldots m} dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^m$, where the "hat" indicates an omission of the term, and let $\omega_\wedge = i_\wedge(\omega)$ be given by ${\omega_\wedge}^i_{1\ldots m}$. Then, for a 1-form ϕ given locally by $\phi_j dx^j$ we have

$$\begin{aligned} {\omega_\wedge}^i_{1\ldots m} \phi_i dx^1 \wedge \ldots \wedge dx^m &= \phi_j dx^j \wedge \omega_{1\ldots\hat{\imath}\ldots m} dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^m \\ &= \textstyle\sum_i (-1)^{i-1} \omega_{1\ldots\hat{\imath}\ldots m} \phi_i dx^1 \wedge \ldots \wedge dx^m \end{aligned}$$

so ${\omega_\wedge}^i_{1\ldots m} = (-1)^{i-1} \omega_{1\ldots\hat{\imath}\ldots m}$ is the local expression for the isomorphism. (Note that here and in the sequel we write the summation symbol explicitly when Einstein's summation convention cannot be used clearly as in the case of the index i.) We conclude that any variational flux may be viewed as an $(m-1)$-form.

4.4 The Cauchy Stress Induced by a Piola-Kirchhoff Stress

Consider the *contraction* vector bundle morphism

$$c\colon L\big(L(T\mathcal{U}, W), \bigwedge^m (T^*\mathcal{U})\big) \oplus_{\mathcal{U}} W \to L\big(T^*\mathcal{U}, \bigwedge^m (T^*\mathcal{U})\big)$$

given by

$$c(B, w)(\phi) = B(w \otimes \phi)$$

for $B \in L\big(L(T\mathcal{U}, W), \bigwedge^m (T^*\mathcal{U})\big)$, $w \in W$, and $\phi \in T^*\mathcal{U}$, where $(w \otimes \phi)(v) = \phi(v)w$. Locally, if B is given by $B^i_{\alpha 1\ldots m}$ then, $c(B, w)$ is is represented by $B^i_{\alpha 1\ldots m} w^\alpha$. Although this is an extension of the usual notation we will still use $w \lrcorner B$ for $c(B, w)$.

Consider a section S^+ of $L\big(L(T\mathcal{U}, W), \bigwedge^m (T^*\mathcal{U})\big)$ and a vector field w. Then, $w \lrcorner S^+$ is a variational flux and $i_\wedge^{-1}(w \lrcorner S^+) \in \bigwedge^{m-1}(T^*\mathcal{U})$ is represented by

$$\sum_i (-1)^{i-1} {S^+}^i_{\alpha 1\ldots m} w^\alpha dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^m$$

where the summation over α is implied. We note that $i_\wedge^{-1}(w \lrcorner S^+)$ depends on w linearly, so we have a section σ of $L\big(W, \bigwedge^{m-1}(T^*\mathcal{U})\big)$ satisfying

$$\sigma(w) = i_\wedge^{-1}(w \lrcorner S^+)$$

Locally σ is represented by

$$\sigma_{\alpha 1\ldots \hat{\imath}\ldots m} = (-1)^{i-1} {S^+}^i_{\alpha 1\ldots m} \quad \text{(no sum over } i)$$

We will use

$$i_\sigma \colon L\big(L(T\mathcal{U}, W), \bigwedge^m (T^*\mathcal{U})\big) \to L\big(W, \bigwedge^{m-1} (T^*\mathcal{U})\big)$$

to denote the obviously linear injective mapping such that $\sigma = i_\sigma \circ S^+$.

We conclude that the mapping

$$i_\sigma \circ \mathcal{I}^* \colon L\big(J^1(W), \bigwedge^m (T^*\mathcal{U})\big) \to L\big(W, \bigwedge^{m-1} (T^*\mathcal{U})\big)$$

is a linear mapping (no longer injective) that gives a Cauchy stress to any given variational stress. Locally it is given by

$$(x^i, S_{\alpha 1\ldots m}, S^j_{\beta 1\ldots m}) \mapsto (x^i, \sigma_{\beta 1\ldots \hat{\imath}\ldots m})$$

where,

$$\sigma_{\beta 1\ldots \hat{\imath}\ldots m} = (-1)^{i-1} {S^+}^i_{\beta 1\ldots m} \qquad \text{(no sum over } i)$$

5 The Divergence of a Variational Stress and the Principle of Virtual Power

5.1 The Divergence of a Variational Flux

Let s be a variational flux. We define the *divergence* $\operatorname{div} s$ of s to be the m-form defined by

$$\operatorname{div} s = d\big(i_\wedge^{-1}(s)\big)$$

Using the same scheme of notation for the local representatives as before, we have

$$\begin{aligned}
(\operatorname{div} s)_{1\ldots m} dx^1 \wedge \ldots \wedge dx^m &= d\Big(\textstyle\sum_i (-1)^{i-1} s^i_{1\ldots m} dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^m\Big) \\
&= \textstyle\sum_i (-1)^{i-1} s^i_{1\ldots m,j} dx^j \wedge dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^m \\
&= s^i_{1\ldots m,i} dx^1 \wedge \ldots \wedge dx^m
\end{aligned}$$

as expected.

5.2 The Divergence of a Variational Stress

For a given variational stress S and a generalized velocity w, consider the variational flux $w \lrcorner \mathcal{I}^*(S)$ and its divergence

$$\operatorname{div}\big(w \lrcorner \mathcal{I}^*(S)\big) = d\Big(i_{\wedge}^{-1}(w \lrcorner \mathcal{I}^*(S))\Big)$$

given locally by

$$(S^i_{\alpha 1 \ldots m} w^\alpha)_{,i} dx^1 \wedge \ldots \wedge dx^m = \big(S^i_{\alpha 1 \ldots m,i} w^\alpha + S^i_{\alpha 1 \ldots m} w^\alpha_{,i}\big) dx^1 \wedge \ldots \wedge dx^m$$

Next, as $S(j^1(w))$, is represented locally by

$$\big(S_{\alpha 1 \ldots m} w^\alpha + S^i_{\alpha 1 \ldots m} w^\alpha_{,i}\big) dx^1 \wedge \ldots \wedge dx^m$$

the difference

$$\operatorname{div}\big(w \lrcorner \mathcal{I}^*(S)\big) - S(j^1(w))$$

is represented locally by

$$\big(S^i_{\alpha 1 \ldots m,i} - S_{\alpha 1 \ldots m}\big) w^\alpha dx^1 \wedge \ldots \wedge dx^m$$

From its local representation it is clear that $\operatorname{div}\big(w \lrcorner \mathcal{I}^*(S)\big) - S(j^1(w))$ is linear in w, hence, there is an element of $L\big(W, \bigwedge^m (T^*\mathcal{U})\big)$ that gives this difference when it is evaluated on a section w. Thus, we define the *generalized divergence* of the variational stress S to be the section $\operatorname{Div}(S)$ of the vector bundle $L\big(W, \bigwedge^m (T^*\mathcal{U})\big)$ satisfying

$$\operatorname{Div}(S)(w) = \operatorname{div}\big(w \lrcorner \mathcal{I}^*(S)\big) - S(j^1(w))$$

for every generalized velocity field w.

5.3 The Principle of Virtual Power

Consider the power expended by the variational stresses. Using the previous results we have

$$\begin{aligned} F_{\mathcal{B}}(w) &= \int_{\mathcal{B}} S(j^1(w)) \\ &= \int_{\mathcal{B}} \operatorname{div}\big(w \lrcorner \mathcal{I}^*(S)\big) - \int_{\mathcal{B}} \operatorname{Div}(S)(w) \end{aligned}$$

Since

$$\begin{aligned} \operatorname{div}\big(w \lrcorner \mathcal{I}^*(S)\big) &= d\Big(i_{\wedge}^{-1}\big(w \lrcorner \mathcal{I}^*(S)\big)\Big) \\ &= d\big(\sigma(w)\big) \end{aligned}$$

where, $\sigma = i_\sigma \circ \mathcal{I}^*(S)$ is the Cauchy stress induced by the variational stress S, we may write the power in the form

$$F_{\mathcal{B}}(w) = \int_{\mathcal{B}} d\big(\sigma(w)\big) - \int_{\mathcal{B}} \operatorname{Div}(S)(w)$$

We may use Stokes' theorem on the first integral to obtain

$$F_{\mathcal{B}}(w) = \int_{\partial\mathcal{B}} i^*_{\mathcal{B}}\big(\sigma(w)\big) - \int_{\mathcal{B}} \operatorname{Div}(S)(w)$$

where, $i^*_{\mathcal{B}}$ is the restriction of $(m-1)$-forms on $\mathcal{U}$ to $\partial\mathcal{B}$ so

$$i^*_{\mathcal{B}}\big(\sigma(w)\big) \in \bigwedge^{m-1} (T^*\partial\mathcal{B})$$

We conclude that the force induced on $\mathcal{B}$ by the variational stress S is composed of a body force and a surface force, i.e., it is of the form

$$F_{\mathcal{B}}(w) = \int\limits_{\mathcal{B}} \beta_{\mathcal{B}}(w) + \int\limits_{\partial\mathcal{B}} \tau_{\mathcal{B}}(w)$$

where $\tau_{\mathcal{B}}(w) = i^*_{\mathcal{B}}\big(\sigma(w)\big)$ and $\operatorname{Div} S + \beta_{\mathcal{B}} = 0$.

5.4 Newton's Law of Action and Reaction

Newton's law of action and reaction, stating that $\tau_{\mathcal{B}}(x) = -\tau_{\mathcal{B}'}(x)$ if the bodies $\mathcal{B}$ and $\mathcal{B}'$ are tangent at x and lie on the other side of the common tangent space, follows in the classical case from the linear dependence of the traction on the unit normal to the boundary. In the generalized theory of Cauchy stresses reviewed above, Newton's law is implied by the definition of the inclined restriction.

In the dependence of the surface force $\tau_{\mathcal{B}}(w) = i^*_{\mathcal{B}}\big(\sigma(w)\big)$ that we obtained from Stokes' theorem, this odd dependence is implicit because the value

$$i^*_{\mathcal{B}}\big(\sigma(w)\big)(v_1, \ldots, v_{m-1}) = \sigma(w)(v_1, \ldots, v_{m-1})$$

for any collection of $m-1$ vectors $\{v_1, \ldots, v_{m-1}\}$, does not depend on $\mathcal{B}$.

We recall that the assumed orientation on $\mathcal{U}$ and the outwards pointing vectors defined on the boundary of a body $\mathcal{B}$ determine a unique orientation on $\partial\mathcal{B}$—an orientation for which $\{v_1, \ldots, v_{m-1}\}$ are positively oriented if for any outwards pointing vector v, the vectors $\{v, v_1, \ldots, v_{m-1}\}$ are positively oriented in $\mathcal{U}$. The form $i^*_{\mathcal{B}}\big(\sigma(w)\big)$ gives the power with respect to this orientation on $\partial\mathcal{B}$. If we reverse the outwards pointing vectors, so we consider the body $\mathcal{B}'$ that is in contact with $\mathcal{B}$, the form $i^*_{\mathcal{B}'}\big(\sigma(w)\big)$ indeed does not change. However, its integral gives the power using the inverse orientation than the one corresponding to $i^*_{\mathcal{B}}\big(\sigma(w)\big)$. The definition of the integral of a differential form implies that the results of integration of the form $i^*_{\mathcal{B}}\big(\sigma(w)\big)$ over $\partial\mathcal{B}'$ with its induced orientation will be of the opposite sign then the integral over $\partial\mathcal{B}$ with its induced orientation. Thus, the power density on $\partial\mathcal{B}'$ is the inverse of that on $\partial\mathcal{B}$.

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