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ABSTRACT

In this paper, one proposes adaptive observers for a class of uniformly observable MIMO nonlinear systems with general nonlinear parameterizations. The state and the unknown parameters of the considered systems are supposed to lie in bounded domains which size can be arbitrarily large and the exponential convergence of the observers is shown to result under a well-defined persistent excitation condition. Moreover, the gain of the observers involves a design function that has to satisfy a simple condition which is given. Different expressions of such a function are proposed and it is shown that adaptive high gain like observers and adaptive sliding mode like observers can be derived by considering particular expressions of the design function. The theory is supported by simulation results related to the estimation of the biomass concentration and the Contois model parameters in a bioreactor.

Keywords:
Nonlinear system
Nonlinear parameterization
Adaptive observer
Persistent excitation
High gain observer
Sliding mode observer

1. Introduction

The problem of joint estimation of missing states and constant parameters in linear and nonlinear state-space systems with adaptive observers has motivated a lot of work, for adaptive control, or recently fault detection and isolation in dynamic systems. Some early works on adaptive observers for linear systems can be found in Kreisselmeier (1977) and Lüders and Narendra (1973) while more recent results are reported in Zhang (2002). Since the eighties, many results on nonlinear systems have became available. For example, adaptive observers have been proposed for a class of nonlinear systems which can be linearized with a change of coordinates up to output injection in Bastin and Gevers (1988), Marino and Tomei (1992), Marino and Tomei (1995)and Santosuosso, Marino, and Tomei (2001). More recently, some more general results on nonlinear systems have been reported in Besançon (2000), Cho and Rajamani (1997) and Rajamani and Hedrick (1995). These methods do not require the considered nonlinear systems to be linearizable, instead, they assume the existence of some Lyapunov functions satisfying particular conditions. In relatively recent works (Zhang, 2002; Zhang & Clavel, 2001), the authors proposed an adaptive observer for a class of MIMO linear time-varying systems. A tentative to generalize the observer design to a class of single output uniformly observable nonlinear systems is made in Xu and Zhang (2004). However, the resulting proposed observer was complex since it is synthesized by considering a collection of systems corresponding to several delayed versions of the original system.

All references previously cited deal with linearly parameterized systems. Indeed, very few results are available in the literature that address state and parameter estimation in the presence of nonlinear parameterizations (Kojić & Annaswamy, 2002; Kojić, Annaswamy, Loh, & Lozano, 1999; Loh, Annaswamy, & Skantze, 1999; Skantze, Kojić, Loh, & Annaswamy, 2000). Nonlinear parameterizations are inevitable in many realistic dynamic models, even in the case where only few state variables are considered. Tentatives to obtain linear parameterization in models where unknown parameters occur nonlinearly may give rise to overparameterization and its underlying problems (Loh et al., 1999). When considered, the nonlinear parameterization assumes, as the linear one, that the unknown parameters appear in the model through functions that are known, i.e. these functions do not involve nonmeasured states. Furthermore, in most of the works dealing with nonlinear parameterization, the adaptive control problem is rather considered and the parameter convergence is rarely addressed. In Kojić and Annaswamy (2002), the authors considered a concave/convex parameterization and they showed that the parameter convergence is guaranteed under certain conditions of persistent excitation.

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In this paper, one proposes an approach which allows the design adaptive observers for a class of uniformly observable nonlinear MIMO systems with general nonlinear parameterizations. Two main features of the proposed approach are worth to be mentioned. Firstly, the convergence of the proposed observer is guaranteed under a well-defined persistent excitation condition. Secondly, the structure of the proposed observer is simple and it is able to give rise to different observers among which adaptive high gain like observers (Bornard & Hammouri, 1991; Farza, M'Saad, & Rossignol, 2004; Gauthier, Hammouri, & Othman, 1992) and adaptive sliding mode like observers (Drakunov, 1992; Drakunov & Utkin, 1995; Utkin, 1992). This is achieved through the specification of a design function in the observer gain which is calibrated through the choice of a single design parameter.

This paper is organized as follows. The next section introduces the class of nonlinear systems which shall be considered with a view to adaptive observer design. For clarity purposes, linear parameterization is firstly considered. This also allows us to easily make the link with the nonlinear parameterization. In Section 3, the observer design is detailed. The equations of the proposed adaptive observer are given and a full convergence analysis is made. Besides, different expressions of the observer design function are specified and it is shown that they give rise to different observers. In Section 4, the observer synthesis is extended to the nonlinear parameterization case. A simulation example is given in Section 5 in order to illustrate the theory.

2. Problem formulation

Consider MIMO systems which are diffeomorphic to the following form:

$$\begin{cases}\dot{x} = Ax + g(u,x) + \Psi(u,x)\rho \\ y = Cx = x^1\end{cases} \tag{1}$$

with

$$x = \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^q \end{pmatrix}; \qquad x^k = \begin{pmatrix} x_1^k \\ x_2^k \\ \vdots \\ x_p^k \end{pmatrix}; \qquad \rho = \begin{pmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_m \end{pmatrix};$$

$$g(u,x) = \begin{pmatrix} g^1(u,x^1) \\ g^2(u,x^1,x^2) \\ \vdots \\ g^{q-1}(u,x^1,\dots,x^{q-1}) \\ g^q(u,x) \end{pmatrix}; \; \Psi^{\mathrm{T}}(u,x) = \begin{pmatrix} \Psi_1^{\mathrm{T}}(u,x) \\ \Psi_2^{\mathrm{T}}(u,x) \\ \vdots \\ \Psi_m^{\mathrm{T}}(u,x) \end{pmatrix},$$

$$\Psi_j(u,x) = \begin{pmatrix} \Psi_j^1(u,x^1) \\ \Psi_j^2(u,x^1,x^2) \\ \vdots \\ \Psi_j^{q-1}(u,x^1,\dots,x^{q-1}) \\ \Psi_j^q(u,x) \end{pmatrix}$$

$$A = \begin{bmatrix} 0 & I_{(q-)p} \\ 0 & 0 \end{bmatrix} \qquad C = [I_p, 0_p, \dots, 0_p] \tag{2}$$

where the output $y \in \mathbb{R}^p$; the state $x \in \mathbb{R}^n$ with $x^k \in \mathbb{R}^p$ and $x_i^k \in \mathbb{R}$, $k = 1, \dots, q$ and $i = 1, \dots, p$; the input $u \in \mathbb{R}^s$; $\rho \in \mathbb{R}^m$ is a vector of unknown constant parameters, $\rho_i \in \mathbb{R}$, $i = 1, \dots, m$; $g(u,x) \in \mathbb{R}^n$ with $g^k(u,x) \in \mathbb{R}^p$, $k = 1, \dots, q$; $\Psi(u,x)$ is an $n \times m$ matrix and each $\Psi_j(u,x) \in \mathbb{R}^n$, $j = 1, \dots, m$, denotes its jth column with $\Psi_j^k(u,x) \in \mathbb{R}^p$, $k = 1, \dots, q$. The notation I_k where k is a positive number refers to the $k \times k$ identity matrix while 0_k and $0_{k_1 \times k_2}$, k, k_1, k_2 being positive integers, denote the null matrix with dimension $k \times k$ and $k_1 \times k_2$, respectively. Our objective consists in designing adaptive observers to simultaneously estimate the state and the unknown parameters.

System (1) may seem as being very particular since it assumes a nonprime dimension and all sub-blocks x^k have the same dimension. In fact, it has been shown in Hammouri and Farza (2003) that system (1) is a canonical form that characterizes the following class of uniformly observable nonlinear systems (systems which are observable for any input):

$$\dot{x} = f(u,x), \qquad y = \bar{C}x = x^1$$

$$\text{with } x = \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^q \end{pmatrix}, f(u,x) = \begin{pmatrix} f^1(u,x^1,x^2) \\ f^2(u,x^1,x^2,x^3) \\ \vdots \\ f^{q-1}(u,x) \\ f^q(u,x) \end{pmatrix}$$

$$\text{and } \bar{C} = [I_{n_1}, 0_{n_1 \times n_2}, 0_{n_1 \times n_3}, \dots, 0_{n_1 \times n_q}]$$

where the state $x \in \mathbb{R}^n$ with $x^k \in \mathbb{R}^{n_k}$, $k = 1, \dots, q$ and $n_1 \geq n_2 \geq \dots \geq n_q$, $\sum_{k=1}^q n_k = n$; the input $u(t) \in \mathcal{U}$ the set of bounded absolutely continuous functions with bounded derivatives from $\mathbb{R}^+$ into U a compact subset of $\mathbb{R}^s$; the output $y \in \mathbb{R}^{n_1}$ and $f(u,x) \in \mathbb{R}^n$ with $f^k(u,x) \in \mathbb{R}^{n_k}$. The functions f^k are assumed to satisfy the following condition:

(C) For $k = 1, \dots, q-1$, the map $x^{k+1} \mapsto f^k(u,x^1,\dots,x^k,x^{k+1})$ is one tone from $\mathbb{R}^{n_k}$ into $\mathbb{R}^{n_k}$. Moreover, $\exists \alpha_f, \beta_f > 0$ such that for all $k \in \{1, \dots, q-1\}$, $\forall x \in \mathbb{R}^n$, $\forall u \in U$,

$$\alpha_f^2 I_{n_{k+1}} \leq \left(\frac{\partial f^k}{\partial x^{k+1}}(u,x)\right)^{\mathrm{T}} \frac{\partial f^k}{\partial x^{k+1}}(u,x) \leq \beta_f^2 I_{n_{k+1}}.$$

Now, consider the following injective map:

$$\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}^{n_1 q}, \qquad x = \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^q \end{pmatrix} \mapsto z = \begin{pmatrix} z^1 \\ z^2 \\ \vdots \\ z^q \end{pmatrix}$$

$$= \Phi(u,x) = \begin{pmatrix} x^1 \\ f^1(u,x^1,x^2) \\ \dfrac{\partial f^1}{\partial x^2}(u,x^1,x^2) f^2(u,x^1,x^2,x^3) \\ \vdots \\ \left(\displaystyle\prod_{k=1}^{q-2} \frac{\partial f^k}{\partial x^{k+1}}(u,x)\right) f^{q-1}(u,x) \end{pmatrix}$$

where $z^k \in \mathbb{R}^{n_1}$, $k = 1, \dots, q$. This transformation puts the original system under the following form (see Hammouri & Farza, 2003 for more details):

$$\begin{cases}\dot{z} = \tilde{A}z + \tilde{\varphi}(v,z) \\ y = \tilde{C}z = z^1\end{cases} \tag{3}$$

where the state $z \in \mathbb{R}^{n_1 q}$, $v = [u^{\mathrm{T}}, \dot{u}^{\mathrm{T}}]^{\mathrm{T}}$, $\tilde{\varphi}(v,z)$ has a triangular structure with respect to z and the matrices $\tilde{A}$ and $\tilde{C}$ are: $\tilde{A} = \begin{bmatrix} 0 & I_{(q-1)n_1} \\ 0 & 0 \end{bmatrix}$ and $\tilde{C} = [I_{n_1}, 0_{n_1}, \dots, 0_{n_1}]$. It is clear that in the case where the parameter vector ρ is known, system (3) is under form (1) with $n_1 = p$.

Let us now come back to system (1). Recall that our objective consists in proposing adaptive observers which allow us to jointly estimate the missing states as well as the unknown parameters. Please notice that though system (1) deals with linear parameterization, it includes most of the models considered in the literature with view to adaptive observer design (Besançon, 2000;

Santosuosso et al., 2001). Indeed, the unknown parameters appear in system (1) through additive functions which are unknown since they may depend on the nonmeasured state.

The observer design requires some assumptions which will be stated in due courses. At this step, one assumes the following:

(A1) The state $x(t)$, the control $u(t)$ and the unknown parameters ρ are bounded, i.e. $x(t) \in X$, $u(t) \in U$ for $t \geq 0$ and $\rho \in \Omega$ where $X \subset \mathbb{R}^n$, $U \subset \mathbb{R}^s$ and $\Omega \in \mathbb{R}^m$ are compact sets.
(A2) The matrix $\Psi(u, x)$ is continuous on $U \times X$.
(A3) The functions $g(u, x)$ and $\Psi(u, x)$ are Lipschitz with respect to x uniformly in u where $(u, x) \in U \times X$.

Please notice that since the state is confined to the bounded set X, one can extend the nonlinearities $g(u, x)$ and $\Psi(u, x)$ into $\tilde{g}(u, x)$ and $\tilde{\Psi}(u, x)$ in such a way that the restrictions of $\tilde{g}(u, x)$ and $\tilde{\Psi}(u, x)$, respectively, coincide with $g(u, x)$ and $\Psi(u, x)$ on X and that $\tilde{g}(u, x)$ and $\tilde{\Psi}(u, x)$ become global Lipschitz, i.e. Lipschitz on the whole space $\mathbb{R}^n$. Indeed, let $\sigma : \mathbb{R}^n \longrightarrow X, x \mapsto \sigma(x)$ be any smooth bounded saturation function that coincides with x on X, i.e. $\sigma(x) = x$ for all $x \in X$ (see e.g. Conlon, 1992; Shim, 2000; Shim, Son, & Seo, 2001). One defines the respective Lipschitz extensions, $\tilde{g}$ and $\tilde{\psi}$, of g and Ψ as follows:

$$\begin{cases} \tilde{g}(u, x) = g(u, \sigma(x)) \\ \tilde{\Psi}(u, x) = \Psi(u, \sigma(x)). \end{cases}$$

Now, consider the following dynamical system:

$$\begin{cases} \dot{x} = Ax + \tilde{g}(u, x) + \tilde{\Psi}(u, x)\rho \\ y = Cx = x^1. \end{cases} \tag{4}$$

It is clear that system (4) coincides with system (1) for $(x, u, \rho) \in X \times U \times \Omega$. Therefore, it does not make any difference that we consider system (4) instead of (1) for the observer synthesis. Indeed, system (4) shall be considered in the next section. Please notice that for any bounded input $u \in U$, $\tilde{g}(u, x)$ and $\tilde{\Psi}(u, x)$ are, by construction, globally Lipschitz with respect to x and are bounded for all $x \in \mathbb{R}^n$.

3. Observer design with linear parameterization

Before giving our candidate observer, one introduces the following notations:

(1) Let Δ_θ be the (block) diagonal matrix defined by:

$$\Delta_\theta = \text{diag}\left[I_p, \frac{1}{\theta}I_p, \ldots, \frac{1}{\theta^{q-1}}I_p\right] \tag{5}$$

where $\theta > 0$ is a real number. Easy computations allow us to check the following identities:

$$\Delta_\theta A \Delta_\theta^{-1} = \theta A \quad \text{and} \quad C\Delta_\theta^{-1} = C. \tag{6}$$

(2) Let S be the unique solution of the algebraic Lyapunov equation :

$$S + A^{\mathrm{T}}S + SA - C^{\mathrm{T}}C = 0 \tag{7}$$

where A and C are given by (2). It has been shown in Gauthier et al. (1992) that S is Symmetric Positive Definite (SPD) and that the matrix $(A - S^{-1}C^{\mathrm{T}}C)$ is Hurwitz. Moreover, the matrix S has been given explicitly in Busawon, Farza, and Hammouri (1998) and in particular one has: $S^{-1}C^{\mathrm{T}} = \begin{bmatrix} C_q^1 I_p \\ C_q^2 I_p \\ \vdots \\ C_q^q I_p \end{bmatrix}$.

(3) $\forall \tilde{y} \in \mathbb{R}^p$, let $K(\tilde{y}) \in \mathbb{R}^p$ be a function satisfying the following property:

$$\tilde{y}^{\mathrm{T}}K(\tilde{y}) \geq \frac{1}{2}\tilde{y}^{\mathrm{T}}\tilde{y}. \tag{8}$$

Let us now consider the following dynamical system:

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x} + \tilde{g}(u, \hat{x}) + \tilde{\Psi}(u, \hat{x})\hat{\rho}(t) \\ \quad - \theta\Delta_\theta^{-1}\left(S^{-1} + \Upsilon(t)P(t)\Upsilon^{\mathrm{T}}(t)\right)C^{\mathrm{T}}K(C\tilde{x}) \\ \dot{\hat{\rho}}(t) = -\theta P(t)\Upsilon^{\mathrm{T}}(t)C^{\mathrm{T}}K(C\tilde{x}) \\ \dot{\Upsilon}(t) = \theta(A - S^{-1}C^{\mathrm{T}}C)\Upsilon(t) + \Delta_\theta\tilde{\Psi}(u(t), \hat{x}(t)) \\ \quad \text{with } \Upsilon(0) = 0 \\ \dot{P}(t) = -\theta P(t)\Upsilon^{\mathrm{T}}(t)C^{\mathrm{T}}C\Upsilon(t)P(t) + \theta P(t) \\ \quad \text{with } P(0) = P^{\mathrm{T}}(0) > 0 \end{cases} \tag{9}$$

where $\hat{x} = \begin{bmatrix} \hat{x}^1 \\ \hat{x}^2 \\ \vdots \\ \hat{x}^q \end{bmatrix} \in \mathbb{R}^n$ with $\hat{x}^k \in \mathbb{R}^p, k = 1, \ldots, q$; $\tilde{x} = \hat{x} - x$ where x is the unknown trajectory of system (1); $\hat{\rho} = \begin{bmatrix} \hat{\rho}_1 \\ \hat{\rho}_2 \\ \vdots \\ \hat{\rho}_m \end{bmatrix} \in \mathbb{R}^m$; S, C and Δ_θ are respectively given by (7), (2) and (5); $K(C\tilde{x})$ is a function satisfying condition (8); u and y are respectively the input and the output of system (1); $\theta > 0$ is a real number and the notation $P(0) = P^{\mathrm{T}}(0) > 0$ means that the initial condition of the Ordinary Differential Equation (ODE) governing P is chosen SPD.

Before stating our main result, one assumes the additional assumption:

(A4) The inputs u are such that for any trajectory $\hat{x}$ of system (9) starting from $\hat{x}(0) \in X$, the matrix $C\Upsilon(t)$ is persistently exciting i.e.

$\exists\delta_1, \delta_2 > 0; \exists T > 0; \forall t \geq 0$:

$$\delta_1 I_m \leq \int_t^{t+T} \Upsilon^{\mathrm{T}}(\tau)C^{\mathrm{T}}C\Upsilon(\tau)\mathrm{d}\tau \leq \delta_2 I_m.$$

Please notice that assumption (A4) gives a certain excitation condition which is stated in a classical way (Narendra & Annaswamy, 1989). However, this assumption does not state how to generate the input u that ensures the realization of this condition. In fact, up to our knowledge, excepting some particular cases (such as linear systems), the problem of characterizing the set of inputs ensuring the persistent excitation condition is still open.

One now states the following :

Theorem 3.1. *Assume that system (1) satisfies assumption (A1) to (A4). Then, system (9) is an adaptive observer for system (1) with an exponential error convergence for relatively high values of θ.*

The proof of this theorem is given in the next section. However, before detailing this proof, one shall give some comments and facts which will be used throughout the proof.

(1) One notices that the matrix $\Upsilon(t)$ is bounded. This comes from the facts that the matrix $\tilde{\Psi}$ is bounded and the matrix $\theta(A - S^{-1}C^{\mathrm{T}}C)$ is Hurwitz. Moreover, the corresponding upper bound does not depend on θ for $\theta \geq 1$. To prove this, let us change the time scale by setting $\tau = t/\theta$ and let $\bar{\Upsilon}(t) = \Upsilon\left(\frac{t}{\theta}\right)$. From Eqs. (9), one has:

$$\dot{\bar{\Upsilon}}(t) = (A - S^{-1}C^{\mathrm{T}}C)\bar{\Upsilon}(t) + \frac{1}{\theta}\Delta_\theta\tilde{\Psi}\left(u\left(\frac{t}{\theta}\right), \hat{x}\left(\frac{t}{\theta}\right)\right). \tag{10}$$

According to the expression of Δ_θ given in (5), the matrix $\frac{1}{\theta}\Delta_\theta\Psi(u\left(\frac{t}{\theta}\right), \hat{x}\left(\frac{t}{\theta}\right))$ is bounded and the corresponding upper bound does not depend on θ or $\theta \geq 1$. Since the matrix $A - S^{-1}C^{\mathrm{T}}C$ is Hurwitz one directly concludes to the boundedness of $\bar{\Upsilon}(t)$ (or equivalently to that of $\Upsilon(t)$) with an upper bound independent of θ.

In fact, the matrix Υ is a filtered version of $\tilde{\Psi}$. The initial condition of Υ is taken equal to zero in order to make shorter the transient behaviour.

(2) Under Assumption (A4), the matrix $P(t)$ governed by the ODE described in (9) is SPD and bounded. Moreover, itscorresponding upper and lower bounds are independent of θ.

To prove this, let us again change the time scale by setting $\tau = t/\theta$ and let $\bar{P}(t) = P\left(\frac{t}{\theta}\right)$. Then, one has:

$$\dot{\bar{P}}(t) = -\bar{P}(t)\Upsilon^{\mathrm{T}}\left(\frac{t}{\theta}\right)C^{\mathrm{T}}C\Upsilon\left(\frac{t}{\theta}\right)\bar{P}(t) + \bar{P}(t)$$

$$\text{with } \bar{P}(0) = \bar{P}^{\mathrm{T}}(0) > 0. \tag{11}$$

Under assumption (A4), it has been shown in Zhang and Clavel (2001) that $\bar{P}$ is SPD and bounded and the corresponding bounds (obviously) do not depend on θ. The same result is trivially valid for $P(t)$.

(3) Please notice that the time derivative of $\hat{x}$ given in (9) can be written as follows:

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + \tilde{g}(u,\hat{x}) + \tilde{\Psi}(u,\hat{x})\hat{\rho}(t)\\ &\quad - \theta\Delta_\theta^{-1}S^{-1}C^{\mathrm{T}}K(C\tilde{x}) + \Delta_\theta^{-1}\Upsilon(t)\dot{\hat{\rho}}(t).\end{aligned} \tag{12}$$

The equation of $\dot{\hat{x}}$ consists in a copy of the model (4) with a corrective term. The corrective term is composed by two terms. The first one, $-\theta\Delta_\theta^{-1}S^{-1}C^{\mathrm{T}}K(C\tilde{x})$ is rather classical and is met in classical high gain state observers (Farza, M'Saad, & Sekher, 2005). The second term, $\Delta_\theta^{-1}\Upsilon(t)\dot{\hat{\rho}}(t)$ is similar to the expression used for updating the unknown parameters, i.e. the term used in the expression of $\dot{\hat{\rho}}$. The idea of using such a structure for the corrective term has been used by Zhang (2002) for synthesizing an adaptive observer for a class of MIMO linear time-varying systems. In this work, the same structure is used to synthesize an adaptive observer for the considered class of MIMO nonlinear uniformly observable systems.

3.1. Convergence analysis

Set $\tilde{x}(t) = \hat{x} - x$ and $\tilde{\rho}(t) = \hat{\rho}(t) - \rho$. From (9) and (12), one has

$$\begin{aligned}\dot{\tilde{x}} &= A\tilde{x} - \theta\Delta_\theta^{-1}S^{-1}C^{\mathrm{T}}K(C\tilde{x}) + \Delta_\theta^{-1}\Upsilon(t)\dot{\hat{\rho}}(t) + \tilde{g}(u,\hat{x})\\ &\quad - \tilde{g}(u,x) + (\tilde{\Psi}(u,\hat{x}) - \tilde{\Psi}(u,x))\rho + \tilde{\Psi}(u,\hat{x})\tilde{\rho}\end{aligned} \tag{13}$$

$$\dot{\tilde{\rho}} = -\theta P(t)\Upsilon^{\mathrm{T}}(t)C^{\mathrm{T}}K(C\tilde{x}). \tag{14}$$

Set $\bar{x} = \Delta_\theta\tilde{x}$. Using the identities (6), one obtains:

$$\begin{aligned}\dot{\bar{x}} &= \theta A\bar{x} - \theta S^{-1}C^{\mathrm{T}}K(C\bar{x}) + \Upsilon(t)\dot{\hat{\rho}}(t) + \Delta_\theta\tilde{\Psi}(u,\hat{x})\tilde{\rho}\\ &\quad + \Delta_\theta(\tilde{\Psi}(u,\hat{x}) - \tilde{\Psi}(u,x))\rho + \Delta_\theta\left(\tilde{g}(u,\hat{x}) - \tilde{g}(u,x)\right).\end{aligned} \tag{15}$$

Now, as in Zhang (2002), set: $\eta(t) = \bar{x}(t) - \Upsilon(t)\tilde{\rho}(t)$. For writing convenience and as long as there is no ambiguity, the time variable t shall be omitted in the sequel. Using the fact that Υ is governed by the ODE given in (9), one can show that:

$$\begin{aligned}\dot{\eta}(t) &= \theta A\eta + \theta S^{-1}C^{\mathrm{T}}C\Upsilon\tilde{\rho} - \theta S^{-1}C^{\mathrm{T}}K(C\bar{x}) + \Delta_\theta(\tilde{\Psi}(u,\hat{x})\\ &\quad - \tilde{\Psi}(u,x))\rho + \Delta_\theta\left(\tilde{g}(u,\hat{x}) - \tilde{g}(u,x)\right).\end{aligned} \tag{16}$$

Set $V_1(\eta(t)) = \eta^{\mathrm{T}}(t)S\eta(t)$, $V_2(\tilde{\rho}(t)) = \tilde{\rho}^{\mathrm{T}}(t)P^{-1}(t)\tilde{\rho}(t)$ where $P(t)$ is given in (9) and let $V(\eta(t),\bar{\rho}(t)) = V_1(\eta(t)) + V_2(\tilde{\rho}(t))$ be a Lyapunov candidate function. Using (7), one gets:

$$\begin{aligned}\dot{V}_1(t) &= -\theta\eta^{\mathrm{T}}S\eta + \theta\eta^{\mathrm{T}}C^{\mathrm{T}}C\eta + 2\theta\eta^{\mathrm{T}}C^{\mathrm{T}}C\Upsilon\tilde{\rho} - 2\theta\eta^{\mathrm{T}}C^{\mathrm{T}}K(C\bar{x})\\ &\quad + 2\eta^{\mathrm{T}}S\Delta_\theta(\tilde{\Psi}(u,\hat{x}) - \tilde{\Psi}(u,x))\rho\\ &\quad + 2\eta^{\mathrm{T}}S\Delta_\theta\left(\tilde{g}(u,\hat{x}) - \tilde{g}(u,x)\right).\end{aligned} \tag{17}$$

It is obvious that

$$\|\bar{x}\| \leq \|\eta\| + \|\Upsilon(t)\|\|\tilde{\rho}\|. \tag{18}$$

By the Mean Value Theorem, one gets:

$$\begin{aligned}\Delta_\theta\left(\tilde{g}(u,\hat{x}) - \tilde{g}(u,x)\right) &= \Delta_\theta\frac{\partial\tilde{g}}{\partial x}(u,\xi)(\hat{x} - x)\\ &= \Delta_\theta\frac{\partial\tilde{g}}{\partial x}(u,\xi)\Delta_\theta^{-1}\bar{x}\end{aligned} \tag{19}$$

where $\xi \in \mathbb{R}^n$. Since $\tilde{g}$ is globally Lipschitz, the matrix $\frac{\partial\tilde{g}}{\partial x}(u,\xi)$ is bounded. Moreover, according to the triangular structure of $\tilde{g}$, this matrix is lower triangular and each entry of the following matrix $\Delta_\theta\frac{\partial\tilde{g}}{\partial x}(u,\xi)\Delta_\theta^{-1}$ is polynomial in $\frac{1}{\theta}$. As a result, for $\theta \geq 1$ and from (19), one obtains

$$\|\Delta_\theta\left(\tilde{g}(u,\hat{x}) - \tilde{g}(u,x)\right)\| \leq \left\|\Delta_\theta\frac{\partial\tilde{g}}{\partial x}(u,\xi)\Delta_\theta^{-1}\right\|\|\bar{x}\| \leq c_1\|\bar{x}\| \tag{20}$$

where c_1 is a constant which does not depend on θ for $\theta \geq 1$.

Using (20) and (18) , one obtains:

$$\|\Delta_\theta\left(\tilde{g}(u,\hat{x}) - \tilde{g}(u,x)\right)\| \leq c_1\|\eta\| + c_2\|\tilde{\rho}\| \tag{21}$$

where $c_2 = c_1\sup\{\Upsilon(t) \parallel t \geq 0\}$. Therefore, one has:

$$\begin{aligned}&2\eta^{\mathrm{T}}S\Delta_\theta\left(\tilde{g}(u,\hat{x}) - \tilde{g}(u,x)\right)\\ &\quad\leq 2\|S\|\|\Delta_\theta\left(\tilde{g}(u,\hat{x}) - \tilde{g}(u,x)\right)\|\|\eta\|\\ &\quad\leq c_3\|\eta\|^2 + c_4\|\eta\|\|\tilde{\rho}\| \leq c_5V_1 + c_6\sqrt{V_1}\sqrt{V_2}\end{aligned} \tag{22}$$

where $c_3 = 2c_1\|S\|$, $c_4 = 2c_2\|S\|$, $c_5 = \frac{c_3}{\lambda_{\min}(S)}$ and $c_6 = \frac{c_4}{\sqrt{\lambda_{\min}(S)\lambda_{\min}(P)}}$ are positive constants which do not depend on $\theta \geq 1$, $\lambda_{\min}(\cdot)$ denoting the smallest eigenvalue of $(\cdot)$.

Since each column of the matrix $\tilde{\Psi}$ assumes a triangular structure and since ρ is bounded, the arguments developed above are still be valid for bounding $2\eta^{\mathrm{T}}S\Delta_\theta\left(\tilde{\Psi}(u,\hat{x}) - \tilde{\Psi}(u,x)\right)\rho$ and indeed by proceeding in a similar way as above, one obtains:

$$2\eta^{\mathrm{T}}S\Delta_\theta\left(\tilde{\Psi}(u,\hat{x}) - \tilde{\Psi}(u,x)\right)\rho \leq c_7V_1 + c_8\sqrt{V_1}\sqrt{V_2} \tag{23}$$

where c_7 and c_8 are positive constants (depending on the bounds of ρ) which do not depend on $\theta \geq 1$.

Using (22) and (23), inequality (17) can be written as follows:

$$\begin{aligned}\dot{V}_1(t) &\leq -\theta V_1 + \theta\eta^{\mathrm{T}}C^{\mathrm{T}}C\eta + 2\theta\eta^{\mathrm{T}}C^{\mathrm{T}}C\Upsilon\tilde{\rho}\\ &\quad - 2\theta\eta^{\mathrm{T}}C^{\mathrm{T}}K(C\bar{x}) + k_1V_1 + k_2\sqrt{V_1}\sqrt{V_2}\end{aligned} \tag{24}$$

with $k_1 = c_5 + c_7$ and $k_2 = c_6 + c_8$.

Let us now derive the time derivative of V_2. One has:

$$\begin{aligned}\dot{V}_2(t) &= 2\tilde{\rho}^{\mathrm{T}}P^{-1}(t)\dot{\tilde{\rho}} - \tilde{\rho}^{\mathrm{T}}P^{-1}(t)\dot{P}(t)P^{-1}(t)\tilde{\rho}\\ &= -\theta V_2 - 2\theta\tilde{\rho}^{\mathrm{T}}\Upsilon^{\mathrm{T}}C^{\mathrm{T}}K(C\Delta_\theta^{-1}\bar{x}) + \theta\tilde{\rho}^{\mathrm{T}}\Upsilon^{\mathrm{T}}(t)C^{\mathrm{T}}C\Upsilon(t)\tilde{\rho}\\ &= -\theta V_2 - 2\theta\tilde{\rho}^{\mathrm{T}}\Upsilon^{\mathrm{T}}C^{\mathrm{T}}K(C\tilde{x}) + \theta\tilde{\rho}^{\mathrm{T}}\Upsilon^{\mathrm{T}}(t)C^{\mathrm{T}}C\Upsilon(t)\tilde{\rho}.\end{aligned} \tag{25}$$

Hence, using (24) and (25), one obtains

$$\begin{aligned}\dot{V}(t) &= \dot{V}_1(t) + \dot{V}_2(t)\\ &\leq -(\theta - k_1)V_1 - \theta V_2 + k_2\sqrt{V_1}\sqrt{V_2} + \theta\eta^{\mathrm{T}}C^{\mathrm{T}}C\eta\\ &\quad + 2\theta\eta^{\mathrm{T}}C^{\mathrm{T}}C\Upsilon\tilde{\rho} - 2\theta\eta^{\mathrm{T}}C^{\mathrm{T}}K(C\bar{x})\\ &\quad - 2\theta\tilde{\rho}^{\mathrm{T}}\Upsilon^{\mathrm{T}}C^{\mathrm{T}}CK(C\bar{x}) + \theta\tilde{\rho}^{\mathrm{T}}\Upsilon^{\mathrm{T}}(t)C^{\mathrm{T}}C\Upsilon(t)\tilde{\rho}\\ &= -(\theta - k_1)V_1 - \theta V_2 + k_2\sqrt{V_1}\sqrt{V_2}\\ &\quad + \theta\left(\bar{x}^{\mathrm{T}}C^{\mathrm{T}}C\bar{x} - 2\,(C\bar{x})^{\mathrm{T}}K(C\bar{x})\right)\\ &\leq -(\theta - k_1)V_1 + k_2\sqrt{V_1}\sqrt{V_2} - \theta V_2.\end{aligned} \tag{26}$$

The last inequality is obtained according to the inequality (8). Now, set $V_1^\star = (\theta - k_1)V_1$, $V_2^\star = \theta V_2$ and $V^\star = V_1^\star + V_2^\star$. Please notice that $V^\star \geq (\theta - k_1)V$.

Inequality (26) yields to

$$\begin{aligned}\dot{V}(t) &\leq -V^\star + \frac{k_2}{2\sqrt{\theta(\theta - k_1)}}V^\star\\ &\leq -(\theta - k_1)\left(1 - \frac{k_2}{2\sqrt{\theta(\theta - k_1)}}\right)V.\end{aligned} \tag{27}$$

Now, it suffices to choose θ such that $\left(1 - \frac{k_2}{2\sqrt{\theta(\theta-k_1)}}\right) > 0$. This ends the proof.

3.2. Some particular observers

Some particular expressions of the function $K(\tilde{y})$ that satisfy conditions (8) are given and discussed in this section.

3.2.1. Adaptive high gain observers

Consider the following expression of $K(\tilde{y})$:

$$K_{HG}(\tilde{y}) = k\tilde{y} \tag{28}$$

where $k \geq \frac{1}{2}$ is a real number. One can easily check that expression (28) satisfies conditions (8) for relatively high values of k. Notice that when no parameter needs to be estimate, it is easy to see that the resulting observer is of the high gain variety. More specifically, the proposed observer with $K(\tilde{y})$ specialized as in (28) is in fact an adaptive version of the well-known high gain state observer (Farza et al., 2004; Gauthier et al., 1992; Hammouri & Farza, 2003).

3.2.2. Adaptive sliding mode like observers

Here, the function K is specified as follows:

$$K(\tilde{y}) = k\,\mathrm{sign}(\tilde{y}) \tag{29}$$

where $k > 0$ is a real number and 'sign' is the usual signum function. It is easy to see that condition (8) is trivially satisfied by (29). However, expression (29) is rarely used in practice due to the chattering phenomena intrinsic to the signum function. Rather, one uses continuous functions having similar properties as the function signum. For example, consider the following function:

$$K_{\mathrm{Tanh}}(\tilde{y}) = k_1 \mathrm{Tanh}(k_0\tilde{y}) \tag{30}$$

where Tanh denotes the hyperbolic tangent function and $k_1, k_0 > 0$ are real numbers. It is easy to see that condition (8) is satisfied for relatively high values of k_1. Moreover, one has $\lim_{k_0\to+\infty} \mathrm{Tanh}(k_0\tilde{y}) = \mathrm{sign}(\tilde{y})$.

Similarly to the hyperbolic tangent function, one can easily check that the inverse tangent function $K_{ArcTan}(\tilde{y})$ also constitutes valid expressions for $K(\tilde{y})$. Besides, one can consider new valid expressions for $K(\tilde{y})$, for example by adding $K_{\mathrm{Tanh}}(\tilde{y})$ to $K_{HG}(\tilde{y})$ (Filipescu, Dugard, & Dion, 2003).

4. Observer design with nonlinear parameterization

In this section, the above observer design shall be extended to the following class of nonlinear system with nonlinear parameterization:

$$\begin{cases} \dot{x} = Ax + \varphi(u, x, \rho) \\ y = Cx = x^1 \end{cases} \tag{31}$$

with

$$x = \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^q \end{pmatrix}, \qquad x^k \in \mathbb{R}^p, \quad k = 1, \ldots, q; \qquad \rho = \begin{pmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_m \end{pmatrix},$$

$\rho_i \in \mathbb{R}, i = 1, \ldots, m;$

$$\varphi(u, x, \rho) = \begin{pmatrix} \varphi^1(u, x^1, \rho) \\ \varphi^2(u, x^1, x^2, \rho) \\ \vdots \\ \varphi^{q-1}(u, x^1, \ldots, x^{q-1}, \rho) \\ \varphi^q(u, x, \rho) \end{pmatrix};$$

the matrices A and C are defined as in (2).

It is easy to see that the class of systems (31) includes that of systems (1) with the function φ specialized as follows: $\varphi(u, x, \rho) = \Psi(u, x)\rho + g(u, x)$. Now, the observer design necessitates some hypotheses. As for linear parameterization, one still assumes the boundedness of the inputs, the states as well as the unknown parameters. Thus, assumption (A1) still be assumed while assumption (A2) and (A3) shall be reformulated in order to account for the nonlinear parameterization. Indeed, together with (A1), one assumes the following hypotheses:

(A2′) The function $\varphi(u, x, \rho)$ is Lipschitz with respect to x and ρ, uniformly in u where $(u, x, \rho) \in U \times X \times \Omega$.

(A3′) The nonlinear parameterization $\varphi(u, x, \cdot)$ is one to one from $\mathbb{R}^m$ into $\mathbb{R}^m$.

Before synthesizing the observer, on has to construct a prolongation, $\tilde{\varphi}$, of the nonlinearities φ which coincides with φ on $\in U \times X \times \Omega$ and which is globally Lipschitz. The process of constructing such a prolongation is similar to that considered previously when dealing with the linear parameterization with the difference that one has also to saturate the unknown parameter ρ since it appears nonlinearly in the system. Indeed, the prolongation $\tilde{\varphi}$ can be defined in a similar manner as in the linear parameterization case, i.e.

$$\tilde{\varphi}(u, x, \rho) = \varphi(u, \sigma(x), \sigma^\rho(\rho)) \tag{32}$$

$\sigma : \mathbb{R}^n \longrightarrow X, x \mapsto \sigma(x)$ and $\sigma^\rho : \mathbb{R}^m \longrightarrow \Omega, \rho \mapsto \sigma^\rho(\rho)$ are smooth bounded saturation functions and are such that $\sigma(x) = x$ and $\sigma^\rho(\rho) = \rho$ for all $x \in X$ and $\rho \in \Omega$.

Again, as in the linear parameterization case, consider the following dynamical system:

$$\begin{cases} \dot{x} = Ax + \tilde{\varphi}(u, x, \rho) \\ y = Cx = x^1. \end{cases} \tag{33}$$

It is clear that system (33) coincides with system (31) for $(x, u, \rho) \in X \times U \times \Omega$. Therefore, it does not make any difference that we consider system (33) instead of (31) for the observer synthesis. Indeed, system (4) shall be considered in the next section. Please notice that for any bounded input $u \in U$, the function $\tilde{\varphi}(u, x, \rho)$ is by construction globally Lipschitz with respect to x and ρ and is bounded for all $(x, \rho) \in \mathbb{R}^n \times \mathbb{R}^m$.

Now, consider the following dynamical system:

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + \tilde{\varphi}(u, \hat{x}, \hat{\rho}) - \theta\Delta_\theta^{-1}\left(S^{-1} + \Upsilon(t)P(t)\Upsilon^{\mathrm{T}}(t)\right)C^{\mathrm{T}}K(C\tilde{x}) \\ \dot{\hat{\rho}}(t) = -\theta P(t)\Upsilon^{\mathrm{T}}(t)C^{\mathrm{T}}K(C\tilde{x}) \\ \dot{\Upsilon}(t) = \theta\left(A - S^{-1}C^{\mathrm{T}}C\right)\Upsilon(t) + \Delta_\theta \dfrac{\partial\tilde{\varphi}}{\partial\rho}(u, \hat{x}, \hat{\rho}) \\ \quad \text{with } \Upsilon(0) = 0 \\ \dot{P}(t) = -\theta P(t)\Upsilon^{\mathrm{T}}(t)C^{\mathrm{T}}C\Upsilon(t)P(t) + \theta P(t) \\ \quad \text{with } P(0) = P^{\mathrm{T}}(0) > 0 \end{cases} \tag{34}$$

where $\hat{x} = \begin{pmatrix} \hat{x}^1 \\ \hat{x}^2 \\ \vdots \\ \hat{x}^q \end{pmatrix}$; $\hat{\rho} = \begin{pmatrix} \hat{\rho}_1 \\ \hat{\rho}_2 \\ \vdots \\ \hat{\rho}_m \end{pmatrix}$; S, C and Δ_θ are respectively given by (7), (2) and (5) ; $K(C\tilde{x})$ is a function satisfying condition (8); u and y are respectively the input and the output of system (31) and $\theta > 0$ is a real number.

Again, due to the nonlinear parameterization, one has to reformulate assumption (A4) which is restated as follows:

(**A4**′) The inputs u are such that for any trajectory of system (34) starting from $\left(\hat{x}(0), \hat{\rho}(0)\right) \in X \times \Omega$, the matrix $C\Upsilon(t)$ is persistently exciting i.e.

$$\exists\delta_1, \delta_2 > 0; \exists T > 0; \forall t \geq 0 : \delta_1 I_m \leq \int_t^{t+T} \Upsilon^{\mathrm{T}}(\tau)C^{\mathrm{T}}C\Upsilon(\tau)\mathrm{d}\tau \leq \delta_2 I_m.$$

One now states the main result.

Theorem 4.1. *Under assumptions* (A1), (A2′), (A3′) *and* (A4′), *system* (34) *is an adaptive observer for system* (33) *with an exponential error convergence for relatively high values of* θ.

Proof of Theorem 4.1. The proof is similar to that of Theorem 3.1. Indeed, set $\tilde{x}(t) = \hat{x} - x$ and $\tilde{\rho}(t) = \hat{\rho}(t) - \rho$. Then, one can show that

$$
\begin{aligned}
\dot{\tilde{x}} &= A\tilde{x} + \tilde{\varphi}(u, \hat{x}, \hat{\rho}) - \tilde{\varphi}(u, x, \rho) \\
&\quad - \theta \Delta_\theta^{-1} S^{-1} C^{\mathrm{T}} K(C\tilde{x}) + \Delta_\theta^{-1} \Upsilon(t) \dot{\tilde{\rho}} \\
&= A\tilde{x} + \frac{\partial \tilde{\varphi}}{\partial \rho}(u, \hat{x}, \hat{\rho})\tilde{\rho} + \left(\frac{\partial \tilde{\varphi}}{\partial \rho}(u, \hat{x}, \rho_\xi) - \frac{\partial \tilde{\varphi}}{\partial \rho}(u, \hat{x}, \hat{\rho}) \right) \tilde{\rho} \\
&\quad + \frac{\partial \tilde{\varphi}}{\partial x}(u, \xi, \rho)\tilde{x} - \theta \Delta_\theta^{-1} S^{-1} C^{\mathrm{T}} K(C\tilde{x}) + \Delta_\theta^{-1} \Upsilon(t) \dot{\tilde{\rho}}
\end{aligned}
$$

where $\rho_\xi \in \mathbb{R}^m$ and $\xi \in \mathbb{R}^n$ according to the Mean Value Theorem.

Set $\bar{x} = \Delta_\theta \tilde{x}$. One obtains:

$$
\begin{aligned}
\dot{\bar{x}} &= \theta A\bar{x} - \theta S^{-1} C^{\mathrm{T}} K(C\bar{x}) + \Delta_\theta \frac{\partial \tilde{\varphi}}{\partial \rho}(u, \hat{x}, \hat{\rho})\tilde{\rho} + \Upsilon(t)\dot{\tilde{\rho}}(t) \\
&\quad + \Delta_\theta \left(\frac{\partial \tilde{\varphi}}{\partial \rho}(u, \hat{x}, \rho_\xi) - \frac{\partial \tilde{\varphi}}{\partial \rho}(u, \hat{x}, \hat{\rho}) \right) \tilde{\rho} \\
&\quad + \Delta_\theta \frac{\partial \tilde{\varphi}}{\partial x}(u, \xi, \rho) \Delta_\theta^{-1} \bar{x}.
\end{aligned}
$$

Define $\eta = \bar{x} - \Upsilon \tilde{\rho}$. Proceeding as in the previous section, one can show that:

$$
\begin{aligned}
\dot{\eta} &= \theta A\eta + \theta S^{-1} C^{\mathrm{T}} C \Upsilon \tilde{\rho} - \theta S^{-1} C^{\mathrm{T}} K(C\bar{x}) \\
&\quad + \Delta_\theta \left(\frac{\partial \tilde{\varphi}}{\partial \rho}(u, \hat{x}, \rho_\xi) - \frac{\partial \tilde{\varphi}}{\partial \rho}(u, \hat{x}, \hat{\rho}) \right) \tilde{\rho} \\
&\quad + \Delta_\theta \frac{\partial \tilde{\varphi}}{\partial x}(u, \xi, \rho) \Delta_\theta^{-1} (\eta + \Upsilon \tilde{\rho}).
\end{aligned}
$$

Set $V_1(\eta(t)) = \eta^{\mathrm{T}}(t) S \eta(t)$, $V_2(\bar{\rho}(t)) = \tilde{\rho}^{\mathrm{T}}(t) P^{-1}(t) \tilde{\rho}(t)$ and let $V(\eta(t), \tilde{\rho}(t)) = V_1(\eta(t)) + V_2(\tilde{\rho}(t))$ be the Lyapunov candidate function. Again, proceeding as in the previous section, one can show that:

$$
\begin{aligned}
\dot{V}_1(t) &= -\theta V_1 + \theta^{\mathrm{T}} C^{\mathrm{T}} C\eta + 2\theta \eta^{\mathrm{T}} C^{\mathrm{T}} C \Upsilon \, \tilde{\rho}(t) - 2\theta \eta^{\mathrm{T}} C^{\mathrm{T}} K(C\bar{x}) \\
&\quad + 2\eta^{\mathrm{T}} S \Delta_\theta \left(\frac{\partial \tilde{\varphi}}{\partial \rho}(u, \hat{x}, \rho_\xi) - \frac{\partial \tilde{\varphi}}{\partial \rho}(u, \hat{x}, \hat{\rho}) \right) \tilde{\rho} \\
&\quad + 2\eta^{\mathrm{T}} S \Delta_\theta \frac{\partial \tilde{\varphi}}{\partial x}(u, \xi, \rho) \Delta_\theta^{-1} (\eta + \Upsilon \tilde{\rho}). \qquad (35)
\end{aligned}
$$

According to Assumption (A2′), $\frac{\partial \tilde{\varphi}}{\partial x}(u, \cdot, \cdot)$ and $\frac{\partial \tilde{\varphi}}{\partial \rho}(u, \cdot, \cdot)$ are bounded. Moreover, according to the triangular structure of $\tilde{\varphi}$ with respect to x, the matrix $\Delta_\theta \frac{\partial \tilde{\varphi}}{\partial x}(u, \xi, \rho)$ is also bounded and the corresponding upper bound does not depend on θ for $\theta \geq 1$. Thus, one has:

$$
2\eta^{\mathrm{T}} S \Delta_\theta \frac{\partial \tilde{\varphi}}{\partial x}(u, \xi, \rho) \Delta_\theta^{-1} (\eta + \Upsilon \tilde{\rho}) \leq k_1 V_1 + k_2' \sqrt{V_1} \sqrt{V_2} \qquad (36)
$$

$$
2\eta^{\mathrm{T}} S \Delta_\theta \left(\frac{\partial \tilde{\varphi}}{\partial \rho}(u, \hat{x}, \rho_\xi) - \frac{\partial \tilde{\varphi}}{\partial \rho}(u, \hat{x}, \hat{\rho}) \right) \tilde{\rho} \leq k_2'' \sqrt{V_1} \sqrt{V_2} \qquad (37)
$$

where $k_1, k_2', k_2'' > 0$ are real numbers which do not depend on θ for $\theta \geq 1$.

Combining (35)–(37), one obtains:

$$
\begin{aligned}
\dot{V}_1(t) &\leq -\theta V_1 + \theta \eta^{\mathrm{T}} C^{\mathrm{T}} C\eta + 2\theta \eta^{\mathrm{T}} C^{\mathrm{T}} C \Upsilon \, \tilde{\rho} \\
&\quad - 2\theta \eta^{\mathrm{T}} C^{\mathrm{T}} K(C\bar{x}) + k_1 V_1 + k_2 \sqrt{V_1} \sqrt{V_2} \qquad (38)
\end{aligned}
$$

where $k_2 = k_2' + k_2''$.

The time derivative of V_2 can be derived in a same manner as in the previous section and according to (25), one has:

$$
\dot{V}_2(t) = -\theta V_2 - 2\theta \tilde{\rho}^{\mathrm{T}} \, \Upsilon^{\mathrm{T}} C^{\mathrm{T}} K(C\bar{x}) + \theta \tilde{\rho}^{\mathrm{T}} \Upsilon^{\mathrm{T}}(t) C^{\mathrm{T}} C \Upsilon(t) \tilde{\rho}. \qquad (39)
$$

The remaining part of the proof is identical to that of Theorem 3.1 (compare (38) and (39) with (24) and (25), respectively). □

5. Example: State and kinetic parameters estimation in a bioreactor

In this section, the performances of the proposed observers are illustrated through a typical bioreactor model which belongs to the class of systems (31), i.e. with nonlinear parameterization. Please notice that from practical point of view, the prolongation techniques are rarely used as shown by several experimental processes where observers, synthesized under assumptions similar to those considered in this paper, have been successfully applied and no prolongation has been computed (see e.g. Busawon, Yahoui, Hammouri, & Grellet, 2001; Deza, Busvelle, & Gauthier, 1992; Farza, Hammouri, Jallut, & Liéto, 1999; Viel, Busvelle, & Gauthier, 1995). Indeed, one shall illustrate the theory described in the above sections through the following example where no prolongation is considered.

We consider a simple microbial culture which involves a single biomass x_2 growing on a single substrate x_1. The bioprocess is supposed to be continuous with a dilution rate $D(t)$ and an input substrate concentration $s_{\text{in}}(t)$. The specific growth rate is assumed to follow the Contois model (Bailey & Ollis, 1986). The mathematical dynamical model of the process is constituted by the following two mass balance equations associated to x_1 and x_2, respectively:

$$
\begin{cases}
\dot{x}_1(t) = -k \dfrac{\mu^\star x_1 x_2}{K_C x_2 + x_1} + D(t)(s_{\text{in}}(t) - x_1(t)) \\
\dot{x}_2(t) = \dfrac{\mu^\star x_1 x_2}{K_C x_2 + x_1} - D(t) x_2(t)
\end{cases} \qquad (40)
$$

where x_1 and x_2 respectively denote the concentration of the substrate and the biomass, $\mu^\star$ and K_C are the Contois law parameters and k is the known yield coefficient. The substrate concentration is assumed to be measured and the objective is to estimate $x_2(t)$ and the Contois law parameters.

System (40) has been considered in Gauthier et al. (1992) where the authors exhibited a compact set $X \in \mathbb{R}^2$ which is positively invariant under the dynamics of (40). Moreover, it was shown that the following function $\Phi : X \longrightarrow \Phi(X)$, $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} z_1 & = & x_1 \\ z_2 & = & -k \frac{\mu^\star x_1 x_2}{K_C x_2 + x_1} \end{pmatrix}$ is a diffeomorphism from X onto its image.

System (40) can be written in the new coordinates as follows:

$$
\begin{cases}
\dot{z}_1 = z_2 + D(s_{\text{in}} - z_1) \\
\dot{z}_2 = \rho_1 (1 + \rho_2 \dfrac{z_2}{z_1})^2 z_2 - \rho_2 \left(\dfrac{z_2}{z_1} \right)^2 (z_2 + D s_{\text{in}}) - D z_2 \\
y = z_1
\end{cases} \qquad (41)
$$

where $\rho_1 = \mu^\star$ and $\rho_2 = \frac{K_C}{k\mu^\star}$. It is easy to see that system (41) is under form (31) and as a result the estimation of z, ρ_1 and ρ_2 can be achieved using an observer of the form (34). Please notice that the original Contois law parameters can be recovered from ρ_1 and ρ_2 as follows: $\mu^\star = \rho_1$ and $K_C = k\rho_1\rho_2$.

In what follows, one shall give simulation results obtained when the observer design function is specified as in Eq. (28). In fact, many other simulations have been carried out with other expressions such as those given in (29) and (30) and the obtained results were quite similar to those given hereafter. In fact, from the observer design point of view, the introduction of the design function K allows us to clearly show that sliding mode observers belong to the class of high gain observers with a particular specification of this function. In practice, its choice is not very crucial as shown by the several numerical simulations that have been carried out (and not shown here). However, we have recently proposed a high gain output feedback controller which was derived by exploiting the concept of duality

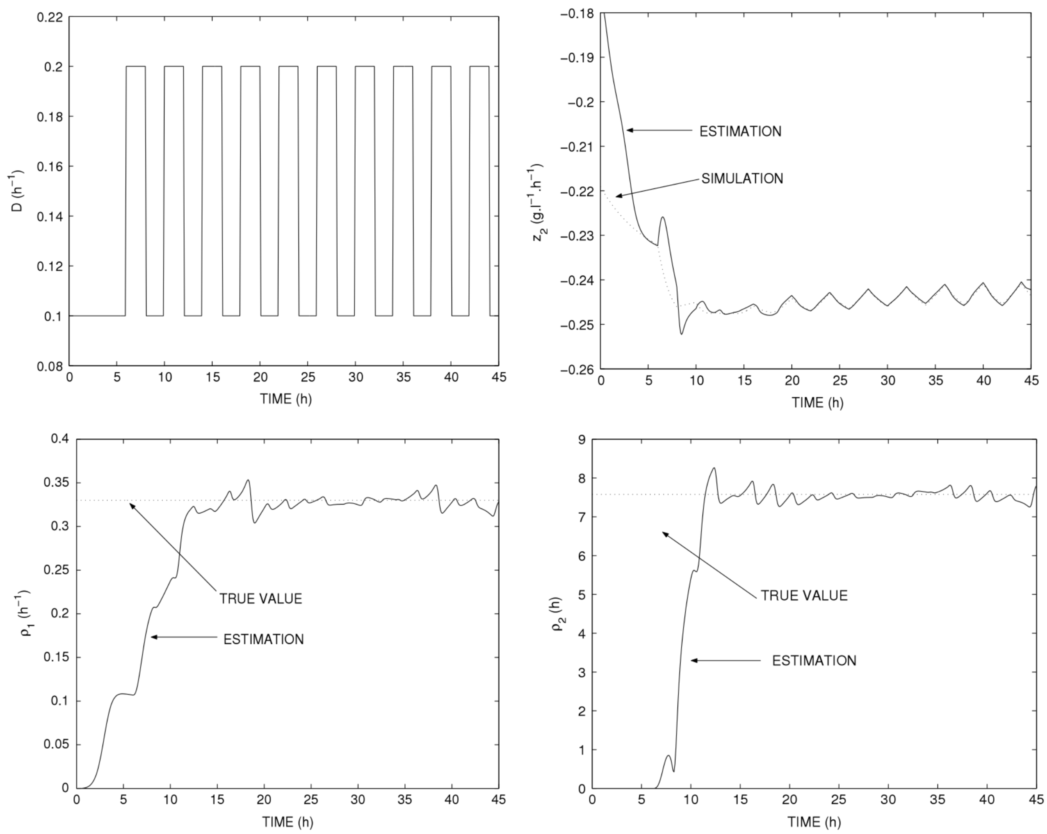

Fig. 1. Time evolution of the dilution rate with estimation results for $\theta = 2$.

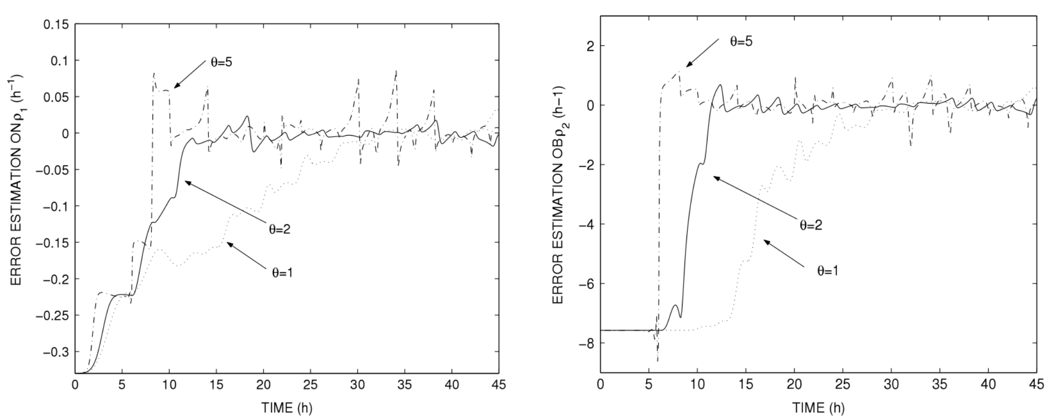

Fig. 2. The estimation error for different values of the observer design parameter.

Observability/Controllability (Hajji, Farza, M'Saad, & Kamoun, 2008). The underlying controller is expressed as a function that has similar properties as the design function K. This offers the possibility to take into account physical controller constraints by designing auto saturated controllers through specific choices of the design function, namely functions given by (29) or (30).

The model and observer simulations have been carried out using a constant input substrate concentration and a dilution rate which varies as a square wave signal as shown in Fig. 1. The parameter values used in simulation are:

$$\mu^{\star} = 0.33\,h^{-1}; \qquad K_C = 5\,g.g^{-1};$$
$$k = 20\,g.g^{-1}; \qquad s_{\text{in}} = 5\,g.l^{-1}.$$

The resulting values of the parameters ρ_1 and ρ_2 are $\rho_1 = 0.33\,h^{-1}$ and $\rho_2 = 7.5758\,h$. In order to simulate practical situations, the measurements of z_1 issued from the model simulation have been corrupted by noise measurements with a zero mean value and a variance $\sigma^2 = 10^{-3}$.

The tuning of the design parameter θ is achieved using a trial-and-error method bearing in mind the well-known property related to its choice: as for standard high gain state observers, such choice is a compromise between fast convergence of the observer, obtained through relatively high values of θ, and a well behaviour with respect to the measurement noise which is obtained when the values of θ are chosen relatively small.

The results presented below are obtained by fixing the value of θ to 2. We have reported in Fig. 1 the estimation of ρ_1, ρ_2 and z_2 obtained with this value. In Fig. 2, we have reported the estimation error on ρ_1 and ρ_2 obtained with three values for θ, i.e. $\theta = 1$, $\theta = 2$ and $\theta = 5$. Although all these results can be considered as quite satisfactory, those obtained with the value 2 are preferable since they constitute a good compromise between fast convergence (obtained with the value 5) and a good behaviour with respect to measurement noise (obtained with the value 1).

6. Conclusion

The main motivation of this paper was to design adaptive observers for a class of MIMO uniformly observable nonlinear systems with linear and nonlinear parameterizations. Of fundamental importance, the exponential convergence of the observers was shown to be guaranteed for both parameterizations under well-defined persistent excitation conditions. Another feature of the proposed observers lies in their ability to give rise to different observers having different structures. Indeed, it has been shown that high gain adaptive observers and sliding mode like adaptive observers can be derived from the set of proposed observers. Simulation results carried out under realistic conditions have been reported and they demonstrated the good capabilities of the observer in providing good estimates of the missing states and the unknown parameters.

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