Low frequency coupling and mode interference in an inhomogeneous lattice of finite length

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Summary
The propagation through a lattice made with two waveguides periodically coupled by perforations is studied at low-frequencies. A degree of inhomogeneity is introduced with parametrically opened diaphragms inserted into one waveguide of the lattice. Analytical results obtained thanks to the fourth-order transfer matrix formalism illustrate three physical phenomena. The first phenomena is the effect of the perforations, the second is the effect of the inhomogeneity of the lattice, and the third is the effect of the interference between propagating modes when the system have finite length. These results give analytical expressions for the insertion loss, the characteristic impedance or propagation constants, that can be of practical interest for understanding and designing non local acoustic treatments for automotive and turbofan engines. The cases of a *strongly inhomogeneous* lattice and *almost homogeneous* lattice and their transition to homogeneous and branched resonator cases are discussed.

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1. Introduction

This work aims to describe the acoustic propagation at low frequencies in a system of two coupled waveguides, with a particular focus on understanding the effect introduced by inhomogeneity of the lattice, that is made of two waveguides filled with different propagation media. The coupling is carried out periodically, with lateral perforations disposed regularly in the axial direction of the waveguides (see Fig.1). This initial system was chosen to represent by means of a discrete model, a variety of situations encountered in practice for noise mitigation by wall treatments for automotive exhaust and aircraft turbofan engine nacelles.

The analytical model for an elementary cell of the periodic lattice and for a lattice of finite length is detailed in [1]. Conventional tools for propagation in periodic media [2] make possible to establish the analytical expressions that characterize an elementary cell of the inhomogeneous lattice (eigenvectors, eigenmodes, and associated characteristic impedances, phase velocities ...). In the following, expressions for the fourth-order transfer matrix and the dispersion equation of the lattice are recalled in section 2 and 3. The coupling between the two waveguides by means of lateral perforations is described rigorously by a perforation matrix [3].

By introducing boundary conditions at the ends of the lattice, the influence of the properties of one elementary cell on the potential noise mitigation (insertion loss) of a lattice of finite length is also illustrated.

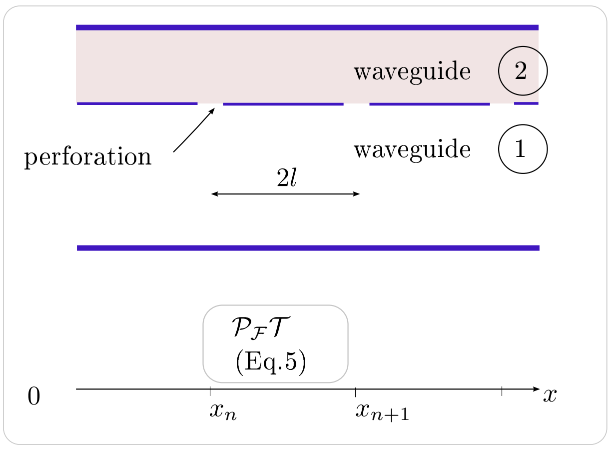

Figure 1. Inhomogeneous lattice, with two waveguides (with different propagation media), periodically coupled by lateral perforations

Starting from well known (and extreme) lattice configurations used as references, namely coupled waveguides filled with air (totally homogeneous lattice) and

Helmholtz resonators branched on Waveguide 1 (totally inhomogeneous lattice), the discrete model is used to unveil how the degree of inhomogeneity modifies the properties of the lattice and its associated insertion loss. The degree of inhomogeneity of the lattice is introduced here by means of parametrically opened diaphragms inserted into Waveguide 2. The cases of a *strongly inhomogeneous* lattice and *almost homogeneous* lattice and their transition to extreme (references) case are discussed.

2. Fourth order transfer matrix of one cell of a periodic lattice

We consider two waveguides periodically coupled along their axes by lateral perforations as shown in Fig.1. In this section, we write the transfer matrix of an elementary cell (of length $2l$) of the periodic lattice. The two waveguides are filled with different propagation media, and the resulting lattice is called an inhomogeneous lattice.

Kergomard et al. [3] showed that coupling between plane waves in guides 1 and 2 introduced by a lateral perforation situated at a given abscissa x_n, can be described in an exact manner by a perforation matrix of fourth order. At a given frequency, the plane wave amplitudes of the acoustic pressures and velocity on the left of a lateral perforation

$$\mathcal{V}_\mathrm{L} = \begin{pmatrix} \mathbf{V}_{1L} \\ \mathbf{V}_{2L} \end{pmatrix} = \begin{pmatrix} p_{1L} \\ v_{1L} \\ p_{2L} \\ v_{2L} \end{pmatrix}, \tag{1}$$

can be related to the same quantities on the right $\mathcal{V}_R$ of the perforation, using a fourth order perforation matrix $\mathcal{P}_\mathcal{F}$ in order to write:

$$\mathcal{V}_L = \mathcal{P}_\mathcal{F} \;\; \mathcal{V}_R \,. \tag{2}$$

The matrix $\mathcal{P}_\mathcal{F}$ takes the form [3] (anti-symmetrical orientation):

$$\mathcal{P}_\mathcal{F} = \begin{pmatrix} (\gamma_1 + \gamma_2\mathbf{M}) & \gamma_2(\mathbf{I}-\mathbf{M}) \\ \gamma_1(\mathbf{I}-\mathbf{M}) & (\gamma_2+\gamma_1\mathbf{M}) \end{pmatrix}, \tag{3}$$

with

$$\gamma_{1,2} = \frac{S_{1,2}}{S_1+S_2},\; \mathbf{M} = \mathbf{I} + \frac{2Z_aY_s}{1-Z_aY_s}\begin{pmatrix} 1 & Y_s^{-1} \\ Z_a^{-1} & 1 \end{pmatrix},$$

where S_1 and S_2 are the cross sections of the waveguides, and Z_a and Y_s introduce respectively the series (specific) impedance and shunt (specific) admittance of the lateral perforation. $\mathbf{I}$ is the second order identity matrix.

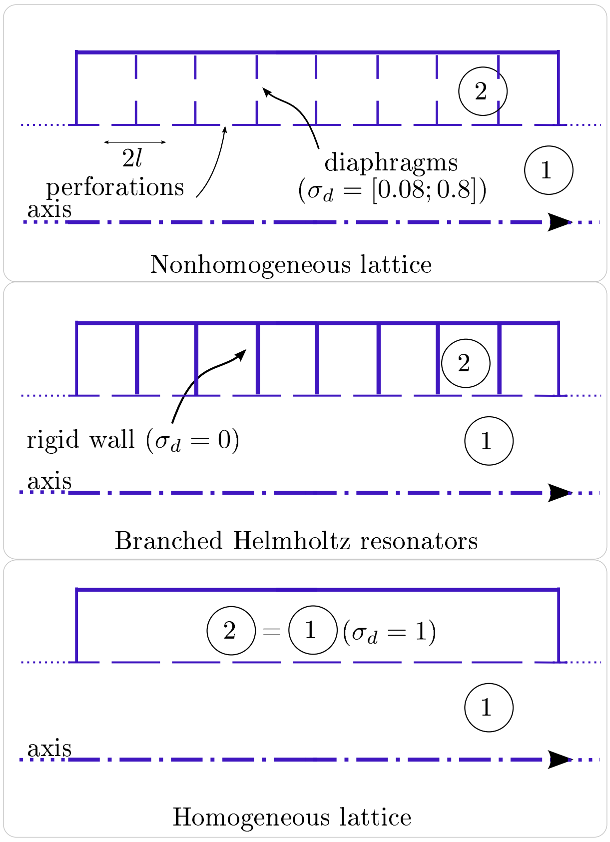

Figure 2. Cylindrical Lattices of finite length: inhomogeneous lattice (top), branched Helmholtz resonators with closed cells in Waveguide 2 (center) and homogeneous lattice (bottom).

The propagation of plane waves along the uncoupled portion of the waveguides, i.e. between abscissa x_{n+1} and x_n (length $2l$, see Fig.1), is described by a classical fourth order transfer matrix $\mathcal{T}$:

$$\mathcal{T} = \begin{pmatrix} \mathbf{T}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{T}_2 \end{pmatrix} = \begin{pmatrix} A_1 & B_1 & 0 & 0 \\ C_1 & D_1 & 0 & 0 \\ 0 & 0 & A_2 & B_2 \\ 0 & 0 & C_2 & D_2 \end{pmatrix}. \tag{4}$$

The complete transfer matrix relating the plane wave amplitudes on the left of two successive perforations of the periodic lattice (see Fig.1) is written as the product of the transfer and the perforation matrix:

$$\mathcal{V}_{L,n} = \mathcal{P}_\mathcal{F}\mathcal{T} \;\; \mathcal{V}_{L,n+1} \,. \qquad \text{(and also } \mathcal{V}_{\mathrm{R},n} = \mathcal{T}\,\mathcal{P}_\mathcal{F} \;\; \mathcal{V}_{R,n+1}) \tag{5}$$

Since the expressions for the transfer matrix $\mathcal{T}$ and the perforation matrix $\mathcal{P}_\mathcal{F}$ given above are general, they can describe propagation at low frequency in a great variety of physical situations. For instance situations where Waveguides 1 and 2 are non reciprocal (for instance by the presence of flow), exhibit viscothermal losses (equivalent fluid modeling a porous material), or include discontinuities like diaphragms (as for example in [4]).

3. Dispersion within the reciprocal periodic lattice

Kergomard and Pachebat [1] showed that thanks to the block-wise expressions of the perforation matrix (Eq.3) and the transfer matrix (Eq.4), the eigenvectors and eigenvalues of the transfer matrix $\mathcal{P}_{\mathcal{F}}\mathcal{T}$ can be obtained analytically.

In the following, the lateral perforations radius is supposed to be small compared to the wavelength. The series impedance associated to anti-symmetrical profile of the flow velocity across the perforation can be ignored, by choosing $Z_a = 0$ [3] in the expression of the perforation matrix (Eq.3) . Under this particular assumption, the characteristic polynomial $\det(\mathcal{P}_{\mathcal{F}}\mathcal{T} - \lambda\mathcal{I})=0$ gives the dispersion equation [1]

$$\frac{1}{Y_s} = 2\lambda\left[\gamma_2\frac{B_1}{\Delta_1} + \gamma_1\frac{B_2}{\Delta_2}\right], \tag{6}$$

where $\Delta_{1,2} = \det(\mathbf{T}_{1,2} - \lambda\mathbf{I}) = \lambda^2 - \lambda(A_{1,2} + D_{1,2}) + \det\mathbf{T}_{1,2}$ and $\mathbf{I}$ is the identity matrix of order 2.

In the following, we also assume that the waveguides are symmetrical and reciprocal. These physical properties imply for the transfer matrices of Waveguides 1 and 2: $A_{1,2} = D_{1,2}$ and $det(\mathbf{T}_{1,2}) = 1$. Thus we can rewrite the dispersion equation of order four in λ (Eq.6), as a second order polynomial in $\cosh\Gamma = (\lambda + 1/\lambda)/2$:

$$\frac{2}{Y_p} = \frac{\overline{B}_1}{\cosh\Gamma - A_1} + \frac{\overline{B}_2}{\cosh\Gamma - A_2}, \tag{7}$$

where $Y_p = 2Y_sS_1S_2/(S_1 + S_2)$ is the acoustic admittance of the lateral perforation,and $\overline{B}_{1,2} = B_{1,2}/S_{1,2}$.

The discriminant Δ of the quadratic equation in $\cosh\Gamma$ (Eq.7) can be written by defining a *Coupling coefficient* $\mathcal{C}$, as follows:

$$\begin{aligned}\Delta &= (A_1 - A_2)^2\left[1 + 2\frac{\overline{B}_1 - \overline{B}_2}{\overline{B}_1 + \overline{B}_2}\mathcal{C} + \mathcal{C}^2\right]\\ &\text{where } \mathcal{C} = \frac{1}{2}Y_p\frac{\overline{B}_1 + \overline{B}_2}{A_1 - A_2}\ .\end{aligned} \tag{8}$$

The two solutions of the dispersion equation (Eq.7) are:

$$\left\{\begin{aligned}\cosh\Gamma &= \tfrac{1}{2}\left(A_1 + A_2\right.\\ &\quad \left. + \tfrac{1}{2}Y_p\left[\overline{B}_1 + \overline{B}_2\right] - \sqrt{\Delta}\right)\\ \cosh\Gamma' &= \tfrac{1}{2}\left(A_1 + A_2\right.\\ &\quad \left. + \tfrac{1}{2}Y_p\left[\overline{B}_1 + \overline{B}_2\right] + \sqrt{\Delta}\right)\end{aligned}\right. . \tag{9}$$

Nb of cells n_c	15
Nb of perf/cell n_s	11
Cell Length $2l$	$8.5\,10^{-3}$ m
Guide 1 radius r_1	$2.54\,10^{-2}$ m
Guide 2 radius r_2	$5.08\,10^{-2}$ m
Perf. radius r_s	$1.25\,10^{-3}$ m
Perf. open ratio σ	2.1
Diaph. open ratio σ_d	[0.08;0.8]
Diaph. radius r_d	$1.25\,10^{-2}$ m
Nb of diaph./cell n_d	[1; 10]

Table I. Geometrical parameters of the lattice shown in Fig.2

The first eigenmode $\cosh\Gamma$ corresponds to an average plane mode propagating within the lattice with no influence of the lateral perforations (at any abscissa in the lattice, pressures within Waveguides 1 and 2 are equal in amplitude and phase). The second eigenmode $\cosh\Gamma'$ is called the *flute* mode (pressures within Waveguides 1 and 2 are equal in amplitude and opposite in phase) [3, 1]. Expressions for the *flute* mode can be found for a continuous model in [5] and [6]. The *flute* mode is always strongly evanescent at very low frequency (large Y_p). The strong coupling between Waveguides 1 and 2 occurs when the coupling coefficient $\mathcal{C}$ (Eq.8) is large: the media within Waveguides 1 and 2 are not very different ($A_1 - A_2$ is small), or when the perforation effect is strong (large Y_p).

After some algebra (see [1] for details), and starting from the analytical expressions of the eigenvectors and eigenvalues of the transfer matrix $\mathcal{P}_{\mathcal{F}}\mathcal{T}$ of one cell, it is possible to obtain analytically the transfer matrix $(\mathcal{P}_{\mathcal{F}}\mathcal{T})^{n_c}$ for a periodic set of n_c cells, to take into account the boundary conditions of a finite length lattice (via the expression of $(\mathcal{P}_{\mathcal{F}}\mathcal{T})^{n_c}$ as an impedance matrix), and to obtain the expression of the insertion loss of the finite length lattices represented in Fig.2. The analytical approach proposed, in addition to providing a physical interpretation of the results, avoids numerical problems that usually appear when one eigenmode is strongly evanescent.

4. Application to an automotive muffler

In order to illustrate the effect of inhomogeneity of the lattice on the propagation and attenuation at low frequencies, we apply the above results to a particular geometry shown in Fig.2 (top). This geometry includes diaphragms within Waveguide 2. The diaphragm opening is used as a parameter in order to explore a wide variety of situations. With no diaphragms, the lattice is homogeneous (Fig.2, bottom), and the geometrical dimensions (see Tab.I) are chosen according to the *long resonator* studied numerically by Sullivan and Crocker [7]. On the opposite,

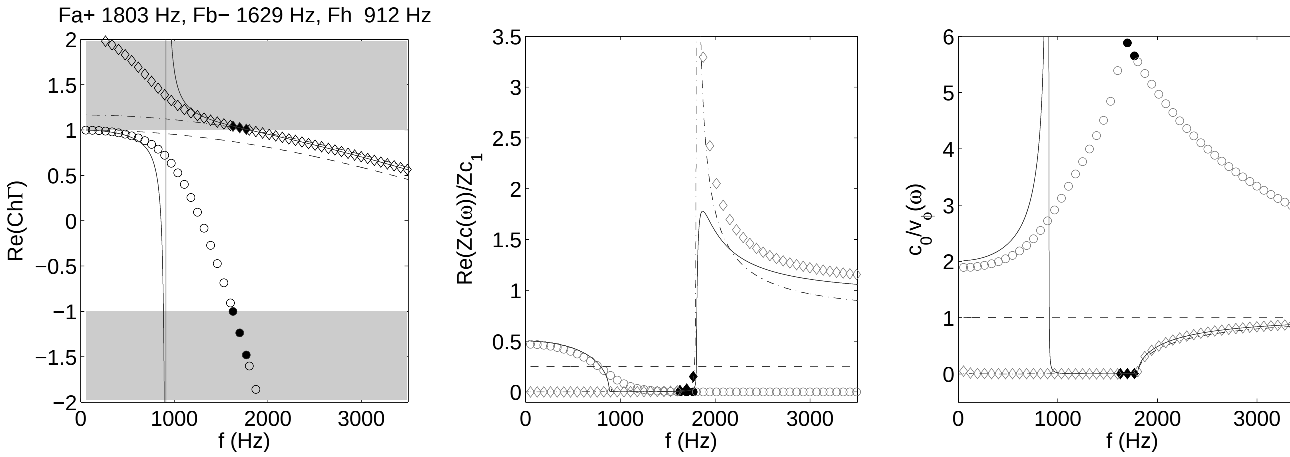

Figure 3. Properties of the two eigenmodes of an elementary cell of the *strongly inhomogeneous* lattice ($\sigma_d = 0.08$): solutions of the dispersion equation (left), characteristic impedance (center) and relative phase velocity (right). Plane mode (marker o) and flute mode (marker $\diamondsuit$). Filled marker indicate the stop band $[F_b^-; F_a^+]$ (no propagating eigenmodes). Solutions for two extreme lattice configurations are shown for reference: homogeneous lattice ($\sigma_d = 1$, dashed and dash-dot lines), and branched Helmholtz resonators ($\sigma_d = 0$, solid line).

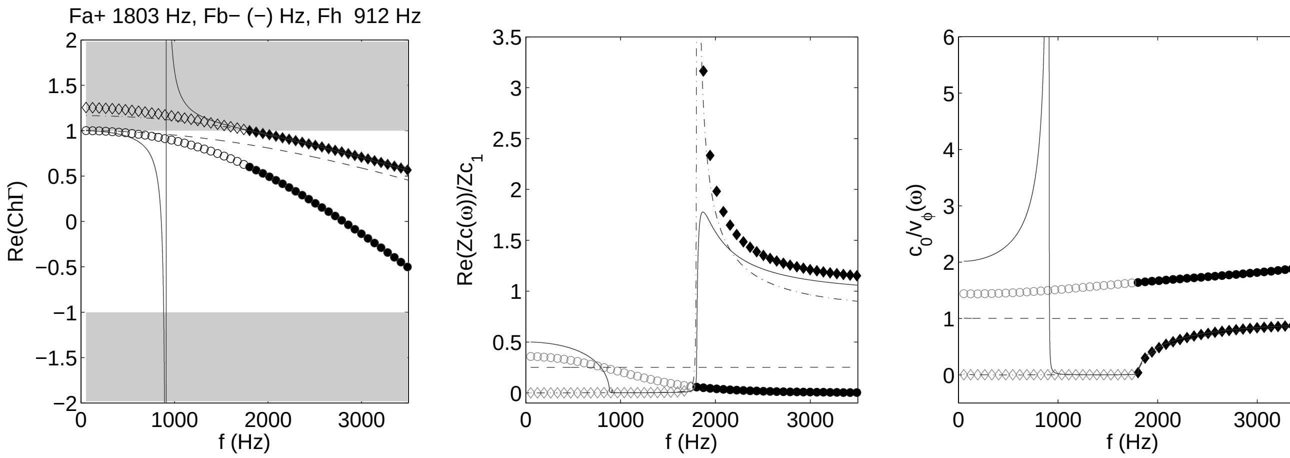

Figure 4. Properties of the two eigenmodes of an elementary cell of the *almost homogeneous* lattice ($\sigma_d = 0.8$): solutions of the dispersion equation (left), characteristic impedance (center) and relative phase velocity (right). Plane mode (marker o) and flute mode (marker $\diamondsuit$). Filled marker indicate the band $[F_a^+; F_b^-]$ with two propagating eigenmodes. Solutions for two extreme lattice configurations are shown for reference: homogeneous lattice ($\sigma_d = 1$, dashed and dash-dot lines), and branched Helmholtz resonators ($\sigma_d = 0$, solid line).

with totally closed diaphragms, the lattice is a series of branched resonators (Fig.2, center). But thanks to the approach presented above, we can also explore two intermediate situations by varying the diaphragm opening: an *almost homogeneous* lattice (wide open diaphragms) and a *strongly inhomogeneous* lattice (diaphragms with very small opening).

Waveguides 1 and 2 are co-axial cylinders filled with ambient air. The classical transfer matrix for a plane mode (pressure-velocity) along one uncoupled portion of length l of lossless Waveguide 1 or 2 (no lateral perforation) is:

$$\mathbf{t_{1,2}} = \begin{pmatrix} \cos(kl) & jZ_c^0 \sin(kl) \\ j\sin(kl)/Z_c^0 & \cos(kl) \end{pmatrix}$$

where $Z_c^0 = \rho_0 c_0$ and $k = \omega/c_0$ are the characteristic impedance and wavenumber of the medium (air) filling Waveguides 1 and 2, with ρ_0 its density and c_0 its sound velocity.

The transfer matrix $\mathbf{T}_1$ and $\mathbf{T}_2$ of Eq.4 are given by $\mathbf{T}_1 = (\mathbf{t}_1)^2$, and $\mathbf{T}_2 = \mathbf{t}_2 \mathbf{D}\, \mathbf{t}_2$ where the acoustic mass impedance Z_d in the matrix

$$\mathbf{D} = \begin{pmatrix} 1 & Z_d \\ 0 & 1 \end{pmatrix} \qquad (10)$$

introduces the inhomogeneity due to the presence of the diaphragms within Waveguide 2. The impedance of one diaphragm is chosen as $Z_d = j\omega\rho S_2/2r_d \left(1 - \sqrt{\pi r_d^2/S_2}\right)$. The open ratio of the diaphragms for one cell is defined as $\sigma_d = n_d r_d^2/(r_2^2 - r_1^2)$ where n_d is the number of diaphragms around the circumference of one elementary cell. Similarly, for the lateral perforations, the impedance $1/Y_s$ of one perforation is chosen as $1/Y_s = j\omega\rho/2r_s$. The open ratio

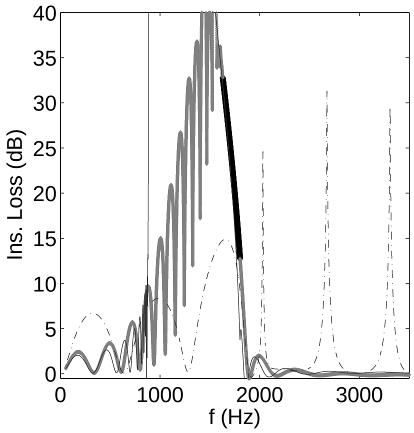

Figure 5. Insertion Loss of the *strongly inhomogeneous* lattice ($\sigma_d = 0.08$) of finite length. Black line indicate a stop band (no propagating eigenmodes). Solutions for two extreme lattice configurations are shown for reference: homogeneous lattice ($\sigma_d = 1$, dash-dot line), and branched Helmholtz resonators ($\sigma_d = 0$, thin solid line).

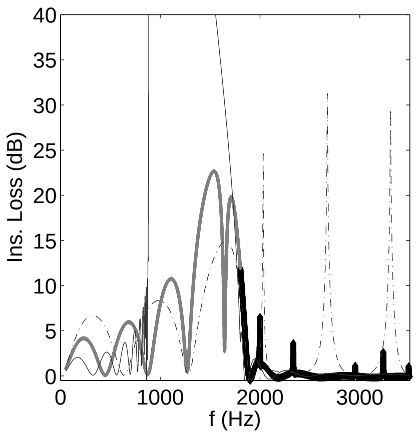

Figure 6. Insertion Loss of the *almost homogeneous* lattice ($\sigma_d = 0.8$) of finite length. Black line indicate a band with two propagating eigenmodes. Solutions for two extreme lattice configurations are shown for reference: homogeneous lattice ($\sigma_d = 1$, dash-dot line), and branched Helmholtz resonators ($\sigma_d = 0$, thin solid line).

of the lateral perforations for one cell is defined as $\sigma_s = n_s r_s^2/(4lr_1)$ where n_s is the number of lateral perforations (all situated at x_n $(n = 1..n_c)$) of elementary cell number n.

5. Results

We first examine how two classical configurations can be interpreted thanks to the proposed approach. The first configuration is an homogeneous lattice and the second is a lattice with Helmholtz resonators branched on Waveguide 1. Then we will study the effect of inhomogeneity of the lattice on the propagation and insertion loss at low frequencies.

At the limit where propagation media are identical within Waveguides 1 and 2 ($A_1 = A_2$, Fig.2, bottom), $\mathcal{C}$ tends to infinity and the two eigenmodes (Eq.9) reduce for a homogeneous lattice to the following expression:

$$\begin{cases} \cosh\Gamma = A \\ \cosh\Gamma' = A + \frac{1}{2}Y_p(\overline{B}_1 + \overline{B}_2) \end{cases}. \tag{11}$$

The corresponding homogeneous plane mode Γ and homogeneous flute mode Γ' are represented in Fig.3 (left) with dashed lines and dash-dot lines respectively. Analytical expressions for the normalized characteristic impedance of the plane mode $Z_c(\omega)/(\rho_0 c_0/S_1)$ and the flute mode $Z'_c(\omega)/(\rho_0 c_0/S_1)$, and their relative phase velocity $c_0/v_\phi(\omega) = Im(\Gamma)c_0/(2l\omega)$ and $c_0/v'_\phi(\omega) = Im(\Gamma')c_0/(2l\omega)$ can also be obtained (not presented). Their variation is shown in Fig.3 (center) and Fig.3 (right), using the same dashed and dash-dot lines. Fig.3 shows, as already reported in [1], that a homogeneous lattice is a particular case with a non dispersive plane mode ($c_0/v_\phi(\omega)$ is constant) which is always propagating (-1<$\cosh\Gamma < 1$). The flute mode $\cosh\Gamma'$ tends asymptotically, when increasing frequency above its cut-on frequency $F_a^+ = 1803$Hz (F_a^+ is given by $\cosh\Gamma' = 1$), towards physical characteristics similar to the plane mode. This similarity for $f > F_a^+$ between homogeneous plane mode and flute mode results in longitudinal interferences, driven by the total length of the finite lattice, as detailed in [1], associated to resonant attenuation peaks of the insertion loss of the homogeneous lattice for $f > F_a^+$ (Fig.5, dash-dot line).

In the case of branched Helmholtz resonators, the diaphragms are totally closed. The impedance Z_d in Eq.10 tends to infinity and $A_1 \ll A_2$, $\overline{B}_1 \ll \overline{B}_2$. The unique mode propagating through the lattice is given by:

$$\cosh\Gamma = A_1 + \frac{1}{2Z_h}\overline{B}_1\,, \tag{12}$$

where $Z_h = 1/Y_p + \overline{B}_2/2(A_2 - 1)$ is the input impedance of the Helmholtz resonator of one cell of the lattice (Fig.2, center). The series of branched resonators acts as a local reacting treatment on Waveguide 1, with a maximum effect (see Fig.3 left, thin solid line) at the Helmholtz resonance $F_h = 912$ Hz (given by $Z_h = 0$). The lattice exhibits a stop band with $|\cosh\Gamma| > 1$ (when ignoring viscothermal losses) within the interval $[F_h; F_a^+]$, that corresponds also to a very small relative characteristic impedance $Z_c(\omega)/(\rho_0 c_0/S_1)$ (see Fig.3 center, thin solid line) and relative phase velocity $c_0/v_\phi(\omega)$ (see Fig.3 right, thin solid line). The existence of this stop band (no propagating mode) induces a cumulative attenuation: the insertion loss shown in Fig.5 (thin solid line) is directly proportional to the number of elementary cells of the finite lattice (here $n_c = 15$, see Tab.I), and is very small outside the stop band.

In the case of a *strongly inhomogeneous* lattice, the open area of the diaphragms between two neighboring cells of the lattice represents only 8% of S_2 (that is $\sigma_d = 0.08$). At low frequencies and for the plane mode (marker o), Fig.3 shows that the propagation constant $\cosh\Gamma$ Fig.3 (left), the characteristic impedance Fig.3 (center), and the relative phase velocity Fig.3 (right), are very similar to the branched Helmholtz resonator solution (thin solid line). Conversely, at frequencies above F_a^+(given by $\cosh\Gamma' = 1$), the flute mode (marker $\diamondsuit$) is very close to the branched Helmholtz resonator solution (thin solid line). This small amount of aperture (8%) into the resonators wall is sufficient to reduce dramatically the stop band from $[F_h; F_a^+]$ (branched Helmholtz resonators) to $[F_b^-; F_a^+]$ (where F_b^- is given by $\cosh\Gamma = -1$). The stop band is indicated with black filled markers on Fig.3. Concerning the insertion loss (Fig.5), *the strongly inhomogeneous* lattice (thick line) differs significantly from the branched Helmholtz resonator case (thin solid line) only for frequencies below F_b^-. In particular within the stop band $[F_b^-; F_a^+]$, the two insertion losses are very close, and the *strongly inhomogeneous* lattice may probably be described as a locally reacting liner, at least for frequencies within or close to the stop band. This point might be further investigated.

If one continues to open the diaphragms (increasing σ_d), the cut-off frequency of the plane mode F_b^- increases, as opposed to F_a^+ which does not depend on σ_d. F_b^-tends to infinity for 100% opened diaphragms (homogeneous lattice). For an *almost homogeneous* lattice with an open area of the diaphragms that represents 80% of S_2 (that is $\sigma_d = 0.8$), F_b^- is situated above the maximum frequency under study $F_{max} = 3500$ Hz. Below F_{max} the system has no stop band. It exhibits two propagating eignemodes within $[F_a^+; F_{max}]$ (indicated with black filled markers on Fig.4), as for a homogeneous lattice (dashed and dash-dot lines). But contrary to a homogeneous lattice, even with diaphragms almost totally opened ($\sigma_d = 0.8$), the plane mode (marker o) is dispersive: $c_0/v_\phi(\omega)$ not is constant, see Fig.4 (right). The propagation constant and the relative phase velocity of the plane mode and the flute mode (marker $\diamondsuit$) are not close to each other. As a consequence, the resonant attenuation peaks of the insertion loss above F_a^+ observed for the homogeneous lattice (Fig.6 dash-dot line), due the longitudinal interference of two similar eigenmodes, are strongly reduced in the case of the *almost homogeneous* lattice (thick line).

6. Conclusion

The particular type of inhomogeneity (diaphragms) used here strongly affects the phase velocity of the plane eigenmode, and also modifies the stop band and two-modes band of the infinite lattice.

From a practical point of view, this approach illustrates how the inhomogeneity of the infinite lattice modifies the insertion loss of the lattice of finite length. It appears that globally, the insertion loss of both homogeneous lattice and branched Helmholtz resonators, is lowered when diaphragms are introduced.

As a consequence, even though diaphragms was a simple way to introduce parametrically the inhomogeneity within the lattice, it may not be a practical way to improve the performance of the lattice as an acoustic treatment. Nevertheless, lattice configurations including other types of inhomogeneity like for instance porous materials (described as equivalent viscothermal fluid) are in the scope of the method.

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